Schild's null strings in flat and curved backgrounds

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Schild's null (tensionless) strings are discussed in certain flat and curved backgrounds. We find closed, stationary, null strings as natural configurations existing on the horizons of spacetimes which possess such null hypersurfaces. Examples of these are obtained in Schwarzschild and Rindler spacetimes. A dynamic null string is also identified in Rindler spacetime. Furthermore, a general prescription (with explicit examples) is outlined by means of which null string configurations can be obtained in a large class of cosmological backgrounds. [S0556-2821(96)01210-6]

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In a posthumously published paper, nineteen years ago, Schild [1] first introduced the notion of a null or tensionless string (see also Karlhede and Lindström [2]). By replacing the Nambu-Goto Lagrangian with its square, he wrote down the action and equations of motion for the null string following the standard treatment for null geodesics. The null string was quantized by Lizzi *et al.* [3]. Surprisingly, they found that consistent quantization did not imply a critical dimension. Subsequently, it was realized that the absence of a critical dimension was essentially an artifact of the ordering of operators in the quantum theory. Whereas Lizzi *et al.* preferred Weyl ordering (the absence of oscillator modes being the rationale), others [4] used normal ordering and obtained $D=26$ as the critical dimension. Simultaneously, a Hamiltonian analysis was also carried out [5] and supersymmetric generalizations were discussed [6]. The latest, carefully done Hamiltonian Becchi-Rouet-Stora-Tyutin (BRST) analysis by Gustaffson *et al.* [7] (see also [8,9]) which employs the use of a smeared δ function in the canonical commutation relations, concludes that the critical dimension of the bosonic null string is $D=2$. A review of the literature on null p-branes and super p-branes up to 1993 can be found in Bandos and Zheltukhin [10].

However, null strings have almost never been discussed in a curved Lorentzian background. The only reference to a curved background can be found in the concluding portion of Schild's paper. Stachel [11], on the other hand, has obtained an energy-momentum tensor satisfying the conservation law with matter made out of a cloud of null strings (null string dust). This sort of matter can therefore be used as source for the Einstein equations and exact metrics generated by such a source. More recently, however, it has been shown [12] that the energy-momentum tensor describing a fluid of null strings can act as a source for metrics representing Friedman-Robertson-Walker (FRW) universes in both its matter and radiation-dominated epochs. In this article our aim is as follows. We first set up the equations of motion of a null string in a curved background and then obtain exact string configurations in a variety of backgrounds which include the Rindler, Schwarzschild, and some cosmological spacetimes.

Let us begin with the curved background action written down by Schild [1]. This is given as

$$S=\int \Sigma^2 d\sigma d\tau, \tag{1}$$

where $\Sigma^2=g_{\mu\alpha}g_{\nu\beta}\Sigma^{\mu\nu}\Sigma^{\alpha\beta}$ and

$$\Sigma^{\mu\nu}=\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\sigma}-\frac{\partial x^\mu}{\partial\sigma}\frac{\partial x^\nu}{\partial\tau}. \tag{2}$$

Here, $x^\mu(\tau,\sigma)$ is the embedding function for the null world sheet in a general background spacetime.

Note that Σ^2 is essentially the determinant of the metric induced on the world sheet by the background spacetime. If the world sheet is null then there must exist one null tangent vector. This leads to the fact that the determinant of the induced metric is identically zero.

The field equations and constraints that arise out of variations of the action in (1) are given as

$$\ddot{x}^\mu+\Gamma^\mu_{\rho\lambda}\dot{x}^\rho\dot{x}^\lambda=0, \tag{3}$$

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0, \quad g_{\mu\nu}\dot{x}^\mu x'^\nu=0, \tag{4}$$

where the overdots and primes denote differentiations with respect to τ and σ, respectively.

The constraint equations are not invariant for arbitrary reparametrizations of σ and τ. In fact, the allowed reparametrizations are $\tau_1=f(\sigma,\tau)$ and $\sigma_1=g(\sigma)$. Thus, the $\sigma\sigma$ element of the induced world sheet metric has to be independent of τ. Then, only one has a reparametrization-invariant theory where the class of reparametrizations allowed are restricted in comparison to the usual timelike bosonic string [13]. In fact, one can show that this restricted set of transformations which leave the degenerate character of the metric invariant form the Caroll group introduced by Levy Leblond [14] many years ago.

It should be mentioned that one can also use the action proposed in [2], instead of the one we have used [see Eq. (1)]. The classical equations of motion and constraints which

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are of primary concern in this paper are the same for both these actions as long as one makes the gauge choice $x^{\mu\prime}\dot{x}_\mu=0$, $\phi x'^2=$const where ϕ is the Lagrange multiplier field (analogue of the einbein for the massless particle).

The field equations are essentially geodesic equations. The constraint ($g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0$) implies that we should look for only null geodesics. Thus, knowing the null geodesics in a background spacetime would naturally lead to null string configurations provided all the constraints are satisfied. This is reminiscent of the bosonic case where the stationary string equations can also be transformed into geodesic equations in an *unphysical* Riemannian space [15].

We now move on towards solving these equations of motion and constraints in specific background spacetimes in order to obtain specific string configurations.

a. Minkowski spacetime. To begin, let us deal with the almost trivial case of Minkowski spacetime as the background. The null string equations of motion and constraints yield a solution of the form

$$x^\mu=a^\mu\tau+b^\mu(\sigma) \tag{5}$$

with the constraints

$$a^\mu a_\mu=0 \quad \text{and} \quad a^\mu b'_\mu(\sigma)=0. \tag{6}$$

A simple choice for a^μ and b^μ could be

$$a^\mu\equiv(1,1,0,0), \quad b^\mu\equiv(0,0,b^2(\sigma),b^3(\sigma)). \tag{7}$$

The string here is located on the lightcone of Minkowski spacetime ($x=t$). It can be closed or open depending on the choice of $b^2(\sigma)$ and $b^3(\sigma)$.

b. Rindler spacetime. The metric in Rindler spacetime is given as

$$ds^2=-a^2x^2dt^2+dx^2+dy^2+dz^2. \tag{8}$$

The null string equations of motion and constraints in this background turn out to be

$$\ddot{t}+2\frac{\dot{x}}{x}\dot{t}=0, \tag{9}$$

$$\ddot{x}+2a^2x\dot{t}^2=0, \tag{10}$$

$$\ddot{y}=0, \quad \ddot{z}=0, \tag{11}$$

$$-a^2x^2\dot{t}^2+\dot{x}^2+\dot{y}^2+\dot{z}^2=0. \tag{12}$$

Let us choose a generic string configuration as

$$t=t(\tau), \quad x=x(\tau), \quad y=y(\sigma), \quad z=z(\sigma). \tag{13}$$

A solution of the equations of motion and constraints is

$$t=\frac{1}{2a}\ln\tau, \quad x=A_1\tau^{1/2}, \quad y=g_1(\sigma), \quad z=g_2(\sigma), \tag{14}$$

where A_1 can be any constant and $g_1(\sigma)$ and $g_2(\sigma)$ are any two functions of σ. By choosing the functions $g_1(\sigma)$ and $g_2(\sigma)$ suitably, we can have different string configurations. Moreover, as $\tau\to0$ ($t\to-\infty$), $x\to0$, which is the event horizon of Rindler spacetime. The domain $0\leqslant\tau\leqslant\infty$ encompasses the whole of $-\infty\leqslant t\leqslant\infty$. The configuration starts out at the horizon at $t\to-\infty$ and ends at $t\to\infty$ where $x\to\infty$. Notice also that the string equations of motion and constraints have a very simple stationary string solution. This is given as

$$t=\tau, \quad x=0, \quad y=f_1(\sigma), \quad z=f_2(\sigma). \tag{15}$$

This solution exists exclusively at the horizon of the Rindler spacetime for all time t. However, this is quite expected because the horizon being one way membrane is a null hypersurface of the spacetime.

c. Schwarzschild spacetime. We now write down the string equations of motion and constraints for a general static, spherically symmetric metric given by

$$ds^2=-e^{2\Phi(r)}dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{16}$$

The null string equations of motion are

$$\ddot{t}+2\tilde{\Phi}\dot{r}\dot{t}=0, \tag{17}$$

$$\ddot{r}+\frac{\tilde{b}r-b}{2r(r-b)}\dot{r}^2-(r-b)(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=0, \tag{18}$$

$$\ddot{\theta}+\frac{2}{r}\dot{\theta}\dot{r}-\sin\theta\cos\theta\dot{\phi}^2=0, \tag{19}$$

$$\ddot{\phi}+\frac{2}{r}\dot{\phi}\dot{r}+2\cot\theta\dot{\phi}\dot{\theta}=0. \tag{20}$$

The constraint $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0$ reduces to

$$-e^{2\Phi}\dot{t}^2+\frac{\dot{r}^2}{1-b(r)/r}+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2=0. \tag{21}$$

The other constraint is trivially satisfied because of the diagonal nature of the background metric. In addition, the quantity $g_{\mu\nu}x^{\mu\prime}x^{\nu\prime}$ must be independent of τ.

It is easy to see that a closed stationary string of the form

$$t=\tau, \quad r=C, \quad \theta=\frac{\pi}{2}, \quad \phi=C_0\sigma, \tag{22}$$

can exist in *any* such geometry which posseses a horizon (i.e., $g_{00}=0$ somewhere), irrespective of the functional forms of $b(r)$ and $\Phi(r)$. In fact, the closed, stationary string exists exactly at the horizon which is by definition a null hypersurface. Thus, Lorentzian wormholes cannot have a closed, stationary null string anywhere. On the other hand, they do support a closed, stationary timelike string at their throats as has been pointed out in a recent paper by this author [16].

d. Cosmological spacetimes. We now move on towards obtaining null strings in cosmological models. The background metric is

$$ds^2=-dt^2+R^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right], \tag{23}$$

where $k=-1,0,1$ refer to hyperbolic, flat, and three-sphere (S^3) spacelike sections.

The string equations of motion in this background turn out to be

$$\ddot{t}+R\widetilde{R}\frac{\dot{r}^2}{1-kr^2}+R\widetilde{R}r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=0, \tag{24}$$

$$\ddot{r}+2\frac{\widetilde{R}}{R}\dot{r}\dot{t}+\frac{\dot{r}^2kr}{1-kr^2}-(1-kr^2)r(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=0, \tag{25}$$

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}+2\frac{\widetilde{R}}{R}\dot{t}\dot{\theta}-\sin\theta\cos\theta\dot{\phi}^2=0, \tag{26}$$

$$\overline{2R^2r^2\sin^2\theta\dot{\phi}}=0, \tag{27}$$

and the constraint equation is

$$-\dot{t}^2+R^2\left[\frac{\dot{r}^2}{1-kr^2}+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right]=0. \tag{28}$$

Our ansatz for a string solution would be

$$t=t(\tau),\quad r=r(\tau),\quad \theta=\frac{\pi}{2},\quad \phi=C_0\sigma. \tag{29}$$

This represents a closed string which is dynamic (nonstationary). The condition on $g_{\mu\nu}x^{\mu\prime}x^{\nu\prime}$ (i.e., its being independent of τ) constrains the choice of $r(\tau)$ and $R(t)$. We must have

$$r(\tau)R(t)=\frac{1}{C}, \tag{30}$$

where C is some constant. Therefore, the radius $r(\tau)$ of the string is inversely related to the scale factor governing the evolution of the spacelike sections of the cosmological model. Note that by virtue of this choice, the degenerate-induced metric on the null world sheet is the same for all forms of $R(t)$ and $r(\tau)$.

With the ansatz for a string configuration and subsequently that of $r(\tau)$, we end up with two equations which we have to solve in order to obtain $R(t)$ and $t(\tau)$:

$$\widetilde{R}=\pm\sqrt{C^2R^2-k}, \tag{31}$$

$$\ddot{t}+\frac{\widetilde{R}}{R}\dot{t}^2=0, \tag{32}$$

where the overdot denotes differentiation with respect to t.

We now treat the $k=-1,0,1$ cases separately. In all the results below we have chosen to use the plus sign on the right-hand side (rhs) of Eq. (28).

i. $k=-1$. This is the Universe with a hyperbolic spacelike section. The scale factor equation (28) has a solution given as

$$R(t)=\frac{1}{C}\sinh(Ct+C_1), \tag{33}$$

where C_1 is an arbitrary integration constant. Thus, the Universe begins with a big bang (at $t=-C_1/C$) and expands ever after.

The $r(\tau)$ and $t(\tau)$ turn out to be

$$r(\tau)=\left[\left(\frac{C^2}{C_0}\tau\right)^2-1\right]^{-1/2}, \tag{34}$$

$$t(\tau)=\frac{1}{C}\left(\cosh^{-1}\frac{C^2}{C_0}\tau-C_1\right), \tag{35}$$

where C_0 is an arbitrary integration constant. The closed null string collapses at $\tau\to\pm\infty$. However, it is necessary that $\tau>C_0/C^2$.

ii. $k=0$. Here the spacelike sections are flat. Equation (28) has a solution which represents the inflationary universe, with a scale factor given as

$$R(t)=\exp(Ct). \tag{36}$$

The string configuration is given as

$$t(\tau)=\frac{1}{C}\ln(C\tau+C_1), \tag{37}$$

$$r(\tau)=\frac{1}{C(C\tau+C_1)}. \tag{38}$$

Thus, as $\tau\to\infty$, $t\to\infty$, and $r\to0$. In this case also, the closed null string collapses but only as $\tau\to\infty$.

iii. $k=1$. This represents a Universe with S^3 spacelike sections. The scale factor has a solution given as

$$R(t)=\frac{1}{C}\cosh(Ct+C_1). \tag{39}$$

Note that the scale factor is such that the Universe is nonsingular (there is no big bang here).

The closed null string solution turns out to be

$$r(\tau)=\left[\left(\frac{C^2}{C_0}\tau\right)^2+1\right]^{-1/2}, \tag{40}$$

$$t(\tau)=\frac{1}{C}\left(-C_1+\operatorname{arcsinh}\frac{C^2}{C_0}\tau\right). \tag{41}$$

In this case also, the string collapses only as $\tau\to\pm\infty$.

It can be seen quite easily that equations similar to (28) and (29) [with the assumption $r(\tau)R(t)=C$] hold for a more general class of cosmological spacetime metrics generically represented as

$$ds^2=-dt^2+R^2(t)\left(\frac{dr^2}{1-b(r)/r}+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right). \tag{42}$$

Equation (29) remains unaltered whereas (28) becomes

$$\widetilde{R}=\pm CR\sqrt{1-b(r)/r}. \tag{43}$$

Thus, for a given $b(r)$, we can solve (40) to find $R(t)$ and thereby determine $t(\tau)$ and $r(\tau)$. We now illustrate this with two representative examples.

(i) First, let us choose $b(r)=b_0^2/r$. This represents an evolving version of Ellis geometry [17]. The scale factor equation has a solution given by

$$R(t)=\frac{4e^{Ct+C_1}}{1+4\alpha^2 e^{2Ct+2C_1}}. \tag{44}$$

The string solution turns out to be

$$t=\frac{1}{C}\left\{\ln\frac{1}{2\alpha}\tan\frac{\alpha C\tau}{2}-C_1\right\}, \tag{45}$$

$$r(\tau)=\frac{\alpha}{C\,\sin\alpha C\tau}. \tag{46}$$

In the above, $\alpha^2=b_0^2C^2$.

This represents a string which is significantly large at small τ $(\tau\to 0)$ but becomes smaller and smaller as one approaches $\tau=\pi/2\alpha C$ where $r=b_0$ and $t=(1/C)(\ln(1/2\alpha)-C_1)$, $R(t)=1/\alpha$. Since $r(\tau)$ and $t(\tau)$ are both periodic in τ, the solution is valid only in the domain $n\pi/\alpha C<\tau<(n+1)\pi/\alpha C$.

(ii) This case involves the evolving version of the horizonfree Schwarzschild wormhole [18] for which we have $b(r)=b_0$.

The scale factor equation has a solution given by

$$R(t)=\frac{4e^{Ct+C_1}}{(1+\beta e^{Ct+C_1})^2}. \tag{47}$$

The string configuration is

$$t=\frac{1}{C}\left[\ln\left(\frac{1}{\beta}\left\{\frac{4}{\beta C(C_2-\tau)}-1\right\}\right)-C_1\right], \tag{48}$$

$$r(\tau)=\frac{\beta}{4}\frac{[4/\beta C(C_2-\tau)]^2}{[4/\beta C(C_2-\tau)]-1}, \tag{49}$$

where $\beta=b_0C$.

This configuration is also defined only for $\tau<C_2-4/\beta C$. One can try out various choices for $b(r)$ and easily obtain string configurations and their corresponding scale factors by utilizing the relations described in the previous paragraphs [Eqs. (29) and (40)].

Alternatively, in Eqs. (29) and (40), one can use $R(t)$ as the input and derive the resulting functional form of $b(r)$ (i.e., the features of the spacelike hypersurface). For example, assuming $R(t)\sim t^\nu$ [which is motivated by the standard matter ($\nu=\frac{2}{3}$) and radiation-dominated ($\nu=\frac{1}{2}$) scale factors], we obtain expressions for $t(\tau)$, $r(\tau)$, and $b(r)$:

$$t(\tau)=\left(\frac{\nu+1}{C_0}\right)^{1/(\nu+1)}(\tau-C_1)^{1/(\nu+1)},$$

$$r(\tau)=\frac{1}{C}\left(\frac{\nu+1}{C_0}\right)^{-\nu/(\nu+1)}(\tau-C_1)^{-\nu/(\nu+1)}, \tag{50}$$

$$b(r)=r(1-\nu^2C^{(2-2\nu)/\nu}r^{2/\nu}). \tag{51}$$

For $\nu=\frac{1}{2}$ (radiation-dominated FRW), $\nu=\frac{2}{3}$ (matter-dominated FRW), and $\nu=1$ (Milne), the $b(r)$ turns out to be

$$b(r)=r(1-\tfrac{1}{4}C^2r^4)\quad(\nu=\tfrac{1}{2}), \tag{52}$$

$$b(r)=r(1-\tfrac{4}{9}C^2r^3)\quad(\nu=\tfrac{2}{3}), \tag{53}$$

$$b(r)=r(1-r^2)\quad(\nu=1). \tag{54}$$

Unfortunately, none of these corresponds to the standard or well-known spacelike hypersurfaces we encounter in the context of cosmology.

To conclude, let us now summarize the results obtained.

(1) We have written down the null string equations of motion and constraints in general curved spacetimes. Specializing to specific backgrounds, we have also obtained explicit null string configurations.

(2) In spacetimes with event horizons, there is always a closed, stationary string on the horizon. In Rindler spacetime, we have been able to construct an explicit example of a dynamic string configuration. It is possible that such configurations also exist in general black hole spacetimes but we have not succeeded in finding one.

(3) In cosmological backgrounds, a general prescription has been outlined following which one can construct a dynamic string configuration in a fairly large class of such spacetimes. We have found string configurations in the inflationary or de Sitter universes and some other cases involving some interesting but not so popular scale factors. Examples of string configurations in evolving wormhole spacetimes have also been discussed towards the end.

It remains to be seen whether one can check the perturbative stability of these string configurations. However, to do such an analysis, we have to set up a formalism for null strings along the lines of the one for the timelike ones (see Guven [19], Larsen and Frolov [20], Carter [21], and Capovilla and Guven [22]).

The existence of a closed, stationary null string on the black hole horizon may also have some nontrivial implications on the interpretation of black-hole entropy from null string theory. Finally, it would be worthwhile to work out in detail the conditions under which a general curved background is allowed in null string theory. This is analogous to the β-function equations for the timelike string. However, we have to be careful in the case of the null string because we *cannot* use the inverse of the string tension as a perturbation parameter here. On the contrary, since null strings are tensionless and therefore correspond to the extreme high energy limit of the tensionful theory, one might be tempted to ask—does null string theory have a low-energy limit at all? These and other issues are currently under investigation and will be communicated in future.

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