Averaging of a locally inhomogeneous realistic universe

Toshifumi Futamase
Astronomical Institute, Tohoku University, Sendai, 980-77, Japan
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We present an averaging scheme in general relativity which allows us to study the effect of local inhomogeneity on the global behavior of the universe. The scheme uses 3+1 splitting of spacetime and introduces Isaacson averaging on the spatial hypersurface to get the averaged geometry. As a result of the averaging, the Friedmann-Robertson-Walker (FRW) geometry is derived in the first-order approximation for a wide class of inhomogeneous nonlinear matter distribution. The deviation from the FRW expansion is derived to the next order in terms of the anisotropic distribution of an effective stress-energy tensor. Using a simple model of inhomogeneity we show that the average effect of the inhomogeneity behaves like a negative spatial curvature term and thus has a tendency to extend the age of the universe.

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I. INTRODUCTION

The recent observation of the isotropy of the cosmic microwave background radiation [1] indicates that the universe is remarkably isotropic over the horizon scale if one interprets the observed dipole anisotropy as the result of the peculiar motion in the Earth. Thus it is natural to describe the large scale spatial geometry of the universe by a homogeneous and isotropic metric, namely, the Friedmann-Robertson-Walker (FRW) model. Homogeneity and isotropy are two principles on which the standard big bang model is based. However, the universe is neither isotropic nor homogeneous on local scales and the local metric may not be approximated by a FRW metric. It has been naively regarded that the FRW model is a large scale average of a locally inhomogeneous real universe. There have been some studies which make some sense of such a naive expectation, where a simple spatial averaging is introduced to determine the averaged expansion law of the universe [2–4]. Although these studies have shown some interesting results, such as the back reaction of the local inhomogeneity on the global expansion [5], the averaging is not treated mathematically rigorously. An attempt to make a rigorous statement for averaging out the Einstein equation has been proposed by Zalaletdinov [6,7], but the scheme has not yet proved useful for a realistic situation such as our universe.

The averaging problem in general relativity in cosmological circumstances has not only theoretical but also practical importance. It has been sometimes questioned whether the FRW model is appropriate for the study of the propagation of light rays in the real universe [8]. The smoothed out FRW metric coincides nowhere with the real metric on which light rays propagate. This has fundamental importance in observational cosmology. In fact there have been many studies about the effects of the local inhomogeneity on the distance-redshift relation [9–11].

The aim of this present paper is to present a reasonably rigorous and yet practical scheme for the averaging problem in general relativity in view of the application for cosmology. For this purpose we employ the 3+1 formalism of general relativity [12]. It allows us to project four-dimensional tensorial quantities onto the tensorial quantities defined on the spacelike hypersurface. To make the scheme practical we further simplify the basic equations by applying an approximation based on two small parameters. In the course of the approximation, the background geometry is introduced. Then the Isaacson averaging [13] is performed on the background spatial hypersurface.

The organization of the paper is as follows. In Sec. II, we shall introduce the scale factor and present the basic equations in 3+1 formalism. In Sec. III, we shall introduce two small parameters to characterize the inhomogeneities and present the basic equations neglecting higher order terms. In Sec. IV, we introduce the spatial averaging according to Isaacson. The averaged equations and the lowest order calculation for the perturbed quantities will be given there. A new condition for the applicability of the present approximation is derived. It improves the condition derived in the previous study [6]. It also will be shown that the averaged equations are invariant under gauge transformations which leave the structure of the background. In Sec. V, we take a simple model of inhomogeneity to calculate explicitly its effect on the global expansion law and point out that its effect behaves as a negative curvature term. Finally we shall give some discussions.

II. THE BASIC EQUATIONS IN 3+1 FORMALISM

Here we shall present the basic equations in our scheme using the 3+1 formalism. Let us first assume that there exists a congruence of timelike geodesics from which the spacetime looks isotropic. We shall call these geodesics the basic observers. These may be defined as the world lines of the observer who sees the isotropy of the cosmic microwave background. We shall take any one of these observers and assume that the geodesic is parametrized by proper time t without loss of generality, so that the tangent vector n^μ is normalized, $n^\mu n_\mu = -1$. This observer's proper time will be the cosmic time and it allows use to foliate the spacetime by his simultaneous surfaces: $t =$ const. We shall assume that the three surfaces form hypersurfaces and define the vector field n^μ as the unit normal to the hypersurfaces.

Then we define the spatial metric $h_{\mu\nu}$ on the surface by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu . \tag{1}$$

This is the projection operator onto the spatial surface perpendicular to n^μ. Then by introducing the lapse function $\alpha = 1\sqrt{-g^{00}}$ and the shift vector $\beta_i = g_{0i}$, the unit normal vector is written as $n_\mu = (-\alpha, \vec{0})$ and the spacetime metric may be written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) , \tag{2}$$

where $\gamma_{ij} = g_{ij}$ is the spatial metric of the hypersurface.

We may define the extrinsic curvature of the hypersurface by

$$K_{ij} = -n_{i|j} \tag{3}$$

where $|$ means the covariant derivative with respect to γ_{ij}. The basic variables in the 3+1 formalism are the spatial metric γ_{ij} and the extrinsic curvature K_{ij}. The lapse function and the shift vectors will be freely specified as the coordinate conditions. The basic equations are then obtained by projecting the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4}$$

onto the normal and tangential direction to the hypersurface:

$$R^{(3)} + K^2 - K^{ij}K_{ij} = 16\pi\rho_H + 2\Lambda , \tag{5}$$

$$K^j{}_{i|j} - K_{|i} = 8\pi J_i , \tag{6}$$

$$\frac{\partial}{\partial t}\gamma_{ij} = -2\alpha K_{ij} + L_\beta \gamma_{ij} , \tag{7}$$

$$\begin{aligned}\frac{\partial}{\partial t}K_{ij} = {} & -\alpha_{|ij} + \alpha(R^{(3)} + KK_{ij} - 2K_{ik}K^k_j) \\ & -8\pi\alpha[S_{ij} + \tfrac{1}{2}\gamma_{ij}(\rho_H - S)] - \alpha\Lambda_{ij} + L_\beta K_{ij} ,\end{aligned} \tag{8}$$

where $|$ means the covariant derivative with respect to γ_{ij}, L_β is the Lie derivative along β, $R^{(3)}_{ij}$ is the Ricci curvature of the hypersurface, and

$$\rho_H = T^{\mu\nu}n_\mu n_\nu , \quad J_i = -T^{\mu\nu}h_{i\mu}n_\nu , \quad S_{ij} = T^{\mu\nu}h_{i\mu}h_{j\nu} . \tag{9}$$

Now we shall introduce the scale factor. Let us consider a one-parameter family of $\gamma_s(\tau)$ of timelike geodesics (world lines of galaxies), and let η^μ be the orthogonal deviation vector from γ_0 which is defined to be our basic observer. Thus η^μ represents a spatial displacement from the basic observer to another galaxy. To see how the distance between galaxies changes, we define the spatial distance by

$$\delta l = (h_{\mu\nu}\eta^\mu\eta^\nu)^{1/2} . \tag{10}$$

Then the change of the distance along γ_0 is calculated to be [11]

$$(\dot{\delta l}) = u^\mu(\delta l)_{;\mu} = -[\tfrac{1}{3}K + \sigma_{ij}e^i e^j]\delta l , \tag{11}$$

where $e^i = \eta^i/\delta l$, with $\eta^i = h^i_\mu \eta^\mu$, is the unit spatial vector and σ_{ij} is the trace-free part of the extrinsic curvature:

$$K_{ij} = \sigma_{ij} + \tfrac{1}{3}K\gamma_{ij} . \tag{12}$$

Thus it is natural to interpret the above Eq. (11) as expressing the cosmic expansion, and the first and second terms on the right hand side represent the isotropic and anisotropic expansion, respectively. Motivated by this interpretation we shall introduce the scale factor as

$$K = -3\frac{\dot{a}}{a} , \tag{13}$$

where $a = a(t)$ is the scale factor of the function of the cosmic time t. This choice of the trace of the extrinsic curvature corresponds to the maximal slicing condition in the asymptotically flat case, namely, with $a = 1$. Thus, instead of using the extrinsic curvature itself, we shall use the scale factor and the trace-free part of the extrinsic curvature as our basic variables.

It turns out that it is convenient to use the following variables instead of the original variables:

$$\tilde{\gamma}_{ij} = a^{-2}\gamma_{ij} , \quad \tilde{\beta}^i = a^{-2}\beta^i , \quad \tilde{R}^{(3)} = a^2 R^{(3)} . \tag{14}$$

Using the above variables the basic equations may be written as

$$\frac{1}{6a^2}\tilde{R}^{(3)} + \frac{\dot{a}^2}{a^2} - \frac{1}{6a^4}\tilde{\gamma}^{ki}\tilde{\gamma}^{lj}\sigma_{ij}\sigma_{kl} = \frac{8\pi}{3}\rho_H + \tfrac{1}{3}\Lambda , \tag{15}$$

$$\tilde{\gamma}^{jk}\sigma_{ki|j} = 8\pi a^2 J_i , \tag{16}$$

$$\frac{\partial}{\partial t}\tilde{\gamma}_{ij} = 2(\alpha - 1)\frac{\dot{a}}{a}\tilde{\gamma}_{ij} - 2\alpha a^{-2}\sigma_{ij} + 2\tilde{\beta}_{(i|j)} , \tag{17}$$

$$\begin{aligned}\frac{\ddot{a}}{a} - (1 - \alpha)\frac{\dot{a}^2}{a^2} = {} & -\frac{4\pi}{3}\alpha(\rho_H + a^{-2}\tilde{\gamma}^{kl}S_{kl}) + \tfrac{1}{3}\alpha\Lambda \\ & + \frac{1}{3a^2}\tilde{\Delta}\alpha - \frac{1}{3a^4}\alpha\tilde{\gamma}^{kl}\tilde{\gamma}^{lj}\sigma_{ij}\sigma_{kl} ,\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{ij} + \alpha\frac{\dot{a}}{a}\sigma_{ij} = {} & \alpha\hat{R}^{(3)}_{ij} - 8\pi\alpha\hat{S}_{ij} - (\alpha_{|ij} - \tfrac{1}{3}\tilde{\gamma}_{ij}\tilde{\Delta}\alpha) \\ & -2\alpha a^{-2}\tilde{\gamma}^{kl}\sigma_{ki}\sigma_{lj} + 2\beta^k{}_{|(i}\sigma_{j)k} \\ & +\beta^k\sigma_{ij|k} ,\end{aligned} \tag{19}$$

where we have defined $\tilde{\Delta}\alpha = \tilde{\gamma}^{ij}\alpha_{|ij}$, and a quantity with a caret means the trace-free part of it, for example, $\hat{S}_{ij} = S_{ij} - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{kl}S_{kl}$. It is straightforward to write down the equations for the conservation of the stress-energy tensor in terms of the above variables. We shall write them down after we have introduced the explicit form of the stress-energy tensor.

If we assume a homogeneous and isotropic distribution for the matter and gravitational fields, then the above equations may easily be solved to obtain the usual FRW model. We may proceed to solve the above set of equations perturbatively taking such a solution as the background solution [14]. This is certainly a possibility. However, such an approach may not be able to treat the situation we are interested in here, namely, the situation with nonlinear matter distribution. Thus we shall take another approach in this paper; namely, we regard the expansion of the universe as generated by the coherent effect of the material distribution and thus the equation which governs the time dependence of the scale factor may be obtained from the above equations by a suitable averaging procedure. In fact, recent numerical results have shown that inhomogeneity in the gravitational field increases the rate of cosmic expansion [5]. We shall express this fact by introducing a spatial averaging of the Einstein equation. Before discussing the averaging, however, we need to introduce an approximation to simplify the above set of basic equations.

III. APPROXIMATION METHOD

We have obtained our basic equations in the previous section. These are regarded as scalar, vector, and tensor equations on the spatial hypersurface. We could define the spatial averaging of these equations at this stage, but the actual calculation may not be straightforward because of the lack of the background geometry. Instead we shall here introduce the background geometry by introducing an approximation scheme appropriate for the present situation, namely, a locally inhomogeneous universe. As explained below, the introduction of the background geometry does not imply a small density contrast.

The approximation we are going to employ is motivated by the following consideration. It seems that the metric perturbation in the present universe remains small almost everywhere even when the density contrast is much larger than unity. Thus it is natural to assume that the metric structure of the universe may be described by a small perturbation from the FRW universe in an appropriate coordinate system.

We shall therefore introduce a small parameter ϵ to characterize the order of the gravitational potential of the material clumps. It will be convenient to introduce another small parameter κ to characterize the ratio between the horizon scale and the scale of the density fluctuation.

The relative size of ϵ and κ depends on the system we have in mind. In this paper we are mainly interested in the nonlinear stage where the typical scale of the density fluctuation is smaller than the horizon scale. Then the metric fluctuation is generated by the density contrast $\delta\rho$ via the Poisson equation and the density contrast $\delta\rho/\rho_b$ may be evaluated from $\Delta\phi/(G\rho_b) \sim \epsilon^2/\kappa^2$, where ρ_b is the averaged density and ϕ is the Newtonian potential generated by the density contrast. Thus the linear and nonlinear stages may be characterized by the conditions $\epsilon \ll \kappa$ and $\epsilon \gg \kappa$, respectively. We wish to describe a spacetime where there are fully nonlinear density fluctuations, but the metric perturbation remains small. Thus the condition $\kappa \ll \epsilon \ll 1$ will be assumed in the following analysis.

For convenience we give some examples of the typical values of ϵ and κ. If we take a supercluster whose size is about $30h^{-1}$ Mpc, then κ will be $\sim 10^{-2}$. The density contrast between the supercluster and the averaged density seems to be of the order of unity. Thus the gravitational potential would be $\epsilon^2 \sim \kappa^2 \sim 10^{-4}$ for such a system. The size of the supercluster seems to be the boundary between the linear and nonlinear regions. Typical values for galaxies are $\epsilon \sim 10^{-3}$ and $\kappa \sim 10^{-4.5}$.

According to the above discussion we shall decompose the lapse function and the spatial metric as

$$\alpha = 1 + \phi \ , \qquad (20)$$

$$\tilde{\gamma}_{ij} = \tilde{\gamma}^{(B)}_{ij} + \tilde{h}_{ij} \ , \qquad (21)$$

where as stated above the quantities ϕ and h_{ij} are both of the order of ϵ^2, and $\tilde{\gamma}^{(B)}$ is the spatial metric of the standard FRW geometry:

$$dl^2 = \gamma^{(B)}_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d^2\Omega \ , \qquad (22)$$

where $k = \pm 1, 0$ is the normalized spatial curvature corresponding to closed, open, and flat spatial sections, and $d^2\Omega$ is the standard metric on the unit two-sphere.

Now we are able to regard tensorial quantities in our equations as tensors with respect to the background spatial metric $\tilde{\gamma}^{(B)}_{ij}$. For example, the spatial Ricci tensor $R^{(3)}$ may be written as

$$R^{(3)}_{ij} = 2k\tilde{\gamma}^{(B)}_{ij} + \tilde{R}^{(L)}_{ij} + \tilde{R}^{(\mathrm{NL})}_{ij} \ , \qquad (23)$$

where

$$\begin{aligned}
\tilde{R}^{(L)} &= \tfrac{1}{2}(\tilde{h}^k{}_{i\|jk} + \tilde{h}^k{}_{j\|ik} - \tilde{h}_{\|ij} - \tilde{\Delta}^{(B)}\tilde{h}_{ij}) \ , \\
\tilde{R}^{(\mathrm{NL})}_{ij} &= \tfrac{1}{4}\tilde{h}^{kl}{}_{\|i}\tilde{h}_{kl\|j} - \tfrac{1}{2}(\tilde{h}^{kl}{}_{\|k} - \tfrac{1}{2}\tilde{h}^{\|l}) \\
&\quad \times(\tilde{h}_{li\|j} + \tilde{h}_{lj\|i} - \tilde{h}_{ij\|l}) \\
&\quad -\tfrac{1}{2}\tilde{h}^k{}_{i\|l}(\tilde{h}^l{}_{j\|k} - \tilde{h}^{\|l}{}_{kj}) \\
&\quad -\tfrac{1}{2}\tilde{h}^{kl}(\tilde{h}_{li\|jk} + \tilde{h}_{lj\|ik} - \tilde{h}_{ij\|kl} - \tilde{h}_{kl\|ij}) \ ,
\end{aligned}$$

where $\tilde{h}^k_l = \gamma^{(B)km}\tilde{h}_{ml}$ and $\tilde{\Delta}^{(B)}\tilde{h} = \tilde{\gamma}^{(B)ij}\tilde{h}_{\|ij}$. The subscript $\|$ means the covariant derivative with respect to the background metric $\gamma^{(B)}_{ij}$. Thus the above quantities are second-rank tensors with respect to the background metric. It should be remarked that the relative order of magnitude of the right hand side of Eq. (23) is $1, \epsilon^2\kappa^{-2}, \epsilon^4\kappa^{-2}$, respectively. We have ignored the third-order terms in $\tilde{h}$.

Our approximation consists of neglecting higher order terms in ϵ and κ under the assumption

$$\kappa \ll \epsilon \ll 1 \ . \tag{24}$$

In doing so, we have to estimate the order of magnitude for various quantities. Since we know that deviation from the background metric is of the order of ϵ^2, and the time derivative gives another factor of ϵ/l in the slow-motion situation, the evolution equation of the spatial metric (17) tells us the following ordering for σ_{ij} and β^i:

$$\sigma_{ij} = O\left(\frac{\epsilon^3}{l}\right) \ , \quad \beta^i = O(\epsilon^3) \ . \tag{25}$$

Thus the quadratic term in σ_{ij} is of the order $O(\epsilon^6/l^2)$ which may be safely ignored in our approximation. Likewise terms like $\tilde{h}\sigma_{ij}$, $\tilde{h}\beta^i$, $\tilde{h}S_{ij}$, and terms of more than third order in the small quantities may be neglected.

Neglecting higher order terms we arrive at the equations

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2}\left[k - \frac{k}{3}\tilde{h} + \tfrac{1}{6}(\tilde{R}^{(L)} + \tilde{R}^{(NL)} - \tilde{h}^{ij}\tilde{R}^{(L)}_{ij})\right] - \frac{\Lambda}{3} + O\left(\frac{\epsilon^4}{L^2}, \frac{\epsilon^6}{l^2}\right) = \frac{8\pi}{3}\rho_H \ , \tag{26}$$

$$\sigma^j{}_{i\|j} + O\left(\frac{\epsilon^5}{l^2}\right) = 8\pi J_i \ , \tag{27}$$

$$\frac{\partial}{\partial t}\tilde{h}_{ij} = 2\phi\frac{\dot{a}}{a}\tilde{\gamma}^{(B)}_{ij} - 2a^{-2}\sigma_{ij} + 2\tilde{\beta}_{(i\|j)} + O\left(\frac{\epsilon^5}{l}, \frac{\epsilon^4}{L}\right) \ , \tag{28}$$

$$\begin{aligned}\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\phi = &-\frac{4\pi}{3}(1+\phi)(\rho_H + a^{-2}\tilde{\gamma}^{(B)kl}S_{kl}) + \tfrac{1}{3}(1+\phi)\Lambda \\ &+\frac{1}{3a^2}[\tilde{\Delta}^{(B)}\phi - (\tilde{h}^{kl}{}_{\|l} - \tfrac{1}{2}\tilde{h}^{\|k})\phi_{\|k} - \tilde{h}^{kl}\phi_{\|kl}] + O\left(\frac{\epsilon^6}{l^2}, \frac{\epsilon^4}{L^2}\right) \ ,\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial}{\partial t}\sigma_{ij} = &-2k\hat{\tilde{h}}_{ij} + (1+\phi)\hat{\tilde{R}}^{(L)}_{ij} + \hat{\tilde{R}}^{(\mathrm{NL})}_{ij} - \tfrac{1}{3}(\tilde{h}_{ij}\tilde{R}^{(L)}_{ij} - \gamma^{(B)}_{ij}\tilde{h}^{kl}\tilde{R}^{(L)}_{ij}) \\ &-\frac{\dot{a}}{a}\sigma_{ij} - 8\pi\hat{S}_{ij} - (\phi_{\|ij} - \tfrac{1}{3}\gamma_{ij}\tilde{\Delta}^{(B)}\phi) - \tfrac{1}{2}\hat{\tilde{h}}_{ij\|k}\phi^{\|k} \\ &+(\tilde{h}^k{}_{(i\|j)} - \tfrac{1}{3}\gamma^{(B)}_{ij}\tilde{h}^{kl}{}_{\|l})\phi_{\|k} + \tfrac{1}{2}(\tilde{h}_{ij}\tilde{\Delta}^{(B)}\phi - \gamma^{(B)}_{ij}\tilde{h}^{kl}\phi_{\|kl}) + O\left(\frac{\epsilon^5}{l^2}, \frac{\epsilon^4}{L^2}\right) \ ,\end{aligned} \tag{30}$$

where we have defined $v^2 = \tilde{\gamma}_{ij}v^iv^j$, and in the above equations the trace and trace-free parts are defined in terms of the background metric: for example,

$$\hat{\tilde{R}}^{(L)}_{ij} = \tilde{R}^{(L)}_{ij} - \tfrac{1}{3}\gamma^{(B)}_{ij}\tilde{R}^{(L)} \ , \tag{31}$$

$$\hat{S}_{ij} = S_{ij} = \tfrac{1}{3}\tilde{\gamma}^{(B)}_{ij}\tilde{S} \ , \tag{32}$$

with $\tilde{R}^{(L)} = \tilde{\gamma}^{(B)ij}\tilde{R}^{(L)}_{ij}$ and $\tilde{S} = \tilde{\gamma}^{(B)ij}S_{ij}$.

In the above equations we have not yet introduced an explicit form for the material stress-energy tensor $T^{\mu\nu}$. For consistency with the approximation introduced above, one should understand that the higher order terms in ρ_H and S_{ij} are ignored in the above equations.

In the following argument it is convenient to have equations for the trace-free part and trace part of the perturbed metric and the second fundamental form σ_{ij} with respect to the background metric rather than the full metric:

$$\sigma_{ij} = \hat{\sigma}_{ij} + \tfrac{1}{3}\tilde{\gamma}^{(B)}_{ij}\tilde{\sigma} \ , \tag{33}$$

where $\tilde{h} = \tilde{\gamma}^{(B)}_{ij}\tilde{h}_{ij}$ and $\tilde{\sigma} = \tilde{\gamma}^{(B)}_{ij}\sigma_{ij}$. However, since we imposed the maximal slicing condition, i.e.,

$$\tilde{\gamma}^{ij}\sigma_{ij} = 0 \ ,$$

it can be shown that the trace part of σ_{ij} is of the order $O(\tilde{h}\sigma)$ which may be neglected in our approximation and we may simply regard σ_{ij} as the trace-free part $\hat{\sigma}_{ij}$ with respect to the background metric. Equation (29) takes the form

$$\frac{\partial}{\partial t}\hat{\tilde{h}}_{ij} = -2a^{-2}\sigma_{ij} + 2(\tilde{\beta}_{(i\|j)} - \tfrac{1}{3}\tilde{\gamma}^{(B)}_{ij}\beta^k{}_{\|k}) \ , \tag{34}$$

$$\frac{\partial}{\partial t}\tilde{h} = 6\phi\frac{\dot{a}}{a} + 2\beta^k{}_{\|k} \ . \tag{35}$$

IV. DERIVATION OF FRW GEOMETRY BY AVERAGING

In the previous section we obtained our basic equations as perturbed equations around the background FRW model. The perturbed quantities are classified as scalar, vector, and tensor with respect to the background spatial geometry. Thus is would be natural to introduce the following averaging over the background spatial hypersurface according to Isaacson [10]:

$$\langle Q_{ij}(x)\rangle = \int g_i^{k'}(x,x')g_k^{l'}(x,x')Q_{k'l'}(x')f(x,x')d^3x' \ , \tag{36}$$

where $g_i^{k'}(x,x')$ is the bivector of geodesic parallel displacement and $f(x,x')$ is a weighting function which falls smoothly to zero when x and x' differ more than the averaging region, and with

$$\int f(x,x')d^3x' = 1 \ . \tag{37}$$

Since our background spatial geometry is that of the FRW metric, there will be no problem about the global existence of a unique geodesic from x needed in the construction of $g_i^{k'}$.

Then the following rules derived by Isaacson apply also here.

(1) Under integrals, the divergence becomes reduced by a factor of κ. Thus we may drop the divergence and similar terms.

(2) Under integrals we may "integrate by parts" if we ignore terms reduced by a factor of κ.

(3) Covariant derivatives commute on the perturbed quantities if we ignore terms reduced by a factor of κ^2.

Before taking the average, we have to specify the averaged properties of our basic variables. We shall assume

$$\langle\phi\rangle = 0 \ . \tag{38}$$

Under this assumption, we obtain the averaged equations

$$\frac{\dot a^2}{a^2} + k^2 = \frac{8\pi}{3}\langle\rho_H\rangle + \frac{\Lambda}{3} - \frac{1}{a^2}\langle\tilde R^{(\mathrm{NL})} - \tilde h^{ij}\tilde R_{ij}^{(L)}\rangle + O\left(\frac{\epsilon^4}{L^2},\frac{\epsilon^6}{l^2}\right) \ , \tag{39}$$

$$\langle J_i\rangle = O\left(\frac{\epsilon^5}{l^2}\right) \ , \tag{40}$$

$$\frac{\partial}{\partial t}\langle\hat{\tilde h}_{ij}\rangle = -2a^{-2}\langle\sigma_{ij}\rangle + O\left(\frac{\epsilon^5}{l},\frac{\epsilon^4}{L}\right) \ , \tag{41}$$

$$\frac{\partial}{\partial t}\langle\tilde h\rangle = O\left(\frac{\epsilon^5}{l},\frac{\epsilon^4}{L}\right) \ , \tag{42}$$

$$\frac{\ddot a}{a} = -\frac{4\pi}{3}\langle(1+\phi)(\rho_H + a^{-2}\tilde\gamma^{(B)kl}S_{kl})\rangle + \tfrac{1}{3}\Lambda - \frac{1}{3a^2}\langle(\tilde h^{kl}{}_{\|l} - \tfrac{1}{2}\tilde h^{\|k})\phi_{\|k} + \tilde h^{kl}\phi_{\|kl}\rangle + O\left(\frac{\epsilon^6}{l^2},\frac{\epsilon^4}{L^2}\right) \ , \tag{43}$$

$$\begin{aligned}\frac{\partial}{\partial t}\langle\sigma_{ij}\rangle = {}& -\frac{\dot a}{a}\langle\sigma_{ij}\rangle - 2k\langle\hat{\tilde h}_{ij}\rangle - 8\pi\langle\hat S_{ij}\rangle + \langle\phi\hat{\tilde R}_{ij}^{(L)} + \hat{\tilde R}_{ij}^{(\mathrm{NL})} - \tfrac{1}{3}(\tilde h_{ij}\tilde R_{ij}^{(L)} - \gamma_{ij}^{(B)}\tilde h^{kl}\tilde R_{ij}^{(L)})\rangle \\ & + \langle -\tfrac{1}{2}\hat{\tilde h}_{ij\|k}\phi^{\|k} + (\tilde h^k{}_{(i\|j)} - \tfrac{1}{3}\gamma_{ij}^{(B)}\tilde h^{kl}{}_{\|l})\phi_{\|k} + \tfrac{1}{2}(\tilde h_{ij}\tilde\Delta^{(B)}\phi - \gamma_{ij}^{(B)}\tilde h^{kl}\phi_{\|kl})\rangle + O\left(\frac{\epsilon^5}{l^2},\frac{\epsilon^4}{L^2}\right) \ .\end{aligned} \tag{44}$$

The equations for the local perturbations are obtained from the original equations (26)–(30) by subtracting the above averaged equations. The above averaged equations take simpler forms on inspecting the local equations.

At this stage we need an explicit form of the stress-energy tensor. For simplicity we shall take the dust model which will be a good approximation of the present universe. The generalization for other cases should be straightforward:

$$T^{\mu\nu} = \rho u^\mu u^\nu \ , \tag{45}$$

where u^μ may be calculated to be

$$u^\mu = \frac{dx^\mu}{\sqrt{-ds^2}} = \frac{1}{\alpha}\gamma\frac{dx^\mu}{dt} \tag{46}$$

with $\gamma = [1 - \alpha^{-2}\gamma_{ij}(v^i+\beta^i)(v^j+\beta^j)]^{-1/2}$, and $v^i = dx^i/dt$ is the coordinate velocity which is supposed to be of the order of ϵ in our approximation. The stress-energy tensor may also be decomposed as a scalar, vector, and tensor with respect to the background metric.

Then the conservation laws for the stress-energy tensor up to the relevant order are

$$\frac{\partial}{\partial t}(\rho A) = -3\frac{\dot a}{a}\rho A + \tilde\nabla_i^{(B)}(\rho v^i A) - \left(\dot\phi + \frac{\partial\phi}{\partial x^i}\right)\rho v^i + O\left(\frac{\epsilon^7}{l}, \frac{\epsilon^4}{L}\right) , \tag{47}$$

$$\frac{\partial}{\partial t}(\rho v_i A) = -5\frac{\dot a}{a}\rho v_i A + \tilde\nabla_j^{(B)}(\rho v^j v_i A) - a^{-2}\rho\frac{\partial\phi}{\partial x^i} + O\left(\frac{\epsilon^6}{l}, \frac{\epsilon^4}{L}\right) , \tag{48}$$

where $v_i = \tilde\gamma_{ij}^{(B)}v^j$, $v^2 = \tilde\gamma_{ij}^{(B)}v^iv^j$, $A = 1a^2v^2 + \frac{1}{2}\tilde h - \phi$, and

$$\tilde\nabla_i^{(B)} = \frac{1}{\sqrt{\tilde\gamma^{(B)}}}\frac{\partial}{\partial x^i}\sqrt{\tilde\gamma^{(B)}} .$$

Neglecting higher order terms, the local equations obtained from (27) and (30) takes the simple form

$$-\frac{k}{3}\tilde h + \frac{1}{6}\tilde R^{(L)} + O\left(\frac{\epsilon^4}{L^2}, \frac{\epsilon^6}{l^2}\right) = \frac{8\pi}{3}a^2[\delta\rho + a^2\delta(\rho v^2)] , \tag{49}$$

$$\tilde\nabla^{(B)}\phi - \frac{\dot a^2}{a^2}\phi = 4\pi a^2[\delta\rho + \delta(\rho\phi + 2a^2\rho v^2)] + \tfrac{1}{3}\phi\Lambda + O\left(\frac{\epsilon^6}{l^2}, \frac{\epsilon^4}{L^2}\right) , \tag{50}$$

where we have defined the density fluctuation as

$$\delta\rho - \rho - \rho_b \tag{51}$$

with $\langle\rho\rangle \equiv \rho_b$ the averaged density.

Remembering the explicit expression for $\tilde R^{(L)}$, namely,

$$\tilde R^{(L)} = \tilde h^{kl}{}_{\|kl} - \tilde\Delta^{(B)}\tilde h , \tag{52}$$

the following relation is obtained for the consistency between the above two equations at the lowest order:

$$\tilde h_{ij} = -2\phi\tilde\gamma_{ij}^{(B)} + O(\epsilon^4, \kappa\epsilon^2) , \tag{53}$$

$$\phi = \phi_N + O(\epsilon^4, \kappa\epsilon^2) , \tag{54}$$

where the higher order terms are the usual post-Newtonian terms of order ϵ^4 and the terms of order $\kappa\epsilon^2$ arising from the effect of the background curvature. We shall not need the explicit expression for the first post-Newtonian term, namely, $O(\epsilon^4)$, below because we are only interested in the lowest order nontrivial correction to the usual expansion law. Note that the covariant derivative with respect to the background metric contains Christoffel symbols of order $1/L$. The lowest order term in ϕ is of order ϵ^2 and is satisfied by the Poisson equation

$$\Delta\phi = 4\pi a^2\delta\rho , \tag{55}$$

where Δ is the flat Laplacian. Thus it is the usual Newtonian potential. The effects of the spatial curvature appear in the next order $\kappa\epsilon^2$.

This relation is then used for the calculation of $\tilde R_{ij}^{(L)}$ and $\tilde R_{ij}^{(\mathrm{NL})}$ and we obtain the averaged equations

$$\frac{\dot a^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3}(\rho_b + \langle\rho\phi_N + 2a^2\rho v^2\rangle) + \frac{\Lambda}{3} + \frac{5}{3a^2}\langle\phi_N^{,k}\phi_{N,k}\rangle + O\left(\frac{\epsilon^4}{Ll}, \frac{\epsilon^6}{l^2}\right) , \tag{56}$$

$$\frac{\ddot a}{a} = -\frac{4\pi}{3}(\rho_b + \langle\rho\phi_N + 2a^2\rho v^2\rangle) + \frac{\Lambda}{3} - \frac{1}{a^2}\langle\phi_N^{,k}\phi_{N,k}\rangle + O\left(\frac{\epsilon^6}{l^2}, \frac{\epsilon^4}{Ll}\right) , \tag{57}$$

$$\frac{\partial}{\partial t}\langle\sigma_{ij}\rangle = -\frac{\dot a}{a}\langle\sigma_{ij}\rangle - 8\pi\langle\hat S_{ij}\rangle - 2\langle\phi_{N,i}\phi_{N,j} - \tfrac{1}{3}\gamma_{ij}^{(B)}\phi_N^{,k}\phi_{N,k}\rangle + O\left(\frac{\epsilon^5}{l^2}, \frac{\epsilon^4}{Ll}\right) . \tag{58}$$

One can show the consistency between Eqs. (56) and (57) by using the equations of motion (47) and (48).

Now we shall consider the conditions for the validity of the above approximation. In this respect it is important to notice that the averaged equations above contain only partial derivatives, not covariant derivatives, by making use of the rule (2) above, namely, "integration by parts,"

$$\langle\phi_N\tilde\Delta^{(B)}\phi_N\rangle = -\langle\phi_N^{,k}\phi_{N,k}\rangle + O\left(\kappa\frac{\epsilon^4}{l^4}\right) .$$

Thus the averaged quantities such as $\langle\phi_N^{,k}\phi_{N,k}\rangle$ in the source terms of Eqs. (56)–(58) are of the order $O(\epsilon^4/l^2)$. For the validity of the approximation, it is necessary that neglected terms of order ϵ^6/l^2 and ϵ^4/Ll should be smaller than the terms not neglected. If one compares terms like $\rho_b = G\rho_b \sim 1/L^2$ with the neglected terms, we have

$$\frac{1}{L^2} \gg \frac{\epsilon^6}{l^2}, \frac{\epsilon^4}{Ll} ;$$

namely, we have the following restriction for the validity of the above approximation method:

$$\epsilon^3 \ll \kappa . \tag{59}$$

This condition improves that derived in the earlier studies [2,3]. This was possible because of the careful definition of the spatial averaging.

Finally it should be noted that these expressions agree exactly with the previous results in which the harmonic gauge is employed. Here we have used the maximal slicing condition for the lapse function. We have not specified any particular gauge for the shift vector because their contributions may be negligibly small or vanish in the course of the averaging. In fact we could also choose the nonmaximal slicing condition and take the following form for the trace of the second fundamental form:

$$K = -3\left(\frac{\dot{a}}{a} + \sigma\right) . \tag{60}$$

Then Eq. (35) becomes

$$\frac{\partial}{\partial t}\phi = -\phi\frac{\dot{a}}{a} + \tfrac{1}{3}\beta^k_{\|k} + \alpha\sigma . \tag{61}$$

This shows that σ is of the order $O(\epsilon^3/l)$ as long as $\alpha = 1 + O(\epsilon)$. We can also assume that the average of σ vanishes:

$$\langle\sigma\rangle = 0 . \tag{62}$$

This is because the nonzero average of σ is absorbed into the newly defined scale factor. Then it is easily shown that the terms containing σ are of higher orders which we have ignored. The choice of α will also not modify the averaged equations since the terms containing α are averaged out or of sufficiently higher order. Thus the averaged equations up to the order we are interested in this paper are invariant against the following family of gauge transformations:

$$\alpha = 1 + O(\epsilon) , \quad \beta^i = O(\epsilon^3) . \tag{63}$$

These are the transformations which leave the structure of the background space.

V. AVERAGED EXPANSION OF A LOCALLY INHOMOGENEOUS UNIVERSE

In this section we shall take a simple model of inhomogeneity to calculate explicitly its effect on the global expansion law. For simplicity we shall consider an Einstein–de Sitter background ($k = 0, \Lambda = 0$). In this case our averaging reduces to a simple volume averaging. We shall here recover the velocity of light c and the gravitational constant G in the equation, and moreover, work with the physical frame; namely, the physical velocity is taken as $\vec{u} = (1/a)\vec{v}$. The following analysis is essentially the same as that of Bildhauer and Futamase [15]. Here we shall correct some mistakes in the earlier treatment and include it for completeness.

To the lowest order, the local equations for inhomogeneous distributed matter are

$$\frac{\partial\rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla\cdot(\rho\vec{u}) = 0 , \tag{64}$$

$$\frac{\partial\vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla)\vec{u} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla\phi_N , \tag{65}$$

$$\Delta\phi_N = 4\pi G a^2(\rho - \rho_b) . \tag{66}$$

The averaged metric takes the form

$$\langle ds^2\rangle = -c^2dt^2 + a^2(\delta_{ij} + \langle h_{ij}\rangle)dx^i dx^j , \tag{67}$$

where the scale factor and $\langle h_{ij}\rangle$ are determined as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left(\rho_b + \left\langle\frac{\rho\vec{u}^2}{c^2}\right\rangle + \frac{5}{8\pi G}\langle(\nabla\phi_N)^2\rangle\right) , \tag{68}$$

$$\langle h_{ij}\rangle = 16\pi G\int_{t_{in}}^{t} dt' a^{-3}\int_{t_{in}}^{t'} dt'' a^{-1}\langle\hat{\tau}_{ij}\rangle , \tag{69}$$

with

$$\langle\tau^{ij}\rangle = a^2\left(\left\langle a^2\rho\frac{u^i u^j}{c^2}\right\rangle + \frac{1}{8\pi G}\langle 2\phi^i_N\phi^j_N - \delta^{ij}(\nabla\phi_N)^2\rangle\right) , \tag{70}$$

$$\hat{\tau}^{ij} = \tau^{ij} - \tfrac{1}{3}\delta^{ij}\tau^k_k. \tag{71}$$

The model we chose is the Zeldovich approximate ansatz of pancake theory [16]. This is constructed by a transformation from Eulerian coordinates $\vec{q}$ to Lagrangian coordinates $\vec{X}$, depending on the initial conditions for the peculiar velocity $\vec{U}(\vec{X})$. Then our model for inhomogeneity to the lowest order may be written as

$$\vec{q} = \vec{F}(\vec{X},t) = \vec{X} + \tfrac{3}{2}t_{in}\vec{U}(\vec{X})g(t) , \tag{72}$$

$$\vec{u} = \tfrac{3}{2}at_{in}\vec{U}(\vec{X})\dot{g}(t) , \tag{73}$$

$$\vec{\nabla}\phi = \frac{6\pi G\rho_b a}{c^2}(1+g)t_{in}\vec{U}(\vec{X}) , \tag{74}$$

$$\rho(\vec{X},t) = \rho_b\frac{1+\delta_{in}(\vec{X})}{\det[F_{i,j}(\vec{X},t)]} , \tag{75}$$

where

$$\vec{U}(\vec{X}) = \vec{\nabla}s_{in}(\vec{X}) , \tag{76}$$

$$\delta_{in}(\vec{X}) = -\frac{c^2}{4\pi G\rho_{b_{in}}t_{in}}\Delta s_{in}(\vec{X}) , \tag{77}$$

$$g(t) = a(t) - 1 , \tag{78}$$

and

$$a(t) = \left(\frac{t}{t_{in}}\right)^{2/3} . \tag{79}$$

A potential for the initial velocity is given by $s_{in}(\vec{X})$ and the initial value for the density contrast is denoted by $\delta_{in}(\vec{X})$.

Then the Hubble equation (68) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\rho_b + \rho_c) \tag{80}$$

with

$$\rho_c = \frac{9a^2\rho_b t_{\rm in}^2 \dot{g}^2}{4c^2}\left\langle \frac{1+\delta_{\rm in}}{\det(F_{i,j})}|\vec{U}|^2\right\rangle_q + \frac{45\pi G a^2 \rho_{\rm in}^2 t_{\rm in}^2 (1+g)^2}{2c^4}\langle|\vec{U}|^2\rangle_q \,. \tag{81}$$

Here $\langle\cdots\rangle_q$ is the spatial average over q space. Using background relationships such as $t_{\rm in} = 2/[3H_0(1+z_{\rm in})^{3/2}]$ and $\rho_b = c^2/6\pi G t^2$, $\rho_{\rm inhom}$ may be written as

$$\rho_c = \frac{\rho_{b0}a_0}{c^2}\frac{a_0^2}{a^2}\left(\left\langle \frac{1+\delta_{\rm in}}{\det(F_{i,j})}|\vec{U}|^2\right\rangle_q + \frac{15}{4}\langle|\vec{U}|^2\rangle_q\right) = \rho_{c0}\frac{a_0^2}{a^2} \,. \tag{82}$$

In this expression, after transformation to X space, the determinant cancels, and the first term is a time-independent number. In the second term $\vec{U}(\vec{X}) = \vec{u}(\vec{X}, t_{\rm in}) = \vec{u}(\vec{q}, t_{\rm in})$ is used to perform the q integration. Thus the averages are time-independent numbers. This means that the scale factor dependence of the correction term behaves like a (negative) curvature term. Therefore the age of the universe gets greater than in the exact homogeneous model, but not more than the age of the open FRW model.

If we specify the initial peculiar-velocity potential, the correction term may be explicitly calculated. As a simple example, let us take the plane-wave model

$$s_{\rm in}(\vec{X}) = \frac{2}{3t_{\rm in}}\left(\frac{d}{4\pi}\right)^2 \sum_{\vec{n}} \frac{1}{|\vec{n}|^2}\sin\left(\frac{2\pi}{d}\vec{n}\cdot\vec{X}\right) \,, \tag{83}$$

where $\sum_{\vec{n}}$ is a sum over the different modes and d is the comoving length of the inhomogeneity. The normalization constant $2/(3t_{\rm in}) = H_0(1+z_{\rm in})$ is chosen so that the amplitude of the density contrast at the starting time $t_{\rm in}$ of the nonlinear stage equals 1.

Then the correction to the averaged density ρ_c may be explicitly evaluated as

$$\rho_c = \frac{\rho_{b0}}{36\pi^2}10^{-6}(1+z_{\rm in})^4 h_0^2 \left(\frac{d}{Mpc}\right)^2 M \,, \tag{84}$$

where

$$M = \left\langle \frac{1+\delta_{\rm in}}{\det(F_{i,j})}\cos^2\left(\frac{2\pi\vec{n}\cdot\vec{X}}{d}\right)\right\rangle_q + \frac{15}{4}\left\langle \cos^2\left(\frac{2\pi\vec{n}\cdot\vec{X}}{d}\right)\right\rangle_q \,. \tag{85}$$

If we take the simple example $\{\vec{n}\} = \{(1,0,0), (0,1,0), (0,0,1)\}$, then we have $M = \frac{57}{8}$.

VI. DISCUSSION

We have proposed an averaging scheme in general relativity which is mathematically well defined and practical for application to cosmology. After decomposing the spacetime into 3+1 geometry and introducing an approximation scheme, the locally inhomogeneous spatial section is averaged in the sense of Isaacson.

The resultant spacetime is the FRW model in the lowest order of approximation and the equations governing the averaged expansion coincide with previous studies by the author. Moreover, these equations are invariant under the gauge transformation which keeps the background structure fixed. We have also improved the condition for the approximation to be valid. The new condition (59) appears to be satisfied almost everywhere in the present universe. We have shown that the effect of local inhomogeneity behaves like a negative curvature in a simple situation. Thus, even if the averaged space has zero spatial curvature, the age of such an inhomogeneous universe is greater than that of the spatially flat FRW universe, but less than that of the open FRW universe. Further study is necessary to gain some insight into the age problem since the age estimation suffers from various biasing effects.

It would be interesting to use the present formalism to see the effects of local inhomogeneity on the isotropy of the cosmic background radiation. A preliminary study has been done using a simple model of the inhomogeneity and an induced anisotropy is used to impose an upper limit on the redshift when the density contrast gets into the nonlinear stage [17]. A similar analysis should be done using a more realistic analytical model of the inhomogeneity [18].

Since the present approximation makes use of the 3+1 formulation of general relativity which is used in numerical study of the Einstein equation, our approximation may be examined by the numerical analysis developed in this framework.

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