Description of chaos in simple relativistic systems

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Chaos is investigated in the context of general relativity and gravitation. We show how quantitative and global measures of chaos can be obtained from qualitative and local ones. After averaging—first, over all two-directions, and second, along the trajectory—the rate of separation of nearby trajectories (Lyapunov-like exponents) can be obtained. This gives us a tool to the invariant chaos description. The sign of the Ricci scalar serves as a criterion of the local instability in simple mechanical systems (systems with a natural Lagrange function). We also show how to reduce relativistic simple mechanical systems to the classical ones. Timelike and null geodesics in multi-black-hole cosmological spacetimes are considered. The role of relativistic systems in general relativity is emphasized. [S0556-2821(96)05310-6]

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I. INTRODUCTION

Our investigations are focused on developing geometrical methods to study the chaotic behavior of dynamical systems in general relativity and gravitation. Standard detectors of chaos pose difficulties since they depend on the choice of the particular coordinate system. To avoid such problems chaos should be described in an invariant way.

As Wintner [1] points out, the Maupertuis principle is an interesting curiosity in geometrical mechanics. It was not formulated by Maupertuis but rather by Jacobi, Euler, and Lagrange. In our approach [2], we use the Maupertuis principle which allows us to reduce the Hamiltonian flows to geodesic flows on a Riemannian or pseudo-Riemannian space with the Jacobi metric

$$\hat{g}=2|E-V|g=2Wg \tag{1}$$

where V is a potential and g is a metric which is taken from the kinetic energy form $T=\frac{1}{2}g(v,v)$, $v^{\alpha}=dq^{\alpha}/d\tau$, $\alpha=1,\dots,N$. The Hamiltonian of our system is of the form

$$\mathscr{H}=T+V=\frac{1}{2}g^{\alpha\beta}p_{\alpha}p_{\beta}+V(q)=E=\text{const},$$

where $p_{\alpha}=g_{\alpha\beta}\dot{q}^{\beta}$.

We consider the geodesic deviation equation

$$\frac{D^2n}{ds^2}=-\text{grad}_nV_u(n) \tag{2}$$

where u and n are tangent and normal vectors to the geodesic, respectively, $[\text{grad}_n]^i=\partial/\partial n^i$, and $V_u(n)$ is the geodesic deviation potential.

In general, we can precisely determine neither the tangent vector u nor the deviation vector n for the specific initial value of the parameter s along the geodesic. Therefore it seems reasonable to introduce an averaged potential $\overline{V_u}(n)$ which is found by choosing vectors u and n at random; i.e., every direction determined by the bivector $\underline{u\wedge n}$ is equally probable. The averaged deviation potential $\overline{V_u}(n)$ is

$$\overline{V_u}(n)=\frac{1}{2N(N-1)}\hat{R}\hat{g}(n,n)\hat{g}(u,u) \tag{3}$$

where $N=\dim\mathscr{M}$, $\mathscr{M}$ is a configuration space; and $\hat{R}$ is a Ricci scalar calculated with respect to the Jacobi metric $\hat{g}$.

The problem of determining the average separation rate of nearby geodesics has been reduced to determining the normal separation vector $n=\hat{n}^iE_i$, $i=1,\dots,N$. After introducing the orthonormal Fermi basis $\{E_1,\dots,E_{N-1},E_N=u$ and $\nabla_uE^a=0$, $\hat{g}(E_i,E_j)=\delta_{ij}\}$ Eq. (2) is reduced to the form

$$\frac{d^2\hat{n}^{\alpha}}{ds^2}=-\frac{\hat{R}(W(q(s)))}{N(N-1)}\hat{n}^{\alpha}\text{sgn}(E-V) \tag{4}$$

where $s:ds/d\tau=2|E-V|$ is the natural parameter along the geodesics.

From (4) one can see that the full information concerning the averaged local instability of the geodesic flow is contained in the product $\hat{R}\hat{g}(u,u)$ which is calculated with respect to the Jacobi metric $\hat{g}$ [2]. Therefore the local behavior of nearby geodesics can be obtained from the metric without integrating the equations of motion. The generalized Maupertuis principle allows us to construct a model of Lagrange dynamics in the same sense as the famous Poincaré construction is a model of the Lobatchevsky geometry. Of course, there is an isomorphism between the original dynamics and its model. In our case, the dynamics is modeled by the congruence of geodesics on the conformally singular manifold with the Jacobi metric. However, the existence of singularities of the Ricci scalar (see [1], Sec. 239) does not exclude this approach: by constructing the so-called singular

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covering (the covering space is without singularities), we can suitably regularize the problem (see [3]). Moreover, one can prove the existence and uniqueness of the degenerate geodesics passing through conformally singular points [4]. One can also show that the existence of singularity in the Ricci scalar does not mean the existence of singularity in the solution of the deviation equation [5].

It should be noticed that, for classical mechanical systems $(E-V \geqslant 0)$, the deviation of nearby geodesics, calculated along the normal vector, is described by a nonautonomous equation (the harmonic oscillator with the negative potential energy depending on time). The equilibrium position is, of course, unstable. By comparison with the autonomous case, one can see that the solutions of the deviation equation on a manifold with negative sectional curvature grow not more slowly than the exponential of the covered distance — $e^{\lambda s}$ (where λ is equal to the square root of the absolute value of the sectional curvature in this two-direction, for which this value is the smallest one). As the result of the above, the normal components of the deviation of the nearby geodesics behave similarly to a ball near unstable equilibrium [6].

The norm $\|n\|^2$ of the deviation vector n measures the actual distance between nearby geodesics of congruence (and not only between points on them). With the help of the components of the normal vector, one can determine the principal Lyapunov-like exponent. Such a detector of chaotic behavior is qualitative (cf. [2]). However, chaos cannot be created by an averaging procedure. The negative value of the Ricci scalar is a sufficient (but not necessary) condition of the local instability of the geodesic flow. Evidently, the Ricci scalar as an invariant of the internal geometry is a gauge invariant detector of chaos. This means that such a detector is invariant with respect to any change of coordinate systems. However, the problem is that the standard Lyapunov exponents are global and quantitative measures of chaos rather than qualitative and local ones. The deviation equation is in fact local, but the Lyapunov-like exponents derived from it are defined globally.

Deterministic chaos in general relativity is obscure as a result of the gauge freedom of this theory. In particular, the property of sensitive dependence on initial conditions — the key ingredient of deterministic chaos — should be invariant with respect to time reparametrization. The sensitive dependence on initial conditions means that long-time prediction of motion in the phase space is impossible since small initial perturbations grow arbitrarily large as the system evolves in time. This property is a necessary but not sufficient condition for the existence of deterministic chaos. It is also a sufficient condition if the geodesic congruence is defined on a compact manifold [7]. Unfortunately, the compactness of a phase space (or finiteness of its volume) is rather difficult to prove. The choice of the bounded orbits may be made from the corresponding condition imposed on the effective potential method or, equivalently, as we do, from the compactness (boundedness) of the admitted domain of motion.

Additionally, as is well known, the Lyapunov stability is not invariant with respect to changing the coordinates and to rescaling the time variable [8,9]. If the system is locally unstable then nearby geodesics diverge exponentially, which implies sensitive dependence on initial conditions.

To make the presentation self-contained and to establish the notation, the standard definitions and formulas will be given. Our main results are the following: (1) a quantitative measure of separation of nearby trajectories is obtained via the averaging procedure of the qualitative separation measure; (2) a method of reducing relativistic mechanical systems to classical ones is proposed; (3) a general formula for the Ricci scalar for systems with the potential function of the form $V=V(r_{ij})$, where r_{ij} is the distance between particles, is given.

It should be noticed that it is meaningless to apply our approach to a certain class of problems, for instance, to the problem of motion of a test particle in the external gravitational field of general relativity. Such test particles and photons already move along geodesics in a given spacetime, and their dynamics is geometrized from the very beginning. In this case, the geodesic motion is determined by the Hamilton function which coincides with the kinetic energy form.

II. RICCI SCALAR FOR CLASSICAL SIMPLE MECHANICAL SYSTEMS

Mechanical systems with the natural Lagrangian and positive kinetic energy form $(E-V \geqslant 0)$ will be called classical simple mechanical systems. For these systems the Ricci scalar of the space with Jacobi metric (1) is given by the formula [2]

$$\hat{R}=\frac{N-1}{8(E-V)^3}[4(E-V)\Delta V-(N-6)N_{V^2}] \tag{5}$$

where

$$\Delta V=g^{ij}\nabla_i\nabla_j V,$$

$$N_{V^2}=g^{ij}\nabla_i V\nabla_j V,$$

and ∇_i is the covariant differentiation with respect to the metric g. Let us consider a classical mechanical system $(g_{ij}=\text{const}\times\delta_{ij}$ and $\nabla_i=\partial_i)$ with the potential energy V taken from $\mathbf{R}^{3N}$ in the form

$$V(\vec{x}_1,\ldots,\vec{x}_N)=\sum_{i=1}^{N} V(r_{ij}), \tag{6}$$

$$r_{ij}=r_{ji}=|\vec{x}_i-\vec{x}_j|=\left[\sum_{k=1}^{3}(x_i^k-x_j^k)^2\right]^{1/2}.$$

The Ricci scalar for the potential in the general form (6) can be determined if we notice that

$$\frac{\partial}{\partial\vec{x}_k}V(\vec{x}_1,\ldots,\vec{x}_N)=\frac{\partial}{\partial\vec{x}_k}\left[\sum_{1\leqslant s<k}V(r_{sk})+\sum_{k<s\leqslant N}V(r_{ks})\right]$$

and

$$\frac{\partial V(r_{sk})}{\partial \vec{x}_k}=V'(r_{sk})\frac{\partial r_{sk}}{\partial \vec{x}_k}=V'(r_{ks})\frac{\partial r_{ks}}{\partial \vec{x}_k}$$

where

$$\frac{\partial r_{sk}}{\partial \vec{x}_k}=\frac{\vec{x}_k-\vec{x}_s}{r_{rs}}=\frac{\partial r_{ks}}{\partial \vec{x}_k}.$$

From the above formulas we obtain

$$\frac{\partial}{\partial \vec{x}_k}V(\vec{x}_1,\ldots,\vec{x}_N)=\sum_{s\neq k}(\vec{x}_k-\vec{x}_s)\frac{V'(r_{ks})}{r_{ks}}=\sum_{s\neq k}\vec{r}_{ks}\frac{V'(r_{ks})}{r_{ks}} \tag{7}$$

where $\vec{r}_{ks}=\vec{x}_k-\vec{x}_s$.

It is convenient to introduce the following notation. Let

$$\vec{M}_k(\vec{x}_1,\ldots,\vec{x}_N)\equiv\sum_{s\neq k}\vec{r}_{ks}\frac{V'(r_{ks})}{r_{ks}}; \tag{8}$$

then we have

$$\mathrm{grad}V=(\vec{M}_1,\ldots,\vec{M}_N)$$

and

$$\|\mathrm{grad}V\|^2=\|\vec{M}_1\|^2+\cdots+\|\vec{M}_N\|^2. \tag{9}$$

From formula (5) one can see that the Ricci scalar is determined if $\|\mathrm{grad}V\|^2$ and the Laplacian $\Delta V=\Sigma_{k=1}^{N}(\partial/\partial\vec{x}_k)\vec{M}_k$ are given.

After simple calculations we obtain

$$\Delta V=\sum_{k=1}^{N}\left\{\sum_{s\neq k}^{N}\left[V''(r_{ks})+2\frac{V'(r_{ks})}{r_{ks}}\right]\right\}. \tag{10}$$

In the case of the Newtonian gravitational potential

$$V(r_{ij})=-G\frac{m_i m_j}{r_{ij}}$$

we have $\Delta V=0$.

It is easy to see that total force acting on the system at the point $(\vec{x}_1,\ldots,\vec{x}_N)$ is equal to zero because

$$\frac{\partial V}{\partial \vec{x}_1}+\cdots+\frac{\partial V}{\partial \vec{x}_N}=0$$

or

$$\begin{aligned}\vec{M}_1+\vec{M}_2+\cdots+\vec{M}_N&=\sum_{s\neq 1}\vec{r}_{1s}\frac{V'(r_{1s})}{r_{1s}}+\sum_{s\neq 2}\vec{r}_{2s}\frac{V'(r_{2s})}{r_{2s}}\\&\quad+\cdots+\sum_{s\neq N}\vec{r}_{Ns}\frac{V'(r_{Ns})}{r_{Ns}}\\&=\vec{r}_{12}\left[\frac{V'(r_{12})}{r_{12}}-\frac{V'(r_{21})}{r_{21}}\right]\\&\quad+\vec{r}_{13}\left[\frac{V'(r_{13})}{r_{13}}-\frac{V'(r_{31})}{r_{31}}\right]+\cdots\\&\quad+\vec{r}_{N-1N}\left[\frac{V'(r_{N-1N})}{r_{N-1N}}\right.\\&\quad\left.-\frac{V'(r_{NN-1})}{r_{NN-1}}\right]=0.\end{aligned}$$

For a higher derivative theory of gravitation, the perturbation expansion gives rise to divergences which can be avoided with the help of a renormalization procedure. In this type of theory, the Newtonian limit of gravitational potential is of the form [10]

$$V(r)=\frac{1}{r}-\frac{4}{3}\frac{e^{-m_2 r}}{r}+\frac{1}{3}\frac{e^{-m_0 r}}{r}.$$

Special cases of the potential function (6) are the Debye-Hückel potential and the Lennard-Jones potential. The latter has the form

$$V(r_{ij})=4\epsilon\left[\left(\frac{\sigma}{r_{ij}}\right)^{12}-\left(\frac{\sigma}{r_{ij}}\right)^{6}\right]. \tag{11}$$

It is a realistic model for interaction between the atoms of inert gases such as argon. It is convenient to choose σ as the units of mass and time such that $4\epsilon=1$.

After substitution of the Lennard-Jones potential form into formulas (7), (8), and (9) we obtain

$$V'(r_{ij})=24\sigma^6\epsilon\left(\frac{1}{r_{ij}^7}-\frac{2\sigma^6}{r_{ij}^{13}}\right),$$

$$V''(r_{ij})=-24\sigma^6\epsilon\left(\frac{7}{r_{ij}^8}-\frac{26\sigma^6}{r_{ij}^{14}}\right),$$

and

$$\Delta V=-24\sigma^6\epsilon\sum_{k=1}^{N}\left[\sum_{s=1,s\neq k}^{N}\left(\frac{5}{r_{ks}^8}-\frac{22\sigma^6}{r_{ks}^{14}}\right)\right], \tag{12}$$

as well as

$$\begin{aligned}N_{V^2}&=\|\mathrm{grad}V\|^2\\&=2\sum_{1\leq i<j\leq N}[V'(r_{ij})]^2\\&\quad+2\sum_{i=1}^{N}\left[\sum_{1\leq k<s\leq N;k,s\neq i}\frac{\vec{r}_{ik}}{r_{ik}}\frac{\vec{r}_{is}}{r_{is}}V'(r_{ik})V'(r_{is})\right].\end{aligned} \tag{13}$$

Formulas (5) and (13) show that for $N>6$ (the number of particles is greater than 2) the Ricci scalar is negative if $\Delta V<0$. In the case of the Lennard-Jones potential, it means that if $\forall s\neq k, r_{sk}>\sqrt[5]{22\sigma^6/5}$, then the system is locally unstable.

The idea of determining the relaxion time for the N-body system in terms of the Ricci scalar was formulated by Krylov [11] for the potential (6), but with no solid justification. In our approach the Ricci scalar is the average measure of the sectional curvature over all two-directions for any type of potential function [2]. Our paper is an attempt to develop Krylov's idea for the case of relativistic dynamical systems.

III. RICCI SCALAR AND LOCAL INSTABILITY FOR RELATIVISTIC DYNAMICAL SYSTEMS

Systems with a natural Lagrangian and an indefinite kinetic energy form will be called relativistic simple dynamical (mechanical) systems. They find their natural applications in general relativity and gravitation. In such applications the corresponding Jacobi metric is, in general, pseudo-Riemannian. The configuration space admissible for motion of the Hamiltonian system is a manifold with singularities. In relativistic mechanical systems timelike, spacelike, and null vectors must be distinguished. If, for example, the signature of the Jacobi metric is $(-+\cdots+)$ we define directions corresponding to the minus sign as spacelike $(\hat{g}_{\alpha\beta}n^\alpha n^\beta<0)$, those corresponding to the plus sign as timelike $(\hat{g}_{\alpha\beta}n^\alpha n^\beta>0)$, and null when $\hat{g}_{\alpha\beta}n^\alpha n^\beta=0$. Let us notice that the Hamiltonian constraint $\mathscr{H}=0$ in general relativity determines the tangent vector u which is spacelike in the region $E-V<0$, timelike in the region $E-V>0$, and null on the boundary set $E=V$.

The above discussed criterion of the local instability "on the average" must be modified in the case of a relativistic mechanical system. In this case, the local instability of the geodesic flow means that

$$\hat{R}<0 \quad \text{if } \hat{g}(u,u)=1,$$

$$\hat{R}>0 \quad \text{if } \hat{g}(u,u)=-1,$$

$$\|u\|^2=\hat{g}(u,u)=\text{sgn}(E-V),$$

$$ds/dt=2|E-V|,$$

$$\hat{g}=2|E-V|g,$$

$$\Updownarrow$$

$$\hat{R}<0,$$

$$\|u\|^2=\hat{g}(u,u)=1, \quad ds/dt=2(E-V), \quad \hat{g}=2(E-V)g.$$

The reduction of the motion to a geodesic flow via the Maupertuis procedure admits freedom in the choice of sign of ds/dt [i.e., monotonicity of the function $s(t)$]. While in the first case the function $s(t)$ is strictly monotonic (reversible) in the second case, it is nonmonotonic (irreversible).

From the above criteria one can see that, for a conformal factor with module, we get a global formulation of the Maupertuis-Jacobi principle, whereas in the second case (no module) the approach is strictly local.

The above criteria also show that it is not the sign of the Ricci scalar $\hat{R}$ that decides the sign of the right-hand side of (4) but rather the sign of $\hat{R}\text{sgn}(E-V)$.

The deviation equation (4) can be interpreted as the equation of a harmonic oscillator in $n^\alpha, \alpha=1,\ldots,N-1$, coordinates and with the time dependent potential energy

$$\widetilde{V}=\frac{\hat{R}(s)}{2N(N-1)}\hat{g}(u,u)\sum_{\alpha=1}^{N-1}(n^\alpha)^2.$$

The local instability of solutions of the geodesic equation is equivalent to the condition

$$\widetilde{V}<0 \quad \text{or} \quad [\text{grad}_n V_u(n)]^i<0$$

where

$$V_u(n)=\frac{1}{2}K_{u;n}\hat{g}(u,u)\hat{g}(n,n).$$

Here $K_{u;n}$ is the sectional curvature in the $(u;n)$ direction [2]. The above criterion of local instability is not equivalent to Arnold's condition: $V_u(n)<0$ [6] (neither is it identical with our condition $\widetilde{V}<0$). It can be easily seen that if the norms of the tangent and normal vectors are positive then both criteria coincide. In general, the sectional curvature and the sign of the norm of the tangent vector decide the local instability of the geodesic flow.

IV. REDUCTION OF A RELATIVISTIC SIMPLE MECHANICAL SYSTEM TO A CLASSICAL ONE

Let us consider geodesic motion of an uncharged test mass $(m=1)$ in the gravitational field of two fixed masses M_1 and M_2. In this problem gravitational attraction of two fixed masses is balanced by the electrostatic repulsive force and there is the Majumdar-Papapetrou solution to the Einstein-Maxwell equations [12,13]. The line element is

$$ds^2=-\frac{1}{U^2(x,y,z)}dt^2+U^2(x,y,z)(dx^2+dy^2+dz^2)$$

where

$$U(x,y,z)=1+\frac{M_1}{r_1}+\frac{M_2}{r_2},$$

$$r_i=\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2},$$

and this line element represents the static solutions of the Einstein-Maxwell equations for two fixed centers located at the points (x_i,y_i,z_i), $i=1,2$. In general, the Majumdar-Papapetrou form of solutions is true if $U=U(x,y,z)$ satisfies the Laplace equation in flat space: $\Delta U=U_{xx}+U_{yy}+U_{zz}=0$.

The geodesic motion is determined by the Hamiltonian of relativistic simple dynamical systems (see Table I). In this case the motion of a test particle or a photon in spacetimes of general relativity is geodesic. Therefore the Maupertuis-

TABLE I. Examples of relativistic simple dynamical systems.

Mechanical system	Hamilton (Lagrange) function	Remarks
Friedmann-Robertson-Walker cosmology coupled to real free massive scalar field	$\mathscr{H}=\frac{1}{2}(-p_1^2+p_2^2)+\frac{1}{2}(-q_1^2+q_2^2+m^2q_1^2q_2^2)=0$	Nonintegrable (chaotic) [22,21]
Friedmann-Robertson-Walker model with curvature-squared terms in action	$\mathscr{H}=-\frac{1}{4}(p_1^2-p_2^2)+\frac{1}{4}(q_1^2-q_2^2)+\frac{q_1^2}{8\beta}(-q_1+q_2)^2$	Nonintegrable (chaotic) [23,24]
Cosmological models (Friedmann or Bianchi I) with minimally coupled scalar fields	$\mathscr{H}=\frac{1}{2}\frac{p_\phi^2}{a^3}-\frac{2\pi}{3m_p^2}\frac{p_a^2}{a}+\frac{1}{2}m^2a^3\phi^2-\frac{3m_p^2}{8\pi}k_\gamma a^{3-2\gamma}$	Nonchaotic [25,26]
Multidimensional cosmology with topology $\mathbf{R}\times M_1^3\times M_2^{n-3}$ equivalent to classical model with minimally coupled scalar field	$\mathscr{L}=-\frac{1}{2}n(n-1)\dot{q}_1^2+\frac{1}{2}\dot{q}_2^2+\frac{\overline{R}}{2}e^{-2q_1}+V(q_2)$	Integrable [9]
Mixmaster model as disturbed periodic Toda lattice	$\mathscr{H}=\frac{1}{2}\Sigma_{i,j=1}^n a_{ij}p_ip_j+\Sigma_{k,l=1}^n b_{kl}\exp(\Sigma_{i=1}^n\alpha_k q_i+\Sigma_{i=1}^n\alpha_l q_i)$	Integrable? chaotic? [27]
Free particle in Majumdar-Papapetrou spacetime	$\mathscr{H}=\frac{1}{2m}[-U^2p_t^2+U^{-2}(p_x^2+p_y^2+p_z^2]$	Chaotic [28]
Charged particle in uniform magnetic field and linearly polarized gravitational wave	$\mathscr{L}=\frac{1}{2}\pi_0^2-\frac{1}{2}\pi_1^2-\frac{1}{2}\frac{(x^1)^2}{1-\alpha\sin[\nu(x^1-x^0)]}\equiv\frac{1}{2}$	Chaotic [29]
Multidimensional cosmology with topology $\mathbf{R}\times\mathbf{M}_1\times\cdots\times\mathbf{M}_n$	$\mathscr{L}=\frac{1}{2}G_{ij}\dot{x}^i\dot{x}^j-V(x)$	Chaotic? [30]
General relativity with scalar field in Arnowitt-Deser-Misner (ADM) formulation	$\mathscr{H}=N\left\{G_{ij\alpha\beta}\Pi^{ij}\Pi^{\alpha\beta}-\sqrt{g}\,{}^3R+\frac{\sqrt{g}}{2}\left[\frac{\Pi_\phi^2}{g}+V(\phi)\right]\right\}$	Integrable? chaotic? [19]

Jacobi procedure is not necessary here. Let us note that in this class of problems there is no trouble with the metric's singularity.

For the Majumdar-Papapetrou metric there is the first integral which corresponds to the Killing field of the time coordinate t — $\partial/\partial t$ — and one can reduce the problem to the three-dimensional case. In this case the problem of geodesics (timelike and zero) is equivalent to the problem of geodesics on the Riemannian space with the metric

$$\overline{ds}^2=U^2(x,y,z)[U^2(x,y,z)+h](dx^2+dy^2+dz^2)$$
$$=2Wg_{\alpha\beta}dx^\alpha dx^\beta$$

where $h<0$ is the constant of motion, $U=U(x,y,z)$ satisfies the Laplace equation (we will not specify the concrete form of this function), and $U^2+h\geqslant 0$ — the region admissible for motion.

On such a metric manifold (Riemannian), the geodesic motion is determined by the Hamiltonian

$$\mathscr{H}=T=\frac{1}{2}g^{ij}p_ip_j, \quad p_i=g_{ij}\dot{x}^j.$$

The above procedure of reducing relativistic simple dynamical systems to classical simple mechanical systems by using first integrals of the equation of motion in general works for static spacetimes with the metric

$$ds^2=g_{00}dt^2+g_{ij}dx^idx^j, \quad i,j=1,2,3,$$

and

$$g_{00}<0, \quad g_{0i}=0.$$

In this case three-dimensional Riemannian space takes the form of the conformally Euclidean space with the metric

$$\overline{ds}^2=-\frac{1-hg_{00}}{g_{00}}\sum_{i=1}^N (dx^i)^2\equiv 2Wg_{ab}dx^adx^b,$$
$$-\frac{1-hg_{00}}{g_{00}}\geqslant 0, \tag{14}$$

and for $h\equiv 0$ we obtain the null-geodesic problem in the static spacetime.

As was mentioned elsewhere [2] the Ricci scalar plays the role of the average measure of the local instability of the geodesic flow. If three-dimensional space is compact and the Ricci scalar is negative, and the entire phase space is a chaotic invariant set, then the geodesic flow of the manifold is chaotic on ''average'' (if the sectional curvature in all two-directions and at any point is negative the geodesic flows of compact manifolds are chaotic). The sign of the sectional curvature can be characterized by the so-called invariant curvature polynomials [2].

In general, for the metric (14), the Ricci scalar takes the form

$$\hat{R}=\frac{N-1}{8W^3}[-4W\Delta W-(N-6)(\nabla W)^2].$$

Because the conformal factor $2W=f(U)$ is a function of harmonic functions we obtain the very simple final results

$$\hat{R}=-\frac{(N-1)(\nabla U)^2}{f(U)}\left[(\ln f)''+\frac{N-2}{4}(\ln f)'^2\right], \quad '\equiv\frac{d}{dU},$$

for null geodesics,

$$\hat{R}_{h=0}=0,$$

and for spacelike ($h>0$) or timelike ($h<0$) geodesics,

$$\hat{R}_{h\neq 0}=-\frac{(N-1)U^2(\nabla U)^2}{f^3(U)}h(2U^2+h).$$

The sign of the Ricci scalar for timelike geodesics is strictly positive whereas for spacelike geodesics it is strictly negative. This fact means, in our terminology, that the system has the property of a sensitive dependence on initial conditions only for spacelike geodesics (tachyons) but we must remember that the negativeness of the Ricci scalar as a criterion of local instability is sufficient but not necessary.

V. AVERAGE SEPARATION RATES ALONG A GEODESIC

For simplicity let us consider the averaged deviation equations (over all two-directions) (4) in the case $N=2$. The generalization of our considerations to the N-dimensional case may be done automatically. Let

$$\ddot{x}=f(s)x,\quad x\in\mathbf{R}^2, \tag{15}$$

or

$$\dot{x}^1=x^2,$$

$$\dot{x}^2=f(s)x^1,$$

where an overdot denotes differentiation with respect to s; $x^1=n^1$, $x^2=\dot{n}^1$, and we have

$$f(s)=-\frac{\hat{R}(q(s))}{N(N-1)}=-\frac{\hat{R}(q(s))}{2}.$$

We assume that solutions of Eqs. (15) can be given by the Picard functional series of successive approximations of $\Phi(s)$. This solution is a 2×2 matrix,

$$\Phi_0(s)=\mathbb{1},$$

$$\Phi_{k+1}(s)=\mathbb{1}+\int_0^s A(s')\Phi_k(s')ds', \tag{16}$$

where

$$A(s)\equiv\begin{bmatrix}0 & 1\\ f(s) & 0\end{bmatrix}.$$

In our case $(N=2)$ we have

$$\Phi_0(s)=\mathbb{1},$$

$$\Phi_1(s)=\mathbb{1}+\int_0^s A(s')\Phi_0(s')ds', \tag{17}$$

and

$$\Phi_1(s)=\mathbb{1}+\int_0^s\begin{bmatrix}0 & 1\\ f(s') & 0\end{bmatrix}ds'=\mathbb{1}+\begin{bmatrix}0 & s\\ \int_0^s f(s')ds' & 0\end{bmatrix}.$$

Now, we introduce the following definition of the Lyapunov-like exponents averaged over the parameter s:

$$\lambda=:\lim_{s\to\infty}\frac{1}{s}\int_0^s f(s')ds'. \tag{18}$$

Let us note that that they have quantitative character.

We assume that limit (18) exists and then we obtain

$$\Phi_1(s)\cong\mathbb{1}+\begin{bmatrix}0 & s\\ s\lambda & 0\end{bmatrix} \tag{19}$$

as $s\to\infty$.

Analogously, we obtain

$$\Phi_2(s)=\mathbb{1}+\int_0^s\begin{bmatrix}0 & 1\\ f(s') & 0\end{bmatrix}\begin{bmatrix}1 & s'\\ s'\lambda & 1\end{bmatrix}ds'\cong\mathbb{1}+\begin{bmatrix}(s^2/2)\lambda & s\\ s\lambda & \int_0^s s'f(s')ds'\end{bmatrix}. \tag{20}$$

By integrating by parts we obtain

$$\int_0^s s'f(s')ds'=s'\int_0^{s'}f(s_1)ds_1\Big|_{s'}^{s}-\int_0^s\left[\int_0^{s'}f(s_1)ds_1\right]ds'$$
$$=s\int_0^s f(s')ds'-\int_o^s s'\left[\frac{1}{s'}\int_0^{s'}f(s_1)ds_1\right]ds'$$
$$\cong\lambda s^2-\lambda\frac{s^2}{2}=\lambda\frac{s^2}{2}.$$

By induction one can show that

$$\Phi_{2n}(s)=\begin{bmatrix}\sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!} & \sum_{k=0}^{n-1}\frac{\lambda^k s^{2k+1}}{(2k+1)!}\\ \lambda\sum_{k=0}^{n-1}\frac{\lambda^k s^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!}\end{bmatrix}, \tag{21}$$

and

$$\Phi_{2n+1}(s)=\begin{bmatrix}\sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!}\\ \lambda\sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!}\end{bmatrix}. \tag{22}$$

To see this it is enough to notice that

$$\Phi_{2n+1}(s)=\mathbb{1}+\int_0^s\begin{bmatrix}0 & 1\\ f(s') & 0\end{bmatrix}\Phi_{2n}(s)ds$$

$$=\mathbb{1}+\int_0^s\begin{bmatrix}0 & 1\\ f(s') & 0\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{n}\frac{\lambda^k s'^{2k}}{(2k)!} & \sum_{k=0}^{n-1}\frac{\lambda^k s'^{2k+1}}{(2k+1)!}\\ \lambda\sum_{k=0}^{n-1}\frac{\lambda^k s'^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n}\frac{\lambda^k s'^{2k}}{(2k)!}\end{bmatrix}ds'$$

$$=\mathbb{1}+\int_0^s\begin{bmatrix}\lambda\sum_{k=0}^{n-1}\frac{\lambda^k s'^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n}\frac{\lambda^k s'^{2k}}{(2k)!}\\ \sum_{k=0}^{n}\frac{\lambda^k f(s')s'^{2k}}{(2k)!} & \sum_{k=0}^{n-1}\frac{\lambda^k f(s')s'^{2k+1}}{(2k+1)!}\end{bmatrix}ds'$$

$$=\mathbb{1}+\begin{bmatrix}\lambda\sum_{k=0}^{n-1}\frac{\lambda^k s^{2k+2}}{(2k+2)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!}\\ \sum_{k=0}^{n}\frac{\lambda^k}{(2k)!}\int_0^s f(s')s'^{2k}ds' & \sum_{k=0}^{n-1}\frac{\lambda^k}{(2k+1)!}\int_0^s f(s')s'^{2k+1}ds'\end{bmatrix}$$

$$=\mathbb{1}+\begin{bmatrix}\sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!}\\ \lambda\sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n-1}\frac{\lambda^{k+1} s^{2k+2}}{(2k+2)!}\end{bmatrix}$$

$$=\begin{bmatrix}\sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!}\\ \lambda\sum_{k=0}^{n}\frac{\lambda^k s^{2k+1}}{(2k+1)!} & \sum_{k=0}^{n}\frac{\lambda^k s^{2k}}{(2k)!}\end{bmatrix}.$$

Finally we obtain

$$\lim_{n\to\infty}\Phi_n(s)\cong\begin{bmatrix}\cosh(\sqrt{\lambda}s) & \frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}s)\\ \sqrt{\lambda}\sinh(\sqrt{\lambda}s) & \cosh(\sqrt{\lambda}s)\end{bmatrix}$$

and general solutions of (15) are of the form

$$\lambda>0:\quad x(s)\cong\cosh(\sqrt{\lambda}s)x_0+\frac{1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}s)\dot{x}_0,$$

$$\dot{x}(s)\cong\sqrt{\lambda}\sinh(\sqrt{\lambda}s)x_0+\cosh(\sqrt{\lambda}s)\dot{x}_0, \tag{23}$$

$$\lambda<0:\quad x(s)\cong\cos(\sqrt{|\lambda|s})x_0+\frac{1}{\sqrt{|\lambda|}}\sin(\sqrt{|\lambda|s})\dot{x}_0,$$

$$\dot{x}(s)\cong-\sqrt{|\lambda|}\sin(\sqrt{|\lambda|}x_0+\cos(\sqrt{|\lambda|s})\dot{x}_0. \tag{24}$$

The exact solutions of the averaged geodesic deviation equations over all two-directions have a very simple interpretation, namely, the first order averaging procedure gives us the Ricci scalar as an average measure of the local instability, whereas the second order averaging procedure gives us the average scalar curvature

$$\lambda=-\lim_{s\to\infty}\frac{1}{s}\int_0^s\frac{\hat{R}(s')}{N(N-1)}ds'.$$

In this way, the parameter λ may be treated as a quantitative measure of the average rate of separation of trajectories (over all two-directions and the natural parameter s along the geodesic).

If the dynamical process is ergodic then

$$\lambda=-\lim_{s\to\infty}\frac{1}{s}\int_0^s\frac{\hat{R}(s')}{N(N-1)}ds'$$

$$=-\frac{1}{(\mathrm{Vol}M^N)}\int_{M^N}\hat{R}d(\mathrm{Vol}M^N);$$

i.e., the average over ‘‘time’’ can be replaced by the average over the finite configuration space M^N. It is worth noticing that in this ergodic case the principal Lyapunov-like exponent λ is proportional to the ‘‘gravitational action.’’ For the two-dimensional, compact configuration space,

$$\lambda=-\frac{1}{VM^2}4\pi\kappa(s^2),$$

if we assume ergodicity. This means that the local instability of geodesic flows on two-dimensional compact Riemannian spaces is determined by their topological characteristics (Euler characteristic κ). On the other hand, the classification of the two-dimensional compact manifolds is known and their Euler characteristics are equal to $\kappa(s^2)=2-2g$, where g is the number of handles. This coincidence certainly reflects a deep connection between deterministic chaos (as a nonlinear effect) and the global topological methods.

It is interesting to observe that if $\int^s \hat{R} ds$ goes to infinity as $s\rightarrow\infty$ we obtain

$$\lambda=-\frac{\hat{R}(W(q(\infty)))}{N(N-1)}.$$

The above formula means that in the case of asymptotically free systems, $W\rightarrow$const as $s\rightarrow\infty$ and we obtain $\lambda=0$ [this is also the case if $\hat{R}(W(q(s)))$ is quasiperiodic]. This reflects the fact that the system is integrable. Moreover, in the Bianchi type IX cosmology near the initial singularity trajectories concentrate in the neighborhood of $V=0$ [14]. Gutkin proved the important theorem characterizing a class of ingrable systems with exponential potentials. He has shown that the system which is asymptotically free is integrable [15]. This theorem can be used to demonstrate that the models of this subclass which allow Kasner's asymptotics in the neighborhood of the initial or final singularities are integrable (for instance, the Bianchi class A model with massless scalar field) [16].

The application of the above chaos criterion, which is now global and quantitative, to the classical ($E-V\geqslant 0$) and relativistic ($E-V\geqslant 0, E-V<0$) mechanical systems will be investigated in subsequent papers. In Table I some relativistic simple mechanical systems are presented. Let us notice that the invariant qualitative criterion of chaos is a new "quality" and the question "What does it mean?" is in principle an open problem. It seems that the chaotic behavior of the Bianchi type IX model does not correspond to any standard concept of deterministic chaos. On the other hand, the invariant chaos description is necessary in the context of gauge theories such as the theory of relativity and its cosmological applications [17].

If the kinetic energy form is indefinite we have to work with the pseudo-Riemannian manifold with the singular boundary set $\Theta: |E-V|=0$ on which the Jacobi metric degenerates. The Maupertuis principle, in this case, reduces the problem of motion to the study of global geodesics in the general sense (γ is a geodesic on $M\backslash\Theta$). Singularities of $\hat{g}^{\alpha\beta}$ are indeed obstacles to both analytical and numerical computations [18] but no problem arises in discussing the dynamics near those points in terms of the original dynamical systems. It is worthwhile mentioning that the Maupertuis principle for the case of relativistic dynamical systems was implicitly used by Misner in his minisuperspace construction [14]. In this case, Misner's supertime coincides with our parameter s.

Another important circumstance is that the Jacobi geodesics can be uniquely prolonged through the singularity [3,4]. Owing to this fact we can draw global conclusions concerning the behavior of dynamics (not just piecing together some local geodesics, as is done in the typical case).

VI. THE MIXMASTER DYNAMICS

It can be shown that the Bianchi IX [B(1X)] cosmology [19] is well approximated by a series of Bianchi I solutions (Kasner epochs) connected by bounces (Bianchi II solutions). Therefore one can understand the dynamics of the Mixmaster cosmology (near the initial singularity) as the dynamics of a particle in the potential moving in a three-dimensional space.

Singularities in the Jacobi metric for the B(IX) model in the Balinskii-Khalatnikov-Lifshitz (BKL) approximation belong to the mild singularity class such that the metric is continuous and the curvature is of δ-function type. The potential function can be approximated by a δ function which is zero everywhere except for moments s_n of change of Kasner epochs. It allows us to interpret the solutions of the geodesic equation in terms of generalized functions. The most important consequence of this is the possibility to uniquely prolong geodesics through this conformal singular point. Then the Ricci scalar takes the form

$$\hat{R}\equiv\delta(s-s_n)=\begin{cases}0 & \text{at Kasner epoch change } \; s\neq s_n,\\ \infty & \text{during Kasner epoch } \; s=s_n,\end{cases}$$

where $s_n=2\Sigma_{i=1}^{n}\beta_i(u)$ represents the total length of n Kasner epochs in terms of Maupertuis time s, and in the Belinski-Lifshitz-Khalatnikov approximation we obtain

$$\overline{\lambda}=-\lim_{s\rightarrow\infty}\frac{1}{s}\int_{s_0}^{s}\hat{R}(s')ds'=-\lim_{s\rightarrow\infty}\frac{1}{s}\int^{s}\sum_{n}\delta(s-s_n)ds\cong\frac{1}{2\overline{\beta}},$$

$$\overline{\beta}=\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i=1}^{n}\beta_i,$$

where n is the number of Kasner epochs in the interval (s_0,s); $2\beta_i$ represents the length of the ith Kasner epoch in the parameter s, and $2\overline{\beta}$ the average length of the Kasner epoch in the Maupertuis time s; the length $2\beta_i$ is related to the dominating Kasner exponent p_i, $i=1,2,3$, such that $p_i<p_j<p_k$ when $\beta_i=|2p_i(u)|$, $\Sigma_{i=1}^{3}p_i=\Sigma_{i=1}^{3}p_i^2=1$. As is well known, p_i can be parametrized by one parameter u (for the specific terminology and a review of chaos in general relativity we recommend Ref. [20]). Thus the infinite number

of δ-type contributions to $\overline{\lambda}$ can lead to finite quantity of the average rate of separation of nearby geodesics.

Therefore, in every transition from one Kasner epoch to another one bit of information is lost (on the average) because in every Kasner transition the normal separation vector increases e times:

$$n^{\alpha}(s) \propto e^{(1/2\overline{\beta})\overline{\Delta s}} = e^{(1/2\overline{\beta})2\overline{\beta}} = e.$$

In the first BKL approximation the volume of the configuration space with the Jacobi metric is finite (whereas in the second BKL approximation it is infinite). This fact means that the information loss is not associated with a change of epoch, but rather with a change of era.

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