Statistics of the microwave background anisotropies caused by squeezed cosmological perturbations

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It is likely that the observed large-angular-scale anisotropies in the microwave background radiation are induced by cosmological perturbations of quantum-mechanical origin. Such perturbations are now placed in squeezed vacuum quantum states and, hence, are characterized by large variances of their amplitude. The statistical properties of the anisotropies should reflect the underlying statistics of the squeezed vacuum quantum states. The theoretical variances for the temperature angular correlation function are derived and described quantitatively. It is shown that they are indeed large and must be present in observational data if the anisotropies are truly caused by perturbations of quantum-mechanical origin. Unfortunately, these large theoretical statistical uncertainties will make the extraction of cosmological information from the measured anisotropies a much more difficult problem than we wanted it to be. [S0556-2821(96)01312-4]

I. INTRODUCTION

The line of reasoning in this paper is as follows.

We see the anisotropies in the microwave background radiation at the largest angular scales [1]. Observers convincingly argue that this is a genuine cosmological effect.

If the large-angular-scale anisotropy in the microwave background is really produced by cosmological perturbations (density perturbations, rotational perturbations, gravitational waves), then their today's wavelengths are of the order of and longer than today's Hubble radius l_H. Strictly speaking, all wavelengths give contributions to the anisotropy at every given angular scale. But if the spectrum of the perturbations is not excessively ''red'' or ''blue,'' the dominant contribution is provided by wavelengths indicated above. For instance, the major contribution to the quadrupole anisotropy is provided by wavelengths somewhat longer than l_H.

In the expanding Universe, the wavelengths of perturbations increase in proportion to the cosmological scale factor. The wavelengths that are longer than some length scale today have always been longer than that scale in the past. Moreover, the wavelengths of the perturbations of our interest are much longer than the Hubble radius defined at the previous times, when one goes back in time up to the era of primordial nucleosynthesis — the earliest era of which we have observational data. It is hard to imagine (although it does not seem to be logically impossible) that cosmological perturbations of our interest, with such long wavelengths, could have been generated by local physical processes during the interval of time between the era of primordial nucleosynthesis and now. We are bound to conclude that these perturbations were generated in the very early Universe, before the era of primordial nucleosynthesis. There is still 80 orders of magnitude, in terms of energy density, to go from

the era of primordial nucleosynthesis to the Planck era; a lot of things could have happened in between.

The law of evolution of the very early Universe is not known, but it is likely that it could have been significantly different from the law of expansion of the radiation-dominated Universe. If so, some amount of cosmological perturbations must have been generated quantum mechanically, as a result of parametric interaction of the quantized perturbations with strong variable gravitational field of the very early Universe. Gravitational waves have been generated inevitably, while density and rotational perturbations — if we were lucky; see [2] and references therein. (If the cosmological scale factor has always been the one of the radiation-dominated Universe, we must stop here, because the parametric coupling vanishes in this case, and cosmological perturbations cannot be amplified classically and cannot be generated quantum mechanically.) The amount and spectrum of the generated perturbations depend on the law of evolution of the very early Universe (the strength and variability of the gravitational pump field), and this is how we can learn about what was going on there. In particular, the law of evolution of the very early Universe could have been of inflationary type.

If the cosmological perturbations were generated quantum mechanically, they should now be placed in the squeezed vacuum quantum states [3] (for an introduction to squeezed states see, for example, Refs. [4,5] and the pioneering works quoted there). Squeezing of cosmological perturbations might have degraded by now at short wavelengths but should survive at long wavelengths, especially in the case of gravitational waves.

Now, the squeezed vacuum quantum states can only be squeezed in the variances of phase which unavoidably means increased variances in amplitude. The statistical properties of the squeezed vacuum quantum states are significantly different from the statistical properties of the ''most classical'' quantum states — coherent states. This is well illustrated by the fact that the variance of the number of quanta in a strongly squeezed vacuum quantum state is much larger than

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the variance of the number of quanta in the coherent state with the same mean number of quanta $\langle N \rangle$, $\langle N \rangle \gg 1$. For a squeezed vacuum state the variance is $\langle N^2 \rangle - \langle N \rangle^2 = 2 \langle N \rangle \times (\langle N \rangle + 1) \gg \langle N \rangle$ while for a coherent state it is $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$. In cosmology, the mean number $\langle N \rangle$ is a characteristic of the expected mean square amplitude of cosmological perturbations, while the variance $\langle N^2 \rangle - \langle N \rangle^2$ is a characteristic of statistical uncertainties in the amplitude. These two characteristics are independent properties of a quantum state or a stochastic process. Theoretical models may agree on $\langle N \rangle$ and disagree on variance or agree on variance and disagree on $\langle N \rangle$.

The statistical properties of squeezed cosmological perturbations will inevitably be reflected in statistical properties of the microwave background anisotropies caused by them. Squeezing is a phase-sensitive phenomenon, and to fully extract its properties the quantum optics experimenters use the phase-sensitive detecting techniques based on a local oscillator. In cosmology, we are very far from being able to build a local oscillator, except maybe, in the distant future, for short gravitational waves. In addition, in our study of the microwave background anisotropies, we are interested in such long-wavelength perturbations that it would take billions of years to wait for seeing the time-dependent oscillations of variances in the quadrature components of the perturbation field. On the other hand, the amount of cosmological squeezing is enormously greater than what is achieved in quantum optics laboratory experiments. In cosmology, we can only rely on the phase-insensitive, direct detection. One can expect that the underlying large variances of the amplitude of cosmological perturbations should result in large statistical deviations from the mean values for the microwave anisotropies.

A detailed study and proof of this statement is the purpose of this paper.

At this point it is necessary to comment on the possible numerical values of $\langle N \rangle$ (and, hence, of the amplitude) for cosmological perturbations of different nature which can be generated quantum mechanically in one and the same cosmological model. A consistent quantum theory provides us, of course, with both, the mean value of N and its variance. According to the calculations of Ref. [2], the contribution of quantum-mechanically generated gravitational waves to the large-angular-scale anisotropy is somewhat greater (even in the limit of the de Sitter expansion) than the contribution of quantum-mechanically generated density perturbations. It is argued in [6,7,30] that the ''standard'' inflationary formula for density perturbations, which requires in this limit an arbitrarily large excess of density perturbations over gravitational waves, is based on errors. However, in this paper, we will not discuss any longer the relative contributions to $\delta T / T$ supplied by cosmological perturbations of different nature (density perturbations, rotational perturbations, or gravitational waves). Instead, we will concentrate on their statistical properties which are determined by their common origin — quantum mechanics and the phenomenon of squeezing. These properies are related to the variance of N rather than to its mean value. Our discussion will be equally well applicable to the perturbations of any nature if they have the same origin.

II. THE GENERAL EQUATIONS FOR QUANTIZED COSMOLOGICAL PERTURBATIONS

Here, we will briefly summarize some basic information about quantized cosmological perturbations (see [2,6] and references therein). The squeezed field operator derived in this section is a basic mathematical construction for our further discussion of statistical properties.

The metric of the homogeneous isotropic universe can be written in the form

$$ds^2 = -a^2(\eta)(d\eta^2 - \gamma_{ij} dx^i dx^j), \tag{1}$$

where γ_{ij} is the spatial metric. For reasons of simplicity, we will be considering only spatially flat universes, that is $\gamma_{ij} = \delta_{ij}$.

Following Lifshitz, it is convenient to write the perturbed metric in the form

$$ds^2 = -a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij}) dx^i dx^j], \tag{2}$$

where h_{ij} are functions of η time and spatial coordinates. By writing the perturbed metric in this form we do not lose anything in the physical content of the problem, but we gain considerably in the mathematical tractability of the perturbed Einstein equations. While searching for solutions, it is certainly preferable to work with a simpler form of equations. If one prefers the ''gauge-invariant formalisms,'' one is welcome to take the found solution and compute with its help whichever gauge-invariant quantity one likes. These quantities, being gauge invariant, have the same values in all gauges.

The components h_{ij} of the perturbed gravitational field can be classified in terms of scalar, vector, and tensor eigenfunctions of the Laplace differential operator. The components of the perturbed energy-momentum tensor can also be classified in the same manner. After that, the linearized Einstein equations reduce to a set of ordinary differential equations, separately for scalar (density perturbations), vector (rotational perturbations), and tensor (gravitational waves) parts.

The number of independent unknown functions of time that can potentially be present (on grounds of the classification scheme) in the perturbed Einstein equations is always greater than the number of independent equations. It is six functions and four equations for density perturbations, three functions and two equations for rotational perturbations, and two functions and one equation for gravitational waves. In order to make the system of equations closed, it is necessary to say something about the perturbed components of the energy-momentum tensor or to specify from the very beginning the form of the energy-momentum tensor. The popular choices are perfect fluids and scalar fields. Even for gravitational waves, it is not a totally trivial question what their definition is (see, for example, Ref. [8]). However, after everything is being set, and as soon as the scale factor $a(\eta)$ (the background solution) is known, the general solution to the perturbed equations can be found. In practice, exact solutions are being found piecewise, at the intervals of evolution where the energy-momentum tensor has simple prescribed forms.

We can now write the quantum-mechanical operator for the perturbations of the gravitational field h_{ij} in the universal form

$$h_{ij} = \frac{C}{a(\eta)} \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3n \sum_{s=1}^{2} \overset{s}{p}_{ij}(\mathbf{n}) \frac{1}{\sqrt{2n}} [\overset{s}{c}_{\mathbf{n}}(\eta)e^{i\mathbf{n}\cdot\mathbf{x}}$$

$$+ \overset{s\dagger}{c}_{\mathbf{n}}(\eta)e^{-i\mathbf{n}\cdot\mathbf{x}}]. \tag{3}$$

We will start the explanation of Eq. (3) from the polarization tensors $\overset{s}{p}_{ij}$. Let us introduce, in addition to the unit wave-vector $\mathbf{n}/n$, two more unit vectors l_i, m_i, orthogonal to each other and to $\mathbf{n}$:

$$\frac{n_i}{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad l_i = (\sin\phi, -\cos\phi, 0),$$

$$m_i = \pm(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{4}$$

$+$ for $\theta < \pi/2$, $-$ for $\theta > \pi/2$.

The two independent polarization tensors, $s=1,2$, for each class of perturbations, can be written as follows. For gravitational waves,

$$\overset{1}{p}_{ij}(\mathbf{n}) = (l_i l_j - m_i m_j), \quad \overset{2}{p}_{ij}(\mathbf{n}) = (l_i m_j + l_j m_i).$$

For rotational perturbations,

$$\overset{1}{p}_{ij}(\mathbf{n}) = \frac{1}{n}(l_i n_j + l_j n_i), \quad \overset{2}{p}_{ij}(\mathbf{n}) = \frac{1}{n}(m_i n_j + m_j n_i).$$

For density perturbations,

$$\overset{1}{p}_{ij}(\mathbf{n}) = \sqrt{\frac{2}{3}}\delta_{ij}, \quad \overset{2}{p}_{ij}(\mathbf{n}) = -\sqrt{3}\frac{n_i n_j}{n^2} + \frac{1}{\sqrt{3}}\delta_{ij}.$$

The polarization tensors of each class satisfy the conditions $\overset{s}{p}_{ij}\overset{s'}{p}{}^{ij} = 2\delta_{ss'}$, $\overset{s}{p}_{ij}(-\mathbf{n}) = \overset{s}{p}_{ij}(\mathbf{n})$. In practical handling of the density perturbations it proves convenient to use sometimes, in addition to the *scalar* polarization component $\overset{1}{p}_{ij}$, the *longitudinal-longitudinal* component (proportional to $n_i n_j$) instead of $\overset{2}{p}_{ij}$. The explicit functional dependence of the polarization tensors is needed for the calculation of various angular correlation functions.

The evolution of the creation and annihilation operators $\overset{s}{c}_{\mathbf{n}}(\eta)$, $\overset{s\dagger}{c}_{\mathbf{n}}(\eta)$, for each class of perturbations and for each polarization state, is defined by the Heisenberg equations of motion:

$$\frac{dc_{\mathbf{n}}(\eta)}{d\eta} = -i[c_{\mathbf{n}}(\eta), H], \quad \frac{dc_{\mathbf{n}}^{\dagger}(\eta)}{d\eta} = -i[c_{\mathbf{n}}^{\dagger}(\eta), H]. \tag{5}$$

The dynamical content of the problem is determined by the Hamiltonian H. Its form depends on the class of perturbations and additional assumptions about the energy-momentum tensor which we have to make, as was discussed above.

Under the simplest assumptions about gravitational waves (waves interact only with the background gravitational field, there is no anisotropic material sources), the Hamiltonian for each polarization component takes on the form

$$H = nc_{\mathbf{n}}^{\dagger}c_{\mathbf{n}} + nc_{-\mathbf{n}}^{\dagger}c_{-\mathbf{n}} + 2\sigma(\eta)c_{\mathbf{n}}^{\dagger}c_{-\mathbf{n}}^{\dagger} + 2\sigma^*(\eta)c_{\mathbf{n}}c_{-\mathbf{n}}, \tag{6}$$

where the coupling function $\sigma(\eta)$ is $\sigma(\eta) = (i/2)(a'/a)$.

For rotational perturbations, assuming that the primeval matter is capable of supporting torque oscillations, assuming that the oscillations are minimally coupled to gravity, and assuming that the torsional velocity of sound is equal to the velocity of light, the Hamiltonian for each polarization component reduces to exactly the same form (6) with the same coupling function $\sigma(\eta)$.

For density perturbations, we consider specifically a minimally coupled scalar field with arbitrary scalar field potential as a model for matter in the very early Universe, and perfect fluids at the later eras. The quantization is based on the *scalar* polarization component (the function of time responsible for another polarization state is not independent). There is only one independent sort of creation and annihilation operators in this case. The operators $\overset{s}{c}_{\mathbf{n}}(\eta)$, $\overset{s\dagger}{c}_{\mathbf{n}}(\eta)$ are expressible in terms of the operators $d_{\mathbf{n}}(\eta)$, $d_{\mathbf{n}}^{\dagger}(\eta)$ for which the Hamiltonian has again the same form (6) but with the coupling function $\sigma(\eta) = (i/2)(a\sqrt{\gamma})'/a\sqrt{\gamma}$, where

$$\gamma = 1 + (a/a')' = -\dot{H}/H^2$$

and H is the Hubble parameter. For density perturbations, it is the operators $d_{\mathbf{n}}(\eta)$, $d_{\mathbf{n}}^{\dagger}(\eta)$ that participate in Eqs. (5) and (6).

Now, let us turn to the constant C in Eq. (3). Its value is determined by the normalization of the field of each class to the "half of the quantum in each mode." Under the assumptions listed above, one derives $C = \sqrt{16\pi}l_{\mathrm{Pl}}$ for gravitational waves, $C = \sqrt{32\pi}l_{\mathrm{Pl}}$ for rotational perturbations, and $C = \sqrt{24\pi}l_{\mathrm{Pl}}$ for density perturbations, where l_{Pl} is the Planck length, $l_{\mathrm{Pl}} = (G\hbar/c^3)^{1/2}$.

The form of the Hamiltonian (6) dictates the form of the solution (Bogoliubov transformation) to Eq. (5):

$$c_{\mathbf{n}}(\eta) = u_n(\eta)c_{\mathbf{n}}(0) + v_n(\eta)c_{-\mathbf{n}}^{\dagger}(0),$$

$$c_{\mathbf{n}}^{\dagger}(\eta) = u_n^*(\eta)c_{\mathbf{n}}^{\dagger}(0) + v_n^*(\eta)c_{-\mathbf{n}}(0), \tag{7}$$

where $c_{\mathbf{n}}(0)$, $c_{\mathbf{n}}^{\dagger}(0)$ are the initial values of the operators taken long before the interaction with the pump field became important $[\sigma(\eta)/n \to 0]$ and which define the vacuum state $c_{\mathbf{n}}(0)|0\rangle = 0$. The complex functions $u_n(\eta)$, $v_n(\eta)$ obey coupled first-order differential equations following from Eq. (5) and satisfy the condition $|u_n|^2 - |v_n|^2 = 1$ which guarantees that the commutator relationship $[c_{\mathbf{n}}(0), c_{\mathbf{m}}^{\dagger}(0)] = \delta^3(\mathbf{n}-\mathbf{m})$ is satisfied at all times, $[c_{\mathbf{n}}(\eta), c_{\mathbf{m}}^{\dagger}(\eta)] = \delta^3(\mathbf{n}-\mathbf{m})$. If one introduces the function $\mu_n(\eta) = u_n(\eta) + v_n^*(\eta)$, one recovers from the equations for $u_n(\eta)$, $v_n(\eta)$ the classical equations of motion. For gravitational waves,

$$\mu_n'' + \left[n^2 - \frac{a''}{a} \right] \mu_n = 0. \tag{8}$$

For rotational perturbations,

$$\mu_n'' + \left[n^2 \frac{v_t^2}{c^2} - \frac{a''}{a} \right] \mu_n = 0, \tag{9}$$

where v_t is the torsional velocity of sound which we assumed above to be c. For the scalar field density perturbations,

$$\mu_n'' + \left[n^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] \mu_n = 0. \tag{10}$$

If the pump field is such that the γ function is independent of time, Eq. (10) reduces to exactly the same form as Eq. (8) for gravitational waves.

In the Schrödinger picture, the initial vacuum quantum state $|0_\mathbf{n}\rangle \, |0_{-\mathbf{n}}\rangle$ evolves into a two-mode squeezed vacuum quantum state. In our problem, each of the two-mode squeezed vacuum quantum states is a product of two identical one-mode squeezed vacuum quantum states which correspond to the decomposition of the real field h_{ij} over real spatial harmonics $\sin \mathbf{n} \cdot \mathbf{x}$ and $\cos \mathbf{n} \cdot \mathbf{x}$. In the Heisenberg picture, the initial vacuum quantum state does not evolve in time and is the same now.

By using Eq. (7) one can present the field (3) in the form

$$h_{ij}(\eta, \mathbf{x}) = C \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 n \sum_{s=1}^{2} \overset{s}{p}_{ij}(\mathbf{n}) \frac{1}{\sqrt{2n}} [\overset{s}{h}_n(\eta) e^{i\mathbf{n}\cdot\mathbf{x}} \overset{s}{c}_\mathbf{n}(0)$$

$$+ \overset{s*}{h}_n(\eta) e^{-i\mathbf{n}\cdot\mathbf{x}} \overset{s\dagger}{c}_\mathbf{n}(0)], \tag{11}$$

where the functions $\overset{s}{h}_n(\eta) = [1/a(\eta)] [\overset{s}{u}_n(\eta) + \overset{s*}{v}_n(\eta)]$. For gravitational waves and rotational perturbations, the functions $\overset{s}{h}_n$ are simply $\overset{s}{h}_n = \overset{s}{\mu}_n / a$ where $\overset{s}{\mu}_n$ are solutions to Eqs. (8) and (9) with appropriate initial conditions. For density perturbations, the functions $\overset{s}{h}_n$ are derivable from solutions to Eq. (10) in accord with the relationship between c and d operators. In addition, for density perturbations, we should regard $\overset{1}{c}_\mathbf{n}(0) = \overset{2}{c}_\mathbf{n}(0)$, $\overset{1\dagger}{c}_\mathbf{n}(0) = \overset{2\dagger}{c}_\mathbf{n}(0)$ in Eq. (11). In all cases, for a given cosmological model, that is for a model in which the scale factor $a(\eta)$ is known from the very early times and up to now, the functions $\overset{s}{h}_n$ can be found from the classical equations of motion with appropriate initial conditions.

It follows from Eq. (11) that the mean quantum-mechanical value of the field h_{ij} is zero at every spatial point and at every moment of time, $\langle 0|h_{ij}|0\rangle = 0$. One can also calculate variances of the field, that is the expectation values of its quadratic combinations. One useful quantity is $h_{ij} h^{ij}$. By manipulating with the product of two expressions (11), using the summation properties of the polarization tensors, and remembering that the only nonvanishing correlation function is

$$\langle 0|\overset{s}{c}_\mathbf{n}(0) \, \overset{s'\dagger}{c}_{\mathbf{n}'}(0)|0\rangle = \delta_{ss'} \delta^3(\mathbf{n} - \mathbf{n}'),$$

one can derive the formula

$$\langle 0|h_{ij}(\eta, \mathbf{x}) h^{ij}(\eta, \mathbf{x})|0\rangle = \frac{C^2}{2\pi^2} \int_0^{\infty} n \sum_{s=1}^{2} |\overset{s}{h}_n(\eta)|^2 dn. \tag{12}$$

Equation (12) shows that the variance is independent of the spatial point $\mathbf{x}$ but does depend on time.

The expression under the integral in formulas such as Eq. (12) is usually called the power spectrum (in this case, it is the power spectrum of the quantity $h_{ij} h^{ij}$):

$$P(n) = \frac{C^2}{2\pi^2} n \sum_{s=1}^{2} |\overset{s}{h}_n(\eta)|^2. \tag{13}$$

In cosmology, it is common to use the power spectrum defined in terms of the logarithmic frequency interval, that is the function

$$P_Z(n) = \frac{C^2}{2\pi^2} n^2 \sum_{s=1}^{2} |\overset{s}{h}_n(\eta)|^2 \tag{14}$$

(Z from Zeldovich). We are mostly interested in the power spectrum of cosmological perturbations in the present Universe, at the matter-dominated stage. This spectrum is never smooth as a function of frequency (wave number) n. Squeezing of the phase variances and the associated standing wave pattern of the field make the spectrum an oscillating function of n for each moment of time. In their turn, the oscillations in the power spectrum will produce oscillations in the distribution of the higher-order multipoles of the angular correlation function for the temperature anisotropies. However, the spectrum is smooth for sufficiently long waves. At a given moment of time, this applies to all perturbations whose wavelengths are of the order of and longer than the Hubble radius defined at that time. Moreover, the smooth part of the spectrum is power-law dependent on n if the scale factor $a(\eta)$ of the very early Universe (the pump field) was power-law dependent on η time.

Let us assume that the scale factor at the initial stage of expansion was

$$a(\eta) = l_0 |\eta|^{1+\beta}, \tag{15}$$

where l_0 and β are constants. If the evolution is governed by a scalar field, the Einstein equations require the constant β to be $\beta \leq -2$. The value $\beta = -2$ corresponds to the de Sitter expansion. At later times, the scale factor changed to the laws of the radiation-dominated and matter-dominated universes. From solutions for $\overset{s}{h}_n(\eta)$ traced up to the matter-dominated stage, one can find

$$\sum_{s=1}^{2} |\overset{s}{h}_n(\eta)|^2 \sim \frac{1}{l_0^2} n^{2\beta+2} \quad \text{and} \quad P_Z(n) \sim \frac{l_{\text{Pl}}^2}{l_0^2} n^{2(\beta+2)}.$$

It is convenient to introduce the *characteristic* amplitude $h(n)$ of the metric perturbations defining this amplitude as

the standard deviation (square root of variance) of the perturbed gravitational field per logarithmic frequency interval. In the long-wavelength limit under discussion, this quantity is universally expressed (both, for gravitational waves and for density perturbations) by the formula [20]

$$h(n) \sim \frac{l_{Pl}}{l_0} n^{\beta+2}. \tag{16}$$

Note that the functional form of $h(n)$ is the same for gravitational waves and density perturbations, the difference is in the numerical coefficient (omitted in this discussion) which is somewhat in favor of gravitational waves [2,6]. The numerical level of $h(n)$ is mainly controlled by the constant l_0.

The spectra of other quantities can be found in the same manner. For instance, in case of density perturbations, one can derive the spectrum of perturbations in the matter density $\delta\rho/\rho$. Since the relationship between $\delta\rho/\rho$ and the metric perturbations involves the factor $(n\eta)^2$ and, hence, involves two extra powers of n, $(\delta\rho/\rho)(n) \sim n^2 h(n)$, one finds

$$\left\langle 0 \left| \frac{\delta\rho}{\rho} \frac{\delta\rho}{\rho} \right| 0 \right\rangle \sim \int_0^\infty P_Z^\rho(n) \frac{dn}{n},$$

where $P_Z^\rho(n) \sim (l_{Pl}^2/l_0^2) n^{2(\beta+4)}$ and

$$\frac{\delta\rho}{\rho}(n) \sim \frac{l_{Pl}}{l_0} n^{\beta+4}. \tag{17}$$

It follows from Eq. (16) that $h(n)$ is independent of n if $\beta = -2$. This independence corresponds to the original Zeldovich's definition of the "flat" spectrum: all waves enter the Hubble radius with the same amplitude. If the gravitational field perturbations $h(n)$, regardless of their wavelength, have equal amplitudes upon entering the Hubble radius, the matter density perturbations $(\delta\rho/\rho)(n)$ do also have equal amplitudes [the extra factor $(n\eta)^2$ is of the order of 1 at the time when a given wave n enters the Hubble radius]. For models of the very early Universe governed by a scalar field, the spectral index $\beta+2$ in Eq. (16) can never be positive.

Formula (16) and the associated formula (17) should be compared with the "standard" inflationary formula which requires that the amplitudes of density perturbations taken at the time of entering the Hubble radius should go to infinity in the limit of the de Sitter inflation, $\beta \to -2$. It should also be noted that a "disgusting convention" (the term is borrowed from Ref. [9]) is often being used according to which one and the same Harrison-Zeldovich spectrum is described by the spectral index $n_t = 0$ for gravitational waves and by the spectral index $n_s = 1$ for density perturbations. Of course, there is no need in this convention. In both cases, the metric perturbations with the Harrison-Zeldovich spectrum are described by the same spectral index (zero), see Eq. (16).

We will finish this section with a short discussion of coherent states. There is no natural mechanism for the generation of cosmological perturbations in coherent states, but if there were one it would be reflected in many parts of the theory. The interaction part of the Hamiltonian (6) would be

linear (not quadratic) in the creation and annihilation operators. The analogue of Eq. (7) would read

$$c_{\mathbf{n}}(\eta) = e^{-in\eta} c_{\mathbf{n}}(0) + \alpha_n(\eta),$$

$$c_{\mathbf{n}}^\dagger(\eta) = e^{in\eta} c_{\mathbf{n}}^\dagger(0) + \alpha_n^*(\eta), \tag{18}$$

where the complex function $\alpha_n(\eta)$ is determined by the coupling function in the Hamiltonian. On the position — momentum diagram, the evolution (18) of the field operators corresponds to *displacing* the vacuum state without *squeezing* whereas the evolution (7) corresponds to *squeezing* the vacuum state without *displacing*. In terminology of mechanics, coherent states are produced by a force acting on the oscillator whereas squeezed vacuum states are produced by a parametric influence. In coherent states, the mean value of the field is not zero. The correlation functions (at least, for some quantities) would also be different from the squeezed state case. In cosmological context, this would eventually be reflected in the differing statistical properties of perturbations and induced microwave background anisotropies.

The calculations of the next section are based in an essential way on the field operator (11) for the squeezed vacuum perturbations.

III. QUANTUM-MECHANICAL EXPECTATION VALUES FOR THE MICROWAVE BACKGROUND ANISOTROPIES

The microwave background anisotropies are a subject of intense study [21].

In absence of cosmological perturbations, the temperature of the microwave background radiation seen in all directions on the sky would be the same T. Let us denote a direction on the sky by a unit vector $\mathbf{e}$. The presence of cosmological perturbations makes the temperature seen in the direction $\mathbf{e}$ differing from T. The temperature perturbation produced by density perturbations or gravitational waves can be described by the formula [10]

$$\frac{\delta T}{T}(\mathbf{e}) = \frac{1}{2} \int_0^{w_1} \frac{\partial h_{ij}}{\partial \eta} e^i e^j dw, \tag{19}$$

where $\partial h_{ij}/\partial\eta$ is taken along the integration path $x^i = e^i w$, $\eta = \eta_R - w$, from the event of reception $w = 0$ to the event of emission $w = w_1 = \eta_R - \eta_E$. The formula for rotational perturbations is more complicated than (19) [10], and we will leave rotational perturbations aside.

For quantized cosmological perturbations, the $\frac{\delta T}{T}(\mathbf{e})$ becomes a quantum-mechanical operator. Using Eq. (11) we can write this operator as

$$\frac{\delta T}{T}(\mathbf{e}) = \frac{C}{2} \frac{1}{(2\pi)^{3/2}} \int_0^{w_1} dw \int_{-\infty}^\infty d^3 n \sum_{s=1}^2 \overset{s}{p}_{ij}(\mathbf{n}) e^i e^j$$

$$\times [\overset{s}{c}_{\mathbf{n}}(0) \overset{s}{f}_n(w) e^{iw\mathbf{n}\cdot\mathbf{e}} + \overset{s\dagger}{c}_{\mathbf{n}}(0) \overset{s*}{f}_n(w) e^{-iw\mathbf{n}\cdot\mathbf{e}}]$$

$$\tag{20}$$

where

$$\overset{s}{f_n}(w) \equiv \frac{1}{\sqrt{2n}} \frac{dh_n^s}{d\eta} \Bigg|_{\eta = \eta_R - w}.$$

Having defined the observable $(\delta T/T)(\mathbf{e})$ and knowing the quantum state $|0\rangle$ we can compute various quantum-mechanical expectation values. In the laboratory quantum mechanics, the verification of theoretical predictions expressed in terms of the expectation values would require experiments on many identical systems. An immediate generalization of this principle to cosmology would require speculations about outcomes of experiments performed in ''many identical universes.'' Without having access to ''many universes'' we can only rely on the mean (expected) values of the observables and on the probability distribution functions as indicators of what is likely or not to be observed in our own single Universe. We will return to this point in Sec. IV.

The expected value of the temperature perturbation to be observed in every fixed direction on the sky is zero:

$$\left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}) \right| 0 \right\rangle = 0.$$

One particular measured temperature map is the result of the measurement performed over one particular realization of the random process describing cosmological perturbations of quantum-mechanical origin. For this realization, the temperature perturbations may, should, and in fact are, present. Many measurements will not help (except reducing the instrumental noises) in the sense that they all should give identical results, because the time scale of the perturbations under discussion is so enormously larger than an interval of time between the experiments. If the Cosmic Background Explorer (COBE) map is correct, we will have to live with this map practically forever.

Let us now compute the expected angular correlation function for the temperature perturbations seen in two given directions on the sky, $\mathbf{e}_1$ and $\mathbf{e}_2$. This correlation function is defined as the mean value for the product of $(\delta T/T)(\mathbf{e}_1)$ and $(\delta T/T)(\mathbf{e}_2)$:

$$K(\mathbf{e}_1, \mathbf{e}_2) = \left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \right| 0 \right\rangle. \tag{21}$$

By manipulating with the product of two expressions (20) one can derive the formula

$$K(\mathbf{e}_1, \mathbf{e}_2) = \frac{1}{4} C^2 \frac{1}{(2\pi)^3} \int_0^{w_1} dw \int_0^{w_1} d\overline{w} \int_{-\infty}^{\infty} d^3 n \, e^{i\mathbf{n} \cdot (\mathbf{e}_1 w - \mathbf{e}_2 \overline{w})}$$

$$\times \sum_{s=1}^{2} (p_{ij}(\mathbf{n}) e_1^i e_1^j)(p_{ij}(\mathbf{n}) e_2^i e_2^j) \overset{s}{f_n}(w) \overset{s*}{f_n}(\overline{w}). \tag{22}$$

The next step is the formidable task of taking the integrals over angular variables in three-dimensional wave-vector $\mathbf{n}$ space. However, it can be done (see Ref. [11] for gravitational waves, Ref. [12] for rotational perturbations, and Ref. [2] for density perturbations). The final expression reduces, without making any additional assumptions whatsoever, to the form

$$K(\mathbf{e}_1, \mathbf{e}_2) = K(\delta) = l_{\mathrm{Pl}}^2 \sum_{l=l_{\min}}^{\infty} K_l P_l(\cos\delta). \tag{23}$$

We see that the correlation function depends only on the angle δ between the directions $\mathbf{e}_1, \mathbf{e}_2$, not directions themselves. The coefficient l_{Pl}^2 is taken from C^2, other numerical coefficients are included in K_l. The quantities K_l involve the integration of $\overset{s}{f_n}(w)$ over the parameter w and the remaining integration over the wave numbers n. The numerical values of K_l depend on a chosen sort of cosmological perturbations and a chosen cosmological model; so far, the formula (23) is totally general. $P_l(\cos\delta)$ are the Legendre polynomials. The lowest multipole $l_{\min}$ follows automatically from the theory and it turns out to be, not surprisingly, $l_{\min}=0$ for density perturbations, $l_{\min}=2$ for gravitational waves (and $l_{\min}=1$ for rotational perturbations). For the separation angle $\delta=0$, Eq. (23) reduces to the variance of $\delta T/T(\mathbf{e})$, that is

$$\left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}) \frac{\delta T}{T}(\mathbf{e}) \right| 0 \right\rangle = K(0) = l_{\mathrm{Pl}}^2 \sum_{l=l_{\min}}^{\infty} K_l. \tag{24}$$

Formula (23) gives the expected value of the observable $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$. If the experimenter measured this observable in ''many universes'' and averaged the measured numbers, he/she would get the result (23). Moreover, formula (23) says that if the experimenter made the measurements at any other pair of directions, but with the same separation angle δ, he/she would again get, after the averaging over ''many universes,'' the same result (23). Without having access to ''many universes,'' we can ask what is the theoretical standard deviation of the quantity $(\delta T/T)(\mathbf{e}_1) \times (\delta T/T)(\mathbf{e}_2)$. [In practice, for deriving $K(\delta)$ we need a kind of ergodic hypothesis allowing us to replace the averaging over ''universes'' by the averaging over pixels on a single map.] The variance $V(\mathbf{e}_1, \mathbf{e}_2)$ of this quantity is, by definition,

$$V(\mathbf{e}_1, \mathbf{e}_2) = \left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \right| 0 \right\rangle$$

$$- \left[\left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \right| 0 \right\rangle \right]^2. \tag{25}$$

The standard deviation is the square root of this number.

The calculation of $V(\mathbf{e}_1, \mathbf{e}_2)$ requires us to deal with the product of four expressions (20). However, the mean values of the products of four creation and annihilation operators are easy to handle. One can show that $V(\mathbf{e}_1, \mathbf{e}_2)$ depends only on the separation angle δ and

$$V(\mathbf{e}_1,\mathbf{e}_2) = V(\delta) = \left[\left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2) \right| 0 \right\rangle\right]^2$$

$$+ \left[\left\langle 0 \left| \frac{\delta T}{T}(\mathbf{e})\frac{\delta T}{T}(\mathbf{e}) \right| 0 \right\rangle\right]^2, \qquad (26)$$

that is

$$V(\delta) = K^2(\delta) + K^2(0). \qquad (27)$$

The standard deviation for the observable $\frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2)$ is

$$\sigma(\delta) = [V(\delta)]^{1/2} = \sqrt{K^2(\delta) + K^2(0)}. \qquad (28)$$

In a similar fashion one can derive the higher-order correlation functions for two directions $\mathbf{e}_1$, $\mathbf{e}_2$ and the correlation functions for larger number of directions, but we will not need this information.

In the limit $\delta = 0$, Eqs. (25) and (26) say that

$$\left\langle 0 \left| \left[\frac{\delta T}{T}(\mathbf{e}) \right]^4 \right| 0 \right\rangle = 3 \left[\left\langle 0 \left| \left(\frac{\delta T}{T}(\mathbf{e}) \right)^2 \right| 0 \right\rangle \right]^2. \qquad (29)$$

The familiar factor 3 relating the fourth-order moment with the square of the second-order moment (given that the first-order moment is equal to zero) is the reflection of the underlying Gaussian nature of the squeezed vacuum wave functions associated with the Hamiltonian (6).

By examining Eq. (28) one can conclude that for each separation angle δ the standard deviation of the angular correlation function is very big. Even at those separation angles at which $K(\delta)$ vanishes, the standard deviation is as big as the variance for $(\delta T/T)(\mathbf{e})$ itself. However, the value of the standard deviation for a given variable is not very informative *per se*, as long as the probability density function for this variable is not known. If the probability density function (PDF) were normal, we could say that the probability to find a result outside of 1σ interval is 32%. Without knowing the PDF we could resort to the Chebyshev inequality, but it would only tell us that this probability is less than 1. To get more information about possible deviation of the angular correlation function from its mean values we will consider in Sec. IV a classical random model which will reproduce the expectation values calculated above and will allow us to construct the PDF for the variable $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$. On the other hand, the quantum-mechanical calculations of Sec. III will shed light on the classical model. As is known, ''quantum mechanics helps us understand classical mechanics,'' see on this subject a paper by Zeldovich signed by the pseudonym Paradoksov [13].

IV. CLASSICAL MODEL FOR THE STATISTICS OF THE MICROWAVE BACKGROUND ANISOTROPIES

A distribution of the microwave background temperature over the sky is a real function of the angular coordinates. Assuming that $\delta T/T$ is a sufficiently smooth function on a sphere, one can expand it over the set of orthonormal complex spherical harmonics $Y_{lm}(\theta,\phi)$ [14]:

$$\frac{\delta T}{T}(\mathbf{e}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [a_{lm}Y_{lm}(\mathbf{e}) + a_{lm}^* Y_{lm}^*(\mathbf{e})]. \qquad (30)$$

We want to formulate a statistical hypothesis about the coefficients a_{lm}, so it is better to write them first in terms of real (r) and imaginary (i) components:

$$a_{lm} = a_{lm}^r + ia_{lm}^i, \quad a_{lm}^* = a_{lm}^r - ia_{lm}^i,$$

$$Y_{lm} = Y_{lm}^r + iY_{lm}^i, \quad Y_{lm}^* = Y_{lm}^r - iY_{lm}^i,$$

$$\frac{\delta T}{T}(\mathbf{e}) = 2\sum_{l=0}^{\infty} \sum_{m=-l}^{l} [a_{lm}^r Y_{lm}^r(\mathbf{e}) - a_{lm}^i Y_{lm}^i(\mathbf{e})]. \qquad (31)$$

Our statistical hypothesis is as follows: (i) all members of the set of random variables $\{a_{lm}^r, a_{lm}^i\}$ are statistically independent, (ii) each individual variable is normally distributed and has a zero mean, (iii) all variables with the same index l have the same standard deviation σ_l. All said is expressed by the probability density function (PDF) for individual variables,

$$f(a_{lm}^r) = \frac{1}{\sqrt{2\pi}\sigma_l} e^{-(a_{lm}^r)^2/2\sigma_l^2},$$

$$f(a_{lm}^i) = \frac{1}{\sqrt{2\pi}\sigma_l} e^{-(a_{lm}^i)^2/2\sigma_l^2}, \qquad (32)$$

and by the PDF for the entire set of variables, which is simply a product of all PDF's for all individual variables,

$$f(\{a_{lm}^r, a_{lm}^i\}) = \sqcap_{l,m} f(a_{lm}^r) f(a_{lm}^i). \qquad (33)$$

Having postulated the PDF's, we can now compute the expectation values of certain functions of the random variables. Below, the angular brackets will denote the expectation values calculated with the help of the PDF (33), unless other definition is stated.

Obviously, all linear functions have a zero mean:

$$\langle a_{lm}^r \rangle = 0, \quad \langle a_{lm}^i \rangle = 0. \qquad (34)$$

For quadratic combinations we have

$$\langle a_{l_1 m_1}^r a_{l_2 m_2}^r \rangle = \sigma_{l_1}^2 \delta_{l_1 l_2} \delta_{m_1 m_2},$$

$$\langle a_{l_1 m_1}^i a_{l_2 m_2}^i \rangle = \sigma_{l_1}^2 \delta_{l_1 l_2} \delta_{m_1 m_2},$$

$$\langle a_{l_1 m_1}^r a_{l_2 m_2}^i \rangle = 0,$$

$$\langle a_{l_1 m_1}^i a_{l_2 m_2}^r \rangle = 0. \qquad (35)$$

All triple products have zero means. Among quartic combinations, only those can survive which have four indices (r), or four indices (i), or two indices (r) and two indices (i). Two representative expressions are

$$\langle a^r_{l_1 m_1} a^r_{l_2 m_2} a^r_{l_3 m_3} a^r_{l_4 m_4}\rangle = \sigma^2_{l_1} \sigma^2_{l_3} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{l_3 l_4} \delta_{m_3 m_4}$$
$$+ \sigma^2_{l_1} \sigma^2_{l_2} \delta_{l_1 l_3} \delta_{m_1 m_3} \delta_{l_2 l_4} \delta_{m_2 m_4}$$
$$+ \sigma^2_{l_1} \sigma^2_{l_2} \delta_{l_1 l_4} \delta_{m_1 m_4} \delta_{l_2 l_3} \delta_{m_2 m_3}, \tag{36}$$

$$\langle a^r_{l_1 m_1} a^r_{l_2 m_2} a^i_{l_3 m_3} a^i_{l_4 m_4}\rangle = \sigma^2_{l_1} \sigma^2_{l_3} \delta_{l_1 l_2} \delta_{m_1 m_2} \delta_{l_3 l_4} \delta_{m_3 m_4}. \tag{37}$$

Other quartic combinations can be obtained by the replacement $(r) \leftrightarrow (i)$ in Eqs. (36) and (37) [or by permutation of pairs (lm) in case of Eq. (37)]. The higher-order correlations can be derived in a similar way, but we will not need them.

In our further calculations related to the random variables $(\delta T/T)(\mathbf{e})$ and $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$ it is easier to deal with the complex coefficients a_{lm}, so we will first translate the above relationships to them. By using the available information one can derive

$$\langle a_{lm}\rangle = 0, \quad \langle a_{l_1 m_1} a^*_{l_2 m_2}\rangle = 2\sigma^2_l \delta_{l_1 l_2} \delta_{m_1 m_2},$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a^*_{l_3 m_3} a^*_{l_4 m_4}\rangle = 4\sigma^2_{l_1} \sigma^2_{l_2}(\delta_{l_1 l_3} \delta_{m_1 m_3} \delta_{l_2 l_4} \delta_{m_2 m_4}$$

$$+ \delta_{l_1 l_4} \delta_{m_1 m_4} \delta_{l_2 l_3} \delta_{m_2 m_3}). \tag{38}$$

The mean values of the complex conjugated quantities are given by the same formulas (38). Other nonvanishing quartic combinations can be obtained from the one in Eq. (38) by the permutation of pairs (lm).

Now, even before deriving the PDF's for the random variables $(\delta T/T)(\mathbf{e})$ and $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$, we can find some expectation values. It is clear from the definition (30) and Eq. (38) that

$$\left\langle \frac{\delta T}{T}(\mathbf{e})\right\rangle = 0. \tag{39}$$

When calculating the angular correlation function one should remember that

$$\sum_{m=-l}^{l} Y_{lm}(\mathbf{e}_1) Y^*_{lm}(\mathbf{e}_2) = \frac{2l+1}{4\pi} P_l(\cos\delta) \tag{40}$$

(note the origin of the factor $2l+1$ which will accompany us often). By taking the product of two expressions (30) and using Eqs. (38) and (40) one can find the angular correlation function

$$\left\langle \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2)\right\rangle = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma^2_l (2l+1) P_l(\cos\delta). \tag{41}$$

If the separation angle δ is zero, we obtain

$$\left\langle \frac{\delta T}{T}(\mathbf{e}) \frac{\delta T}{T}(\mathbf{e})\right\rangle = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma^2_l (2l+1). \tag{42}$$

[One may notice that the mean value of the random variable a^2_l defined as $a^2_l = \Sigma^l_{m=-l} a_{lm} a^*_{lm}$ is $\langle a^2_l\rangle = 2(2l+1)\sigma^2_l$, that is the same expression which enters Eq. (41). This may suggest an interpretation of the quantity $\langle a^4_l\rangle - \langle a^2_l\rangle^2$ as the variance of the multipole moments. One should be careful with this interpretation, see Appendix.]

We can also find the fourth-order expectation values. The product of four expressions (30) in conjunction with Eq. (38) gives

$$\left\langle \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2)\right\rangle - \left\langle \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2)\right\rangle^2$$
$$= \left\langle \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2)\right\rangle^2 + \left\langle \frac{\delta T}{T}(\mathbf{e}) \frac{\delta T}{T}(\mathbf{e})\right\rangle^2. \tag{43}$$

If $\delta = 0$, it follows from Eq. (43) that

$$\left\langle \left[\frac{\delta T}{T}(\mathbf{e})\right]^4\right\rangle = 3\left\langle \left[\frac{\delta T}{T}(\mathbf{e})\right]^2\right\rangle^2. \tag{44}$$

Up to the difference in the meaning of the angular brackets, the formulas (39), (43), and (44) reproduce the analogous results of the previous section. Moreover, from comparison of Eqs. (23) and (24) with Eqs. (41) and (42) we can relate the quantities K_l, derivable from a given cosmological model plus perturbations, with the abstract quantities σ_l.

We can now engage in our major enterprise — the construction of the PDF for the random variable $v\equiv(\delta T/T) \times(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$. We will start from the PDF for the random variable $z\equiv(\delta T/T)(\mathbf{e})$. When it is necessary to distinguish directions $\mathbf{e}_1$ and $\mathbf{e}_2$, we will use the notations z_1 and z_2.

The variable z is a function of the variables $\{a^r_{lm}, a^i_{lm}\}$ whose PDF's are known, Eqs. (31) and (32). There exist regular methods (see, for example, an excellent book [15]) allowing one to derive rigorously the PDF of a function. However, in our case where the function is linear and all PDF's are normal, we can partially rely on a guesswork. Combining formulas and guessing, we can write

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-z^2/2\sigma^2_z}, \tag{45}$$

where

$$\sigma^2_z = \frac{1}{\pi} \sum_{l=0}^{\infty} \sigma^2_l (2l+1). \tag{46}$$

The PDF (45) certainly leads to Eqs. (39), (42) and (44). Moreover, it allows us to say that the probability to find z outside of $1\sigma_z$ interval is approximately 32%:

$$P(|z|>\sigma_z)\approx 0.32.$$

We now introduce two variables, z_1 and z_2, and ask about the PDF in the two-dimensional space (z_1,z_2). Again, partially relying on a guesswork, we find that

$$f(z_1,z_2)=\frac{1}{2\pi\sigma_z^2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\sigma_z^2(1-\rho^2)}[z_1^2+z_2^2\right.$$

$$\left.-2\rho z_1 z_2]\right\},\qquad (47)$$

where

$$\rho\sigma_z^2=\frac{1}{\pi}\sum_{l=0}^{\infty}(2l+1)\sigma_l^2 P_l(\cos\delta),\quad |\rho|\leqslant 1.\qquad (48)$$

[See Eq. (5.11.1) in Ref. [15]]. First, we can check that the marginal distributions are correct. For $f(z_1)$, one obtains

$$f(z_1)=\int_{-\infty}^{\infty}f(z_1,z_2)dz_2=\frac{1}{\sqrt{2\pi}\sigma_z}e^{-z_1^2/2\sigma_z^2}$$

and one obtains a similar expression for $f(z_2)$. Second, one can check that

$$\langle z_1^2\rangle=\sigma_z^2,\quad \langle z_2^2\rangle=\sigma_z^2,\quad \langle z_1 z_2\rangle=\rho\sigma_z^2,$$

where the angular brackets mean the integration with the PDF (47). These equalities are Eqs. (42) and (41) which we must have obtained.

Finally, we shall derive the PDF for the variable $v=z_1 z_2$. We will do this in some detail following the prescriptions of [15].

Let us introduce the two new variables (z_1,v) instead of (z_1,z_2) according to the transformation

$$z_1=z_1,\quad z_2=\frac{v}{z_1}.$$

The Jacobian of this transformation is $\mathcal{J}=1/z_1$. The PDF $f(v)$ is the result of the integration

$$f(v)=\frac{1}{2\pi\sigma_z^2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\frac{1}{|z_1|}\exp\left\{-\frac{1}{2\sigma_z^2(1-\rho^2)}\left[z_1^2+\frac{v^2}{z_1^2}\right.\right.$$

$$\left.\left.-2\rho v\right]\right\}dz_1.$$

The integral over z_1 can be taken with the help of 3.471.9 from [16]. The resulting PDF can be written in the form

$$f(v)=\begin{cases}\dfrac{1}{\pi\sigma_z^2\sqrt{1-\rho^2}}e^{\rho v/\sigma_z^2(1-\rho^2)}K_0\left(\dfrac{v}{\sigma_z^2(1-\rho^2)}\right) & \text{for } v>0,\\[3ex]\dfrac{1}{\pi\sigma_z^2\sqrt{1-\rho^2}}e^{\rho v/\sigma_z^2(1-\rho^2)}K_0\left(\dfrac{-v}{\sigma_z^2(1-\rho^2)}\right) & \text{for } v<0,\end{cases}\qquad (49)$$

where K_0 is the modified Bessel function of its argument [17].

The function $f(v)$ is quite complicated, and the distribution is obviously not normal. The function $f(v)$ goes to zero for $v\to\pm\infty$ and diverges logarithmically at the point $v=0$. Even a verification of the normalization condition

$$\int_{-\infty}^{\infty}f(v)dv=1\qquad (50)$$

is not trivial. However, with the help of 6.621.3, 9.131.1, 9.121.7, 1.624.9, and 1.623.2 from [16] one can prove the validity of Eq. (50).

The mean value and the standard deviation of the variable v are known, see Eqs. (41) and (43):

$$\langle v\rangle=\rho\sigma_z^2,\quad \sigma_v=[\langle v^2\rangle-\langle v\rangle^2]^{1/2}=\sigma_z^2\sqrt{\rho^2+1}.\quad (51)$$

We already knew that the standard deviation is big. We now see again that $\sigma_v=\sigma_z^2$ at the separation angles at which the angular correlation function vanishes, $\rho=0$, and $\sigma_v=\sqrt{2}\sigma_z^2$ at zero separation angle, $\rho=1$. For other separation angles, σ_v lies between these two numbers.

Now, that we know the PDF, we can assign probabilities to the different ranges of the variable v. For instance, we can calculate the probability that the measured v will be found, say, outside of the $\lambda\sigma_v$ interval surrounding the mean value of v, where λ is an arbitrary fixed number. The probability of our interest is

$$P(|v-\langle v\rangle|>\lambda\sigma_v)=\int_{-\infty}^{\sigma_z^2(\rho-\lambda\sqrt{\rho^2+1})}f(v)dv$$

$$+\int_{\sigma_z^2(\rho+\lambda\sqrt{\rho^2+1})}^{\infty}f(v)dv.\qquad (52)$$

To get a qualitative estimate of the associated theoretical

uncertainties for the observable v, we will ask a slightly different question. What should the number λ be in order to have the 0.32 chance of finding v outside the $\lambda\sigma_v$ interval and, hence, the 0.68 chance to find it inside the interval?

To evaluate the size of the disaster, we will start from the case $\rho=0$. In this case, the PDF (49) is symmetric with respect to the origin $v=0$ (this is why $\langle v\rangle$ is zero in this case) and

$$P(|v|>\lambda\sigma_z^2)=\frac{2}{\pi}\int_\lambda^\infty K_0(x)dx. \qquad (53)$$

We want this number to be approximately equal to 0.32. Judging from the Fig. 9.7 in Ref. [17], one half of the area under the $K_0(x)$ function is accumulated when integrating from approximately $x=1/2$ and up to infinity. This means that λ should approximately be equal to 1/2 .

If $\rho\neq0$ the evaluation of P is more complicated. For $\rho\neq0$, the function (49) is not symmetric with respect to the origin $v=0$. It has larger values at positive v's if $\rho>0$ (this is why $\langle v\rangle>0$ in this case) and it has larger values at negative v's if $\rho<0$ (this is why $\langle v\rangle<0$ in this case). The graph of the function $e^x K_0(x)$ plotted in Fig. 9.8 in Ref. [17] is helpful. A qualitative analysis shows again that λ is approximately equal to 1/2. (More accurate estimates can, of course, be reached by numerical methods.)

At any rate, the $\frac{1}{2}\sigma_v$ interval gives approximately the same probability estimates as if the distribution (49) were normal.

V. CONCLUSIONS

A particular cosmological model plus perturbations give unambiguous predictions with regard to the expectation values of the measurable quantities. Differing models give different predictions. We want to distinguish them observationally and to learn about physics of the very early Universe. However, the quantum-mechanical origin of the cosmological perturbations is reflected in the theoretical statistical uncertainties surrounding the expectation values. One important measurable quantity is the angular correlation function of the microwave background anisotropies. Its mean value at the zero separation angle was denoted σ_z^2 in this paper. It was shown that the standard deviation for the correlation function is very big. The 68% confidence level corresponds, approximately, to $\frac{1}{2}\sigma_z^2$ at the separation angles where the correlation function vanishes, to $(\sqrt{2}/2)\sigma_z^2$ at the zero separation angle, and to intermediate numbers for other separation angles.

The angular correlation function has actually been measured. It is presented in Fig. 3 in the paper [18]. The authors surround the measured points by a narrow shaded region which they address as follows: ''The shaded region is the 68% confidence region … including cosmic variance and instrument noise.'' It is not quite clear what the authors of Ref. [18] (see also Ref. [19]) mean by ''cosmic variance,'' but if they mean the theoretical statistical uncertainties for the correlation function variable v, these uncertainties are significantly larger than what is plotted. According to the calculations presented above, the half width of the shaded

region should be approximately 600 $(\mu K)^2$ near the points where the correlation function vanishes and approximately 840 $(\mu K)^2$ near the point marking the zero separation angle.

The conclusion is a bit disappointing. Apparently, the microwave background anisotropies tell us something important about the very early Universe, but the **channel of information** is so noisy that it will be hard to understand the message.

APPENDIX

The set of random variables $\{a_{lm}^r,a_{lm}^i\}$ defined by Eqs. (32) and (33) lives its own independent life regardless of whether or not the variables are considered random coefficients in the expansion of some function over spherical harmonics. Being such, it allows introduction of new functions and calculation of their expectation values. One interesting variable is defined by the equation

$$a_l^2=\sum_{m=-l}^l a_{lm}a_{lm}^*=\sum_{m=-l}^l |a_{lm}|^2=\sum_{m=-l}^l [(a_{lm}^r)^2+(a_{lm}^i)^2]. \qquad (A1)$$

By using Eq. (38) one can calculate the expectation value of a_l^2:

$$\langle a_l^2\rangle=(2l+1)2\sigma_l^2. \qquad (A2)$$

The factor $2(2l+1)$ reflects the number of independent ''degrees of freedom'' associated with the index l in the representation (30). One can also introduce the variable a_l^4 and calculate its expectation value:

$$\langle a_l^4\rangle=(2l+1)(l+1)8\sigma_l^4=\langle a_l^2\rangle^2\frac{2(l+1)}{2l+1}. \qquad (A3)$$

The difference $\langle a_l^4\rangle-\langle a_l^2\rangle^2$ is, by definition, the variance of the variable a_l^2. From Eqs. (A3) and (A2) one finds

$$\langle a_l^4\rangle-\langle a_l^2\rangle^2=\frac{1}{2l+1}\langle a_l^2\rangle^2. \qquad (A4)$$

This formula, as it stands, expresses a well-known fact: the variance of the random variable χ^2 defined as the sum of squares of n independent random variables (degrees of freedom) with the same normal density, is $n/2$ times smaller than the square of the mean value of χ^2 [15]. There is nothing ''cosmological'' or ''inflationary'' in this fact. Knowing the PDF's for the set $\{a_{lm}^r,a_{lm}^i\}$ one can calculate the higher-order correlation functions for the variable a_l^2 and its distribution function [22–28]. In the recent literature, formula (A4) became known as the ''cosmic variance.'' Formula

(A4) and a possibility (or lack of) to extract complete information about a stochastic process from its single realization are, in general, different issues. For ergodic processes, the existence of a definitely true relationship (A4) prevents in no way the extraction of complete information about the process from a single realization [29].

It is important to realize that it is the mean value of the random variable a_l^2, and not a_l^2 itself, that enters the expected angular correlation function in front of the Legendre polynomials and which is often called the multipole moment. The a_l^2 is a random variable and its variance has meaning, the $\langle a_l^2 \rangle$ is a number and its variance has no meaning. Specifically, one can notice that the angular correlation function (41) can be written in the form

$$\left\langle \frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2) \right\rangle = \frac{1}{2\pi}\sum_{l=0}^{\infty} \langle a_l^2 \rangle P_l(\cos\delta). \quad (A5)$$

On this ground, there may be a temptation to write the random variable $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$ in the form

$$\frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2) = \frac{1}{2\pi}\sum_{l=0}^{\infty} a_l^2 P_l(\cos\delta) \quad (A6)$$

and to interpret Eq. (A4) as the variance for the multipole moments of the correlation function. One should resist to this temptation.

Let us show that the definition (A6) is incorrect despite the fact that it gives correct expectation value (A5). It follows from the definition (A6) that

$$\frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2)\frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2)$$

$$= \frac{1}{4\pi^2}\sum_{l=0}^{\infty} a_l^4 [P_l(\cos\delta)]^2$$

$$+ \frac{1}{4\pi^2}\sum_{l,l'=0,l\neq l'}^{\infty} a_l^2 P_l(\cos\delta) a_{l'}^2 P_{l'}(\cos\delta). \quad (A7)$$

Using (A3) and (A5) and remembering that a_l^2 and $a_{l'}^2$ are statistically independent for $l \neq l'$, one can find the expectation value of the quantity (A7):

$$\left\langle \frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2)\frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2) \right\rangle$$

$$= \frac{1}{4\pi^2}\sum_{l=0}^{\infty} \langle a_l^4 \rangle [P_l(\cos\delta)]^2$$

$$+ \frac{1}{4\pi^2}\sum_{l,l'=0,l\neq l'}^{\infty} \langle a_l^2 \rangle \langle a_{l'}^2 \rangle P_l(\cos\delta)P_{l'}(\cos\delta)$$

$$= \frac{1}{4\pi^2}\sum_{l=0}^{\infty} \langle a_l^4 \rangle [P_l(\cos\delta)]^2 + \frac{1}{4\pi^2}\left[\sum_{l=0}^{\infty} \langle a_l^2 \rangle P_l(\cos\delta)\right]^2$$

$$- \frac{1}{4\pi^2}\sum_{l=0}^{\infty} \langle a_l^2 \rangle^2 [P_l(\cos\delta)]^2$$

$$= \left\langle \frac{\delta T}{T}(\mathbf{e}_1)\frac{\delta T}{T}(\mathbf{e}_2) \right\rangle^2 + \frac{1}{4\pi^2}\sum_{l=0}^{\infty} \frac{1}{2l+1}\langle a_l^2 \rangle^2 [P_l(\cos\delta)]^2. \quad (A8)$$

It follows from (A8) that the variance of the variable $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$ would read [if (A6) were correct]:

$$\frac{1}{\pi^2}\sum_{l=0}^{\infty} (2l+1)\sigma_l^4 [P_l(\cos\delta)]^2. \quad (A9)$$

This expression should be compared with the correct variance following from Eq. (43):

$$\frac{1}{\pi^2}\left[\sum_{l=0}^{\infty} (2l+1)\sigma_l^2 P_l(\cos\delta)\right]^2 + \frac{1}{\pi^2}\left[\sum_{l=0}^{\infty} (2l+1)\sigma_l^2\right]^2. \quad (A10)$$

Formulas (A9) and (A10) disagree even for $\delta = 0$, and even in their first $l=0$ term. This shows that the *ad hoc* definition (A6) is incorrect. The correct definition of the random variable $(\delta T/T)(\mathbf{e}_1)(\delta T/T)(\mathbf{e}_2)$ is the one following from the definition (30) and which we have used in this paper.

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