

Quasinormal modes of maximally charged black holes

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A new algorithm for computing the accurate values of quasinormal frequencies of extremal Reissner-Nordström black holes is presented. The numerically computed values are consistent with the values earlier obtained by Leaver and those obtained through the WKB method. Our results are more precise than other results known to date. We also find the curious fact that the resonant frequencies of gravitational waves with a multipole index l coincide with those of electromagnetic waves with a multipole index $l-1$ in the extremal limit. [S0556-2821(96)00312-8]

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I. INTRODUCTION

The determination of the quasinormal frequencies of black holes has for several years been of interest and is related to experiments which aim at detecting gravitational waves from supernovae or coalescences of binary neutron stars, which are thought to eventually form a black hole. In the late stage of the process of black hole formation, a certain mode of the gravitational wave dominates the emission. This is called the quasinormal mode of a black hole.

The equations governing oscillations of black holes are derived by Regge and Wheeler [1], and Zerilli [2] for the nonrotating uncharged case, Zerilli [3] and Moncrief [4] for the nonrotating charged case, and Teukolsky [5] for the rotating uncharged case. The resonant frequencies of the Schwarzschild black holes were first computed by Chandrasekhar and Detweiler [6], who treated it as a boundary value problem of the second order ordinary differential equation of Regge and Wheeler. Integration of the Regge-Wheeler equation for the quasinormal boundary condition is numerically unstable. To avoid the instability, Leaver [7] presents the numerically stable continued fraction method for Schwarzschild and Kerr black holes. Here, a quasinormal mode function corresponds to a minimal solution of the recurrence relations which are satisfied by the coefficients of a series expansion of a wave function. The minimal solution of the three-term recurrence relation is obtained by the corresponding continued fraction [8]. The continued fraction method can give the values of frequencies with high numerical precision because it uses no approximation, as is the case in the semianalytic WKB method developed in other works [9–12]. Leaver [13] also generalizes the continued fraction method to calculate accurate values of the quasinormal frequencies of Reissner-Nordström black holes, though Kokkotas and Schutz [14] raised the question of the applicability of the continued fraction method to charged black holes.

We are indeed motivated by Leaver's paper [13]. There is another interesting question, that is, whether a continued fraction method can be used for the maximally charged black holes not discussed in his paper. In the extremal limit, the

wave equation has an irregular singular point at the horizon, which makes the series expansion of a solution there invalid, and thus a continued fraction method does not seem to work. In this paper we show that a continued fraction method is applicable even to the extremal black holes if we expand a solution about a suitable ordinary point. In Sec. II, we derive an eigenvalue equation with continued fractions after expanding a wave function about an ordinary point to determine the quasinormal frequencies of the extremal Reissner-Nordström black hole. In Sec. III, we present the numerical results and find an interesting coincidence between the resonant frequencies of gravitational waves with a multipole index l and those of electromagnetic waves with a multipole index $l-1$ in the limit of maximal charge. Section IV is devoted to conclusions and discussion.

II. EIGENVALUE EQUATION FOR QUASINORMAL MODES

Leaver obtained the recurrence relations of Schwarzschild, Kerr [7], and Reissner-Nordström black holes [13] by expanding solutions about the event horizon and then solved the characteristic equations with a continued fraction to get these resonant frequencies accurately. However, his method is not valid when a black hole is extremal. The wave equation has an irregular singular point at the horizon about which the expansion of the solution is not available because a radius of convergence for a series expansion about an irregular singular point generally vanishes. Therefore, we have to expand the solution about a suitable ordinary point in the extremal black holes.

We start from the Zerilli-Moncrief equation for perturbations on Reissner-Nordström backgrounds of mass M and charge Q written as

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_s(r) \right] Z_s(r) = 0, \quad (1)$$

where

$$\frac{dr}{dr_*} = \frac{\Delta}{r^2}, \quad (2)$$

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$$V_s = \frac{\Delta}{r^5} \left[Ar - q_s + \frac{p_s}{r} \right], \tag{3}$$

$$\Delta = r^2 - 2Mr + Q^2, \tag{4}$$

$$q_0 = 2, \tag{5}$$

$$q_1 = 3M - \sqrt{9M^2 + 4Q^2(A-2)}, \tag{6}$$

$$q_2 = 3M + \sqrt{9M^2 + 4Q^2(A-2)}, \tag{7}$$

$$p_0 = 2, \tag{8}$$

$$p_1 = p_2 = 4, \tag{9}$$

$$A = l(l+1), \tag{10}$$

and $s=0, 1, 2$ are for a massless scalar field, an odd parity electromagnetic perturbation, and an odd parity gravitational perturbation, respectively. The new parameter p_s is introduced here in order to deal with the case of the scalar field as well. Note that the definitions of q_1 and q_2 are exchanged with each other from the book of Chandrasekhar [15], though the most of the other notations follow that book. The scaling of t and r in this paper is such that $c = G = M = 1$. We also use $\rho = -i\omega$ as a frequency variable for convenience.

Now we follow Leaver's procedure of getting a recurrence relation which plays a central role in a continued fraction method. The tortoise coordinate in the extremal limit becomes

$$r_* = r + \ln(r-1)^2 - \frac{1}{r-1}, \tag{11}$$

which makes a difference in the basic equations between extremal and nonextremal black holes. The Zerilli-Moncrief equation (1) is therefore rewritten in terms of the r coordinate as

$$r(r-1) \frac{d^2 Z_s}{dr^2} + 2 \frac{dZ_s}{dr} - \left[\frac{\rho^2 r^5}{(r-1)^3} + \frac{Ar}{r-1} - \frac{q_s}{r-1} + \frac{p_s}{r(r-1)} \right] Z_s = 0. \tag{12}$$

This equation has irregular singular points at $r=1$ and $r=\infty$ as we mentioned in the previous section, which makes the series expansion about the horizon invalid because it is hopeless that the series expansion is convergent. Hence, it is necessary to improve Leaver's treatment of obtaining the quasinormal modes of the extremal black holes.

The boundary conditions of a quasinormal wave function are given by

$$Z_s \sim e^{-\rho r_*} \quad \text{as } r_* \rightarrow \infty, \tag{13}$$

$$Z_s \sim e^{\rho r_*} \quad \text{as } r_* \rightarrow -\infty, \tag{14}$$

which mean there only exists purely outgoing wave at infinity and purely ingoing wave at the black hole horizon. These are also transformed to

$$Z_s \sim r^{-2\rho} e^{-\rho r} \quad \text{as } r \rightarrow \infty, \tag{15}$$

$$Z_s \sim e^{\rho r} (r-1)^{2\rho} e^{-\rho \frac{1}{r-1}} \quad \text{as } r \rightarrow 1, \tag{16}$$

which determine a prefactor of a quasinormal wave function in the form

$$Z_s = e^{\rho r} e^{-\rho \frac{1}{r-1}} (r-1)^{2\rho} r^{-4\rho} e^{-2\rho(r-1)} \sum_{n=0}^{\infty} a_n u^n, \tag{17}$$

where u is specified by a point where we expand a solution. We use a series expansion about an ordinary point, which has a finite radius of convergence at least up to the nearest singular point. Thus, we use $u=(r-2)/r$ instead of $u=(r-1)/r$ which is used in Ref. [13]. In our choice of u the black hole horizon and infinity correspond to $u=-1, 1$, respectively. The distances from the point $u=0$ to two singular points $u=-1, 1$ are equal, which can make it possible to examine the boundary conditions at both sides simultaneously. It is clear that the quasinormal mode boundary conditions are satisfied if and only if $\sum a_n u^n$ converges at both boundaries, horizon, and infinity. This is equivalent to the condition that both $\sum a_n$ and $\sum (-)^n a_n$ are finite, that is, both $\sum a_{2n}$ and $\sum a_{2n+1}$ are convergent simultaneously. To examine the convergence of each summation, we first obtain the recurrence relation of coefficients a_n of the expansion (17), and then attempt to obtain the recurrence relation only for $b_n = a_{2n+1}$ and that of $c_n = a_{2n}$. The quasinormal mode boundary conditions are satisfied when the odd and even sequences are minimal solutions of the corresponding recurrence relations, respectively.

After substituting the above series expansion (17) into Eq. (12) we obtain a five-term recurrence relation:

$$\alpha_1 a_2 + \gamma_1 a_0 = 0, \tag{18}$$

$$\alpha_2 a_3 + \gamma_2 a_1 + \delta_2 a_0 = 0, \tag{19}$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} + \epsilon_n a_{n-3} = 0 \quad (n \geq 3), \tag{20}$$

where the recurrence coefficients are given by

$$\alpha_n = n^2 + n, \tag{21}$$

$$\beta_n = 0, \tag{22}$$

$$\gamma_n = -4A - 2n^2 - p_s + 2q_s + n(2 - 24\rho) + 12\rho - 64\rho^2, \tag{23}$$

$$\delta_n = 2(p_s - q_s), \tag{24}$$

$$\epsilon_n = n^2 - p_s - 12\rho + 16\rho^2 + n(-3 + 8\rho). \tag{25}$$

For the scalar field, we can find that $\beta_n = \delta_n = 0$ in the above relation, which means an odd sequence and even sequence are completely decoupled with each other, because the Zerilli-Moncrief equation in terms of u is symmetric under the transformation $u \rightarrow -u$. Therefore, it has only symmetric or antisymmetric solutions as quasinormal mode functions. This is the same as the case of a quantum-mechanical system with a reflection symmetry, where a wave function of

energy eigenstate must be either symmetric or antisymmetric. This makes the calculation of quasinormal frequencies of the scalar field easy. Each sequence is characterized by the three-term recurrence relation (20) starting from the two-term relation (18) or (19). We can obtain two independent characteristic equations below similar to one for the quasinormal frequencies of Schwarzschild black holes:

$$\frac{a_2}{a_0} = -\frac{\gamma_1}{\alpha_1} = \frac{-\epsilon_2}{\gamma_2^-} \frac{\alpha_2 \epsilon_4}{\gamma_4^-} \frac{\alpha_4 \epsilon_6}{\gamma_6^-} \dots, \quad (26)$$

$$\frac{a_3}{a_1} = -\frac{\gamma_2}{\alpha_2} = \frac{-\epsilon_3}{\gamma_3^-} \frac{\alpha_3 \epsilon_5}{\gamma_5^-} \frac{\alpha_5 \epsilon_7}{\gamma_7^-} \dots \quad (27)$$

These two equations are for the modes of even-numbered and those of odd-numbered overtones, respectively.

For gravitational or electromagnetic quasinormal modes, we have to develop a little more complicated way because the presence of the δ_n term makes odd and even sequences dependent on each other, though it seems that they are not much correlated for large n because of the fact that the coefficient δ_n becomes relatively small as n increases compared with other terms. We first need to eliminate the odd or even sequence to examine the convergence of the summation of each sequence. This procedure is presented in Appendix A. For b_n a five-term recurrence relation is obtained,

$$\hat{\alpha}_1 b_2 + \hat{\beta}_1 b_1 + \hat{\gamma}_1 b_0 = 0, \quad (28)$$

$$\hat{\alpha}_2 b_3 + \hat{\beta}_2 b_2 + \hat{\gamma}_2 b_1 + \hat{\delta}_2 b_0 = 0, \quad (29)$$

$$\hat{\alpha}_n b_{n+1} + \hat{\beta}_n b_n + \hat{\gamma}_n b_{n-1} + \hat{\delta}_n b_{n-2} + \hat{\epsilon}_n b_{n-3} = 0 (n \geq 3), \quad (30)$$

and for c_n we get the relation

$$\bar{\alpha}_2 c_3 + \bar{\beta}_2 c_2 + \bar{\gamma}_2 c_1 + \bar{\delta}_2 c_0 = 0, \quad (31)$$

$$\bar{\alpha}_n c_{n+1} + \bar{\beta}_n c_n + \bar{\gamma}_n c_{n-1} + \bar{\delta}_n c_{n-2} + \bar{\epsilon}_n c_{n-3} = 0 \quad (n \geq 3), \quad (32)$$

where each recurrence coefficient is also given in Appendix A.

As mentioned previously, the quasinormal mode boundary conditions are satisfied if and only if both sequences are minimal solutions of recurrence relations simultaneously. In order to use a continued fraction method, both five-term recurrence relations are transformed to three-term recurrence relations by successive Gaussian elimination steps which are exhibited in Appendix B. Then, the ratios of the first two terms of both sequences for minimal solutions are determined by the corresponding continued fractions in

$$\frac{a_4}{a_2} = \frac{c_2}{c_1} = \frac{-\bar{\gamma}_2''}{\bar{\beta}_2''} \frac{\bar{\alpha}_2'' \bar{\gamma}_3''}{\bar{\beta}_3''} \frac{\bar{\alpha}_3'' \bar{\gamma}_4''}{\bar{\beta}_4''} \dots, \quad (33)$$

$$\frac{a_3}{a_1} = \frac{b_1}{b_0} = \frac{-\hat{\gamma}_1''}{\hat{\beta}_1''} \frac{\hat{\alpha}_1'' \hat{\gamma}_2''}{\hat{\beta}_2''} \frac{\hat{\alpha}_2'' \hat{\gamma}_3''}{\hat{\beta}_3''} \dots, \quad (34)$$

where the quantities with double prime are recurrence coefficients of transformed three-term recurrence relations in Appendix B.

In contrast to the scalar field case, the even and odd sequences are coupled to each other through the nonvanishing δ_n terms. By dividing Eq. (20) for $n=3$ by Eq. (20) for $n=5$, we get the relation between the ratio of the first two terms of odd sequence and that of even sequence as

$$\begin{aligned} \frac{a_3}{a_1} &= \frac{\alpha_5 \frac{a_6}{a_4} + \gamma_5 + \epsilon_5 \frac{a_2}{a_4} \frac{a_2}{a_4}}{\alpha_2 \frac{a_4}{a_2} + \gamma_3 + \epsilon_3 \frac{a_0}{a_2} \frac{a_2}{a_2}} \\ &= \frac{-\frac{\alpha_5}{\bar{\alpha}_2''} \left(\bar{\beta}_2'' + \frac{a_2}{a_4} \bar{\gamma}_2'' \right) + \gamma_5 + \epsilon_5 \frac{a_2}{a_4} \frac{a_2}{a_4}}{\alpha_3 \frac{a_4}{a_2} + \gamma_3 - \epsilon_3 \frac{\alpha_1}{\gamma_1} \frac{a_2}{a_2}}, \end{aligned} \quad (35)$$

where the second equality comes from Eq. (18) and the transformed three-term relation (B30) for $n=2$. The quasinormal mode boundary conditions are satisfied when the continued fractions (33) and (34) satisfy Eq. (35). This is the characteristic equation for the quasinormal frequencies of the extremal Reissner-Nordström black hole.

Equation (35) can be shifted an arbitrary number of times n to yield an equation for the ratio of successive terms of higher n :

$$\frac{a_{n+3}}{a_{n+1}} = \frac{\alpha_{n+5} \frac{a_{n+6}}{a_{n+4}} + \gamma_{n+5} + \epsilon_{n+5} \frac{a_{n+2}}{a_{n+4}} \frac{a_{n+2}}{a_{n+4}}}{\alpha_{n+2} \frac{a_{n+4}}{a_{n+2}} + \gamma_{n+3} + \epsilon_{n+3} \frac{a_n}{a_{n+2}} \frac{a_{n+2}}{a_{n+2}}}. \quad (36)$$

Equation (35) and (36) are completely equivalent since every solution to (35) is also a solution to (36) when we substitute a continued fraction into each ratio of a_i/a_{i-2} , and vice versa. Equation (36) can be used as a check of results obtained from Eq. (35).

III. NUMERICAL RESULTS

The eigenvalue equations (26), (27), and (35) derived in the previous section are solved using the HYBRD subroutine distributed in MINPACK libraries. The three least-damped modes of $(s, l) = (2, 2), (2, 3), (2, 4), (1, 1), (1, 2), (1, 3), (0, 0), (0, 1), (0, 2)$ for the extremal Reissner-Nordström black hole are listed in Table I. We compare our results with Leaver's results [13] of nearly maximally charged case (99.99% charged), which should be divided by two because of the difference of our scaling from his. These modes appear as complex conjugate pairs in ρ because the eigenvalue equations are characterized by only real coefficients. We also compare them with the third order WKB quasinormal frequencies of the extremal Reissner-Nordström black hole. The WKB quasinormal frequencies are obtained using the same formula of Kokkotas and Schutz [14], who calculated the frequencies of charged black holes but did not show those of maximally charged black holes.

Our results are in agreement with other results obtained

TABLE I. The computed quasinormal frequencies are listed. Our results are very consistent with the values of WKB quasinormal frequencies. The relative errors of WKB complex and real gravitational quasinormal frequencies for $l=2$ from our results are (0.04%, 0.03%), (0.07%, 0.1%), and (0.1%, 0.5%) for $n=0, 1, 2$, respectively. Leaver's value has rather large discrepancy of 3% for the real frequency of $n=1$ gravitational wave.

	$s=2, l=2$	$s=2, l=3$	$s=2, l=4$
$n=0$	(-0.083460, 0.43134)	(-0.085973, 0.70430)	(-0.087001, 0.96576)
Leaver	(-0.083645, 0.43098)		
WKB	(-0.08349, 0.43013)	(-0.08596, 0.70398)	(-0.08700, 0.96563)
$n=1$	(-0.25498, 0.40452)	(-0.25992, 0.68804)	(-0.26212, 0.95381)
Leaver	(-0.257055, 0.39309)		
WKB	(-0.25675, 0.40076)	(-0.26014, 0.68701)	(-0.26218, 0.95339)
$n=2$	(-0.44137, 0.35340)	(-0.44007, 0.65624)	(-0.44064, 0.93020)
Leaver	(-0.442035, 0.353515)		
WKB	(-0.44210, 0.35136)	(-0.43986, 0.65575)	(-0.44056, 0.93004)
	$s=1, l=1$	$s=1, l=2$	$s=1, l=3$
$n=0$	(-0.083460, 0.43134)	(-0.085973, 0.70430)	(-0.087001, 0.96576)
Leaver	(-0.08343, 0.431415)	(-0.086205, 0.704075)	
WKB	(-0.08349, 0.43013)	(-0.08596, 0.70398)	(-0.08700, 0.96563)
$n=1$	(-0.25498, 0.40452)	(-0.25992, 0.68804)	(-0.26212, 0.95381)
Leaver	(-0.259705, 0.40602)	(-0.26256, 0.68315)	
WKB	(-0.25675, 0.40076)	(-0.26014, 0.68701)	(-0.26218, 0.95339)
$n=2$	(-0.44137, 0.35340)	(-0.44007, 0.65624)	(-0.44064, 0.93020)
Leaver	(-0.44260, 0.35347)	(-0.4408, 0.655675)	
WKB	(-0.44210, 0.35136)	(-0.43986, 0.65575)	(-0.44056, 0.93004)
	$s=0, l=0$	$s=0, l=1$	$s=0, l=2$
$n=0$	(-0.095844, 0.13346)	(-0.089384, 0.37764)	(-0.088748, 0.62657)
WKB	(-0.10371, 0.12109)	(-0.08936, 0.37570)	(-0.08873, 0.62609)
$n=1$	(-0.33065, 0.092965)	(-0.27614, 0.34818)	(-0.26909, 0.60817)
WKB	(-0.33742, 0.09157)	(-0.27828, 0.34392)	(-0.26944, 0.60677)
$n=2$	(-0.58833, 0.075081)	(-0.48643, 0.29846)	(-0.45820, 0.57287)
WKB	(-0.57164, 0.05056)	(-0.48145, 0.29661)	(-0.45750, 0.57254)

by Leaver and those obtained through the WKB method within the accuracy of a few percent. The difference between WKB frequencies and ours is ascribed to the fact that the WKB method gives only the approximate values. Indeed, the tendency that the discrepancy grows with the mode number suggests the breakdown of the WKB approximation for higher n . The difference between Leaver's results and ours is from the numerical error caused by the breakdown of his series expansion at the limit of maximal charge. Thus, we are sure that our quasinormal frequencies of the extremal Reissner-Nordström black hole are the most accurate. Our results and his are plotted in Fig. 1.

Notice that the quasinormal frequencies of $(s, l) = (2, 2)$ and those of $(1, 1)$ approach each other with increasing Q and coincide in the extremal limit. We show this curious coincidence in the limit of maximal charge explicitly in Fig. 2. The trajectories of the first four lowest modes of the third order WKB gravitational quasinormal frequencies of $l=2, 3, 4, 5$ are plotted. These quasinormal frequencies coincide with the frequencies of electromagnetic wave of $l=1, 2, 3, 4$ in the limit of maximal charge, respectively. The first order WKB quasinormal frequencies are given by $\omega_n^2 = V_{\text{peak}} - i(n+1/2)\sqrt{-2V''_{\text{peak}}}$ using the Zerilli-Moncrief Potential V , where a subscript n is a mode number. The

differences of V_{peak} and V''_{peak} between V_1 for $l=1$ and V_2 for $l=2$ are calculated to $\Delta V_{\text{peak}}/V_{\text{peak}} \approx 2.79565 \times 10^{-16}$ and $\Delta V''_{\text{peak}}/V''_{\text{peak}} \approx 5.46819 \times 10^{-15}$, respectively. This explains the coincidence of a few least-damped quasinormal frequencies of these two modes. Even for higher overtone modes $n \sim 30$, however, we numerically find a remarkable coincidence in the quasinormal frequencies of gravitational waves of l and those of electromagnetic waves of $l-1$ for a wide range of multipole indices, $2 \leq l \leq 10$, through our method.

These two potentials, V_1 with a multipole index l and V_2 with a multipole index $l+1$, indeed give the same transmission and reflection amplitude for each σ , whose evidence is presented in Appendix C. It implies that the quasinormal frequencies for both potentials are identical.

IV. CONCLUSIONS AND DISCUSSION

We improve the continued fraction method of computing quasinormal frequencies of charged black holes in order to determine the quasinormal frequencies in the extremal limit and solve our formulas of eigenvalue equations by using a root search program. Our results of quasinormal frequencies for the maximally charged black holes are consistent with

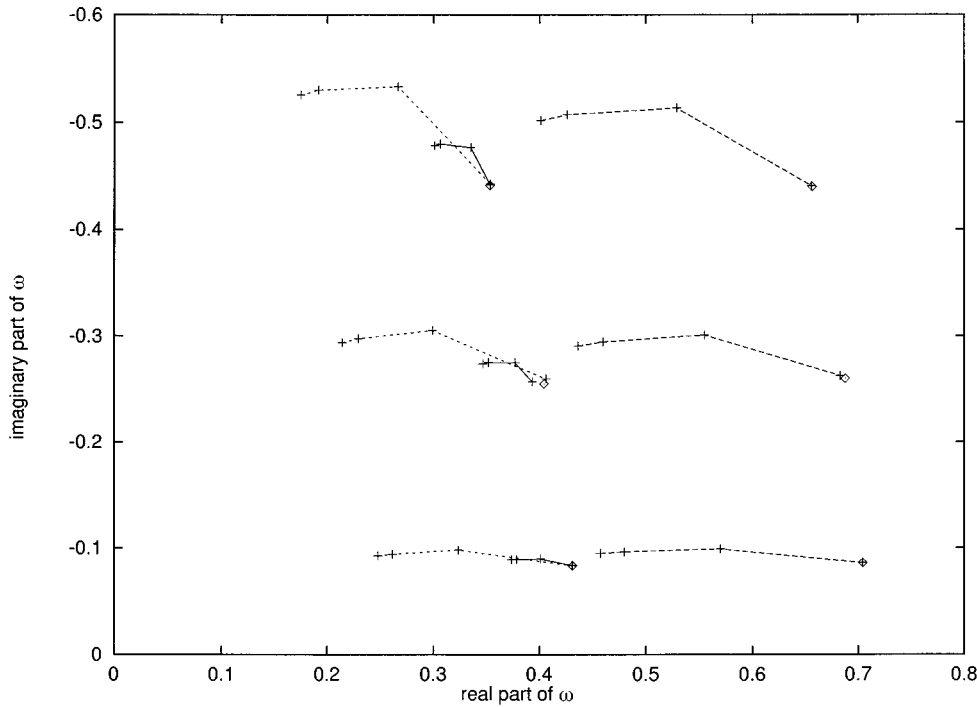


FIG. 1. Quasinormal frequencies for $l=2$ gravitational wave and $l=1, 2$ electromagnetic wave are plotted in the ω plane with the results of Leaver [13]. Each solid line is a trajectory of the first three lowest $l=2$ gravitational mode, parametrized by the charge Q . It has a tendency to coincide at the right end with $l=1$ electromagnetic mode which is shown as a small dashed line. Marks from left to right correspond to $Q=0, 0.4, 0.8, 0.9999$ quasinormal frequencies of Leaver. The frequencies of extremal black holes we computed are plotted as diamonds. Dashed lines are $l=2$ electromagnetic modes.

every previous result [13,14] and are the most accurate of all.

Our procedure is summarized as follows. We expand the solution about the ordinary point and get the recurrence relation of the expansion coefficients. In our method we have to examine the boundary conditions at both boundaries. Then, we need to divide the sequence into an odd sequence and an even sequence to examine convergence of both series. The convergence of the series at both sides occurs if and only if each sequence is minimal. The ratio of successive terms of each sequence is given by a continued fraction. Consequently, the eigenvalue equation is constructed by sub-

stituting the continued fraction into the ratio of successive recurrence relations.

The applicability of our method is free from types of singularities of the equation, in contrast with Leaver's method which is not useful when the equation has two irregular singular points at each of the two boundaries. Though we can use our method to determine quasinormal frequencies of other black holes such as the Schwarzschild, nonextremal Reissner-Nordström, and Kerr black holes, we actually did not follow this procedure in these black holes. For these black holes, his method is easier to obtain the quasinormal

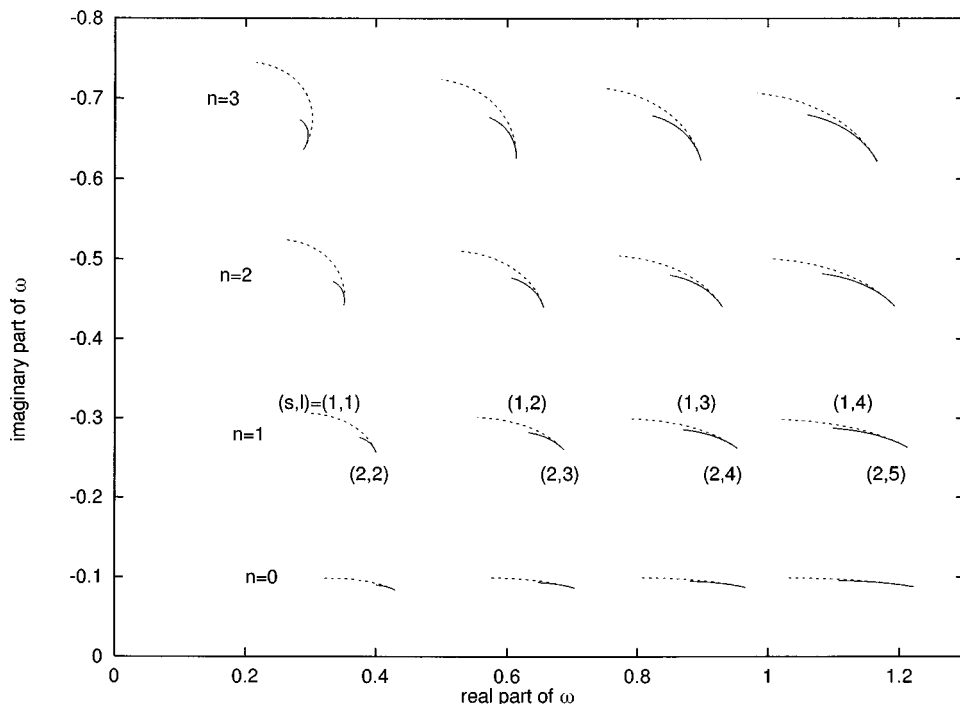


FIG. 2. Solid lines and dashed lines are trajectories of the third order WKB quasinormal frequencies of the gravitational and electromagnetic waves, respectively. Each left end point of lines corresponds to the quasinormal frequency of a charged black hole of $Q=0.8$, and each right end point corresponds to the frequency in the limit of maximal charge. A trajectory of the gravitational quasinormal frequencies depicted in a solid line meets at the right end with a corresponding dashed line, which is a trajectory of the electromagnetic quasinormal frequencies belonging to lower a multipole index by one.

frequencies because the recurrence relation and the eigenvalue equation are simpler than ours. In our method, it is generally difficult to give the explicit forms of the recurrence relations of both even and odd sequences. In that case we will have to eliminate the even or odd sequence by numerical elimination steps. For the extremal Reissner-Nordström black hole, the recurrence relation, Eq. (20), is not so complicated in that it has no β_n term and furthermore, the δ_n term has no dependence on n , which makes our procedure rather easy to accomplish.

We believe our method can be also used for the extremal Kerr black hole which has an irregular singular point at the horizon in the equation for the radial part. Some authors [16–18] have obtained the series of infinite numbers of quasinormal frequencies of the extremal Kerr black hole, which accumulate onto the critical frequency. Leaver also finds a tendency of an accumulation of quasinormal frequencies when the Kerr parameter approaches the maximum value. It is interesting to investigate what will happen to the distribution of quasinormal frequencies in the limit of maximal rotation, where Leaver's method is not valid. We are in preparation for this problem through our method of computing quasinormal frequencies.

We accidentally find a curious coincidence in the quasinormal frequencies of gravitational perturbations with a multipole index l and those of electromagnetic perturbations with $l-1$. These two modes are completely decoupled but they have the same amplitudes of transmission and reflection. In that case the only difference is a phase shift of transmitted or reflected wave that means the quasinormal frequencies for both modes are identical. The situation is very similar to the coincidence of the transmission and reflection amplitudes of odd-parity perturbations with those of even-parity perturbations [15]. At present there is no easy way to understand the hitherto unobserved coincidence but it is interesting that it occurs only in the extremal limit, where the black hole may have an unknown symmetry.

Note added in proof. After we completed the work, we were informed that Andersson had calculated [19] the quasinormal frequencies of nonextremal charged black holes, which are also consistent with our results. His method seems to fail in the extremal limit as does Leaver's method.

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APPENDIX A

Here, we summarize a procedure to obtain a recurrence relation for the odd sequence and that for the even sequence

of a_n . First, we write down the recurrence relation, Eqs. (18)–(20), in the matrix expression and separate the odd sequence from the even sequence as follows:

$$\begin{pmatrix} \gamma_1 & \alpha_1 & & & \\ \epsilon_3 & \gamma_3 & \alpha_3 & & \\ & \epsilon_5 & \gamma_5 & \alpha_5 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_2 \\ a_4 \\ \vdots \end{pmatrix} = - \begin{pmatrix} 0 & & & & \\ & \delta & & & \\ & & \delta & & \\ & & & \delta & \\ & & & & \ddots \end{pmatrix} \times \begin{pmatrix} 0 \\ a_1 \\ a_3 \\ \vdots \end{pmatrix}, \quad (\text{A1})$$

$$\begin{pmatrix} \gamma_2 & \alpha_2 & & & \\ \epsilon_4 & \gamma_4 & \alpha_4 & & \\ & \epsilon_6 & \gamma_6 & \alpha_6 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{pmatrix} = - \begin{pmatrix} \delta & & & & \\ & \delta & & & \\ & & \delta & & \\ & & & \delta & \\ & & & & \ddots \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_2 \\ a_4 \\ \vdots \end{pmatrix}, \quad (\text{A2})$$

where the term δ_n is a constant and therefore we write it as δ here, which makes us able to accomplish the below-mentioned procedure explicitly. If we substitute the right-hand side of Eq. (A2) into the left-hand side of Eq. (A1) like this,

$$\begin{pmatrix} \gamma_1 & \alpha_1 & & & \\ \epsilon_3 & \gamma_3 & \alpha_3 & & \\ & \epsilon_5 & \gamma_5 & \alpha_5 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \gamma_2 & \alpha_2 & & & \\ \epsilon_4 & \gamma_4 & \alpha_4 & & \\ & \epsilon_6 & \gamma_6 & \alpha_6 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & & & & \\ \delta^2 & 0 & & & \\ & \delta^2 & 0 & & \\ & & & \delta^2 & 0 \\ & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{pmatrix}, \quad (\text{A3})$$

the even sequence is eliminated and then we can obtain a recurrence relation including only b_n which was presented in Eq. (30) and the recurrence coefficients are explicitly given by

$$\hat{\alpha}_n = -3n + n^2 + 8n^3 + 4n^4, \quad (\text{A4})$$

$$\hat{\beta}_n = 4n(-n - 2An - 4n^3 + nq_s + 9\rho - 24n^2\rho - 32n\rho^2), \quad (\text{A5})$$

$$\begin{aligned} \hat{\gamma}_n &= 8 + 12A + 4l^2(l+1)^2 + 24n^4 - 6q_s - 4Aq_s + q_s^2 - 70\rho - 48A\rho + 24q_s\rho + 308\rho^2 + 128A\rho^2 - 64q_s\rho^2 - 768\rho^3 + 1024\rho^4 \\ &\quad + n^3(-48 + 224\rho) + n^2(46 + 16A - 8q_s - 336\rho + 864\rho^2) \\ &\quad + 2n(-11 - 8A + 4q_s + 126\rho + 48A\rho - 24q_s\rho - 432\rho^2 + 768\rho^3), \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} \hat{\delta}_n &= 8 + 4A - 16n^4 - (4 - q_s)^2 - 2q_s + 32n^3(2 - 5\rho) + 68\rho + 32A\rho - 16q_s\rho - 440\rho^2 - 32A\rho^2 + 16q_s\rho^2 + 896\rho^3 - 512\rho^4 \\ &\quad + 4n^2(-21 - 2A + q_s + 120\rho - 144\rho^2) + 4n(10 + 4A - 2q_s - 97\rho - 8A\rho + 4q_s\rho + 288\rho^2 - 224\rho^3), \end{aligned} \quad (\text{A7})$$

$$\hat{\epsilon}_n = -15n + 41n^2 - 24n^3 + 4n^4 - 30\rho + 164n\rho - 144n^2\rho + 32n^3\rho + 164\rho^2 - 288n\rho^2 + 96n^2\rho^2 - 192\rho^3 + 128n\rho^3 + 64\rho^4. \quad (\text{A8})$$

Similarly, we get a recurrence relation for c_n and the recurrence coefficients are given by

$$\bar{\alpha}_n = 1 - 5n^2 + 4n^4, \quad (\text{A9})$$

$$\bar{\beta}_n = (-1 + 2n)(2 + 2A - 8n - 4An + 12n^2 - 8n^3 - q_s + 2nq_s + 6\rho + 48n\rho - 48n^2\rho + 32\rho^2 - 64n\rho^2), \quad (\text{A10})$$

$$\bar{\gamma}_2 = 41 + 24A + 4A^2 - 12q_s - 4Aq_s + q_s^2 + 306\rho + 96A\rho - 48q_s\rho + 948\rho^2 + 128A\rho^2 - 64q_s\rho^2 + 1536\rho^3 + 1024\rho^4, \quad (\text{A11})$$

$$\begin{aligned} \bar{\gamma}_n &= 38 + 24A + 4A^2 + 24n^4 - 12q_s - 4Aq_s + q_s^2 - 308\rho - 96A\rho + 48q_s\rho + 956\rho^2 + 128A\rho^2 - 64q_s\rho^2 - 1536\rho^3 + 1024\rho^4 \\ &\quad + 32n^3(-3 + 7\rho) + 2n^2(77 + 8A - 4q_s - 336\rho + 432\rho^2) \\ &\quad + 4n(-29 - 8A + 4q_s + 189\rho + 24A\rho - 12q_s\rho - 432\rho^2 + 384\rho^3), \end{aligned} \quad (\text{A12})$$

$$\bar{\delta}_2 = -8 + 4A + 6q_s - q_s^2 + 12\rho - 12A\rho + 6q_s\rho - 76\rho^2 - 16A\rho^2 + 8q_s\rho^2 - 336\rho^3 - 256\rho^4, \quad (\text{A13})$$

$$\begin{aligned} \bar{\delta}_n &= -42 - 6A - 16n^4 - (4 - q_s)^2 + 3q_s + 32n^3(3 - 5\rho) + 402\rho + 48A\rho - 24q_s\rho - 1160\rho^2 - 32A\rho^2 + 16q_s\rho^2 + 1344\rho^3 - 512\rho^4 \\ &\quad + 4n^2(-51 - 2A + q_s + 180\rho - 144\rho^2) + 4n(45 + 6A - 3q_s - 247\rho - 8A\rho + 4q_s\rho + 432\rho^2 - 224\rho^3), \end{aligned} \quad (\text{A14})$$

$$\bar{\epsilon}_n = 21 - 76n + 83n^2 - 32n^3 + 4n^4 - 152\rho + 332n\rho - 192n^2\rho + 32n^3\rho + 332\rho^2 - 384n\rho^2 + 96n^2\rho^2 - 256\rho^3 + 128n\rho^3 + 64\rho^4. \quad (\text{A15})$$

APPENDIX B

We give the Gaussian elimination steps to transform the five-term recurrence relations into three-term recurrence relations. For b_n the step defined by

$$\hat{\alpha}'_n = \hat{\alpha}_n, \quad (\text{B1})$$

$$\hat{\beta}'_n = \hat{\beta}_n, \quad (\text{B2})$$

$$\hat{\gamma}'_n = \hat{\gamma}_n, \quad (\text{B3})$$

$$\hat{\delta}'_n = \hat{\delta}_n, \quad \text{for } n=1,2, \quad (\text{B4})$$

$$\hat{\alpha}'_n = \hat{\alpha}_n, \quad (\text{B5})$$

$$\hat{\beta}'_n = \hat{\beta}_n - \hat{\alpha}'_{n-1} \hat{\epsilon}_n / \hat{\delta}'_{n-1}, \quad (\text{B6})$$

$$\hat{\gamma}'_n = \hat{\gamma}_n - \hat{\beta}'_{n-1} \hat{\epsilon}_n / \hat{\delta}'_{n-1}, \quad (\text{B7})$$

$$\hat{\delta}'_n = \hat{\delta}_n - \hat{\gamma}'_{n-1} \hat{\epsilon}_n / \hat{\delta}'_{n-1} \quad \text{for } n \geq 3, \quad (\text{B8})$$

transforms the five-term recurrence relation into a four-term recurrence relation and the second elimination step below,

$$\hat{\alpha}''_1 = \hat{\alpha}'_1, \quad (\text{B9})$$

$$\hat{\beta}''_1 = \hat{\beta}'_1, \quad (\text{B10})$$

$$\hat{\gamma}''_1 = \hat{\gamma}'_1, \quad (\text{B11})$$

$$\hat{\alpha}''_n = \hat{\alpha}'_n, \quad (\text{B12})$$

$$\hat{\beta}''_n = \hat{\beta}'_n - \hat{\alpha}''_{n-1} \hat{\delta}'_n / \hat{\gamma}''_{n-1}, \quad (\text{B13})$$

$$\hat{\gamma}''_n = \hat{\gamma}'_n - \hat{\beta}''_{n-1} \hat{\delta}'_n / \hat{\gamma}''_{n-1} \quad \text{for } n \geq 2, \quad (\text{B14})$$

brings us to a final three-term recurrence relation:

$$\hat{\alpha}''_n b_{n+1} + \hat{\beta}''_n b_n + \hat{\gamma}''_n b_{n-1} = 0 \quad \text{for } n \geq 1. \quad (\text{B15})$$

For c_n the successive elimination steps in

$$\bar{\alpha}'_2 = \bar{\alpha}_2, \quad (\text{B16})$$

$$\bar{\beta}'_2 = \bar{\beta}_2, \quad (\text{B17})$$

$$\bar{\gamma}'_2 = \bar{\gamma}_2, \quad (\text{B18})$$

$$\bar{\delta}'_2 = \bar{\delta}_2, \quad (\text{B19})$$

$$\bar{\alpha}'_n = \bar{\alpha}_n, \quad (\text{B20})$$

$$\bar{\beta}'_n = \bar{\beta}_n - \bar{\alpha}'_{n-1} \bar{\epsilon}_n / \bar{\delta}'_{n-1}, \quad (\text{B21})$$

$$\bar{\gamma}'_n = \bar{\gamma}_n - \bar{\beta}'_{n-1} \bar{\epsilon}_n / \bar{\delta}'_{n-1}, \quad (\text{B22})$$

$$\bar{\delta}'_n = \bar{\delta}_n - \bar{\gamma}'_{n-1} \bar{\epsilon}_n / \bar{\delta}'_{n-1} \quad \text{for } n \geq 3, \quad (\text{B23})$$

and

$$\bar{\alpha}''_2 = \bar{\alpha}'_2, \quad (\text{B24})$$

$$\bar{\beta}''_2 = \bar{\beta}'_2, \quad (\text{B25})$$

$$\bar{\gamma}''_2 = \bar{\gamma}'_2 - \alpha_1 \bar{\delta}'_2 / \gamma_1, \quad (\text{B26})$$

$$\bar{\alpha}''_n = \bar{\alpha}'_n, \quad (\text{B27})$$

$$\bar{\beta}''_n = \bar{\beta}'_n - \bar{\alpha}''_{n-1} \bar{\delta}'_n / \bar{\gamma}''_{n-1}, \quad (\text{B28})$$

$$\bar{\gamma}''_n = \bar{\gamma}'_n - \bar{\beta}''_{n-1} \bar{\delta}'_n / \bar{\gamma}''_{n-1} \quad \text{for } n \geq 2, \quad (\text{B29})$$

give us a three-term recurrence relation:

$$\bar{\alpha}''_n c_{n+1} + \bar{\beta}''_n c_n + \bar{\gamma}''_n c_{n-1} = 0 \quad \text{for } n \geq 2. \quad (\text{B30})$$

APPENDIX C

Now, we summarize here the relation between V_1 for a multipole index l and V_2 for a multipole index $l+1$. If we introduce a new function f ,

$$f = \frac{r-1}{r^2}, \quad (\text{C1})$$

which vanishes at both boundaries, $r \rightarrow 1$ and $r \rightarrow \infty$, then two potentials are given by

$$V_1 = +(l+1) \frac{df}{dr_*} - 4f^3 + (l+1)^2 f^2, \quad (\text{C2})$$

$$V_2 = -(l+1) \frac{df}{dr_*} - 4f^3 + (l+1)^2 f^2. \quad (\text{C3})$$

If these two different potentials yield the same amplitudes of transmission and reflection, the following integrals for the two potentials should be the same:

$$I_1 = \int V dr_*, \quad (\text{C4})$$

$$I_2 = \int V^2 dr_*, \quad (\text{C5})$$

$$I_3 = \int (2V^3 + V'^2) dr_*, \quad (\text{C6})$$

$$I_4 = \int (5V^4 + 10VV'^2 + V''^2) dr_*, \quad (\text{C7})$$

$$I_5 = \int (14V^5 + 70V^2V'^2 + 14VV''^2 + V'''^2) dr_*, \quad (\text{C8})$$

$$I_6 = \dots$$

Because of the simple forms, Eqs. (C2) and (C3), we can see the coincidence of these integrals, $I_1 \sim I_{10}$, for the two potentials. The explicit forms of the above integrals, $I_1 \sim I_5$, are given by

$$I_1 = (1 + 6l + 3l^2)/3, \quad (\text{C9})$$

$$I_2 = (4 + 18l + 93l^2 + 84l^3 + 21l^4)/630, \quad (\text{C10})$$

$$I_3 = (23 + 120l + 476l^2 + 1560l^3 + 1820l^4 + 858l^5 + 143l^6)/45045, \quad (\text{C11})$$

$$I_4 = (1648 + 8736l + 32716l^2 + 105868l^3 + 252567l^4 + 316540l^5 + 203490l^6 + 64600l^7 + 8075l^8)/19399380, \quad (\text{C12})$$

$$I_5 = (25160 + 125820l + 437634l^2 + 1336860l^3 + 3430245l^4 + 6563970l^5 + 8099427l^6 + 6085800l^7 + 2687895l^8 + 642390l^9 + 64239l^{10})/1003917915. \quad (\text{C13})$$

[1] T. Regge and J. Wheeler, Phys. Rev. **108**, 1063 (1957).
 [2] F. Zerilli, Phys. Rev. D **2**, 2141 (1971).
 [3] F. Zerilli, Phys. Rev. D **9**, 860 (1974).
 [4] V. Moncrief, Phys. Rev. D **9**, 2707 (1974); **10**, 1057 (1974); **12**, 1526 (1975).
 [5] S. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
 [6] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. London **A344**, 441 (1975).
 [7] E. Leaver, Proc. R. Soc. London **A402**, 285 (1985).
 [8] W. Gautschi, SIAM Rev. **9**, 24 (1967).
 [9] V. Ferrari and B. Mashhoon, Phys. Rev. D **30**, 295 (1984).
 [10] H. Blome and B. Mashhoon, Phys. Lett. **100A**, 231 (1984).
 [11] B. Schutz and C. Will, Astrophys. J. **291**, L33 (1985).

[12] S. Iyer and C. Will, Phys. Rev. D **35**, 3621 (1987); S. Iyer, *ibid.* **35**, 3632 (1987); E. Seidel and S. Iyer, *ibid.* **41**, 374 (1990).
 [13] E. Leaver, Phys. Rev. D **41**, 2986 (1990).
 [14] K. Kokkotas and B. Schutz, Phys. Rev. D **37**, 3378 (1988).
 [15] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon, Oxford, 1983).
 [16] S. Teukolsky and W. Press, Astrophys. J. **193**, 443 (1974).
 [17] S. Detweiler, Astrophys. J. **239**, 292 (1980); Phys. Rev. Lett. **52**, 67 (1983).
 [18] M. Sasaki and T. Nakamura, Gen. Relativ. Gravit. **22**, 1351 (1990).
 [19] N. Andersson, Proc. R. Soc. London **A442**, 427 (1993).