Soluble models in two-dimensional dilaton gravity

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A one-parameter class of simple models of two-dimensional dilaton gravity, which can be exactly solved including back-reaction effects, is investigated at both classical and quantum levels. This family contains the RST model as a special case, and it continuously interpolates between models having a flat (Rindler) geometry and a constant curvature metric with a nontrivial dilaton field. The processes of formation of black hole singularities from collapsing matter and Hawking evaporation are considered in detail. Various physical aspects of these geometries are discussed, including the cosmological interpretation. [S0556-2821(96)02010-3]

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I. INTRODUCTION

Two-dimensional dilaton gravity models reproduced the essential features of the Hawking model of gravitational collapse, with an exact account of back-reaction effects [1,2]. An important question that remains is to what extent these features are universal or are just properties peculiar to a special model. The Callan-Giddings-Harvey-Strominger (CGHS) action

$$S=\int d^2x\sqrt{-g}\{e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2]\}, \tag{1.1}$$

is different from the Einstein-Hilbert action restricted to spherically symmetric configurations, $ds^2=g_{ij}(x^i)dx^i dx^j+e^{-2\tilde{\phi}(x^i)}d\Omega^2$, $i,j=1,2$,

$$S_{\rm EH}=\int d^2x\sqrt{-g}\{e^{-2\tilde{\phi}}[R+2(\nabla\tilde{\phi})^2]+2\}, \tag{1.2}$$

so it is not obvious that the physics of the CGHS model should be similar to the physics of spherically symmetric Einstein gravity. The problem is that the dimensionally reduced Einstein-Hilbert action coupled to matter is not an exactly solvable model. It is therefore important to look for a more general class of exactly solvable two-dimensional models containing a metric and dilaton field in order to have a more universal picture of the dynamics of black hole formation and evaporation, at least in the case of spherical symmetry. Several attempts in this direction have been made, either by modifying the boundary conditions of [2], as in [3], or by starting from more general actions (see e.g., [4,5]).

It is well known that the most general action for a theory containing a metric and a scalar field can be parametrized by couplings which are functions of the scalar field (see, e.g., [6,7,4]). Our purpose here will be to identify, in the general class, a subclass of solvable semiclassical models with desirable physical properties, exploring its possible application to the process of black hole evaporation as well as its cosmological interpretation. The classical part of the action is given by $S_{\rm cl}=S_0+S_M$, where

$$S_0=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left\{e^{-2\phi/n}\left[R+\frac{4}{n}(\nabla\phi)^2\right]+4\lambda^2e^{-2\phi}\right\} \tag{1.3}$$

and

$$S_M=-\frac{1}{4\pi}\sum_{i=1}^{N}\int d^2x\sqrt{-g}(\nabla f_i)^2. \tag{1.4}$$

In the case $n=1$ the model will reduce to the Russo-Susskind-Thorlacius (RST) model. A similar idea was pursued in [4]. In particular, by demanding the model to have asymptotic weak-coupling regions, the authors obtain a general class of models in which (1.3) is contained.

The classical geometries have typically a spacelike curvature singularity with an associated global event horizon, and a curvature scalar which goes to zero at spatial infinity. In the frame in which the dilaton and metric are static, the generic geometry ($n\neq1$) does not asymptotically approach the Minkowski geometry, instead it approaches the Rindler metric. The scale factor goes to zero or to infinity according to whether $n>1$ or $n<1$. Geometries with non-Minkowskian asymptotic behavior are quite common in general theories of two-dimensional (2D) dilaton gravity (with a general dilaton potential), and they also appear in other contexts, such as, e.g., ''black strings'' in four-dimensional string theory (see [8] and references therein), magnetic flux tubes (e.g., the Melvin vortex in four-dimensional Einstein theory [9]), various (2+1)-dimensional models, general gravity theories with dilaton and Maxwell fields [10], etc.[1] It is therefore of inter-

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[1]In the simplest 2D critical string theory (with zero central-charge deficit) there are no asymptotically Minkowskian solutions (the corresponding charged black hole solutions have a nontrivial asymptotic where the scale factor goes to zero).

est to have a simplified context where these geometries can be investigated in detail.

A basic issue of these types of metrics is how to define an invariant mass in the absence of a preferred asymptotic Minkowski frame. The standard Arnowitt-Deser-Misner (ADM) mass is conjugate to the asymptotic Minkowski time. For the present models, the choice of a time scale is somewhat arbitrary in that any two time coordinates differing by a multiplicative constant are equally valid (there will be, however, a natural time choice, namely the one which, for $n=1$, reduces to the Minkowski time). It will be shown here that, once the time coordinate is fixed the invariant mass conjugate to this time is conserved in the process of black hole formation and evaporation. This quantity constitutes a useful parameter which characterizes the geometry. In particular, the zero-curvature ground-state geometry is obtained by setting the mass parameter to zero in the general solution.

A natural physical application of the models considered here is in the cosmological context (see Sec. V). The geometries corresponding to the cases $n>1$, $n<1$, and $n=1$ are two-dimensional analogues of the Robertson-Walker cosmologies with parameters $k=1$, $k=-1$, and $k=0$, providing a description of expanding or contracting universes.

II. EXACTLY SOLVABLE MODELS

The solvability of the model of [2] is related to the fact that, after a suitable field redefinition, the action in the conformal gauge $[g_{\pm\pm}=0,\ g_{+-}=-(1/2)e^{2\rho}]$ can be written in the ''free field'' form [11]

$$S=\frac{1}{\pi}\int d^2x\left(\frac{1}{\kappa}\left(-\partial_+\chi\partial_-\chi+\partial_+\Omega\partial_-\Omega\right)+\lambda^2e^{2(\chi-\Omega)/\kappa}+\frac{1}{2}\sum_{i=0}^{N}\partial_+f_i\partial_-f_i\right), \tag{2.1}$$

where

$$\chi=\kappa\rho+e^{-2\phi}-\frac{1}{2}\kappa\phi,\quad \Omega=e^{-2\phi}+\frac{1}{2}\kappa\phi,\quad \kappa=\frac{N}{12}.$$

The RST model is not, however, the only dilaton-gravity theory that can be cast into the form (2.1). As we will see below, there are indeed inequivalent dilaton-gravity models which reduce to the above action upon a field redefinition.[2]

We would like to find the most general theory whose action can be written in the form (2.1) and which obeys the following basic requirements: (i) it is reparametrization invariant; (ii) it has the correct anomaly term; (iii) it contains a vacuum solution with $R=0$ as well as asymptotically flat solutions; (iv) there are no unphysical fluxes at infinity in the vacuum (in the frame in which the metric and the dilaton field are static). Now we will show that the most general transformation that meets the above requirements is given by

$$\chi=\kappa\rho+e^{-2\phi/n}+\left(\frac{1}{2n}-1\right)\kappa\phi, \tag{2.2}$$

$$\Omega=e^{-2\phi/n}+\frac{\kappa}{2n}\phi, \tag{2.3}$$

where n is a real number. The case $n=1$ corresponds to the model of [2].

Condition (i) requires, in particular, that the cosmological term in Eq. (2.1) be of the form $\sqrt{-g}f(\phi)=(1/2)e^{2\rho}f(\phi)$. The most general transformation between χ,Ω and ρ,ϕ satisfying this condition can be written as

$$\chi=(\kappa+a)\rho+f_1(\phi)+g(\rho,\phi),\quad \Omega=a\rho+g(\rho,\phi). \tag{2.4}$$

We can use the freedom to redefine the dilaton field ϕ so as to have $\chi-\Omega=\kappa(\rho-\tilde{\phi})$, i.e., $\kappa\tilde{\phi}=-f_1(\phi)$ (henceforth $\tilde{\phi}=\phi$). Thus we can write

$$\chi=(\kappa+a)\rho-\kappa\phi+g(\rho,\phi),\quad \Omega=a\rho+g(\rho,\phi). \tag{2.5}$$

Now, in order to obtain the usual anomaly term $-(\kappa/\pi)\int d^2x\partial_+\rho\partial_-\rho$, $g(\rho,\phi)$ must be of the form $g(\rho,\phi)=b\rho+F(\phi)$. The linear term $b\rho$ can be reabsorbed into a redefinition of a. The correct coefficient of the anomaly term is obtained provided $(\kappa+a)^2-a^2=\kappa^2$, i.e., $a=0$. Thus we have $\chi=\kappa\rho-\kappa\phi+F(\phi)$, $\Omega=F(\phi)$, and we must still demand conditions (iii) and (iv). The equations of motion derived from (2.1) are

$$\partial_+\partial_-(\chi-\Omega)=0,\quad \partial_+\partial_-\chi=-\lambda^2e^{2(\chi-\Omega)/\kappa}. \tag{2.6}$$

From Eq. (2.6) one sees that it is always possible to choose a gauge, the ''Kruskal'' gauge, where $\chi=\Omega$. In this gauge it is easy to show that the curvature scalar R is proportional to

$$\partial_+\partial_-\rho=\frac{1}{F'(\phi)}\left(-\lambda^2-\frac{F''(\phi)}{F'^2(\phi)}\partial_+\Omega\partial_-\Omega\right). \tag{2.7}$$

Consider the most general static solutions to Eq. (2.6) [2]:

$$\Omega=\chi=-\lambda^2x^+x^-+Q\ln(-\lambda^2x^+x^-)+\frac{M}{\lambda},\quad Q,M=\text{const.} \tag{2.8}$$

Let us first obtain the asymptotic part of the function $F(\phi)$. For $(-x^+x^-)\to\infty$, we have [see Eq. (2.8)] $\partial_+\Omega\partial_-\Omega\cong-\lambda^2\Omega=-\lambda^2F(\phi)$. From Eq. (2.7) we see that there are zero-curvature solutions provided

$$1=\frac{F''}{F'^2}F. \tag{2.9}$$

The general solution of Eq. (2.9) is $F(\phi)=ce^{m\phi}$. The constant c can be removed upon a proper shift of the dilaton field. The presence of the constant m reveals a whole class of new solutions labeled by $n=-2/m$, with the vacuum ($R=0$) solution given by

[2]Field redefinitions involving Weyl scalings do not give equivalent theories in dilaton-gravity models due to the presence of the anomaly term. The matter interacts with the geometry through the conformal anomaly, which is always constructed in terms of the appropriate physical metric (for further discussions on this point see [6]).

$$e^{2\rho}=e^{2\phi}=\frac{1}{(-\lambda^2x^+x^-)^n}. \tag{2.10}$$

General configurations approach the vacuum solution in the asymptotic region.

Let us note that the condition $R=0$ is satisfied even if linear terms in ϕ (which are subleading at infinity and do not contribute in F'') are added to $F(\phi)$. One thus concludes that $F(\phi)=e^{m\phi}+B\phi$ is the most general function $F(\phi)$ consistent with the existence of zero-curvature solutions. In this way we obtain

$$\chi=\kappa\rho+e^{-2\phi/n}-(\kappa-B)\phi,\quad \Omega=e^{-2\phi/n}+B\phi. \tag{2.11}$$

The value of B is fixed once condition (iv) is imposed. Indeed, consider the constraint equations:

$$\kappa t_\pm=\kappa^{-1}(-\partial_\pm\chi\partial_\pm\chi+\partial_\pm\Omega\partial_\pm\Omega)+\partial_\pm^2\chi + \frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i. \tag{2.12}$$

Consider the $\sigma^\pm$ coordinates, defined through $\pm\lambda x^\pm=e^{\pm\lambda\sigma^\pm}$, in which the vacuum geometry (2.10) is static, $\phi=-(n/2)\lambda(\sigma^+-\sigma^-)$ and $\rho=[(1-n)/2]\lambda(\sigma^+-\sigma^-)$. Equation (2.12) becomes

$$\kappa t_\pm(\sigma^\pm)=-\frac{\lambda^2}{4}[\kappa-2nB]. \tag{2.13}$$

In order to have $t_\pm(\sigma^\pm)=0$ in the vacuum, B must be equal to $\kappa/2n$. The most general model that can be mapped to the action (2.1) obeying conditions (i)–(iv) is thus given by the one-parameter class of models defined by the transformations (2.2) and (2.3). This leads to the action

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}\Bigg[e^{-2\phi/n}\left(R+\frac{4}{n}(\nabla\phi)^2\right)+4\lambda^2e^{-2\phi} -\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2+\kappa\left(\frac{1-2n}{2n}\phi R+\frac{n-1}{n}(\nabla\phi)^2 -\frac{1}{4}R(\nabla^2)^{-1}R\right)\Bigg]. \tag{2.14}$$

In what follows we will investigate the various physical aspects of this model. A recent study of general models, including a discussion of solvability, can be found in [4].[3]

[3]The model (2.14) corresponds, of course, to a specific choice of the coupling functions of the generic model which has been extensively discussed in the literature (see, e.g, in [6,7,4]). For example, in the notation of [4], the action (2.14) can be obtained with the choice $q(\phi)=e^{-(2/n)\phi}+\kappa[(1-2n)/2n]\phi$, $K(\phi)=(4/n)e^{-(2/n)\phi}+\kappa[(n-1)/n]$, $v(\phi)=-2\phi$, $u(\phi)=0$, $c=1/4$. The classical part of the action (2.14) is included in a more specific subclass of models given by Eq. (4.24) of [4], with $\alpha=-n$.

III. THE CLASSICAL THEORY

Let us first consider the classical theory $\hbar\to 0$. Once $\hbar$ is restored in the formulas, the last three terms in Eq. (2.14) go away in this limit, and we are left with action (1.3). The equations of motion derived from this action are

$$g_{\mu\nu}\left[\frac{4}{n}\left(-\frac{1}{2}+\frac{1}{n}\right)(\nabla\phi)^2-\frac{2}{n}\nabla^2\phi-2\lambda^2\exp\left(\frac{2-2n}{n}\phi\right)\right] +\frac{4}{n}\left(1-\frac{1}{n}\right)\partial_\mu\phi\partial_\nu\phi+\frac{2}{n}\nabla_\mu\partial_\nu\phi+e^{2/n\phi}T^M_{\mu\nu}=0, \tag{3.1}$$

$$\frac{R}{n}-\frac{4}{n^2}(\nabla\phi)^2+\frac{4}{n}\nabla^2\phi+4\lambda^2\exp\left(\frac{2-2n}{n}\phi\right)=0, \tag{3.2}$$

$$\nabla^2 f_i=0. \tag{3.3}$$

Equation (3.1) results from the variation of the metric and (3.2) is the dilaton equation of motion. In the conformal gauge $g_{\pm\pm}=0$, $g_{+-}=-(1/2)e^{2\rho}$ the equations of motion become

$$-\frac{4}{n^2}\partial_+\phi\partial_-\phi+\frac{2}{n}\partial_+\partial_-\phi-\lambda^2\exp\left(\frac{2-2n}{n}\phi+2\rho\right)=0, \tag{3.4}$$

$$\frac{2}{n}\partial_+\partial_-\rho+\frac{4}{n^2}\partial_+\phi\partial_-\phi-\frac{4}{n}\partial_+\partial_-\phi+\lambda^2 \times\exp\left(\frac{2-2n}{n}\phi+2\rho\right)=0, \tag{3.5}$$

$$\partial_+\partial_- f_i=0, \tag{3.6}$$

and the constraints

$$e^{-2\phi/n}\left[\frac{4}{n}\left(1-\frac{1}{n}\right)\partial_\pm\phi\partial_\pm\phi+\frac{2}{n}\partial_\pm^2\phi-\frac{4}{n}\partial_\pm\rho\partial_\pm\phi\right] -\frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i=0. \tag{3.7}$$

From Eqs. (3.4) and (3.5) it follows that

$$\frac{2}{n}\partial_+\partial_-(\rho-\phi)=0, \tag{3.8}$$

i.e., $\rho=\phi+f_+(x^+)+f_-(x_-)$. It is always possible to perform a coordinate transformation $x^\pm\to x^{\pm\prime}=f(x^\pm)$, which preserves the conformal gauge and for which $\rho=\phi$. In this (Kruskal-type) gauge the remaining equations take the form

$$\partial_+\partial_-(e^{-2\phi/n})=-\lambda^2,\quad \partial_\pm^2(e^{-2\phi/n})=-\frac{1}{2}\sum_{i=0}^{N}\partial_\pm f_i\partial_\pm f_i, \tag{3.9}$$

so that the general solution is given by

$$e^{-2\phi/n}=e^{-2\rho/n}=-\lambda^2x^+x^-+h_+(x^+)+h_-(x_-), \tag{3.10}$$

where $h_\pm(x^\pm)$ are arbitrary functions of $x^\pm$ subject to the constraints (3.7).

A. Static solutions

In the Kruskal gauge the general static solution is given by [see Eq. (3.10)]

$$e^{-2\phi/n}=-\lambda^2x^+x^-+Q\,\ln(-\lambda^2x^+x^-)+\frac{M}{\lambda},\quad (3.11)$$

i.e., for these solutions there exists a timelike Killing vector at infinity representing time translation invariance with respect to the time coordinate t, where $t=(1/2)\ln(x^+/x^-)$ (see also below). In Sec. III B it will be shown that M can be interpreted as the mass of the black hole. The parameter Q represents a uniform (incoming and outgoing) energy density flux. Indeed, the constraint equations (3.7) applied to the solution (3.11) give $T_{\pm\pm}=Q/x^{\pm 2}$ or, introducing $(\sigma^\pm)$ defined by $\lambda x^\pm=\pm e^{\pm\lambda\sigma^\pm}$, $T_{\pm\pm}=\lambda^2 Q$.

Let us consider the static solution with $Q=0$:

$$ds^2=-\frac{1}{(M/\lambda-\lambda^2x^+x^-)^n}\,dx^+dx^-,$$
$$e^{-2\phi/n}=\frac{M}{\lambda}-\lambda^2x^+x^-.\quad (3.12)$$

The corresponding curvature scalar R is given by

$$R=8e^{-2\rho}\partial_+\partial_-\rho=4M\lambda n\left[\frac{M}{\lambda}-\lambda^2x^+x^-\right]^{n-2}.\quad (3.13)$$

Consider the range[4] $0<n<2$. In this case we get the standard picture of the $n=1$ solutions, i.e., a spacelike singularity located at $x^+x^-=M/\lambda^3$ and an asymptotically flat region for $-x^+x^-\to\infty$ ($x^+\to\infty$ defines the future null infinity I_R^+ and $x^-\to-\infty$ stands for the past null infinity I_R^-). The event horizon is at $x^-=0$. The Penrose diagram is identical to the standard $n=1$ case (see, e.g., [1]).

From Eq. (3.13) we see that for $n=0$ the two-dimensional spacetime is flat. This is not, however, a trivial solution, since the coupling constant $e^{2\phi}$ is nontrivial and it becomes singular on a spacelike line. To take the limit $n\to 0$ we must first rescale the dilaton field $\phi\to\tilde{\phi}=n\phi$. The classical action (1.3) takes the simple form

$$S_0=\frac{1}{2\pi}\int d^2x\sqrt{-g}(e^{-2\tilde{\phi}}R+4\lambda^2).$$

This is precisely what one gets from the CGHS action (1.1) if the metric is redefined by $g_{\mu\nu}\to e^{2\phi}g_{\mu\nu}$. The case $n=0$ represents an unconventional black hole in the sense that there is a spacelike singularity in the coupling (and hence a horizon), but the two-dimensional curvature vanishes[5] (for a recent discussion on this model, see [12]).

For $n=2$ the two-dimensional curvature is constant. However, the same considerations as for the case $n=0$ apply: the dilaton field is singular on a spacelike line and the full geometry still has a black hole interpretation, with an event horizon at $x^-=0$. In Sec. IV we will see that at the quantum level the curvature of the $n=2$ model is no longer constant, and it becomes singular on a curve where the coupling reaches some finite critical value.

Let us now perform the coordinate transformation $(x^+,x^-)\to(\sigma,t)$ by means of the relation $\pm\lambda x^\pm=f(\lambda\sigma)e^{\pm\lambda t}$, where f is a generic function of $\lambda\sigma$. In this new coordinate system the line element and dilaton field take the form

$$ds^2=\frac{1}{[M/\lambda+f^2(\lambda\sigma)]^n}[-f^2(\lambda\sigma)dt^2+f'^2(\lambda\sigma)d\sigma^2],$$
$$\phi=-\frac{n}{2}\ln\left(\frac{M}{\lambda}+f^2(\lambda\sigma)\right).\quad (3.14)$$

A convenient coordinate system that will be used here is $f(\lambda\sigma)=e^{\lambda\sigma}$,

$$ds^2=\frac{e^{2(1-n)\lambda\sigma}}{[1+(M/\lambda)e^{-2\lambda\sigma}]^n}(-dt^2+d\sigma^2),$$
$$\phi=-\frac{n}{2}\ln\left(\frac{M}{\lambda}+e^{2\lambda\sigma}\right).\quad (3.15)$$

This coordinate system is suitable to calculate the mass of the black hole by means of the ADM procedure (see Sec. III B). From Eq. (3.15) we see that the metric does not asymptotically approach the Minkowski metric unless $n=1$. Instead we observe the remarkable fact that for any $n\neq 1$ the geometry approaches the Rindler metric. Indeed, consider first the vacuum solutions (i.e., with $M=0$) in terms of the spatial coordinate x defined by $df/f^n=\lambda dx$, that is

$$f^{1-n}=\lambda(1-n)x,\quad n<1,$$
$$f^{1-n}=\lambda(n-1)(x_1-x),\quad n>1,\quad (3.16)$$

where x_1 corresponds to the point $f=\infty$. In this frame we get, e.g., for $n<1$,

$$ds^2=dx^2-[\lambda(1-n)x]^2dt^2,\quad \phi=-\frac{n}{1-n}\ln[\lambda(1-n)x],\quad (3.17)$$

that is, the Rindler metric. In the special case $n=1$ one obtains $f=e^{\lambda x}$ and the geometry is the familiar linear dilaton vacuum, i.e., the Minkowski metric $ds^2=-dt^2+dx^2$ and $\phi=-\lambda x$.

For $M\neq 0$ we have

[4]When $n>2$ the geometry is very different; for simplicity here this case will be excluded from the discussion.

[5]In the dimensional reduction interpretation, the singularity in $e^{2\tilde{\phi}}$ translates into a curvature singularity of the four-dimensional metric $ds^2=g_{ij}dx^idx^j+e^{-2\tilde{\phi}}d\Omega^2$.

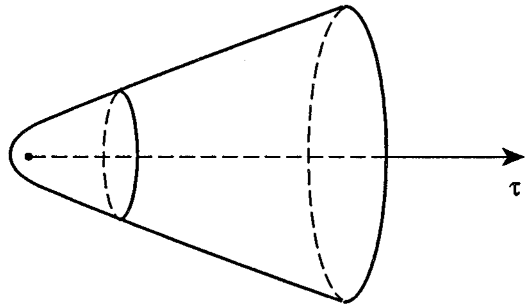

FIG. 1. Euclidean embedding of the metric for $0<n<1$. In the case $n=0$ the metric is that of the plane. For $n<0$ the Euclidean embedding does not exist: the geometry describes a hyperbolic universe that cannot be represented as a two-dimensional surface in three- or higher-dimensional Euclidean space.

$$ds^2=dx^2-F(\lambda x)dt^2, \quad F(\lambda x)=\frac{f^2}{(M/\lambda+f^2)^n}, \tag{3.18}$$

$$\frac{df}{(M/\lambda+f^2)^{n/2}}=\lambda dx. \tag{3.19}$$

Although it is not possible to integrate (3.19) in a closed form for generic n [in the case $n=1$ Eq. (3.19) gives $F(\lambda x)=\tanh^2(\lambda x)$], the geometry can be visualized by examining the form of $F(f^2)$. Near the horizon, $f\cong 0$ and $F(f^2)\cong(\lambda/M)^n f^2\cong 0$. In the asymptotic region, $f\to\infty$ and $F(f^2)=f^{2-2n}$. For $n<1$ the ''cigar'' expands, $F\to\infty$, and for $n>1$ it shrinks (see also Sec. V and Figs. 1–3 therein). In going to the x coordinates, when $n>1$ the point $f=\infty$ is mapped into a finite point x_1, since $f^{1-n}\sim\lambda(n-1)(x_1-x)$ and $F(\lambda x)\sim[\lambda(1-n)(x_1-x)]^2$.

The fact that on the horizon $F(\lambda x)\sim(\lambda x)^2$ for all n shows that the Hawking temperature will be given by $\lambda/2\pi$, irrespective of the value of n. This result is unambiguous once the time scale is fixed, and it will be confirmed below by means of two alternative derivations.

B. ADM mass

In this paragraph we perform the calculation of the ADM mass for these generalized blackhole configurations. We stress once again that in the absence of a (preferred) asymptotic Minkowski time, there is no unique possible definition of ''mass.'' The calculation that follows corresponds to the mass conjugate to the time t introduced before; this is a natural time choice in that it reduces to the Minkowski time for the $n=1$ model. The introduction of this mass parameter is useful since it is a conserved quantity in the process of evaporation characterizing the geometry (see below).

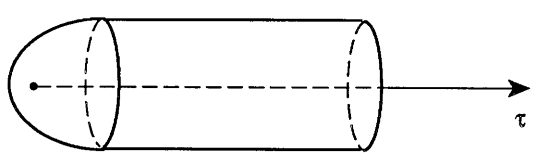

FIG. 2. Standard ''cigar'' geometry for $n=1$.

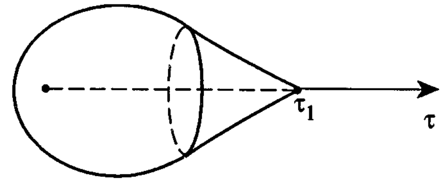

FIG. 3. Euclidean embedding of the metric for $1<n<2$. For $n=2$ the euclidean metric reduces to the metric of the sphere.

If we denote by $A_{\mu\nu}$ the gravitational field equations and by ξ^μ a Killing vector field, then $j_\mu=A_{\mu\nu}\xi^\nu$ should be a conserved current and the corresponding conserved charge density a total divergence. The corresponding charge is determined as a surface term at infinity. In the case $\xi^\mu=(1,0)$, representing time translation invariance, the only conserved quantity is the total energy or mass.

We work in the (σ,t) coordinate system introduced before. In this frame the metric (3.15), which for the moment we write generically as $ds^2=-e^{2\rho}(dt^2-d\sigma^2)$, and the dilaton depends only on σ. The 00 component of Eq. (3.1) now reads

$$A_{00}=e^{-2\phi/n}g_{00}g^{11}\left[\frac{4}{n}\left(-\frac{1}{2}+\frac{1}{n}\right)(\partial_1\phi)^2-\frac{2}{n}\,\partial_1^2\phi+\frac{2}{n}\,\partial_1\rho\partial_1\phi\right]-2g_{00}\lambda^2e^{-2\phi}. \tag{3.20}$$

In the linear approximation A_{00} is good enough to prove the conservation of the charge. Let us expand ρ and ϕ around their vacuum values, i.e., $\phi=-n\lambda\sigma+\delta\phi$ and $\rho=(1-n)\lambda\sigma+\delta\rho$. Note that $\delta\phi=\delta\rho$ [see (3.15)], so that the last term in (3.20) gives no first-order contributions. Using also $g_{00}g^{11}=-1$, we find

$$j_0=e^{2\lambda\sigma}\left(\frac{2}{n}\,\partial_1^2\delta\phi+\frac{6}{n}\,\lambda\partial_1\delta\phi+\frac{4}{n}\,\lambda^2\delta\phi\right)=\partial_\sigma\left[e^{2\lambda\sigma}\left(\frac{2}{n}\,\partial_1\delta\phi+\frac{2}{n}\,\lambda\,\delta\phi\right)\right]. \tag{3.21}$$

This means that

$$\int d\sigma j_0=\left[e^{2\lambda\sigma}\left(\frac{2}{n}\,\partial_1\delta\phi+\frac{2}{n}\,\lambda\,\delta\phi\right)\right]\Bigg|_{\sigma=\infty}. \tag{3.22}$$

Now let us explicitly determine $\delta\phi$. From

$$e^{-2\phi}=\left(\frac{M}{\lambda}+e^{2\lambda\sigma}\right)^n=e^{2\lambda n\sigma}\left(1+\frac{M}{\lambda}\,e^{-2\lambda\sigma}\right)^n\sim e^{2\lambda n\sigma}\left(1+n\,\frac{M}{\lambda}\,e^{-2\lambda\sigma}\right),$$

we get $\delta\phi=-(nM/2\lambda)e^{-2\lambda\sigma}$. Substituting in Eq. (3.22), we finally obtain

$$\int d\sigma\, j_0 = M. \tag{3.23}$$

C. Dynamical formation of black holes

Let us now return to the general solution, Eq. (3.10), and consider the problem of dynamical blackhole formation starting from the vacuum. The discussion is a straightforward generalization of the $n=1$ model of [2], but we briefly outline it in order to fix the notation.

Using the constraints [see Eq. (3.9)] we can express the general solution in terms of physical quantities, such as the Kruskal momentum and energy [2]:

$$P_+(x^+) = \int_0^{x^+} dx^+ T_{++}^M(x^+),$$

$$M(x^+) = \lambda \int_0^{x^+} dx^+ x^+ T_{++}^M(x^+).$$

We get

$$e^{-2\rho/n} = e^{-2\phi/n} = -\lambda^2 x^+[x^- + \lambda^{-2}P_+(x^+)] + \lambda^{-1}M(x^+). \tag{3.24}$$

The corresponding curvature scalar R is given by

$$R = 4\lambda n M(x^+)\left[\frac{M(x^+)}{\lambda} - \lambda^2 x^+[x^- + \lambda^{-2}P_+(x^+)]\right]^{n-2}. \tag{3.25}$$

It exhibits a singularity located at

$$M(x^+) - \lambda^3 x^+[x^- + \lambda^{-2}P_+(x^+)] = 0. \tag{3.26}$$

There is an apparent horizon at $x^- + \lambda^{-2}P_+(x^+) = 0$ and an event horizon at $x^- + \lambda^{-2}P_+(\infty) = 0$.

Consider the case of an incoming shock wave at $x^+ = x_0^+$ represented by the stress tensor $T_{++}^M = \frac{1}{2}\Sigma_{i=0}^N \partial_+ f_i \partial_+ f_i = a\,\delta(x^+ - x_0^+)$. The constraint equation is then easily satisfied by

$$e^{-2\phi/n} = e^{-2\rho/n} = -a(x^+ - x_0^+)\,\theta(x^+ - x_0^+) - \lambda^2 x^+ x^-. \tag{3.27}$$

In the region $x^+ < x_0^+$ the geometry is that of the vacuum, whereas in the region $x^+ > x_0^+$ the geometry is that of the static blackhole configuration discussed previously (with mass parameter $M = a x_0^+ \lambda$).

IV. QUANTUM THEORY

A. Hawking radiation

Let us first discuss the Hawking radiation ignoring back-reaction effects due to the evaporation. We consider the quantization of the N massless scalar fields in the fixed background of a black hole formed by the collapse of an incoming sho ckwave. Since the f_i's are free fields, they admit the decomposition $f_i = f_{iL}(x^+) + f_{iR}(x^-)$, where f_{iL} represents the incoming wave and f_{iR} the outgoing one$_0^+$ the metric is given by

$$e^{-2\rho/n} = -\lambda^2 x^+\left(x^- + \frac{a}{\lambda^2}\right) + a x_0^+. \tag{4.1}$$

Consider the frame $(\sigma_{\rm out}^+, \sigma_{\rm out}^-)$, appropriate for an out observer, defined by

$$\lambda x^+ = e^{\lambda\sigma_{\rm out}^+}, \quad -\lambda\left(x^- + \frac{a}{\lambda^2}\right) = e^{-\lambda\sigma_{\rm out}^-}. \tag{4.2}$$

The "in" vacuum $|0\rangle_{\rm in}$ is defined as being annihilated by the negative frequency modes with respect to the "in" time $(\sigma_{\rm in}^+, \sigma_{\rm in}^-)$, $\pm\lambda x^\pm = e^{\pm\lambda\sigma_{\rm in}^\pm}$. The Hawking radiation will be determined as usual in terms of the Bogolubov transformation between the "in" and "out" coordinate systems. Since this is independent of n the calculation is formally identical to the case $n=1$ so it will not be reproduced here (see, e.g., [13]). One obtains

$$_{\rm in}\langle T_{--}\rangle_{\rm in} = \frac{N\lambda^2}{48}\left[1 - \frac{1}{[1 + (a/\lambda)e^{\lambda\sigma_{\rm out}^-}]^2}\right]. \tag{4.3}$$

Near the horizon, $\sigma_{\rm out}^- \to \infty$, and $_{\rm in}\langle T_{--}\rangle_{\rm in}$ approaches the constant value $N\lambda^2/48$. In this region it can be shown that $_{\rm in}\langle N_w^{\rm out}\rangle_{\rm in} \sim e^{-2\pi w/\lambda}/1 - e^{-2\pi w/\lambda}$ (where $N_w^{\rm out}$ is the number operator of the out modes of frequency w), that is, the outgoing flux of radiation is thermal at the Hawking temperature $T_H = \lambda/2\pi$.

B. Back reaction

The inclusion of exact back-reaction effects can be done as in the $n=1$ case [2], by solving the semiclassical equations of motion corresponding to the effective action including one-loop effects, Eq. (2.14). In terms of χ,Ω the mathematics is identical to the $n=1$ case. However, some physical quantities, such as, e.g., the curvature scalar and the dilaton, have an n-dependent time evolution. Here we will just point out the general features and the main differences with respect to the standard $n=1$ case.

In terms of χ,Ω the vacuum solution is

$$\Omega = \chi = -\lambda^2 x^+ x^- - \frac{\kappa}{4}\ln(-\lambda^2 x^+ x^-). \tag{4.4}$$

The general time-dependent solution that describes the collapse of general incoming mass-less matter and subsequent evaporation is given by

$$\Omega = \chi = -\lambda^2 x^+[x^- + \lambda^{-2}P_+(x^+)] - \frac{\kappa}{4}\ln(-\lambda^2 x^+ x^-) + \lambda^{-1}M(x^+). \tag{4.5}$$

The curvature scalar of the corresponding geometry is

$$R = 4n e^{-2\rho}\frac{1}{e^{-2\phi/n} - \kappa/4}\left(\lambda^2 + \frac{4}{n^2}\,\partial_+\phi\,\partial_-\phi\, e^{-2\phi/n}\right). \tag{4.6}$$

We notice that there is a singularity along the line $\phi=\phi_{\rm cr}=-(n/2)\ln(\kappa/4)$. This line turns out to be timelike if $T_{++}<\kappa/4x^{+2}$, and it becomes spacelike as soon as $T_{++}>\kappa/4x^{+2}$.

Note that in the $n=2$ case, which classically corresponded to a constant curvature, the geometry has undergone an important change: once the one-loop effect has been incorporated, not only is the curvature not constant, but it blows up at $\phi=\phi_{\rm cr}$. This is not a surprise; while at the classical level the curvature was constant, the coupling became strong in a certain region. The quantum-corrected metric approaches the classical (constant curvature) metric asymptotically, but it departs from it in the strong-coupling region.

As in the $n=1$ case it is always possible to impose boundary conditions on the timelike singularity such that the curvature remains finite (in this picture the critical line can be viewed as a boundary of the spacetime, just as the line $r=0$ in the spherically symmetric reduction of 4D Minkowski space). Since the denominator vanishes on the singularity, the curvature will remain finite only if

$$\lambda^2=-\frac{4}{n^2}\,\partial_+\phi\partial_-\phi e^{-2\phi/n}\Big|_{\phi=\phi_{\rm cr}}, \tag{4.7}$$

or $(\nabla\phi)^2=n^2(\kappa/4)^{n-1}\lambda^2$. This can be accomplished by demanding $\partial_+\Omega|_{\phi=\phi_{\rm cr}}=\partial_-\Omega|_{\phi=\phi_{\rm cr}}=0$. It should be remembered that in order to take the limit $n\to 0$, one must first rescale $\phi\to n\phi$.

Energy conservation can be checked just as in the case $n=1$ [2]. We must compute the quantity

$$E_{\rm out}=-\tfrac{1}{2}\,\lambda\int_{-\infty}^{x_1^-}dx^-[x^-+\lambda^{-2}P(x_1^+)]\sum_{i=0}^{N}\partial_-f_i\partial_-f_i, \tag{4.8}$$

where x_1^+ represents the advanced time at which the incoming energy flux stops, and $x_1^-=-\lambda^{-2}P(x_1^+)$. The result of the integration exactly reproduces the total incoming energy.

V. COSMOLOGICAL MODELS

The theory (1.3) is similar to the dimensional reduction of four-dimensional Einstein gravity (1.2) with

$$ds^2=g_{ij}(x^i)dx^idx^j+e^{-2\tilde\phi(x^i)}d\Omega^2,\quad \tilde\phi=\phi/n. \tag{5.1}$$

For $n<2$ the general model (1.3) with the change $\lambda^2\to-\lambda^2$ exhibits interesting cosmological solutions, which may be regarded as toy Kantowski-Sacks models [14], describing ''spatially homogeneous'' spacetimes with general line element

$$ds^2=A(t)(-dt^2+d\sigma^2)+B(t)d\Omega^2. \tag{5.2}$$

$A(t)$ and $B(t)$ are generic functions of t (a discussion in the case of the $n=1$ model can be found in [15]; other generalized dilaton gravity models are discussed in [16]).

Consider a homogeneous distribution of conformal matter, with

$$T_{++}^{\rm matter}=T_{--}^{\rm matter}=\frac{1}{4}\sum_{i=0}^{N}(\partial_tf_i)^2=c.$$

In the conformal gauge, the scale factor and dilaton of the homogeneous solution $ds^2=e^{2\rho(t)}(-dt^2+d\sigma^2)$ have the following general form:

$$e^{2\rho}=\left(e^{2\lambda t}+\frac{c}{\lambda}\,t+m\right)^{-n}e^{2\lambda t},\quad e^{2\phi/n}=\left(e^{2\lambda t}+\frac{c}{\lambda}\,t+m\right)^{-1}. \tag{5.3}$$

When either $c\neq 0$ or $m<0$ the evolution starts at some finite $t=t_0$ where (for $n>0$) $e^{2\phi},e^{2\rho}$ and R are all singular. When $m>0$ and $c=0$ the evolution starts at $t=-\infty$, where the solution is regular. The behavior at $t\to\infty$ does not depend on the values of m and c, as is clear from Eq. (5.3).

Let us discuss in detail the simplest case $m>0$ and $c=0$. The scale factor is

$$e^{2\rho}=(e^{2\lambda t}+m)^{-n}e^{2\lambda t}. \tag{5.4}$$

The curvature and the dilaton field are given by

$$R=-4n\lambda^2m(e^{2\lambda t}+m)^{n-2},\qquad e^{2\phi}=(e^{2\lambda t}+m)^{-n}. \tag{5.5}$$

From (5.4) we note that the coupling $e^{2\phi/n}$ always stays finite. For $t\to-\infty$, $e^{2\rho(t)}\to 0$, $R\to 4n\lambda^2m^{n-1}$, and $e^{2\phi/n}\to m^{-1}$. Then the Universe begins to expand and the subsequent evolution will be dictated by the value of n. As $t\to\infty$ we have

$$e^{2\rho}\to e^{2\lambda(1-n)t}, \tag{5.6}$$

i.e., the scale will increase for $n<1$ and decrease for $n>1$, whereas $e^{2\phi/n}\to 0$ and $R\to 0$ irrespective of the value of n. Thus for $n<1$ the Universe is open and expands forever (Fig. 1), in the case $n=1$ the expansion slows down to zero asymptotically (Fig. 2), and for $n>1$ the Universe is closed: at a certain time the expansion stops and the Universe begins to contract (see Fig. 3). It is interesting to note that in this last case the collapse takes place in a finite proper time, and in the weak coupling region where $e^{2\phi/n}=0$.

Let us introduce the cosmological time τ and consider the Euclidean metric

$$ds^2=d\tau^2+F(\lambda\tau)dx^2,\qquad F(\lambda\tau)=\frac{e^{2\lambda t}}{(m+e^{2\lambda t})^n},\qquad \frac{e^{\lambda t}dt}{(m+e^{2\lambda t})^{n/2}}=d\tau. \tag{5.7}$$

The compact space coordinate x must have period $2\pi/\lambda$ in order for the metric to be free from conical singularities at $\tau=0$. This is clear from the fact that in the region $\tau\cong 0$ ($t\to-\infty$) one has $F(\lambda\tau)\cong\lambda^2\tau^2$. The metrics with $1<n<2$ will then have a conical singularity at $t=\infty$ (Fig. 3). Indeed, at $t\to\infty$, one finds $F(\lambda\tau)=\lambda^2(n-1)^2(\tau_1-\tau)^2$, which implies a conical singularity at $\tau=\tau_1$ ($t=\infty$) with deficit angle equal to $2\pi(2-n)$. For $n=2$ there is no conical singularity and the metric is that of the sphere, $F(\lambda\tau)=\sin^2(\lambda\tau)$.

Thus the simplest cosmologies with $c=0$ and $m>0$ contain expanding and contracting (non-''isotropic'') universes with no initial singularities. In the case $n>1$ the Universe recollapses in spite of the absence of matter energy density ($c=0$). This is due to the fact that for $n>1$, as the weak limit $e^{2\phi/n}\rightarrow 0$ is approached ($t\rightarrow\infty$), the scale factor $e^{2\rho}$ must go to zero in order to compensate the increase of the cosmological term in the action [see Eq. (1.3)].

A final remark concerns the case when $n=-1$. For this model, in the gauge $\rho=\phi$, the functions [see Eq. (5.2)] $A=e^{2\rho}$ and $B=e^{-2\phi/n}$ become the same. An homogeneous solution in this gauge is $ds^2=(m+2\lambda^2x_0^2)(-dx_0^2+dx_1^2)$, $e^{2\phi}=m+2\lambda^2x_0^2$. The four-dimensional metric (5.1) can thus be written as

$$ds^2=-d\tau^2+R^2(\tau)(d\sigma^2+d\Omega^2), \tag{5.8}$$

which means (in this four-dimensional interpretation) that the spatial section of the metric remains constant throughout the evolution. For large τ, the radius of the Universe increases as $R(\tau)\sim\sqrt{\tau}$ so that the Hubble constant $H\equiv(1/R)(dR/d\tau)$ goes to zero as $1/\tau$. This is quite satisfactory, since the behavior $H\sim 1/R^2$ is characteristic of standard radiation-dominated Friedmann-Robertson-Walker ($k=0$) cosmologies.

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