Black hole in thermal equilibrium with a spin-2 quantum field

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(Received 1 February 1996)

An approximate form for the vacuum averaged stress-energy tensor of a conformal spin-2 quantum field on a black hole background is employed as a source term in the semiclassical Einstein equations. Analytic corrections to the Schwarzschild metric are obtained to first order in $\epsilon=\hbar/M^2$, where M denotes the mass of the black hole. The approximate tensor possesses the exact trace anomaly and the proper asymptotic behavior at spatial infinity is conserved with respect to the background metric and is uniquely defined up to a free parameter $\hat{c}_2$, which relates to the average quantum fluctuation of the field at the horizon. We are able to determine and calculate an explicit upper bound on $\hat{c}_2$ by requiring that the entropy due to the back reaction be a positive increasing function in r. A lower bound for $\hat{c}_2$ can be established by requiring that the metric perturbations be uniformly small throughout the region $2M\leq r<r_0$ where r_0 is the radius of perturbative validity of the modified metric. Additional insight into the nature of the perturbed spacetime outside the black hole is provided by studying the effective potential for test particles in the vicinity of the horizon. [S0556-2821(96)04312-3]

PACS number(s): 04.70.Dy, 04.60.−m, 05.70.Ce

I. INTRODUCTION

The physics of black holes provides a fertile ground in which the confluence of gravitation, quantum mechanics, and thermodynamics takes place. Progress in our understanding of this confluence as well as of the specific thermal and mechanical aspects of black holes requires one to construct and study model theories of semiclassical black holes which can provide insights into the kinds of physical effects that may be present in a complete and, as of yet, unrealized description of quantum gravity. A key to one such model theory is the fact that a black hole can exist in (possibly unstable) thermodynamic equilibrium provided it is coupled to thermal quantum fields having a suitable distribution of stress energy. In the semiclassical approach, such fields are characterized by the vacuum average of a stress-energy tensor obtained by the renormalization of a quantum field on the classical background geometry of a black hole. Using such a tensor as a source in the Einstein equation

$$G_{\mu\nu}=8\pi\langle T_{\mu\nu}\rangle_{\rm ren} \tag{1}$$

defines the associated semiclassical back-reaction problem. The solution of (1) encodes the change induced by the stress-energy tensor on the black hole's spacetime metric.

Before a solution can be obtained however, one needs to be able to calculate the expectation value of stress-energy tensors for quantized fields in a suitable vacuum state. This task has proven to be considerably difficult when the background spacetime is that of a static black hole. Indeed, to date, the only exact numerical calculations of $\langle T_{\mu\nu}\rangle_{\rm ren}$ on this background have been carried out for the conformal scalar [1], the U(1) gauge boson [2], and most recently, for the nonconformal scalar field [3]. In each of these cases, excellent analytic approximations to the exact, numerically calculated tensors have been found, and these have been used, in turn, via the solutions of (1), to explore the thermodynamical and mechanical consequences of the back reaction of spin-0 and spin-1 quantum fields on a black hole [4–9]. The case of a massless spin-1/2 fermion has also been investigated in [6] based on an approximate stress-energy tensor. In this way, one has been able to investigate the effects of quantized matter on the geometry of black holes, in a case-by-case fashion, and rather novel spin-dependent effects have been uncovered in the process [7]. While much has been learned from these studies, it is also clear that any discussion of back reaction in quantum field theory in curved spacetime should include the effects of linearized gravitons (whose spin=2), which are expected to contribute to the one-loop effective stress-energy tensor a term of the same order (or perhaps higher order) as those coming from the lower spin fields. A knowledge of how gravitons behave near the singularity at the center of a black hole is likely to be crucial to our understanding of quantum gravity. One would also like to obtain a self-consistent picture of the black hole evaporation problem. However, at the present time, the calculation of such a tensor is confronted by complicated technical difficulties, the solution of which shall require a reconciliation between gauge invariance and renormalization [10]. Namely, while a complete set of solutions for the linear graviton field equations exists only in the radiation gauge, explicit renormalization has been implemented only in the deDonder gauge.

Until such time as the technical difficulties associated with the nonconformal, spin-2 linear graviton can be overcome, it is worthwhile to obtain some idea of the magnitude of the back reaction arising from the spin-2 nature of the

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graviton. This, it turns out, can be achieved provided we are willing to dispense altogether with the inherently nonconformal graviton, and consider instead a conformal spin-2 quantum field. We hasten to point out that we are not suggesting to model the graviton with a conformal field, rather, we seek a concrete means for assessing the importance of spin-2 in the back reaction on a black hole. Nevertheless, it might well turn out that some of the results of this calculation could serve as a guide as to what to expect in the technically complicated graviton case. The back reaction of the conformal spin-2 field can be calculated employing, for example, the approximate stress-energy tensor ansatz constructed by Frolov and Zel'nikov [11] valid for massless, conformal fields in any static spacetime. The main idea of their approach is to approximate $\langle T_{\mu\nu}\rangle_{\rm ren}$ by a tensor expanded in a basis containing the curvature tensor, the Killing vector, and their covariant derivatives, up to some order. The resulting tensor is covariantly conserved with respect to the background metric, possesses the correct (and exact) trace anomaly, and obeys certain important scaling relations and boundary conditions. For a static black hole in a vacuum, a unique (up to a parameter $\hat{c}_2$) tensor is singled out which should provide a reasonable approximation to $\langle T_{\mu\nu}\rangle_{\rm ren}$ in the Hartle-Hawking vacuum state.

That there can arise important features depending solely on the spin of the field coupling to the black hole is supported by results of the back-reaction analyses presented in [5–7]. These do indeed indicate an important dependence on the spin of the quantum field. For example, the energy density of the spin-1 vector boson near the black hole horizon is roughly 120 times greater in magnitude than that of the conformal scalar [5]. A calculation of the radial acceleration of a massive test particle initially at rest just outside the horizon also manifests a curious spin dependence. The acceleration is enhanced for the spin-0 scalar and for the spin-1/2 fermion, but can be reduced for the spin-1 boson [6], for a sufficiently large number (or multiplicity) of U(1) fields. Spin dependence also shows up in the effective potential for test particles in the vicinity of the black hole leading to either an increase or decrease in the black hole's capture cross section [7].

In Sec. II we discuss the relevant features of the Frolov-Zel'nikov approximate stress-energy tensor needed for the present calculation and calculate the spin-dependent parameters needed to apply it to the spin-2 field. The metric perturbations resulting from using this tensor as a source in the semiclassical Einstein equation are calculated in Sec. III. The way in which the black hole mass is renormalized and how the remaining constant of integration gets fixed by the thermodynamic boundary conditions is reviewed briefly. Both upper and lower bounds for $\hat{c}_2$ result from requiring that the metric perturbations due to the back reaction be uniformly small over the entire range of $2M \leq r < r_0$, where r_0 is the radius of perturbative validity of the solutions of (1). In Sec. IV, we compute the entropy ΔS by which the back-reaction of the spin-2 field augments the Bekenstein-Hawking entropy. By requiring that the field increases the thermodynamic entropy of the system, we are able to put an upper bound on the constant $\hat{c}_2$, which measures the magnitude of the quantum fluctuations of the field at the horizon. This particular bound is much more stringent than that coming from the condition of the "smallness" of the solutions. Combining the results from perturbative validity plus well-behaved entropy, we obtain the double inequality $-3080 < \hat{c}_2 < -1366$. We also calculate ΔS for the lower spin conformal fields ($s=0,1/2,1$) and compare these results to previous entropy calculations based on exact stress-energy tensors in order to get some indication for the accuracy of the Frolov-Zel'nikov approximation. Further insight into the nature of the modified spacetime geometry is obtained examining the effective potential for test particle orbits in Sec. V. Our results are summarized briefly in the final section. Units are chosen such that $G=c=k_B=1$ but $\hbar \neq 1$.

II. APPROXIMATE SPIN-2 STRESS-ENERGY TENSOR

For the case of a static black hole background with the metric ($w=2M/r$)

$$ds^2 = -(1-w)dt^2 + (1-w)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2}$$

the tensor ansatz constructed by Frolov and Zel'nikov [11] takes on a relatively simple structure depending on just three spin-dependent constants:

$$T_{\mu\nu} = a_s T^{(\rm tr)}_{\mu\nu} + b_s \tau'_{\mu\nu} + c_s \tau''_{\mu\nu}. \tag{3}$$

Defining the constant tensors

$$\Pi_\mu^{\ \nu} = \delta_\mu^\nu - 4\,\delta_\mu^0 \delta_0^\nu, \tag{4}$$

$$\Psi_\mu^{\ \nu} = \delta_\mu^1 \delta_1^\nu - \delta_\mu^0 \delta_0^\nu, \tag{5}$$

then the various terms in the expansion of $T_\mu^{\ \nu}$ are given by

$$T_\mu^{(\rm tr)\nu} = 48\kappa^4 w^6(\delta_\mu^\nu + 3\Pi_\mu^{\ \nu} - 6\Psi_\mu^{\ \nu}) \tag{6}$$

$$\tau_\mu'^{\ \nu} = \kappa^4[(1+2w+3w^2+4w^3+5w^4+6w^5-105w^6)\Pi_\mu^{\ \nu} + 168w^6\Psi_\mu^{\ \nu}], \tag{7}$$

$$\tau_\mu''^{\ \nu} = \kappa^4 w^3[(4+5w+6w^2+15w^3)\Pi_\mu^{\ \nu} - 12(1+w+w^2 + 2w^3)\Psi_\mu^{\ \nu}], \tag{8}$$

where $\kappa = (4M)^{-1}$ is the surface gravity of the black hole. The constants in (3) are fixed from knowledge of the exact trace anomaly and by boundary conditions to be satisfied at the black hole horizon and at spatial infinity. All three tensors are finite at the horizon ($w=1$). The first tensor $T_\mu^{(\rm tr)\nu}$ is the only one with nonzero trace:

$$T_\mu^\mu = a_s T_\mu^{(\rm tr)\mu} = a_s\left(\frac{48M^2}{r^6}\right). \tag{9}$$

On the other hand, the exact trace anomalies for conformal quantum fields of arbitrary spin on a curved background have been calculated previously [12,13]. The general result can be expressed in terms of a certain linear combination of curvature invariants [14]. For the case of Ricci flat ($R_{\mu\nu}=0$) backgrounds, the trace anomaly simplifies to

$$\langle T^{\mu}_{\mu}\rangle_{\rm ren}=\frac{\hat{a}_s}{(2880\pi^2)}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=\frac{\hat{a}_s}{(2880\pi^2)}\left(\frac{48M^2}{r^6}\right), \tag{10}$$

where the final equality holds for the black hole background. In particular, for spin-2, $\hat{a}_2=212$, and matching coefficients between (9) and (10) yields the identification $a_2=212/(2880\pi^2)$. The second tensor, τ'^{ν}_{μ}, is the only one contributing at far distances from the singularity ($r\to\infty$), and takes on the asymptotic form

$$T^{\nu}_{\mu}\to b_s\tau'^{\nu}_{\mu}=b_s\kappa^4\ {\rm diag}(-3,1,1,1)^{\nu}_{\mu}. \tag{11}$$

The coefficient b_s is determined from the boundary condition that *all* the quantum stress-energy tensors renormalized on a black hole background approach the form of a flat-spacetime radiation stress tensor at the uncorrected Hawking temperature ($T_H=\kappa/2\pi$). The radiation stress tensor in flat space is simply proportional to a constant tensor, the proportionality factor depending on the number of independent helicity states of the field in question. For the case of spin-2, the flat-space limit is given by

$$\langle T^{\nu}_{\mu}\rangle_{\rm ren}\to\left(\frac{\pi^2}{90}\right)h(2)T_H^4\ {\rm diag}(-3,1,1,1)^{\nu}_{\mu}, \tag{12}$$

where $h(2)=2$ is the corresponding number of independent helicity states (this is also the number of independent components of the linear graviton in 3+1 dimensions). Matching coefficients of (11) and (12) in this limit yields $b_2=4/(2880\pi^2)$. The third and final tensor is finite at the horizon and vanishes asymptotically at infinity as r^{-3}. It is, therefore, presumably important only for the intermediate zone near the black hole horizon. Moreover, the coefficient c_s multiplying it can be fixed unambiguously by requiring that [11]

$$T^{\nu}_{\mu}|_{w=1}=\langle T^{\nu}_{\mu}\rangle^{\rm ren}|_{w=1}. \tag{13}$$

The tensor structure of both sides of this equation is identical at the horizon, so only one constant is actually defined. However, the implementation of this boundary condition requires knowledge of the (exact) renormalized quantum stress tensor at the black hole horizon. The exact value of $\langle T^{\nu}_{\mu}\rangle^{\rm ren}$ at the event horizon for the Hartle-Hawking vacuum is known only for the conformal scalar [1], the nonconformal scalar [3], and the electromagnetic field [2]. As it is the spin-2 version of $\langle T^{\nu}_{\mu}\rangle^{\rm ren}$ we are interested in approximating, we shall need a different (but physically equivalent) criterion for establishing the value (or bounds) of c_2. Loose bounds on c_2 may be established by appealing to perturbation theory, as will be made explicit in the next section. However, an improved upper bound results from exploiting the physical properties of the thermodynamic entropy. We will indeed return to this important point later on when we come to discuss the amount by which the spin-2 quantum field auguments the thermodynamical entropy of the black hole. In the meantime, we continue in what follows, keeping c_2 as a free parameter.

III. METRIC PERTURBATIONS

With the explicit components of the approximate spin-2 stress tensor in hand, we may now proceed to solve the back-reaction equation (1) to first order in $\epsilon=\hbar/M^2<1$. As T^{ν}_{μ} is a function only of the radial coordinate, the resulting metric perturbations will be static and spherically symmetric. The most general metric satisfying these conditions involves two independent radial functions, and may be written as [4,6]

$$ds^2=-\left(1-\frac{2m(r)}{r}\right)e^{2\psi(r)}dt^2+\left(1-\frac{2m(r)}{r}\right)^{-1}dr^2+r^2d\Omega^2. \tag{14}$$

Then, the linear perturbations to the metric result from expanding the two metric functions in ϵ as

$$e^{\psi(r)}=1+\epsilon\overline{\rho(r)}, \tag{15}$$

$$m(r)=M[1+\epsilon\overline{\mu(r)}], \tag{16}$$

and the functions $\overline{\rho}$ and $\overline{\mu}$ are solutions of the (linearized) Einstein equations

$$\frac{d\overline{\rho}}{dr}=-\frac{16\pi M^2}{\epsilon w^3}(1-w)^{-1}[T^r_r-T^t_t], \tag{17}$$

$$\frac{d\overline{\mu}}{dr}=\frac{32\pi M^2}{\epsilon w^4}T^t_t. \tag{18}$$

These follow directly from (1) after substituting (14) into the Einstein tensor $G_{\mu\nu}$ and expanding both sides to $O(\epsilon)$ [the stress tensor is itself $O(\hbar)=O(\epsilon)$]. The solutions of these equations involve simple radial integrations which for the present case, upon integrating up from the horizon ($1\leqslant w$), yield

$$\overline{\mu(w)}=-\frac{1}{6K}\{32\hat{a}_2(w^3-1)+\hat{b}_2[-w^{-3}-3w^{-2}-9w^{-1}+12\ln(w)+15w+9w^2-49w^3+38]+\hat{c}_2(3w+3w^2+7w^3-13)\}+C_0K^{-1}, \tag{19}$$

and

$$\overline{\rho(w)}=-\frac{1}{3K}\left[\hat{b}_2\left(-\frac{1}{2}w^{-2}-3w^{-1}+6\ln(w)+10w+\frac{15}{2}w^2+7w^3-21\right)-\hat{c}_2\left(w^3+\frac{3}{2}w^2+2w-\frac{9}{2}\right)\right]+k_0K^{-1}. \tag{20}$$

The existence of the limit $\lim_{w\to 1^+}(T^r_r - T^t_t)/(1-w)$ has been used to render the integration of (17) trivial. Here, $K=3840\pi$, $\hat{c}_2/c_2=\hat{b}_2/b_2=\hat{a}_2/a_2=2880\pi^2$ and C_0 and k_0 are constants of integration. If we set $\mu=\overline{\mu}-C_0K^{-1}$, the mass function can be rewritten to $O(\epsilon)$ as

$$\begin{aligned} m(w) &= M\{1+\epsilon[\mu(w)+C_0K^{-1}]\} \\ &= M(1+\epsilon C_0K^{-1})[1+\epsilon\mu(w)] \\ &= M_{\rm ren}[1+\epsilon\mu(w)], \end{aligned} \tag{21}$$

so that the integration constant C_0 serves to renormalize the (bare) black hole mass, and we, henceforth, write $M\equiv M_{\rm ren}$ in what follows, with the tacit understanding that this stands for the physical black hole mass. The unknown quantities in the perturbed metric are reduced to a single integration constant, k_0, which can be determined after suitable boundary conditions are imposed.

The necessity for imposing boundary conditions has been exhaustively discussed in previous work [4,6,7]. We recapitulate briefly the main points of that discussion here. In the first instance, asymptotic flatness does not fix the value of k_0. To appreciate this point, it suffices to note that $\overline{\rho(r)}\sim(\hat{b}_2/6K)(r/2M)^2$ for $r\to\infty$. Related to this limit is the fact that the stress-energy tensors employed in (1) are asymptotically constant [see (11) and (12)], thus the radiation in a sufficiently large spatial region surrounding the black hole would collapse onto the hole and thereby produce a larger one. This asymptotic constancy of the (renormalized) stress tensors is, of course, not an artifact of any approximation or regularization scheme. It is simply the *physical* condition required in order that an observer at spatial infinity measure the correct value of the Hawking temperature (of the unperturbed black hole). As the perturbations grow (in r) without bound, it is therefore necessary to implant the system consisting of black hole plus thermal quantum fields in a finite cavity with a wall radius $r_0>2M$. The allowed values for r_0 can be determined explicitly by requiring that the metric perturbations remain uniformly small over a certain radial domain. The boundary condition and the region outside the cavity wall are really to be thought of as the ambient spacetime in which the system (black hole+radiation) is embedded. We shall choose microcanonical boundary conditions, specifying thus the total energy $E(r_0)$ at the cavity wall, and match on an exterior metric of Schwarzschild form with an effective mass $M^*=m(r_0)$. Defining ρ by $\overline{\rho}=\rho+k_0K^{-1}$, the continuity of the metric across the wall yields the relation $k_0=-K\rho(r_0)$. The spacetime geometry, including the back reaction, is thus now completely specified for $r\leqslant r_0$ by

$$\begin{aligned} ds^2 = &-\left(1-\frac{2m(r)}{r}\right)\{1+2\epsilon[\rho(r)-\rho(r_0)]\}dt^2 \\ &+\left(1-\frac{2m(r)}{r}\right)^{-1}dr^2+r^2d\Omega^2, \end{aligned} \tag{22}$$

and for $r\geqslant r_0$ by

$$ds^2=-\left(1-\frac{2m(r_0)}{r}\right)dt^2+\left(1-\frac{2m(r_0)}{r}\right)^{-1}dr^2+r^2d\Omega^2. \tag{23}$$

The parameter r_0 and ϵ must be chosen so that the corrections to the back reaction remain suitably small. That is, we must ensure that the effect of T^{ν}_{μ} is a perturbation of the Schwarzschild geometry. This condition will be satisfied provided the proper (orthonormal frame) perturbations, given by

$$h^r_r = w\,\frac{\mu(w)}{1-w}, \tag{24}$$

$$h^t_t=-2[\rho(w_0)-\rho(w)]-h^r_r \tag{25}$$

with $w_0=2M/r_0$, obey

$$\epsilon|h^{\alpha}_{\beta}|\equiv\delta<1. \tag{26}$$

The cavity radius should be chosen so that $r_0\leqslant r_{\rm asymp}$, where $r_{\rm asymp}$ is the asymptotic radius which is the maximum radius for which the metric perturbations remain small. Note that $h^t_t=-h^r_r$ if $r=r_0$. Hereafter, we will take $\delta=\epsilon$ for illustrative purposes. Hence $|h^{\alpha}_{\beta}|=1$. In addition, we shall use $r_0=r_{\rm asymp}$ in the following. Now, to obtain the asymptotic radius $r_{\rm asymp}(=r_0)$ we go to the limit $r\to r_0$ with r_0 tending to a very large but finite value. Taking the leading terms in $|h^r_r|$, we can write

$$\begin{aligned} \lim_{r\to r_0,\, r_0\to \text{large value}} |h^r_r| &\sim \frac{2}{3K}\left(\frac{r}{2M}\right)^2+\frac{13}{6K}\,\hat{c}_2\left(\frac{2M}{r}\right) \\ &=\left(\frac{\delta}{\epsilon}\right)=1. \end{aligned} \tag{27}$$

Solving this equation gives the asymptotic radius $r_{\rm asymp}$.

In order to establish the bounds for $\hat{c}_2$ we must use the requirement of ''smallness'' of the perturbations $|h^{\alpha}_{\beta}|$ in the entire region of r, $2M\leqslant r\leqslant r_0$. In particular, let us check $|h^{\alpha}_{\beta}|$ at the horizon where $r=2M$ (or $w=1$). To obtain the value of h^r_r let us rewrite expression (19) for $\mu(r)$ as

$$\begin{aligned} \mu(w)=-\frac{1-w}{6K}\,&[32\hat{a}_2(-w^2-w-1)+\hat{b}_2(49w^2+40w+25 \\ &-13w^{-1}-4w^{-2}-w^{-3})+\hat{c}_2(-7w^2-10w-13)] \\ &-\frac{1}{6K}\,12\hat{b}_2\,\ln(w). \end{aligned} \tag{28}$$

Now using Eq. (24) we find that the value of h^r_r at the horizon ($w=1$) is

$$h^r_r|_{w=1}=\frac{20\,016+30\hat{c}_2}{72\,382}. \tag{29}$$

Here, we have used the results that

$$\lim_{w\to 1}\frac{\mu(w)}{1-w}=-\frac{1}{6K}(-96\hat{a}_2+96\hat{b}_2-30\hat{c}_2)+\frac{12\hat{b}_2}{6K}. \tag{30}$$

The ln-term gives a positive contribution. The requirement that $\epsilon|h_r^r|=\delta\ll 1$ (or $|h_r^r|=1$) yields the double inequality

$$-3080\leqslant\hat{c}_2\leqslant 1746. \tag{31}$$

Now we may substitute the value $\hat{c}_2=-3080$ into Eq. (27). It is easy to see that the second term in Eq. (27), which contains $\hat{c}_2$, is much less than the first one. So we may neglect the second term, and then the equation takes the form

$$\frac{2}{3K}\left(\frac{r}{2M}\right)^2=1. \tag{32}$$

We have from (32) the following value of the asymptotic radius: $r_0=260M$.

Now let us consider the value of h_t^t at the horizon ($w=1$). We will take $\hat{c}_2=-3080$ and $r_0=260M$. Leaving only the main terms in the expression (25) for h_t^t and using (20) and (29) we obtain

$$\begin{aligned} h_t^t|_{w=1}&=-2\rho(w_0)-h_r^r|_{w=1}\\ &\approx-2\rho(w_0)+1\\ &\approx-\frac{1}{3K}\hat{b}_2w_0^{-2}+\frac{9\hat{c}_2}{3K}+1\\ &\approx-1.64. \end{aligned} \tag{33}$$

Hence, in order that the quantity $\epsilon|h_t^t|$ be small at the horizon, it is necessary that $\epsilon\leqslant 0.6$.

IV. THERMODYNAMICAL ENTROPY

The thermodynamical entropy S of the black hole in thermal equilibrium with the conformal spin-2 quantum field can be computed following the method presented in [6]. From the first law of thermodynamics applied to slightly differing equilibrium systems

$$dE=dQ \quad (dr=0, r\leqslant r_0) \tag{34}$$

and so

$$dS=\frac{dQ}{T}=\frac{dE}{T}=\beta dE, \tag{35}$$

where β is the inverse local temperature [4],

$$\beta(w)=\frac{8\pi M}{\hbar}\{1+\epsilon[\rho(w)-n_2K^{-1}]\}\left(1-\frac{2m(w)}{r}\right)^{1/2}, \tag{36}$$

and

$$E(r)=r-r[g^{rr}(r)]^{1/2} \tag{37}$$

is the quasilocal energy [15], with g^{rr} determined by (22). Choosing M and r as independent variables and fixing r, we can integrate (35) to obtain the total system entropy as a distribution for $r<r_0$:

$$S=\frac{4\pi M^2}{\hbar}+\Delta S, \tag{38}$$

where

$$\Delta S=8\pi\int_1^w\left[\tilde{w}^{-1}(\rho-\mu)+\frac{\partial\mu}{\partial\tilde{w}}-n_2K^{-1}\tilde{w}^{-1}\right]d\tilde{w}, \tag{39}$$

and

$$n_2=\left(\frac{\partial K\mu}{\partial w}\right)_{w=1}=-16\hat{a}_2+14\hat{b}_2-5\hat{c}_2. \tag{40}$$

The quantity ΔS is therefore the amount by which the quantum field changes the Bekenstein-Hawking entropy, $S_{BH}\equiv 4\pi M^2/\hbar$, through its back reaction. S is a function of r and is the total system entropy (black hole plus radiation) contained within the region of radius r. Working out the integral with the explicit forms for ρ and μ calculated above, we find that

$$\begin{aligned} \Delta S/8\pi=&\left(\frac{32\hat{a}_2}{6K}\right)\left(-\frac{2}{3}w^3+2\ln(w)+\frac{2}{3}\right)+\left(\frac{\hat{b}_2}{6K}\right)\left(\frac{4}{3}w^{-3}\right.\\ &+4w^{-2}+12w^{-1}-16\ln(w)-20w-12w^2\\ &\left.+28w^3-\frac{40}{3}\right)+\left(\frac{\hat{c}_2}{6K}\right)[4w-4w^3+8\ln(w)]. \end{aligned} \tag{41}$$

As a general consequence of (39), the horizon is a local extremum with respect to r since

$$\left(\frac{\partial\Delta S}{\partial w}\right)_{M,w=1}=8\pi\left[w^{-1}(\rho-\mu)+\frac{\partial\mu}{\partial w}-n_2w^{-1}K^{-1}\right]_{w=1}=0, \tag{42}$$

as follows from the fact that $\rho(1)=\mu(1)=0$, and the definition of n_2. On physical grounds, we demand that the horizon be a local minimum to prevent the existence of a spherical shell of negative entropy near $r=2M$. The same physical criterion was employed recently to establish limits in the range of the nonminimal coupling constant between the scalar field and the scalar curvature [8]. Computing the second derivative of ΔS at the horizon yields

$$\frac{K}{8\pi}\left(\frac{\partial^2\Delta S}{\partial w^2}\right)_{w=1}=-32\hat{a}_2+\frac{112}{3}\hat{b}_2-\frac{16}{3}\hat{c}_2. \tag{43}$$

Substituting in the values of $\hat{a}_2=212$ and $\hat{b}_2=4$, we find that the horizon will be a local minimum of the entropy if and only if $\hat{c}_2<-1241.5$. Values of $\hat{c}_2$ satisfying this inequality will only guarantee that $\Delta S\geqslant 0$ and increasing at the horizon. In fact, Eq. (41), because it contains a term in $\hat{c}_2$, cannot be non-negative and monotone increasing (as a function of r for fixed M) for all values of $\hat{c}_2$. In order that ΔS be a positive and increasing function everywhere (strictly speaking, for $2M\leqslant r<r_0$), we cannot allow $(\partial\Delta S/\partial r)_M=0$ for *any* value

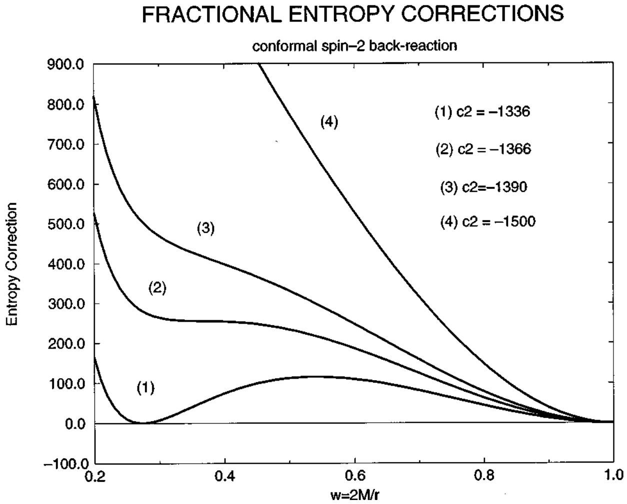

FIG. 1. Entropy correction $(3K/4\pi)\Delta S(w)$ due to the spin-2 back reaction, for various values of $\hat{c}_2$.

of $r>2M$. That is, we do not allow for a spherical layer of negative entropy at any value of r beyond the horizon. Explicit calculation of ΔS indicates that the smallest value (in absolute magnitude) of $\hat{c}_2$ satisfying this condition is $\hat{c}_2=-1366$, which occurs at $r\sim 5.7M$ see, e.g., Fig. 1. We plot the quantity $(3K/4\pi)\Delta S$ versus $w=2M/r$ in Fig. 1 for the values of $\hat{c}_2$ indicated there. Note that large $(r\gg 2M)$ values of the radial coordinate correspond to small values of $w(0<w\leqslant 1)$. One sees in particular that ΔS is positive and increasing at the black hole horizon since $\hat{c}_2<-1241.5$ in all the cases displayed, but that there can (and do) arise local extrema away from the horizon depending on the precise value of $\hat{c}_2$. For example, as indicated in Fig. 1, the value $\hat{c}_2=-1336$ yields a ΔS which increases from the horizon, only to vanish again at $r\sim 7.3M$, after which it increases once more. The entropy due to the spin-2 back reaction will therefore be a positive and monotonically increasing function of r, within the present approximation, provided that $\hat{c}_2<-1366$. Putting this result together with the perturbative bounds obtained earlier, we conclude that $-3080<\hat{c}_2<-1366$, simultaneosly guaranteeing well-behaved entropy as well as the perturbative validity of the solution. Returning to the boundary condition in (13), we see that $\hat{c}_2$ gives a measure of the average quantum fluctuation of the spin-2 field at the black hole horizon. We may thus employ the criterion of physically well-behaved entropy and perturbative validity to establish bounds in the range of spin-2 field fluctuations at the horizon. Working out (13) for $\mu=\nu=0$ (the other components give no new information), we obtain the following bounds for the energy density at the horizon:

$$25.1>\frac{\pi^2}{\kappa^4}\langle T^t_t\rangle^{\text{ren}}_{w=1}>7.3. \tag{44}$$

This should be contrasted to the exact result

$$\frac{\pi^2}{\kappa^4}\langle T^t_t\rangle^{\text{ren}}_{w=1}=0.63, \tag{45}$$

for the spin-1 case and to the corresponding exact result

TABLE I. Spin-dependent constants appearing in Eq. (3).

Spin=s	$\hat{a}_s$	$\hat{b}_s$	$\hat{c}_s$
0	1	2	0
$\frac{1}{2}$	$\frac{7}{4}$	$\frac{7}{2}$	5
1	-13	4	-8
2	212	4	$-3080<\hat{c}_2<-1366$

$$\frac{\pi^2}{\kappa^4}\langle T^t_t\rangle^{\text{ren}}_{w=1}=0.05 \tag{46}$$

for the spin-0 case. The average quantum fluctuations for the spin-2 field at the horizon are at least 100 times larger than for the scalar field case, and at least 10 times greater than for the spin-1 case.

Though in the present work we are primarily interested in the physical properties of the spin-2 back reaction, it is of interest to apply the Killing approximation (3) in order to evaluate ΔS arising from the back reaction of some lower spin conformal fields. Indeed, as the entropy corrections have been calculated independently for the cases spin $s=0,1/2,1$ [6], we have a means of checking the accuracy of this approximation explicitly for the conformal scalar and for the U(1) gauge boson, two cases where exact renormalized stress-energy tensors are known. Stress tensors renormalized on a Schwarzschild background have been obtained in exact form by Howard [1] for the conformal scalar and by Jensen and Ottewill (JO) [2] for the Abelian vector boson. In both cases, there also exist excellent analytic approximations for the full numeric calculations. The analytic form for the conformal scalar was first given by Page (P) [16]. We use these analytic results to calculate $\hat{c}_s$ by means of the horizon boundary condition, Eq. (13). For the massless spin-1/2 fermion, we have used the tensor of Brown, Ottewill, and Page (BOP) [17], but the accuracy of their approximation has not, to our knowledge, been checked against an exact numerical calculation. We collect the various spin-dependent coefficients needed to evaluate ΔS based on the Frolov-Zel'nikov (FZ) approximate tensor, in Table I.

The comparison of the (spin-dependent) entropies calculated from the various stress-energy tensors may be summarized compactly in the following manner:

$$\Delta S_{\text{FZ}}\equiv\Delta S_P \quad (\text{spin}=0), \tag{47}$$

$$\Delta S_{\text{FZ}}\equiv\Delta S_{\text{BOP}} \quad (\text{spin}=1/2), \tag{48}$$

$$\Delta S_{\text{FZ}}-\Delta S_{\text{JO}}=32[w^3-w-2\ln(w)] \quad (\text{spin}=1). \tag{49}$$

The explicit expressions for the corrections calculated from other stress-energy tensors are presented in Table II; the results are taken from [6].

As discussed in [6], the corrections $\Delta S_P\geqslant 0$, $\Delta S_{\text{BOP}}\geqslant 0$, and $\Delta S_{\text{JO}}\geqslant 0$ and are monotonically increasing for all $r\geqslant 2M$. Since the difference $(\Delta S_{\text{FZ}}-\Delta S_{\text{JO}})$ vanishes at the horizon and grows as 64 ln(r) for larger r, ΔS_{FZ} is also a positive and increasing function of r for spin 1. The condition that the horizon be a local minimum of the entropy,

TABLE II. Spin-dependent entropy corrections ΔS based on other stress-energy tensors.

Spin$=s$	Stress tensor	$\Delta S(w)$
0	P	$\frac{8\pi}{K}\left[\frac{52}{9}w^3-4w^2-\frac{20}{3}w+\frac{16}{3}\ln(w)+4w^{-1}+\frac{4}{3}w^{-2}+\frac{4}{9}w^{-3}-\frac{8}{9}\right]$
$\frac{1}{2}$	BOP	$\frac{7}{8}\frac{8\pi}{K}\left[\frac{488}{63}w^3-8w^2-\frac{200}{21}w+\frac{128}{7}\ln(w)+8w^{-1}+\frac{8}{3}w^{-2}+\frac{8}{9}w^{-3}-\frac{16}{9}\right]$
1	JO	$\frac{8\pi}{K}\left[\frac{344}{9}w^3-8w^2+\frac{40}{3}w-96\ln(w)+8w^{-1}+\frac{8}{3}w^{-2}+\frac{8}{9}w^{-3}-\frac{496}{9}\right]$

$\Delta S_{\rm FZ}$ Eq. (41), yields the inequalities $\hat{c}_0<8$, $\hat{c}_{1/2}<14$ and $\hat{c}_1<106$, respectively, which are automatically satisfied by the values displayed in Table I, derived from the exact stress tensors for $s=0,1$ and by the value derived from the approximate BOP tensor for the spinor case. In Fig. 2 we plot the various spin-dependent entropy corrections $(3K/4\pi)\Delta S(w)$ whose functional forms are listed in Table II. We should point out that in the case of the electromagnetic field, an exact value of $\langle T_\mu^\nu\rangle_{\rm ren}$ at the black hole horizon was calculated some time ago in [18,19]. Those results lead, however, to a value for $\hat{c}_1=92h(2)=184$, which clearly violates the entropy positivity bound cited above. The reason for the large discrepancy between the two values of $\hat{c}_1$ (-8 versus 184) is due to an important linearly divergent Christensen subtraction term which had been overlooked in the earlier calculations [2].

V. EFFECTIVE POTENTIAL

Further insight into the nature of the modified black hole metric may be gained by studying the motion of test particles in the vicinity of the horizon. To this end, we compute the effective potential of the perturbed black hole, as this completely characterizes the motion of both massless and massive test particles.

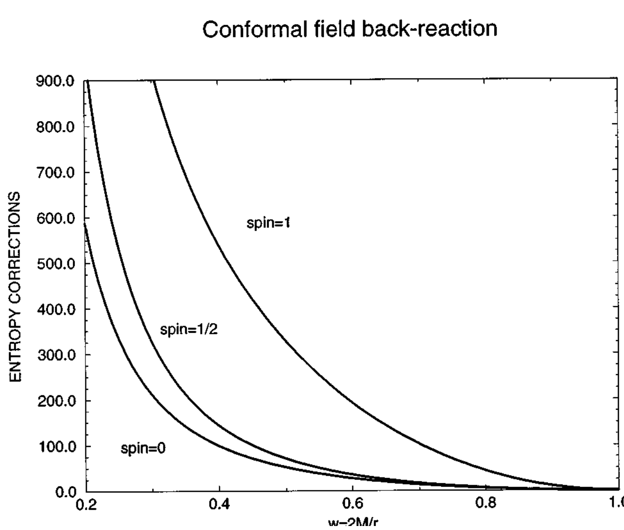

FIG. 2. Entropy corrections $(3K/4\pi)\Delta S(w)$ arising from spin-0, 1/2, 1 back reactions.

We present a derivation for the effective potential for point particles moving in a general static and spherically symmetric background based on a Hamilton-Jacobi approach. The line element and metric of such background spacetimes can always be cast in the form

$$ds^2=g_{tt}(r)dt^2+g_{rr}(r)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{50}$$

The trajectory of a point particle of mass m moving in this background can be obtained from the Hamilton-Jacobi equation

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}+m^2=0, \tag{51}$$

where S denotes the action (not to be confused with the entropy) of the particle. As in every spherically symmetric field of force, the motion occurs in a fixed plane passing through the origin; we take this plane coincident with the slice defined by $\theta=\pi/2$ without any loss of generality. Then, expanding out (51) gives the differential equation for S,

$$g^{tt}\left(\frac{\partial S}{\partial t}\right)^2+g^{rr}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2+m^2=0. \tag{52}$$

By the general procedure for solving the Hamilton-Jacobi equation, we look for an S of the form [20]

$$S(t,r,\phi)=-Et+L\phi+S_r(r), \tag{53}$$

with constant energy E and angular momentum L. Substituting this ansatz into (52) gives an equation for S_r which may be integrated immediately to yield

$$S_r(r)=\int^r(-g^{tt}g_{rr})^{1/2}\left[E^2+g_{tt}\left(m^2+\frac{L^2}{r^2}\right)\right]^{1/2}dr+\delta_r, \tag{54}$$

where δ_r is an arbitrary additive phase constant. The dependence $r=r(t)$ for the radial coordinate of the particle is given by

$$\frac{\partial S}{\partial E}=\gamma=\text{const}, \tag{55}$$

or

$$t=-\gamma+\int^r\frac{(-g^{tt}g_{rr})^{1/2}E}{[E^2+g_{tt}(m^2+L^2/r^2)]^{1/2}}. \tag{56}$$

This can be cast in terms of a differential equation for r as

$$\left(\frac{dr(t)}{dt}\right)=(-g_{tt}g^{rr})^{1/2}\frac{1}{E}\left[E^2+g_{tt}\left(m^2+\frac{L^2}{r^2}\right)\right]^{1/2}, \tag{57}$$

which governs the radii of allowed orbits of particles moving in the gravitational field represented by (22). Hence the function

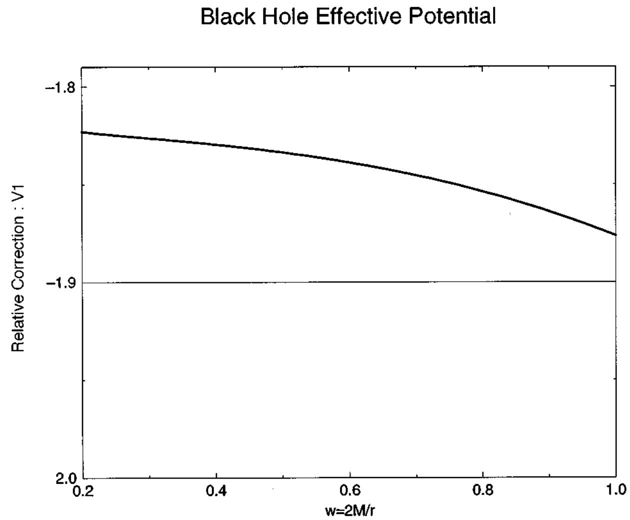

FIG. 3. Relative correction $V_1(w)$ to the black hole effective potential.

$$V(r) \equiv -g_{tt}\left(m^2+\frac{L^2}{r^2}\right), \tag{58}$$

plays the role of the effective potential energy in the sense that the condition $E^2 > V(r)$ determines the admissable range of the particle's motion. Identifying g_{tt} from the solution (22) yields $V(r)$ for the semiclassical black hole:

$$V(w) = (1-w)\{1+\epsilon[2\overline{\rho(w)} - w(1-w)^{-1}\mu(w)]\} \times\left(m^2+\frac{L^2}{4M^2}w^2\right) = V_0(w)[1+\epsilon V_1(w)], \tag{59}$$

and reduces, as it must, to the classical Schwarzschild potential as $\epsilon \to 0$. The relative correction V_1 depends on the cavity radius r_0 and on $r < r_0$. For purposes of illustration, we consider the asymptotic radius $r_0 = 260M$, which is the maximum radius for which the metric perturbations (call them δ) remain uniformly small: that is, $\delta \ll 1$ for $2M \leqslant r < r_0$, as discussed at length in [7]. (The perturbatively valid domains for the spin-1 and spin-2 back reactions are identical, by virtue of the fact that $\hat{b}_1 = \hat{b}_2$). A graphical analysis of V_1 shows that the relative correction is large and negative for all $r \geqslant 2M$ and is insensitive to the particular value of $\hat{c}_2$ used to calculate it. In Fig. 3 we display $V_1(w)$ taking $\hat{c}_2 = -1400$. A glance at this figure shows that $V_1 \sim -1.8$, so that a small amount of back reaction, say $\epsilon = 0.1$, would lead to roughly a 20% decrease in the effective potential. A decreased potential, in turn, implies an increase in the black hole capture cross section, as discussed in [7].

VI. DISCUSSION

The solution of the lowest-order back reaction of a conformal spin-2 field on a Schwarzschild black hole has been calculated based on the approximate stress energy tensor of Frolov and Zel'nikov. The new equilibrium metric has been found to order $O(\hbar)$. By calculating the thermodynamical entropy by which the spin-2 field modifies the Bekenstein-Hawking entropy, we have been able to put an upper limit on one of the constants ($\hat{c}_2$) which parametrizes this approximate stress-energy tensor. This is done by demanding that the fractional correction to the entropy be a positive and monotone increasing function of r, over the entire domain of perturbative validity of the solution. This has led to an upper bound for $\hat{c}_2$. (Conditions that general thermal stress-energy tensors must satisfy in order that $\Delta S \geqslant 0$ have been established recently by Zaslavskii [21].) This bound translates physically into a *lower* limit for the magnitude of the quantum fluctuations of the spin-2 field at the black hole horizon. By demanding that the metric perturbations be small over the range in which the perturbative solution is valid, we have also estimated a lower bound for $\hat{c}_2$, though this particular bound depends on the cavity radius r_0 and on the mass ratio $\epsilon = (M_{\rm Pl}/M)^2 \ll 1$. Nevertheless, these limits translate, via the boundary condition (13), into corresponding bounds on the magnitude of the spin-2 field fluctuations at the black hole horizon. In the context of the spin hierarchy of conformal field back reaction, the spin-2 case is clearly the most important, giving by far, the largest correction. This is manifested in the sequence of spin-dependent horizon-energy densities and is exposed in a striking way in the relative correction to the black hole effective potential. It is hoped that these calculations may shed light on some of the gross features characterizing the graviton back reaction. Certainly, the marked increase in the magnitude of the fluctuations in going from low spins to spin=2 is a feature we expect to persist in the graviton case, as well as the relative increase in the entropy correction.

ACKNOWLEDGMENTS

One of us (D.H.) thanks Juan Pérez-Mercader, Alvaro Domínguez, and Thomas Buchert for useful comments. The work of S.S.V. was supported in part by the Russian Foundation of Fundamental Research Grant No. 96-02-17066a.

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