Nucleation of *p*-branes and fundamental strings

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We construct a solution to the low-energy string equations of motion in five dimensions that describes a circular loop of fundamental string exponentially expanding in a background electric H field. Euclideanizing this gives an instanton for the creation of a loop of fundamental string in a background H field, and we calculate the rate of nucleation. Solutions describing magnetically charged strings and p-branes, where the gauge field comes from Kaluza-Klein reduction on a circle, are also constructed. It is known that a magnetic flux tube in four (reduced) spacetime dimensions is unstable to the pair creation of Kaluza-Klein monopoles. We show that in $(4+p)$ dimensions, magnetic $(p+1)$ ''flux-branes'' are unstable to the nucleation of a magnetically charged spherical p-brane. In ten dimensions the instanton describes the nucleation of a Ramond-Ramond magnetically charged six-brane in type IIA string theory. We also find static solutions describing spherical charged p-branes or fundamental strings held in unstable equilibrium in appropriate background fields. Instabilities of intersecting magnetic flux-branes are also discussed. [S0556-2821(96)02012-7]

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I. INTRODUCTION

Solitons have played an important role in several recent developments in string theory. In particular, they appear to be a key to understanding various nonperturbative aspects of the theory [1–4]. Surprising connections have been found between string states and black holes and between strings and higher-dimensional extended objects, p-branes. There are indications that these objects all play a fundamental role in the theory.

It has been shown that localized solitons such as monopoles can be pair created in a background magnetic field [5]. Recently, there has been considerable interest in the analogous process involving gravity: the pair creation of charged black holes in background electromagnetic fields and by breaking cosmic strings [6–11]. The question naturally arises as to whether extended objects such as p-branes and fundamental strings can also be produced quantum mechanically in appropriate background fields. The special case of p-branes coupled to a, cosmological-constant-inducing, $(p+1)$-form potential in $(p+2)$ spacetime dimensions was previously discussed in [12]. The nucleation of vortex loops, local and global, has also been investigated in four dimensions (see [13] and references therein).

We will present a solution to the low-energy string equations of motion that describes a finite loop of fundamental string in five spacetime dimensions expanding in a background electric-type $H_{\mu\nu\rho}$ field. Analytically continuing this solution yields an instanton corresponding to the nucleation of a single loop of fundamental string. We also find related solutions in $(p+4)$ spacetime dimensions that describe spherical, magnetically charged p-branes expanding in a background magnetic field. Again, analytically continuing the expanding solution gives an instanton for the nucleation of a charged p-brane. Along the way we will construct static versions of the Lorentzian solutions: a loop of fundamental string or spherical magnetic p-brane held in unstable equilibrium in a background field. A further generalization results in solutions describing spherical uncharged branes of any odd (even) dimension either in static unstable equilibrium or expanding in background magnetic fields in odd (even) spacetime dimensions.

The construction of these solutions relies on three observations. We begin by considering Kaluza-Klein theory with a U(1) reduction from D spacetime dimensions to $D-1$. The first observation is that the spatial part of a basic Kaluza-Klein monopole [14,15] can be locally constructed by taking $\mathbf{R}^4$ and considering the U(1) isometry that simultaneously rotates the two orthogonal two-planes by the same angle. This acts freely except for a fixed point at the origin. Dividing out by this action gives us a configuration in three spatial dimensions that is, locally, the Kaluza-Klein monopole. Magnetically charged higher p-branes are given by multiplying this, locally, by p extra trivial directions so that the fixed point set of the induced rotation in $\mathbf{R}^{4+p}$ is the brane. p-branes that are not magnetically charged are formed locally by taking the quotient of $\mathbf{R}^{2k+p}$ by rotations that simultaneously rotate in $k\neq 2$ two-planes. The p-brane is the fixed point set of the rotation. Only in $(p+4)$ spacetime dimensions can the p-brane be charged with respect to the two-form Maxwell field F arising from a circle reduction. The general fixed point set analysis will be described in detail in Sec. II.

The second ingredient in our construction is the better understanding of the pair production of $D=5$ Kaluza-Klein monopoles in magnetic flux tubes that has recently been gained [9]. Remarkably, this process has been shown to be closely related to an instability of Kaluza-Klein magnetic

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fields analogous to that of the ordinary Kaluza-Klein vacuum described by Witten [16]. It has been found that not only the topology but also the metric of the instantons involved are the same in both cases. The topology is $\mathbf{R}^2\times S^3$, and the metric is that of the five-dimensional rotating black hole discovered by Myers and Perry [17]. The idea that emerged from this work is that the monopoles that are pair produced via this instanton arise as the fixed points of the U(1) isometry by which one divides out in performing the Kaluza-Klein reduction. Thus, to construct the higher-dimensional generalizations given here, we take higher-dimensional black holes and divide out by an appropriate U(1) action such that the fixed point sets are the desired p-branes.

Since these are solutions to the vacuum Einstein equations (with the gauge field arising from Kaluza-Klein reduction on an S^1), they are also solutions, to leading order in α', of low-energy string theories in less than ten dimensions when the compactification includes an S^1 factor. In addition, the Ramond-Ramond (RR) gauge field in type-IIA string theory in ten dimensions arises from dimensional reduction from 11 dimensions. We can thus construct an instanton describing the nucleation of a spherical six-brane carrying magnetic RR charge in this theory in ten dimensions.

The third ingredient in our construction is the observation that after dimensionally reducing a D-dimensional vacuum solution via a U(1) isometry to $D-1$ dimensions, one can apply a duality transformation which replaces the Maxwell two-form with a $(D-3)$-form field strength. This yields electric analogues of the magnetic p-branes. For the case $D=6$ (i.e., five reduced spacetime dimensions), the resulting action is precisely the low-energy string effective action involving the metric, dilaton, and three-form H. The duals of the magnetic strings (one-branes) turn out to be fundamental strings.

The layout of the paper is as follows. As we mentioned, Sec. II contains an analysis of the fixed points sets of general U(1) isometries in arbitrary dimensions. We also construct the closely related generalizations of the Melvin magnetic flux tube solution of $D=5$ Kaluza-Klein theory. These are thickened branes of magnetic flux or ''flux-branes,'' which are the appropriate backgrounds for nucleating p-branes. In Sec. III we review various properties of the five-dimensional Kaluza-Klein monopole, monopole-antimonopole pairs, and pair creation. In Sec. IV we present solutions describing spherical, magnetically charged p-branes expanding in magnetic flux-branes. The Euclidean sections of these solutions are the instantons for the nucleation of magnetically charged p-branes in magnetic flux-branes. We also present solutions describing static magnetic p-branes being held in unstable equilibrium by the flux-brane. In addition, we discuss related solitons that do not carry magnetic charge. This is extended in Sec. V to allow more general, and more physical, values of the magnetic field at infinity and we give the production rates for nucleating the charged branes. In Sec. VI we show how dualizing the $D=6$ magnetic string yields the fundamental string in five spacetime dimensions. Thus the duals of our magnetic string solutions describe a loop of fundamental string in static unstable equilibrium in an electric field and also a loop that is expanding in an electric field. The latter, when Euclideanized, is the instanton for the nucleation of a single loop of fundamental string. We calculate the rate for this process. Some concluding remarks are given in Sec. VII. In the Appendix we describe in more detail the calculation of the instanton actions.

Since manifolds of many different dimensions will abound, we will adhere to the convention that D refers to the dimension of a spacetime, d, to the dimension of a Euclidean manifold that is to be considered as a spatial section of spacetime, and that these will always refer to the dimensions of the unreduced geometry.

II. PROPERTIES OF HIGHER-DIMENSIONAL SYMMETRIES

In this section we will give some general results which will be useful later. The theory we start with is vacuum gravity in D dimensions with action, up to boundary terms, given by

$$S=\frac{1}{16\pi G_D}\int d^D x\sqrt{-g_D}R(g_D). \tag{2.1}$$

If a geometry ds_D^2 has a Killing vector $\partial/\partial x^D$ with closed orbits and

$$ds_D^2=\exp\left(-\frac{4}{\sqrt{D-2}}\phi\right)(dx^D+2A_\mu dx^\mu)^2 + \exp\left(\frac{4}{(D-3)\sqrt{D-2}}\phi\right)g_{\mu\nu}dx^\mu dx^\nu, \tag{2.2}$$

then the action can be reexpressed as

$$S=\frac{1}{16\pi G_{D-1}}\int d^{D-1}x\sqrt{-g}\left[R(g)-\frac{4}{D-3}(\boldsymbol{\nabla}\phi)^2 - \exp\left(-4\frac{\sqrt{D-2}}{D-3}\phi\right)F^2\right], \tag{2.3}$$

where $2\pi R G_{D-1}=G_D$ and R is the radius of the compactified dimension. The $(D-1)$-dimensional fields—the dilaton ϕ, gauge potential A_μ, and metric $g_{\mu\nu}$—can be read off from (2.2).

A. Classification of fixed point sets

If the isometry generated by the Killing vector above has fixed points, then ϕ diverges and the metric $g_{\mu\nu}$ will be singular at those points. Let us consider the general classification of fixed points of a U(1) isometry in a d-dimensional Riemannian manifold M. This is a straightforward generalization of the four-dimensional case, which was analyzed in [18]. Let $\mathbf{q}$ be the associated Killing field and consider the tensor

$$q_{\alpha\beta}\equiv q_{\alpha;\beta} \tag{2.4}$$

at a fixed point x where $\mathbf{q}=0$. By virtue of Killing's equation, $q_{\alpha\beta}$ is antisymmetric. Let V denote the kernel of $q_{\alpha\beta}$ and suppose $\dim V=p$. Then vectors in V are directions in the tangent space at the fixed point T_x, which are invariant under the action of the symmetry. Since the exponential map commutes with the symmetry action, it follows that there is a p-dimensional subspace of fixed points. One can show that this subspace is always totally geodesic. In four dimensions, the only possibilities are $p=0$ and $p=2$: The first case is called a ''nut'' and the second a ''bolt.'' In higher dimensions, there are clearly more possibilities. Notice that since the rank of a skew matrix must be even, there can be no isolated fixed points when d is odd. In particular, ''there are no nuts in 11 dimensions'' [19]. In general, when d is odd (even), p is odd (even).

The two-form $q_{\alpha\beta}$ determines an element of the Lie algebra so(d) or, equivalently, a U(1) subgroup of SO(d) that winds around a maximal torus. The windings are determined by the skew eigenvalues of $q_{\alpha\beta}$ in an orthonormal frame. There are at most $[d/2]$ such eigenvalues, where $[r]$ denotes the integer part of r. The eigenvalues must all be rationally related and so determine up to $[d/2]$ integers, n_i, some possibly zero, with no common factor. These integers can be viewed as the number of 2π rotations in different orthogonal two-planes in T_x induced by one orbit of the isometry.

Near a fixed point, M looks locally like $\mathbf{R}^d$ and we can analyze the character of the different actions by identifying the space and the tangent space T_x. Suppose the number of nonzero n_i is k. Then restricting to the $2k$ dimensions acted on by the rotation, we can write the metric as

$$ds^2=\sum_{i=1}^{k}\left(d\rho_i^2+\rho_i^2 d\varphi_i^2\right) \tag{2.5}$$

and

$$\mathbf{q}=\sum_i n_i \frac{\partial}{\partial\varphi_i}. \tag{2.6}$$

Introducing complex coordinates $\{Z^i\equiv\rho_i e^{i\varphi_i}\}$, we can write the circle action as the holomorphic action

$$(Z^1,\ldots,Z^k)\rightarrow(e^{in_1 y}Z^1,\ldots,e^{in_k y}Z^k), \tag{2.7}$$

where y, $0\leqslant y<2\pi$, parametrizes the U(1) subgroup.

The U(1) subgroup acts freely away from the isolated fixed point at $\rho_i=0\ \forall i$. It follows that topologically we have a principal U(1) fibration of the odd-dimensional $(2k-1)$-sphere given by

$$\sum_{i=1}^{k}\rho_i^2=\text{const.} \tag{2.8}$$

In the special case that all the $n_i=1$, this is the Hopf action, giving rise to the Hopf fibration

$$\begin{array}{ccc} S^1 & \stackrel{i}{\rightarrow} & S^{2k-1} \\ & & \downarrow\pi \\ & & \mathrm{CP}^{k-1}. \end{array} \tag{2.9}$$

For the case $k=2$, this is the familiar—from magnetic monopole theory—Hopf fibration of S^3 since $\mathrm{CP}^1\cong S^2$. The sign of n_i can be changed by changing the sign of φ_i; however, changing an odd number of signs changes the orientation of space and gives the ''anti-Hopf'' action.

If the n_i are not all equal to 1 or -1, then $\mathbf{q}$ still acts without fixed points on S^{2k-1}. However, in this case, the quotient metric will have conical singularities. This can be illustrated by the simplest example $k=2$:

$$ds^2=d\rho_1^2+\rho_1^2 d\varphi_1^2+d\rho_2^2+\rho_2^2 d\varphi_2^2. \tag{2.10}$$

Consider the Killing vector

$$\mathbf{q}=n_1\frac{\partial}{\partial\varphi_1}+n_2\frac{\partial}{\partial\varphi_2}, \tag{2.11}$$

and let $\varphi'=\varphi_1$ and $\varphi=\varphi_2-(n_1/n_2)\varphi_1$, which is constant along the orbits of $\mathbf{q}$. Then $\mathbf{q}=n_1(\partial/\partial\varphi')$ and when we reduce and also restrict the metric to the $\rho_1^2+\rho_2^2=1$ surface by setting $\rho_1=\cos\theta$ and $\rho_2=\sin\theta$, we obtain

$$ds_{\mathrm{II}}^2\propto d\theta^2+\frac{\sin^2\theta\cos^2\theta}{\cos^2\theta+(n_1^2/n_2^2)\sin^2\theta}\,d\varphi^2. \tag{2.12}$$

The range of θ is $0\leqslant\theta\leqslant\pi/2$, and the condition that there be no conical singularities at $\theta=0,\pi/2$ is that φ has period $2\pi=2\pi|n_1/n_2|$. Since n_1 and n_2 are coprime, this condition cannot be satisfied unless $n_1^2=n_2^2=1$.

For a Lorentzian manifold, the general classification of fixed point sets is more complicated. The main difficulty is that one cannot always bring the generator of rotations to block diagonal form. Consider so(2,1), for example. Any nonvanishing skew 3×3 matrix has a one-dimensional kernel. The kernel may be timelike, spacelike, or null. In the first two cases, one has a rotation or boost, respectively. These cases admit a block diagonal form with one block the 1×1 zero matrix and the complementary block in the orthogonal two-plane a skew 2×2 matrix. If the kernel is timelike, one has a conventional axis of rotational symmetry. If the kernel is spacelike, the fixed point set is locally like the boost-invariant Boyer axis of a black hole. If the kernel is null, however, corresponding to a so-called null rotation, this reduction cannot be done because there is no uniquely defined orthogonal two-plane. However, it remains true that even for Lorentzian metrics the fixed point sets will be totally geodesic surfaces. Since these fixed points are often located at the center of a soliton, it follows that the soliton obeys the equations of motion of a ''fundamental'' p-brane. This will be true, in particular, for all the p-branes discussed later.

B. Flux-branes

It was shown in [7–9] that a uniform magnetic field in four spacetime dimensions, a generalization of the Melvin solution of Einstein-Maxwell theory [20], can be obtained by a dimensional reduction of a five-dimensional geometry which is flat. This five-dimensional spacetime is obtained by starting with five-dimensional Minkowski spacetime M^5 and identifying points under a combined spatial translation and rotation. When the rotation is zero, one obtains the standard Kaluza-Klein vacuum. When it is nonzero, the field configuration in the reduced space (in which form it was originally discovered [21]) is that of an infinitely long straight magnetic flux tube. The generalization to magnetic ''flux-branes'' in higher dimensions is straightforward. Since time plays no role in the construction, we start with d-dimensional Euclidean space. Higher-dimensional generalizations of the $d=4$

case are obtained by identifying points under an element of the Euclidean group which acts without fixed points on $\mathbf{R}^d$. The question is simply to characterize such elements.

The general element of the Lie algebra of the Euclidean group $e(d)$ is a pair consisting of a translation and a rotation with infinitesimal parameters given by v_α and $\omega_{\alpha\beta}$, respectively. To obtain a nontrivial gauge field, we need $\omega_{\alpha\beta}\neq 0$, and to avoid fixed points we require that the equation

$$\omega\cdot x+v=0 \tag{2.13}$$

have no solution. This is the case if and only if the kernel of ω is nonzero and v has a component in the kernel. Under a change of origin in $\mathbf{R}^d$ by an amount a_α, ω is unchanged, but v changes to $\tilde{v}$ given by

$$\tilde{v}=v+\omega\cdot a. \tag{2.14}$$

One may always choose a so that the translation $\tilde{v}$ lies entirely in the kernel of ω. This means that the general Euclidean motion consists of a translation combined with a rotation in an orthogonal hyperplane.

Since the translation part of the symmetry just fixes the scale of the internal direction, it follows that the classification of these magnetic fields reduces to the classification of rotations in $\mathbf{R}^{d-1}$. A rotation is given by a two-form with $[(d-1)/2]$ skew eigenvalues, and so a magnetic field configuration is specified by $[(d-1)/2]$ real numbers B_i.

We can ask what these field configurations look like in the reduced spacetime. Consider $d=2m+1$ (the d=even case is similar):

$$ds^2=\sum_{i=1}^{m}(d\rho_i^2+\rho_i^2 d\varphi_i^2)+dy^2. \tag{2.15}$$

We identify each point with the point obtained by moving a distance $2\pi R$ along the integral curves of the Killing field,

$$\mathbf{q}=\frac{\partial}{\partial y}+\sum_i B_i\frac{\partial}{\partial\varphi_i}. \tag{2.16}$$

Introducing the new coordinates $\tilde{\varphi}_i=\varphi_i-B_i y$, which are constant along the orbits of $\mathbf{q}$, we find that $\mathbf{q}=\partial/\partial y$ and the above identification just consists of making y periodic with period $2\pi R$ at fixed $\tilde{\varphi}_i$. The flat metric (2.15) now takes the form

$$ds^2=\Lambda\left[dy+\frac{1}{\Lambda}\sum_i B_i\rho_i^2 d\tilde{\varphi}_i\right]^2+\sum_i(d\rho_i^2+\rho_i^2 d\tilde{\varphi}_i^{\,2})-\Lambda^{-1}\left(\sum_j B_j\rho_j^2 d\tilde{\varphi}_j\right)^2, \tag{2.17}$$

where

$$\Lambda=1+\sum_i B_i^2\rho_i^2. \tag{2.18}$$

The reduced metric, dilaton, and gauge potential can be read off from (2.17) (after adding an extra time direction) using (2.2) with $D=d+1$ to give

$$ds_{D-1}^2=\Lambda^{1/(D-3)}\left[-dt^2+\sum_i(d\rho_i^2+\rho_i^2 d\tilde{\varphi}_i^{\,2})-\Lambda^{-1}\left(\sum_j B_j\rho_j^2 d\tilde{\varphi}_j\right)^2\right],$$
$$\exp\left(-\frac{4}{\sqrt{D-2}}\phi\right)=\Lambda,\quad A=\frac{1}{2\Lambda}\sum_i B_i\rho_i^2 d\tilde{\varphi}_i. \tag{2.19}$$

When only one B_i is nonzero, this is a thickened brane of magnetic flux, or ''flux-brane,'' of dimension $(d-3)$ and was found by Gibbons and Maeda [21]. The amount of flux passing through a one-dimensional loop, γ, is given by integrating the one-form potential A around the loop [note that in $(d-1)$ spatial dimensions a circle surrounds a $(d-3)$-brane]: $\text{flux}=\int_\gamma A$. Each nonzero parameter B_i adds another orthogonal flux-brane to the configuration. The gauge field strength is maximized at the intersection of the centers of the flux-branes, which is the fixed point set of the rotational part of $\mathbf{q}$. When k of the parameters are nonzero, this is a $(d-2k-1)$-hyperplane. The generic configuration in the $(d-1)$-dimensional reduced space is a set of $[(d-1)/2]$ orthogonal $(d-3)$-flux-branes. These intersect in a point when d=odd and in a line when d=even.

III. FIVE-DIMENSIONAL MONOPOLES

In this section we shall review some of the geometrical and topological properties of the basic Kaluza-Klein monopole [14,15], relating them to the fixed point set analysis of the previous section. We also present a novel interpretation of the four-dimensional Euclidean Schwarzschild solution.

A. Basic monopole

The single-monopole solution is a five-dimensional spacetime which is a metric product $\mathbf{R}\times M$ of a time factor with coordinate t and Euclidean Taub-Newman-Unti-Tamburino (NUT) space M. M is Ricci flat, self-dual, topologically $\mathbf{R}^4$, and admits an isometric circle action. The metric is, explicitly,

$$ds^2=-dt^2+V^{-1}(dy+2A_\varphi d\varphi)^2+V(dr^2+r^2 d\Omega), \tag{3.1}$$

with

$$A_\varphi=2m(1-\cos\theta),\quad V=1+\frac{4m}{r}. \tag{3.2}$$

The period of y is $2\pi R$ with $R=8m$. The Killing vector associated with the U(1) isometry is $\mathbf{q}=\partial/\partial y$. If the circle action were free, the topology of M would be that of a, possibly twisted, circle bundle

$$\begin{array}{ccc} S^1 & \rightarrow & M \\ & & \downarrow \\ & & \Sigma \end{array} \tag{3.3}$$

over some complete nonsingular three-manifold Σ. One could then think of the S^1 factor globally as an internal manifold. Because of the fixed point $r=0$ at which g_{yy} vanishes and hence the length of the circle fibers goes to zero, this description is only valid away from $r=0$.

At the center of the monopole, the manifold is smooth and locally indistinguishable from the flat metric on $\mathbf{R}^4$. As discussed in the previous section, at a fixed point, the circle action may be thought of as a rotation in two orthogonal two-planes in $\mathbf{R}^4$, characterized by two integers n_1 and n_2. For a single Kaluza-Klein monopole, we have $n_1=n_2=1$, i.e., the Hopf action on small S^3's surrounding the center. The reduced four-dimensional spacetime is thus free of singularities except at the center of the monopole.

Antimonopole solutions are given by (3.1) with the opposite sign of A_φ. Now the U(1) action near the antipole center is labeled by $n_1=-n_2=1$: It is the anti-Hopf action.

B. Static monopole-antimonopole pairs

It is interesting to ask what the topology of a monopole-antimonopole configuration would be. Physically, one would not expect a static asymptotically vacuum solution since the pair will attract, and so one must either give up the asymptotic vacuum condition or suspend the field equations. In both cases the topology should be the same. We will argue that the topology is in fact $\mathbf{R}^2\times S^2$. Let the spatial four-manifold be M; then, $M=A\cup B\cup C$, where A and B are both four-balls D^4 corresponding to the monopole and antimonopole and C is the nontrivial U(1) bundle over $\mathbf{R}^3\sharp D^3\sharp D^3$ ($\mathbf{R}^3$ with two three-balls removed) which has zero winding over the sphere at infinity and windings $+1$ and -1 over the other two S^2 boundaries [22]. Since the bundle is trivial over the sphere at infinity, we can add in an S^1 there: $M\cup S^1=A\cup B\cup C'$, where C' is a U(1) bundle over $S^3\sharp D^3\sharp D^3=S^2\times D^1$, where the one-ball D^1 is just the one-dimensional interval. The U(1) must have unit twist over one S^2 boundary and unit antitwist over the other S^2 boundary. So $C'=S^3\times D^1$ which is $C'=S^4\sharp D^4\sharp D^4$. So $M\cup S^1=S^4$ and $M=S^4-S^1=\mathbf{R}^2\times S^2$.

Thus we see that the monopole-antimonopole manifold has the same topology as the four-dimensional Euclidean Schwarzschild solution:

$$ds^2=\left(1-\frac{2m}{r}\right)d\tau^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{3.4}$$

where τ has period $2\pi R$, $R=4m$. Indeed, we will see that in a certain precise sense (3.4) can be regarded as containing a monopole-antimonopole pair. First, if we add on a trivial time direction, we can regard (3.4) as a static five-dimensional solution describing a minimal two-sphere in space, a "bubble," poised in unstable equilibrium. If we reduce this five-dimensional solution to four dimensions along the orbits of the Killing vector $\partial/\partial\tau$, we find that the minimal two-sphere or bolt, being a fixed point set of that circle action, looks singular in four dimensions.

Consider now the alternative Killing field

$$\mathbf{q}=\frac{\partial}{\partial\tau}+\frac{1}{R}\frac{\partial}{\partial\varphi}. \tag{3.5}$$

It has fixed points at the north and south poles, $\theta=0,\pi$ of the two-sphere $r=2m$. It can be shown that at one, the action is the Hopf action, and at the other it is the anti-Hopf action. At each pole, the two orthogonal planes in which the rotations act are the tangent spaces to the (r,τ)-section at the horizon ("tip of the cigar") and the (θ,φ) horizon two-sphere. Near infinity, however, $\mathbf{q}$ becomes a linear combination of a *translation* and a rotation. As discussed in Sec. II B, a magnetic field in Kaluza-Klein theory is obtained by taking the quotient of flat space with respect to precisely this type of symmetry. If we therefore take the closed orbits of $\mathbf{q}$ as our internal Kaluza-Klein circles, we can interpret (3.4) in four dimensions as a static monopole-antimonopole pair held apart by a background magnetic field with $B=1/R$. One would expect such a configuration to be unstable, and it is: The negative mode of the four-dimensional Euclidean Schwarzschild solution gives rise to an exponentially growing mode of this five-dimensional solution.

C. Dynamical monopole-antimonopole pairs and pair creation

A key observation of [9] was that one may relate the *five*-dimensional Schwarzschild solution to monopole-antimonopole pair production.[1] Consider the five-dimensional Euclidean Schwarzschild solution, which we write as

$$ds^2=\left[1-\left(\frac{r_H}{r}\right)^2\right]d\tau^2+\left[1-\left(\frac{r_H}{r}\right)^2\right]^{-1}dr^2+r^2[d\alpha^2+\cos^2\alpha(d\theta^2+\sin^2\theta\, d\varphi^2)], \tag{3.6}$$

where τ has period $2\pi R$, with $R=r_H$, and $-\pi/2\leqslant\alpha\leqslant\pi/2$. From the purely five-dimensional point of view, this is an instanton that describes the decay of $M^4\times S^1$, where M^4 is four-dimensional Minkowski space. It has topology $\mathbf{R}^2\times S^3$, and it asymptotically approaches flat $\mathbf{R}^4\times S^1$. It has a zero momentum slice $\alpha=0$, which has topology $\mathbf{R}^2\times S^2$ and contains a minimal two-sphere or "bubble," $r=r_H$: Overall, the zero-momentum slice is very similar to the four-dimensional Euclidean Schwarzschild solution. The subsequent Lorentzian post-decay evolution is obtained by setting $\alpha=it$ in (3.6) and describes the minimal two-sphere expanding. Its area increases like $\cosh^2 t$, and this solution looks like a dynamical version of the static bubble of the previous subsection.

If one reduces along $\partial/\partial\tau$, one may think of this as an instanton for the decay of the vacuum in four dimensions [16]. The fixed point set restricted to the zero-momentum slice is the entire minimal two-sphere or bubble which appears singular in the reduced spacetime. However, one may alternatively reduce along $\mathbf{q}$ with $R=r_H$ in (3.5). In four dimensions, in this case, the instanton describes the decay of a magnetic field via pair creation of a monopole-antimonopole pair. We can see this by subjecting the zero-momentum slice to the fixed point set analysis of the previ-

[1]To obtain arbitrary values of the magnetic field at infinity, one should consider the five-dimensional Kerr solution. We will return to this in Sec. V, but for now, we illustrate the construction using the simpler Schwarzschild solution.

ous subsection. The pole and antipole are the fixed points of the circle action of **q**, and the twisting character of **q** at infinity means the particles are immersed in a magnetic field of strength $B=1/R$. Now, however, in the subsequent Lorentzian evolution, the particles do not stay in their static positions, but accelerate apart.

IV. HIGHER-DIMENSIONAL GENERALIZATIONS

A. Flat charged *p*-branes

The simplest generalization of the basic Kaluza-Klein monopole is to take the product of (3.1) with an arbitrary number p of flat directions before doing the Kaluza-Klein reduction. This gives a magnetically charged p-brane in $(p+4)$ spacetime dimensions, e.g., a magnetically charged string in five dimensions. If the extra dimensions are infinite, the p-brane is also infinite. (We could, of course, consider the extra dimensions to be a torus, in which case the brane is also a torus, but this would change the topology of the reduced spacetime. Moreover, if the torus was large, this solution would approach the infinite-brane, and if it were small, it would reduce, in a Kaluza-Klein sense, to the monopole again.)

An obvious instanton describing the production of a pair of these p-branes in a magnetic field is obtained by taking the product of the five-dimensional Euclidean Schwarzschild solution with the extra flat dimensions and reducing via (3.5). The asymptotic magnetic field configuration of this instanton is a $(p+1)$-dimensional flux-brane. However, if the extra dimensions are infinite, then the action for this instanton is infinite, even relative to the background magnetic field.

B. Spherical charged *p*-branes

It might appear that the $(p+1)$-dimensional flux-brane cannot decay because the instanton that describes pair creation of infinite magnetic p-branes has infinite action. This is, in fact, not the case. We will see in this section that it can decay by the nucleation of a single p-brane with topology S^p.

We start with $p=1$ as an example and first describe a magnetically charged loop of string in static equilibrium in a background magnetic field. As discussed in Sec. II, a magnetic field can be obtained by taking six-dimensional Minkowski spacetime,

$$ds^2=-dt^2+d\tau^2+dr^2+r^2d\varphi^2+dx_idx^i, \tag{4.1}$$

and identifying points by moving a distance $2\pi R$ along the Killing vector

$$\mathbf{q}=\frac{\partial}{\partial\tau}+\frac{1}{R}\frac{\partial}{\partial\varphi}. \tag{4.2}$$

The reduced spacetime describes a two-dimensional magnetic flux-brane with $B=1/R$.

To obtain a charged loop of string in this background, we start with the product of time and the five-dimensional Euclidean Schwarzschild solution

$$ds^2=-dt^2+\left[1-\left(\frac{r_H}{r}\right)^2\right]d\tau^2+\left[1-\left(\frac{r_H}{r}\right)^2\right]^{-1}dr^2+r^2d\Omega_3. \tag{4.3}$$

The metric on the three-spheres can be written[2]

$$d\Omega_3=d\theta^2+\sin^2\theta\, d\varphi^2+\cos^2\theta\, d\chi^2, \tag{4.4}$$

where $0\leqslant\theta\leqslant\pi/2$, $0\leqslant\varphi$, $\chi\leqslant2\pi$. We now reduce down to five dimensions using the symmetry (4.2). The fixed points of this Killing field are at $r=r_H$, $\theta=0$, which is clearly a circle parametrized by χ. For each value of χ, the indices of the symmetry at the fixed point (as discussed in Sec. II) are the same as the usual Kaluza-Klein monopole. This shows that in the reduced spacetime there is a magnetic charge at each fixed point on the string. The solution thus describes a circular charged string. Asymptotically, the solution approaches the two-dimensional flux-brane.

To be explicit, we set $\tilde{\varphi}=\varphi-(\tau/r_H)$ in (4.3) [so $\tilde{\varphi}$ is constant along the orbits of the symmetry (4.2)]. We then compare the resulting metric with (2.2), setting $D=6$ and $x^D=\tau$. The result is

$$e^{-2\phi}=1-\left(\frac{r_H}{r}\right)^2+\left(\frac{r}{r_H}\right)^2\sin^2\theta,\quad A=\frac{r^2\sin^2\theta}{2r_H}e^{2\phi}d\tilde{\varphi}, \tag{4.5}$$

$$ds^2=e^{-2\phi/3}\left(-dt^2+\left[1-\left(\frac{r_H}{r}\right)^2\right]^{-1}dr^2+r^2(d\theta^2+\cos^2\theta\, d\chi^2)\right)+e^{4\phi/3}\sin^2\theta(r^2-r_H^2)d\tilde{\varphi}^2.$$

Near $r=r_H$, the metric takes the form

$$ds^2\approx e^{-2\phi/3}[-dt^2+d\tilde{r}^2+r_H^2(d\theta^2+\cos^2\theta\, d\chi^2)]+e^{4\phi/3}\tilde{r}^2\sin^2\theta\, d\tilde{\varphi}^2, \tag{4.6}$$

where $\tilde{r}^2=r^2-r_H^2$ and

$$e^{-2\phi}\approx\sin^2\theta+\left(\frac{\tilde{r}}{r_H}\right)^2(1+\sin^2\theta). \tag{4.7}$$

The singularity $\tilde{r}=0$, $\theta=0$ is a ring representing the loop of string. Notice that $\tilde{r}=0$, $0<\theta\leqslant\pi/2$ is a regular two-dimensional surface spanning the ring singularity.

We now discuss the nucleation of a closed magnetically charged string in our two-dimensional magnetic flux-brane. The appropriate instanton is the six-dimensional Euclidean Schwarzschild solution

$$ds^2=\left[1-\left(\frac{r_H}{r}\right)^3\right]d\tau^2+\left[1-\left(\frac{r_H}{r}\right)^3\right]^{-1}dr^2+r^2(d\alpha^2+\cos^2\alpha\, d\Omega_3), \tag{4.8}$$

where τ has period $2\pi R$, with $R=2r_H/3$, and $-\pi/2\leqslant\alpha\leqslant\pi/2$. By direct analogy with the five-dimensional instanton discussed in the previous section on the basic

[2]To see this, start with (2.5) and (2.8) with $k=2$. Then set $\rho_1=\sin\theta$, $\rho_2=\cos\theta$.

monopole, this instanton has a zero-momentum surface $\alpha=0$ which contains a minimal three-sphere or ''bubble.'' The subsequent Lorentzian evolution is obtained by setting $\alpha=it$. The minimal three-sphere expands exponentially in t. Reducing (4.8) along $\partial/\partial\tau$, we obtain an instanton describing vacuum decay in this $D=6$ Kaluza-Klein theory. If we reduce (4.8) along the symmetry (4.2), we obtain an instanton which is asymptotically a magnetic flux-brane of strength $B=1/R$. Restricted to the surface $\alpha=0$, the reduction is virtually identical to the one described in the static case above. In particular, one has a charged loop of string. In the subsequent Lorentzian evolution, the loop expands since it lies on the expanding bubble.

For the Schwarzschild instanton, the single parameter R (or r_H) governs both the strength of the asymptotic magnetic field and the charge on the string. We will see in the next section that this value of the magnetic field, $B=1/R$, is unphysically large and we will construct solutions corresponding to the decay of more physical values of B by considering instantons based on higher-dimensional rotating black hole solutions. We will also calculate the rate of nucleation in the semiclassical approximation.

One might wonder why the nucleation of a charged loop of string does not violate charge conservation. The point is that in four (reduced) spatial dimensions, the total magnetic charge of any localized object must be zero. This is because the magnetic charge is obtained by integrating $F_{\mu\nu}$ over a two-sphere. One cannot integrate $F_{\mu\nu}$ over the three-sphere at infinity. This does not contradict the fact that locally the string carries a magnetic charge. The magnitude of the charge q at the fixed points is $R/4$, where R is determined by the periodicity in the compact direction. However, the sign of the charge depends on a choice of orientation. If one charge is chosen to be $+q$, the opposite one is necessarily $-q$. This follows from the fact that the orientation induced on the two-sphere enclosing a point of the string depends on the tangent vector to the string, which points in the opposite direction halfway around the loop. If one takes a slice through the Lorentzian solution, it describes oppositely charged monopoles accelerating apart in a magnetic field. More generally, any p-brane, $p\neq 0$, which locally carries a magnetic charge associated with any r-form, F_r, must have zero net charge when the p-brane is confined to a compact region. The reason is simply that the sphere at infinity will have dimension $(p+r)$, which is larger than the rank of F_r.

The construction of magnetically charged spherical p-branes for $p>1$ is a straightforward extension of these ideas to higher dimensions. Taking the quotient of $(p+5)$-dimensional Minkowski spacetime by the symmetry (4.2) yields a $(p+1)$-dimensional magnetic flux-brane. This can support a static charged spherical p-brane as follows. Consider the Euclidean Schwarzschild solution in $d=(p+4)$ dimensions crossed with a trivial time direction,

$$ds^2=-dt^2+\left[1-\left(\frac{r_H}{r}\right)^{p+1}\right]d\tau^2 +\left[1-\left(\frac{r_H}{r}\right)^{p+1}\right]^{-1}dr^2+r^2 d\Omega_{p+2}, \qquad (4.9)$$

where τ is a periodic coordinate with period $2\pi R$ and $R=2r_H/(p+1)$. The metric on the $(p+2)$-spheres can be written

$$d\Omega_{p+2}=d\theta^2+\sin^2\theta\, d\varphi^2+\cos^2\theta\, d\Omega_p, \qquad (4.10)$$

with $0\leqslant\theta\leqslant\pi/2$. We now reduce along the symmetry (4.2). The fixed points again occur at $r=r_H$ and $\theta=0$ so that on a static slice the set of fixed points is now a sphere S^p. At each fixed point, the behavior of the symmetry in the four directions orthogonal to the S^p is again exactly that of a Kaluza-Klein monopole. So we obtain a static charged spherical p-brane.

To nucleate such a p-brane, we start with the $D=(p+5)$ dimensional Euclidean Schwarzschild solution

$$ds^2=\left[1-\left(\frac{r_H}{r}\right)^{p+2}\right]d\tau^2+\left[1-\left(\frac{r_H}{r}\right)^{p+2}\right]^{-1}dr^2 +r^2(d\alpha^2+\cos^2\alpha\, d\Omega_{p+2}), \qquad (4.11)$$

with $d\Omega_{p+2}$ given by (4.10). Reducing this via (4.2), we obtain an instanton for the nucleation of the p-brane. The subsequent Lorentzian evolution is described by the analytic continuation $\alpha=it$, and in this Lorentzian spacetime the fixed point set of (4.2) is the world volume of a spherical charged p-brane that exponentially expands.

There are many instantons which asymptotically approach the same magnetic field. One can start with the product of a $(p+5-m)$-dimensional Schwarzschild solution and $\mathbf{R}^m$. To maintain the same asymptotic magnetic field, we always reduce under (4.2). These instantons describe the nucleation of charged p-branes with topology $\mathbf{R}^m\times S^{p-m}$. However, the action for all of these instantons is infinite as a result of the infinite volume of $\mathbf{R}^m$.

An interesting application of this construction is to the type-IIA string. The low-energy action of this theory in ten dimensions contains a Ramond-Ramond gauge field which comes from Kaluza-Klein reduction of an 11-dimensional metric. The Kaluza-Klein six-brane we have been considering thus carries RR charge [23]. Starting with the $D=11$ Minkowski space and taking the quotient under (4.2), one obtains a seven-dimensional magnetic flux-brane. This is unstable to the nucleation of a spherical, magnetically charged six-brane. The appropriate instanton is simply the $D=11$ Schwarzschild solution.

C. Uncharged p-branes

One can also construct static p-branes in $d-1$ reduced spatial dimensions where $p+4<d$. These do not carry magnetic charge, but they can arise when flux-branes intersect. As described in Sec. II, intersecting flux-branes are obtained by taking the quotient of Minkowski spacetime under a symmetry which is a translation plus a rotation, where the rotation is not restricted to lie in a single two-plane. We first consider static solitons and then discuss how one can nucleate such objects.

We begin with the product of time and the d-dimensional Euclidean Schwarzschild metric. To describe k orthogonal flux-branes asymptotically, we write the metric on the $(d-2)$-spheres in terms of θ_i,φ_i, $i=1,\dots,k$, and coordinates on a $(d-2-2k)$-sphere by iterating (4.10) k times. We then reduce along the symmetry

$$\mathbf{q}=\frac{\partial}{\partial\tau}+\frac{1}{R}\sum_{i=1}^{k}\frac{\partial}{\partial\varphi_i}, \tag{4.12}$$

where τ has period $2\pi R$. Asymptotically, the solution resembles (2.19) with k nonzero B_i's and describes k orthogonal flux-branes. Recall that in $d-1$ reduced spatial dimensions each flux-brane has dimension $(d-3)$ and k of them will intersect in a surface of dimension $(d-1-2k)$. The vector $\mathbf{q}$ has fixed points at $r=r_H$ and $\theta_i=0$ for all i. This is a spherical $(d-2-2k)$-brane which lies in the intersection of the fluxbranes. For $k\neq 1$, this brane does not carry magnetic charge in the reduced spacetime since it does not have the right dimension. Nevertheless, it is a static, though unstable, localized excitation of the fields.

The configuration of k intersecting fluxbranes is unstable to the nucleation of such uncharged $(d-2-2k)$-branes. The instanton is the $(d+1)$-dimensional Euclidean Schwarzschild solution and the analytic continuation is identical to that for the case of the charged branes. The only difference is that one now reduces along the more general symmetry (4.12).

If $k=1$, this instanton construction reduces, of course, to the one in Sec. IV B. There are two other special values of k of particular interest. If d is even and $k=(d-2)/2$ (the maximal number) then the fixed point set consists of two points. On a small sphere surrounding each point, the symmetry acts like the Hopf fibration (2.9), of S^{d-1} by S^1. Bais and Batenberg [24] have constructed a Ricci-flat space containing a single object of this type. It can be viewed as a generalization of the Taub-NUT solution to higher (even) dimensions. Topologically, the manifold is simply $\mathbf{R}^d$. The metrics admit a circle action, with a single fixed point. Although the length of the circle orbits tends to a constant near infinity, the Bais-Batenberg solutions for $d>4$ cannot be regarded as circle bundles over $\mathbf{R}^{d-1}$ asymptotically. This is because the angular part of the reduced space is not S^{d-2}, but $\mathrm{CP}^{(d-2)/2}$. Thus one could argue that these objects cannot exist in isolation. However, we have seen that they can exist in pairs and can, in fact, be pair created.

The second special case occurs when d is odd and k is again maximal, $k=(d-1)/2$. In this case, the symmetry (4.12) has *no* fixed points. It corresponds to a combination of $\partial/\partial\tau$ and a Hopf rotation of the $(d-2)$-sphere. The reduction then leads to a nonsingular solution of the Einstein-Maxwell-dilaton equations arising from the action (2.3).

V. DIFFERENT ASYMPTOTIC MAGNETIC FIELD VALUES

The solutions we have constructed from the Schwarzschild metric have values of the magnetic fields at infinity which are fixed completely by the radius of the compactified dimension, $B=1/R$. We should point out that this value of B is actually unphysically large in the following sense. Consider the single-flux-brane solution with parameter B. The radius of the compactified direction is not constant, but grows from R to infinity as the distance from the center increases. In order for the Kaluza-Klein reduction to make sense and also for the configuration to be a reasonable approximation of a uniform magnetic field, we should only consider distances from the center of the flux-brane, $\rho<1/B$. This restriction occurs not only in Kaluza-Klein theory, but also when considering pair creation of black holes in magnetic fields in Einstein-Maxwell theory in four dimensions. However, in Kaluza-Klein theory we also have the condition that only distances large with respect to the radius of compactification should be considered in the reduced spacetime. Thus $\rho\gg R$, and so we have the condition $BR\ll 1$.

The value of the asymptotic magnetic fields we have been considering so far is well outside this physical range of validity. We have allowed them up until now for reasons of simplicity: The construction of the solutions describing the spherical charged p-branes using the Schwarzschild metrics is simpler, though qualitatively the same as the construction we will now give of solutions describing the same types of branes, but with arbitrary values of the background magnetic field. We wanted to describe the construction in a simpler setting before giving the more complicated solutions that are of more physical interest. These are given by U(1) reductions of higher-dimensional *rotating* black hole solutions.

Myers and Perry found analogues of the Lorentzian Kerr solution for arbitrary spacetime dimension N [17]. A particular case in $N=8$ dimensions has been obtained independently by Chakrabarti [25] using the special properties of the octonions. As noted by Myers and Perry, a rotating body in $N-1$ spatial dimensions has an associated angular momentum which may be thought of as a two-form ω. Thus there are $[(N-1)/2]$ rotation parameters a_i. These, together with the mass, characterize the solutions. If all the parameters a_i vanish, then one obtains the usual higher-dimensional Schwarzschild solution. One expects that, just as in four spacetime dimensions, the solutions are unique, but to our knowledge there is no proof. The general solution has continuous isometry group $\mathbf{R}\times\mathrm{SO}(2)^{[(N-1)/2]}$, but as more of the parameters become zero, the isometry group is enhanced, becoming $\mathbf{R}\times\mathrm{SO}(N-1)$ in the nonrotating Schwarzschild limit. The case obtained by Chakrabarti has $a_1=a_2=a_3$ and is an example of slightly enhanced symmetry (there are extra discrete symmetries). In the general case one therefore has $[(N-1)/2]$ ignorable azimuthal coordinates φ_i parametrizing the maximal torus of $\mathrm{SO}(N-1)$ and which may be thought of near infinity as rotations in $[(N-1)/2]$ orthogonal two-planes in $\mathbf{R}^{N-1}$.

The Euclidean solutions, for which $a_i=i\alpha_i$ with α_i real, are complete and nonsingular provided that a suitable periodic identification is made. If we denote the Euclidean time by $\tau=it$, then one identifies points by moving a distance $2\pi R$ along the integral curves of the Killing field

$$\mathbf{q}=\frac{\partial}{\partial\tau}+\sum_i\Omega_i\frac{\partial}{\partial\varphi_i}, \tag{5.1}$$

where $R=1/\kappa$ and κ and $i\Omega_i$ are the surface gravity and angular velocities, respectively. General expressions for them may be found in [17].

To construct solutions describing spherical charged p-branes, it suffices to consider the metrics with only one angular momentum parameter nonzero. The N-dimensional Euclidean metric then takes the form

$$ds^2=\left(1-\frac{\mu}{r^{N-5}\Sigma}\right)d\tau^2-\frac{2\mu\alpha\sin^2\theta}{r^{N-5}\Sigma}d\tau\, d\varphi+\frac{\Sigma}{r^2-\alpha^2-\mu r^{5-N}}dr^2+\Sigma\, d\theta^2+\frac{\sin^2\theta}{\Sigma}[(r^2-\alpha^2)\Sigma-\mu r^{5-N}\alpha^2\sin^2\theta]d\varphi^2+r^2\cos^2\theta\, d\Omega_{N-4}, \tag{5.2}$$

where $\Sigma=r^2-\alpha^2\cos^2\theta$ and $0\leqslant\theta\leqslant\pi/2$. The horizon is located at $r=r_H$, where

$$r_H^2=\alpha^2+\frac{\mu}{r_H^{N-5}}. \tag{5.3}$$

The radius of the circle at infinity is

$$R=\frac{1}{\kappa}=\frac{2\mu r_H^{6-N}}{(N-3)r_H^2-(N-5)\alpha^2}, \tag{5.4}$$

while the Euclidean angular velocity is

$$\Omega=\frac{\alpha r_H^{N-5}}{\mu}. \tag{5.5}$$

Their product is thus

$$\Omega R=\frac{2\alpha r_H}{(N-3)r_H^2-(N-5)\alpha^2}. \tag{5.6}$$

This clearly vanishes when $\alpha=0$. Now consider the limit $\alpha\to\infty$. Considering (5.3) and (5.4), we find that to keep R fixed we need $r_H\to\alpha$ and $\mu\to R\alpha^{N-4}$. Then (5.6) implies that ΩR approaches 1. Similarly, the limit $\alpha\to-\infty$ sends $\Omega R\to-1$. Thus ΩR takes values between 1 and -1.

To obtain the unstable, static magnetically charged p-branes, one starts with the product of time and (5.2) with[3] $N=p+4$. If one reduces along

$$\mathbf{q}=\frac{\partial}{\partial\tau}+\Omega\frac{\partial}{\partial\varphi}, \tag{5.7}$$

the fixed point set will be the entire horizon and we will obtain an unstable bubble immersed in a magnetic flux-brane of strength $B=\Omega$. If, instead, we reduce along

$$\mathbf{q}'=\mathbf{q}-\frac{\sigma}{R}\frac{\partial}{\partial\varphi}, \tag{5.8}$$

where $\sigma=\Omega/|\Omega|$, we obtain the charged p-brane. The asymptotic value of the magnetic field is $B=\Omega-(\sigma/R)$. $|B|$ can be made as small as we like by tuning Ω. Note that as $B\to0$, $r_H\to\infty$, and the "size" of the p-brane becomes larger. The limiting solution with $B=0$ is just the noncompact p-brane obtained by taking the product of the standard Kaluza-Klein monopole and $\mathbf{R}^p$.

We can clearly turn on additional flux-branes at infinity by starting with the black hole with several rotation parameters nonzero. If we reduce using (5.1), the fixed point set is the horizon itself and we will obtain an unstable bubble living at the intersection of several flux-branes with $B_i=\Omega_i$. If we reduce using $\mathbf{q}'=\mathbf{q}-(\sigma_j/R)\partial/\partial\varphi_j$ [with $\mathbf{q}$ as in (5.1), $\sigma_j=\Omega_j/|\Omega_j|$, and no sum on j], for any choice of φ_j, then we obtain a magnetically charged spherical p-brane in a background of intersecting flux-branes with $B_i=\Omega_i$, $i\neq j$, and $B_j=\Omega_j-(\sigma_j/R)$. Adding additional rotations to $\mathbf{q}'$ reduces the dimension of the fixed point set, which becomes an uncharged brane.

An instanton describing the nucleation of a charged p-brane can be obtained from (5.2) with $N=p+5$ by reducing along $\mathbf{q}'$ in (5.8). This instanton corresponds to the nucleation of a spherical charged p-brane in a background $(p+1)$-flux-brane of strength $B=\Omega-\sigma/R$. The Lorentzian solution, representing the post-tunneling evolution, is obtained by analytically continuing in one of the ignorable angles in $d\Omega_{N-4}$. This appears to give a static solution rather than the expanding solution we obtained earlier from the Schwarzschild metric. However, the resulting timelike Killing field is really a boost, and the spacetime one obtains from the Kerr solution is qualitatively similar to the one obtained from the Schwarzschild solution and describes the spherical p-brane expanding. (For a detailed discussion of this in the case of five dimensions, see [9].)

In the semiclassical approximation, the rate of nucleation is given by e^{-I}, where I is the Euclidean action of the instanton. The action for the instanton with one angular momentum parameter nonzero is computed in the Appendix and is

$$I=\frac{V_{p+1}}{8(p+2)G_{p+4}}\mu, \tag{5.9}$$

where V_{p+1} is the volume of a unit $(p+1)$-sphere. One can rewrite this in terms of the magnetic field at infinity and the compactification radius, but the expression is complicated and not very illuminating (see the Appendix). However, in the limit where the asymptotic magnetic field $|B|$ is small, one finds

$$\mu=\left(\frac{p+1}{2}\right)^{p+1}\frac{R}{|B|^{p+1}}. \tag{5.10}$$

The compactification radius R is related to the charge on the p-brane by the usual expression for Kaluza-Klein monopoles, $q=R/4$, and this charge is in turn proportional to the mass per unit p-volume or tension of the p-brane. Thus the nucleation rate e^{-I} is increased by either increasing $|B|$ or decreasing the tension, as expected. For $p=0$, (5.9) and (5.10) reduce to the Schwinger result for pair-creating monopoles in a weak magnetic field [8].

We close this section by noting that instantons describing the nucleation of a spherical charged p-brane in intersecting flux-brane backgrounds can also be obtained. One considers the Kerr solution with several nonzero rotation parameters

[3]The case of the $N=4$ Euclidean Kerr solution with a flat time direction added was considered in [15]. There, the solution was interpreted as a dipole, but the presence of the background magnetic field was unnoticed.

and reduces along $\mathbf{q}'=\mathbf{q}-(\sigma_j/R)\partial/\partial\varphi_j$ [with $\mathbf{q}$ as in (5.1) and no sum on j]. The spherical charged p-brane appears and subsequently expands within the jth flux-brane. To nucleate an uncharged brane at the intersection of the flux-branes, one simply adds extra rotations to $\mathbf{q}'$ as discussed in the previous section.

VI. NUCLEATING LOOPS OF FUNDAMENTAL STRING

A. Test string approximation

We begin our discussion of fundamental strings by describing the behavior of a circular test string in flat spacetime coupled to a constant background H field. Since we are going to consider only classical solutions, the spacetime can have any dimension larger than 2. We assume that the only nonzero component of H is $H_{012}=h$, where h is a constant. The string action is

$$S=-\frac{1}{4\pi\alpha'}\int d^2\sigma(\sqrt{\gamma}\gamma^{ab}\partial_\alpha X^\mu\partial_\beta X_\mu + B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\epsilon^{\alpha\beta}), \tag{6.1}$$

with $\epsilon^{01}=-1$ and $0\leqslant\sigma\leqslant\pi$. Choosing the conformal gauge $\gamma=\eta$ yields the equation of motion,

$$\partial^2 X_\mu - \tfrac{1}{2}H_{\mu\nu\rho}\partial_\alpha X^\nu\partial_\beta X^\rho\epsilon^{\alpha\beta}=0, \tag{6.2}$$

and the Virasoro constraints

$$\dot{X}^\mu\dot{X}_\mu + X'^\mu X'_\mu=0, \quad \dot{X}^\mu X'_\mu=0. \tag{6.3}$$

We want to consider solutions describing circular loops, and so we set

$$X^0=f(t), \quad X^1=g(t)\sin 2\sigma, \quad X^2=g(t)\cos 2\sigma, \tag{6.4}$$

with the remaining X^i held constant.

One solution to (6.2) and (6.3) is simply $f=2t/h$, $g=1/h$. This is a static loop of string with a radius inversely proportional to the strength of the background H field. It is easy to see that this solution is unstable: A slightly smaller loop collapses inward, while a slightly larger loop expands outward. A second solution is

$$f=\frac{2\sin 2t}{h\cos 2t}, \quad g=\frac{2}{h\cos 2t}. \tag{6.5}$$

This describes a loop which initially is twice as large as the static one and expands outward. Since g^2-f^2 is constant, the world sheet is a hyperbola, describing constant acceleration. If we analytically continue in X^0 and t, we obtain an instanton describing the nucleation of a loop of string. The Euclidean action for this instanton is straightforward to calculate, with the result

$$I=\frac{8}{3\alpha' h^2} \tag{6.6}$$

for all spacetime dimensions. We now construct analogues of these solutions that include the back reaction of the string on the spacetime fields in five dimensions.

B. Spacetime solutions

Starting with a solution (g,ϕ,F) to the equations of motion obtained from (2.3) and performing the duality transformation

$$\tilde{\phi}=-\phi, \quad F_{D-3}=\exp\left(-4\frac{\sqrt{D-2}}{D-3}\phi\right)*F, \tag{6.7}$$

where F_{D-3} is a $(D-3)$-form field strength and the metric g is left unchanged, we obtain a ''dual solution'' $(g,\tilde{\phi},F_{D-3})$ to the equations of motion coming from the action

$$S=\frac{1}{16\pi G_{D-1}}\int d^{D-1}x\sqrt{-g}\left[R(g)-\frac{4}{D-3}(\nabla\tilde{\phi})^2 - \frac{2}{(D-3)!}\exp\left(-4\frac{\sqrt{D-2}}{D-3}\tilde{\phi}\right)F^2_{D-3}\right]. \tag{6.8}$$

This transformation exchanges magnetic F fields with electric F_{D-3} fields and vice versa. In the dual variables there is no longer a connection with Kaluza-Klein theory in D dimensions and we just have a solution in $D-1$ dimensions. Since the metric is invariant under the duality transformation, all of our previous solutions can be reinterpreted as the corresponding electric objects. This is particularly interesting for $D-1=5$ since, as we shall show, the solution describing the nucleation of a magnetic string in the last section is transformed into a solution describing the nucleation of a five-dimensional fundamental string.

In the case $D-1=5$, the action (6.8) is precisely part of the low-energy effective action of string theory in five dimensions, written in terms of the Einstein metric. If we rescale to the string metric $\tilde{g}=e^{4\tilde{\phi}/3}g$, this action takes the more familiar form

$$S=\frac{1}{16\pi G_5}\int d^5x\sqrt{-\tilde{g}}e^{-2\tilde{\phi}}[R(\tilde{g})+4(\nabla\tilde{\phi})^2-\tfrac{1}{12}H^2], \tag{6.9}$$

where we have used the notation $H\equiv 2F_3$. Thus, for every five-dimensional magnetic solution, there is a dual electric solution which extremizes the standard action (6.9). We now discuss some of these solutions. We will mostly work with the fields appearing in (6.9) and drop the tildes on g and ϕ for the remainder of this section.

To begin, recall that the simplest magnetically charged string in five dimensions was obtained as the product of a Kaluza-Klein monopole (3.1) with a line. Transforming to the dual variables, the solution can be reexpressed in the string frame as

$$ds^2=e^{2\phi}(-dt^2+dy^2)+d\mathbf{x}^2, \qquad e^{-2\phi}=1+\frac{4m}{r}, \quad B_{ty}=e^{2\phi}, \tag{6.10}$$

where y is the coordinate along the line. This is the solution corresponding to the fields about a macroscopic fundamental string in five dimensions [26].

The appropriate background to describe the nucleation of fundamental strings is given by a uniform electric flux-brane of dimension 2. This can be constructed from the $D-1=5$, two-dimensional magnetic flux-brane discussed in Sec. II by writing it in dual variables and rescaling to the string metric. The result is

$$ds^2 = e^{2\phi}(-dt^2 + dx_i^2 + d\rho^2) + \rho^2 d\varphi^2,$$
$$e^{2\phi} = 1 + B^2\rho^2, \quad H_{t12} = 2B, \tag{6.11}$$

where (t,x_i), $i=1,2$, are coordinates along the flux-brane. Note that the components of the H field are simply constant and that the induced metric at the center of the flux-brane, $\rho=0$, is flat. Since the metric and dilaton both depend on B^2 while H depends linearly on B, when $|B|$ is small this solution reduces to the configuration we started with in the test string discussion above.

The solution describing the unstable static loop of magnetic string (4.5) can similarly be dualized. The result is a static loop of fundamental string in the background H field (6.11). Since the Einstein metric is unchanged under duality, the metric (4.5) also describes a finite fundamental string loop. This is the exact analogue of the circular test string at rest.

Since the H field has a nonzero time component, the instanton describing the nucleation of a loop of fundamental string will have imaginary H as expected for an electric-type field. It is constructed by starting with the Lorentzian Myers-Perry-Kerr solution in $D=6$:

$$\begin{aligned} ds^2 = {} & \left(1 - \frac{\mu}{r\Sigma}\right) d\tau^2 - \frac{2\mu\alpha \sin^2\theta}{r\Sigma}\, d\tau\, d\varphi \\ & + \frac{\Sigma}{r^2 - \alpha^2 - \mu r^{-1}}\, dr^2 + \Sigma\, d\theta^2 \\ & + \frac{\sin^2\theta}{\Sigma}\,[(r^2-\alpha^2)\Sigma - \mu r^{-1}\alpha^2 \sin^2\theta] d\varphi^2 \\ & + r^2 \cos^2\theta (d\chi^2 + \cos^2\chi\, d\psi^2). \end{aligned} \tag{6.12}$$

We then reduce to five dimensions using the symmetry $\mathbf{q}'$ of (5.7) and (5.8). Explicitly, we set $\varphi = \tilde{\varphi} + [\Omega - (\sigma/R)]\tau$ and then read off the five-dimensional metric, dilaton, and gauge field by putting the metric in the form (2.2) with $D=6$ and $x^D = \tau$. Next, we analytically continue $\psi = -it$. The resulting metric describes an expanding loop of magnetically charged string. (The Killing vector $\partial/\partial t$ is a boost.) We now apply the duality transformation (6.7) to obtain an expanding loop of fundamental string. Finally, we analytically continue back $t = i\psi$ to obtain the desired instanton.

The Euclidean action is not invariant under the duality transformation (6.7). However, for four-dimensional black holes, it has recently been shown that the rate of pair-creating electrically charged black holes is identical to the rate for creating magnetically charged ones [11,10]. This is because one must include a projection onto states of definite electric charge [27] in calculating the rate which exactly compensates for the difference in the action. We expect that a similar result will hold in the present case as well. The rate will then be given by e^{-I}, where I is given by (5.9) with $p=1$. In the limit of small $|B|$, we can use (5.10) to express this as

$$I = \frac{\pi R}{6 G_5 B^2}. \tag{6.13}$$

It was shown in [26] that for a single macroscopic fundamental string in five dimensions, the dilaton charge in (6.10) should be given by $4m = 2G_5/\pi\alpha'$. This can be related to R by recalling that for the Kaluza-Klein monopole $R = 8m$. Using this and setting $h = 2B$, we recover exactly the action found from the test string instanton (6.6).

VII. DISCUSSION

We have constructed solutions describing magnetically charged p-branes and loops of fundamental string, as well as instantons describing the nucleation of these objects in appropriate background fields. The basic idea was to start with a vacuum solution with a U(1) isometry. The fixed points of the isometry describe a p-brane in the reduced spacetime which can carry magnetic charge. One can then apply a duality transformation to obtain electrically charged solutions. If the reduced spacetime is five dimensional, the resulting theory is precisely part of the low-energy string action and the magnetically charged strings are transformed into fundamental strings.

To construct our solutions we have always started with a Euclidean black hole or Euclidean black hole cross time. However, it is clear that there are many other possibilities which can yield interesting solutions. For example, one can start with a Lorentzian black hole cross a circle.[4] If one considers the symmetry consisting of translation around the circle plus rotation of the black hole, the reduced space describes a black hole in a background magnetic flux-brane. This is the likely end point of a p-brane which is smaller than the static radius and collapses to form a black hole. In five dimensions, we can dualize to obtain a black hole in a background H field.

It was shown in [9] that in the standard five-dimensional Kaluza-Klein theory the dominant decay mode for the weak magnetic fields of physical relevance was via ''bubble nucleation'' analogous to the decay of the Kaluza-Klein vacuum. The same is true for the decay of the magnetic flux-branes described here. Indeed, the earlier analysis is just the special case $p=0$. For every p, the reduction of the $(p+5)$-dimensional Myers-Perry-Kerr instanton, with one nonzero rotation parameter, via $\mathbf{q}$, Eq. (5.7), describes decay of a $(p+1)$-flux-brane via ''bubble nucleation,'' while the shifted reduction via $\mathbf{q}'$, Eq. (5.8), describes nucleation of a charged p-brane. If we take two instantons, with different parameters, one reduced along $\mathbf{q}$ and the other along $\mathbf{q}'$, so that the asymptotic value of the magnetic field B is the same in both cases, then we find that the bubble nucleation has smaller action for small $|B|$.

Having said this, it was also pointed out in [9] that a spin structure argument analogous to that which would stabilize the $D=5$ Kaluza-Klein vacuum [16] would also rule out bubble nucleation, but allow the pair production of monopoles. Roughly, the argument is that the $D=5$ Kaluza-Klein

[4]If the radius of the circle is small compared with the mass of the black hole, then this solution is likely to be stable [28].

Melvin solution admits two spin structures which may be distinguished by asking what phase spinors pick up under parallel transport around the internal circle. There are two instantons for the decay of the same four-dimensional flux tube, one corresponding to bubble nucleation and the other to pair production. Each instanton admits only a single spin structure since it is simply connected, but that spin structure tends at infinity to a different one of the two possibilities. So depending on which spin structure is chosen for the background Melvin spacetime, one or other of the decay channels is ruled out. If $|B|$ is small, then there is a natural choice which is in some sense continuous with the choice that rules out the vacuum decay. This allows pair production, but eliminates the bubble nucleation. In particular, this is what we expect in S^1 compactifications that preserve supersymmetry.

A similar argument can be used to show that there is again a natural choice of spin structure for a single background $(p+1)$-flux-brane which would rule out the decay via bubble nucleation, but allow the decay via production of a spherical charged p-brane. These are the only two possible decay routes. When the background is a configuration of intersecting flux-branes, there are more possibilities for the decay. Suppose we have k intersecting flux-branes in $p+4$ spacetime dimensions. One decay channel that always exists is the bubble nucleation. Then there are k channels which are the nucleation and subsequent expansion of a charged p-brane within each individual flux-brane, and $\binom{k}{2}$ channels which correspond to an uncharged $(p-2)$-brane produced and expanding in the intersection of each pair of flux-branes and so on: $\binom{k}{l}$ possible $(p-2l+2)$-branes produced in the intersection of each subset of l flux-branes. The generalization of the spin structure argument seems to result in the bubble nucleation being ruled out, the $(p-4n)$-brane production being allowed, where n is an integer, and the $[p-2(2n+1)]$-brane production being ruled out. So, for example, Bais-Batenberg ''monopole'' pair production would be allowed only if the reduced spacetime dimension were a multiple of 4.

The situation with the fundamental string is slightly different. After we dualize in five dimensions, the connection with six dimensions is lost and in particular we no longer have a spin structure argument. It seems that there should be an argument to eliminate the dual of the bubble nucleation process while keeping the string production process, and the following is a promising possibility. While the instanton describing bubble nucleation is nonsingular in six dimensions, it is singular in five dimensions, as is its dual. It is not clear whether this singular dual instanton corresponds to a physical decay channel of the H field, but we expect not since, if it is allowed, it suggests that the vacuum itself would also decay via dual-bubble nucleation. Of course, the instanton describing the nucleation of a fundamental string is also singular, but here the singularity is readily interpreted in terms of the string source and almost certainly should be allowed.

The extended objects that we have considered are all extremal, in the sense that their mass per unit p-volume was essentially equal to their charge. It would be interesting to know whether one could nucleate nonextremal extended objects. For black holes in four dimensions, it was found that nonextreme black holes were created in thermal equilibrium with their Hawking temperature equal to the acceleration temperature. It thus seems that a necessary condition to nucleate a nonextremal p-brane is that the Hawking temperature must go to zero in the extremal limit, so that it can equal the acceleration temperature for small acceleration.

Our construction has only yielded fundamental strings in five dimensions. However, from the test string calculation it is clear that there should exist analogous solutions in all dimensions larger than 2. In particular, a four-dimensional solution describing a loop of fundamental string should exist. It would be interesting to find it.

In addition to the fundamental strings, we have mentioned that our solutions have two other string theory interpretations. First, the Kaluza-Klein reduction of $D=11$ leads to six-brane solutions carrying Ramond-Ramond charge of the $D=10$ type-IIA theory. On the other hand, for $D\leqslant 10$, the Kaluza-Klein solutions provide solutions to string theory compactifications which include an S^1 factor. Since these are charged with respect to the U(1) gauge field coming from the metric, they carry Neveu-Schwarz–Neveu-Schwarz (NS-NS) charge in the type-II theory. Let us briefly mention some ways in which we can generalize our solutions. For convenience, we discuss these transformations in terms of the simplest flat charged p-branes. By wrapping the type-IIA six-brane solution around an n-torus, we can obtain $(6-n)$-brane solutions of type-II theory in $10-n$ dimensions that carry RR charge. These will be related to the NS-NS $(6-n)$-branes obtained by Kaluza-Klein reduction by some field redefinitions (part of the continuous group of U-duality transformations [1]). Another way to obtain new solutions is to use the fact that the Kaluza-Klein solutions have a U(1) isometry. In particular, there is the T-duality symmetry which includes interchanging the two U(1)'s coming from the dimensional reduction of the metric and antisymmetric tensor. This transformation takes the ''metric'' p-brane in $p+4$ reduced spacetime dimensions to an ''antisymmetric tensor'' p-brane. These latter objects can be considered to be H-monopoles in four dimensions [29,30] with p flat dimensions added. Equivalently, the $p=5$ solution in nine dimensions can be constructed by taking a periodic array of five-branes in ten dimensions to get a five-brane in nine dimensions. By wrapping these solutions around an S^1, we can then obtain a four-brane in eight dimensions, etc. Finally, new solutions can also be obtained by employing various string-string dualities, which amounts to writing the solutions in suitable dual variables [1,2]. It is natural to expect that all of the above transformations acting on our instantons will produce instantons describing the nucleation of the corresponding objects.

In recent work Polchinski has shown that D-branes, surfaces where first-quantized strings have Dirichlet boundary conditions, are carriers of Ramond-Ramond charges [4]. In particular, the six-brane of the type-IIA theory has a D-brane description. This identification has, as yet, only be made in the static, supersymmetric case. Although our instantons are neither static nor supersymmetric, we still might expect a related D-brane construction. Having such a construction might enable one to go beyond the semiclassical approximation in a controlled manner.

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APPENDIX

We here calculate the action of the Euclidean rotating black hole with one nonzero rotation parameter in arbitrary dimension. The metric is [17]

$$\begin{aligned} ds^2 = &\left(1-\frac{\mu}{r^{D-5}\Sigma}\right)d\tau^2 - \frac{2\mu\alpha\sin^2\theta}{r^{D-5}\Sigma}\,d\tau\,d\varphi \\ &+ \frac{\Sigma}{r^2-\alpha^2-\mu r^{5-D}}\,dr^2 + \Sigma\,d\theta^2 \\ &+ \frac{\sin^2\theta}{\Sigma}[(r^2-\alpha^2)\Sigma - \mu r^{5-D}\alpha^2\sin^2\theta]d\varphi^2 \\ &+ r^2\cos^2\theta\,d\Omega_{D-4}, \end{aligned} \tag{A1}$$

where $\Sigma = r^2 - \alpha^2\cos^2\theta$. The horizon is located at $r = r_H$, where

$$r_H^2 = \alpha^2 + \frac{\mu}{r_H^{D-5}}. \tag{A2}$$

The radius of the circle at infinity is

$$R = \frac{1}{\kappa} = \frac{2\mu r_H^{6-D}}{(D-3)r_H^2 - (D-5)\alpha^2}, \tag{A3}$$

while the Euclidean angular velocity is

$$\Omega = \frac{\alpha r_H^{D-5}}{\mu}. \tag{A4}$$

The Euclidean action is defined with respect to a background geometry:

$$\begin{aligned} I = &-\frac{1}{16\pi G_D}\int d^Dx\sqrt{-g_D}R(g_D) \\ &-\frac{1}{8\pi G_D}\int d^{D-1}x\sqrt{h}(K-K_0), \end{aligned} \tag{A5}$$

where K is the trace of the extrinsic curvature of the boundary and K_0 is the trace of the extrinsic curvature of the boundary embedded in the background geometry. Here the appropriate background is just flat $\mathbf{R}^D$.

The instanton is Ricci flat, and so the boundary term is the only contribution. Let the boundary be given by r=const. The induced metric is

$$\begin{aligned} ds_{D-1}^2 = &\left(1-\frac{\mu}{r^{D-5}\Sigma}\right)d\tau^2 - \frac{2\mu\alpha\sin^2\theta}{r^{D-5}\Sigma}\,d\tau\,d\varphi + \Sigma\,d\theta^2 \\ &+ \frac{\sin^2\theta}{\Sigma}[(r^2-\alpha^2)\Sigma - \mu r^{5-D}\alpha^2\sin^2\theta]d\varphi^2 \\ &+ r^2\cos^2\theta\,d\Omega_{D-4}. \end{aligned} \tag{A6}$$

The determinant is

$$\begin{aligned} \sqrt{h} = &\left(1-\frac{\alpha^2}{r^2}-\frac{\mu}{r^{D-3}}\right)^{1/2}\left(1-\frac{\alpha^2}{r^2}\cos^2\theta\right)^{1/2} \\ &\times r^{D-2}\sin\theta\cos^{D-4}\theta\sqrt{\Omega_{D-4}}. \end{aligned} \tag{A7}$$

The unit normal is

$$n = \left(\frac{r^2-\alpha^2-\mu r^{5-D}}{r^2-\alpha^2\cos^2\theta}\right)^{1/2}\frac{\partial}{\partial r}. \tag{A8}$$

K is calculated via $K\sqrt{h} = n\sqrt{h}$, so that

$$K = \frac{n\sqrt{h}}{\sqrt{h}}. \tag{A9}$$

The background value K_0 is easily computed from this by setting $\mu=0$ since (A1) with μ zero is flat for all values of α. Thus we have

$$(K-K_0)\sqrt{h} = n\sqrt{h} - n\sqrt{h}|_{\mu=0}\,\frac{\sqrt{h}}{\sqrt{h}|_{\mu=0}}. \tag{A10}$$

We want to take the limit $r\to\infty$. In this limit,

$$\lim_{r\to\infty}\left(\frac{\sqrt{h}}{\sqrt{h}|_{\mu=0}}\right) = 1 - \frac{\mu}{2r^{D-3}} \tag{A11}$$

and, hence,

$$\begin{aligned} \lim_{r\to\infty}[(K-K_0)\sqrt{h}] &= \lim_{r\to\infty}\left[(n\sqrt{h} - n\sqrt{h}|_{\mu=0}) + n\sqrt{h}\Big|_{\mu=0}\frac{\mu}{2r^{D-3}}\right] \\ &= \lim_{r\to\infty}\left[\frac{\partial n\sqrt{h}}{\partial\mu}\Big|_{\mu=0} + \frac{1}{2r^{D-3}}n\sqrt{h}\Big|_{\mu=0}\right]\mu \\ &= -\tfrac{1}{2}\mu\sin\theta\cos^{D-4}\theta\sqrt{\Omega_{D-4}}. \end{aligned} \tag{A12}$$

Then,

$$I = \frac{\pi R V_{D-4}}{4(D-3)G_D}\,\mu = \frac{V_{D-4}}{8(D-3)G_{D-1}}\,\mu, \tag{A13}$$

where V_{D-4} is the volume of the unit $(D-4)$-sphere.

We can check that this agrees with the thermodynamic and Smarr formulas given in [17]. The mass M and angular momentum J are given by

$$M=\frac{(D-2)V_{D-2}}{16\pi G_D}\,\mu,\quad J=\frac{2}{D-2}\,Ma, \tag{A14}$$

where $a=i\alpha$ is the Lorentzian rotation parameter. The Smarr relation is

$$\omega J+TS=\frac{D-3}{D-2}\,M, \tag{A15}$$

where S is the entropy, $T=\kappa/2\pi$ the temperature, and ω is the Lorentzian angular velocity. The thermodynamic potential W is

$$W=M-TS-\omega J=\frac{1}{D-2}\,M, \tag{A16}$$

and thus the Euclidean action is

$$I=\frac{W}{T}=\frac{2\pi R}{D-2}\,M, \tag{A17}$$

which agrees with (A13) since $V_{D-2}=2\pi V_{D-4}/(D-3)$.

It is more interesting to express (A13) in terms of the value of the asymptotic magnetic field strength $B=\Omega-(\sigma/R)$ $(\sigma=\Omega/|\Omega|)$ and radius of compactification R. From (A2) and (A4), we can eliminate μ and then solve for α, with the result

$$\alpha=\frac{-1+\sqrt{1+4r_H^2\Omega^2}}{2\Omega}. \tag{A18}$$

Using this and (A4) in (A3), we obtain an expression for R in terms of r_H and Ω. This can be inverted to yield

$$\frac{r_H}{R}=\frac{D-4+[(D-4)^2-(1-R^2\Omega^2)(D-3)(D-5)]^{1/2}}{2(1-R^2\Omega^2)}. \tag{A19}$$

One can thus obtain an expression for $\mu=\alpha r_H^{D-5}/\Omega$ in terms of R and Ω, which unfortunately is extremely complicated. However, it simplifies in two limits. When $\alpha=0$, the instanton is just the Euclidean Schwarzschild solution and one has

$$\mu=\left[\frac{(D-3)R}{2}\right]^{D-3}. \tag{A20}$$

When $|\Omega R|\approx 1$ (so $|B|$ is small), one finds

$$\mu=\left(\frac{D-4}{2}\right)^{D-4}\frac{R}{|B|^{D-4}}. \tag{A21}$$

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