

Search templates for gravitational waves from inspiraling binaries: Choice of template spacing

Benjamin J. Owen*

Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125

(Received 8 November 1995)

Gravitational waves from inspiraling, compact binaries will be searched for in the output of the LIGO-VIRGO interferometric network by the method of “matched filtering”—i.e., by correlating the noisy output of each interferometer with a set of theoretical wave form templates. These search templates will be a discrete subset of a continuous, multiparameter family, each of which approximates a possible signal. The search might be performed *hierarchically*, with a first pass through the data using a low threshold and a coarsely spaced, few-parameter template set, followed by a second pass on threshold-exceeding data segments, with a higher threshold and a more finely spaced template set that might have a larger number of parameters. Alternatively, the search might involve a single pass through the data using the larger threshold and finer template set. This paper extends and generalizes the Sathyaprakash-Dhurandhar (SD) formalism for choosing the discrete, finely spaced template set used in the final (or sole) pass through the data, based on the analysis of a single interferometer. The SD formalism is rephrased in geometric language by introducing a metric on the continuous template space from which the discrete template set is drawn. This template metric is used to compute the loss of signal-to-noise ratio and reduction of event rate which result from the coarseness of the template grid. Correspondingly, the template spacing and total number \mathcal{N} of templates are expressed, via the metric, as functions of the reduction in event rate. The theory is developed for a template family of arbitrary dimensionality (whereas the original SD formalism was restricted to a single nontrivial dimension). The theory is then applied to a simple post¹-Newtonian template family with two nontrivial dimensions. For this family, the number of templates \mathcal{N} in the finely spaced grid is related to the spacing-induced fractional loss \mathcal{L} of event rate and to the minimum mass M_{\min} of the least massive star in the binaries for which one searches by $\mathcal{N} \sim 2 \times 10^5 (0.1/\mathcal{L})(0.2M_{\odot}/M_{\min})^{2.7}$ for the first LIGO interferometers and by $\mathcal{N} \sim 8 \times 10^6 (0.1/\mathcal{L})(0.2M_{\odot}/M_{\min})^{2.7}$ for advanced LIGO interferometers. This is several orders of magnitude greater than one might have expected based on Sathyaprakash’s discovery of a near degeneracy in the parameter space, the discrepancy being due to that paper’s high choice of M_{\min} and less stringent choice of \mathcal{L} . The computational power \mathcal{P} required to process the steady stream of incoming data from a single interferometer through the closely spaced set of templates is given in floating-point operations per second by $\mathcal{P} \sim 3 \times 10^{10} (0.1/\mathcal{L})(0.2M_{\odot}/M_{\min})^{2.7}$ for the first LIGO interferometers and by $\mathcal{P} \sim 4 \times 10^{11} (0.1/\mathcal{L}) \times (0.2M_{\odot}/M_{\min})^{2.7}$ for advanced LIGO interferometers. This will be within the capabilities of LIGO-era computers, but a hierarchical search may still be desirable to reduce the required computing power. [S0556-2821(96)02112-1]

PACS number(s): 04.80.Nn, 04.30.-w, 07.05.Kf, 97.80.-d

I. INTRODUCTION

Compact binary star systems are likely to be an important source of gravitational waves for the broadband laser interferometric detectors now under construction [1], as they are the best understood of the various types of postulated gravity wave sources in the detectable frequency band and their waves should carry a large amount of information. Within our own Galaxy, there are three known neutron-star binaries whose orbits will decay completely under the influence of gravitational radiation reaction within less than one Hubble time, and it is almost certain that there are many more as yet undiscovered. Current estimates of the rate of neutron-star-neutron-star (NS-NS) binary coalescences [2,3] based on these (very few) known systems project an event rate of three per year within a distance of roughly 200 Mpc; and estimates based on the evolution of progenitor, main-sequence binaries [4] suggest a distance as small as roughly 70 Mpc for three

events per year. These distances correspond to a signal strength which is within the target sensitivities of the Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO interferometers [5,6]. However, to find the signals within the noisy LIGO-VIRGO data will require a careful filtering of the interferometer outputs. Because the predicted signal strengths lie so close to the level of the noise, it will be necessary to filter the interferometer data streams in order to detect the inspiral events against the background of spurious events generated by random noise.

The gravitational wave form generated by an inspiraling compact binary has been calculated using a combination of post-Newtonian and post-Minkowskian expansions [7,8] to post²-Newtonian order by the consortium of Blanchet, Damour, Iyer, Will, and Wiseman [9], and will be calculated to post³-Newtonian order long before the LIGO and VIRGO interferometers come on-line (ca. 2000). Because the functional form of the expected signal is so well known, it is an ideal candidate for matched filtering, a venerable and widely known technique which is laid out in detail elsewhere [10] and briefly summarized here.

*Electronic address: owen@tapir.caltech.edu

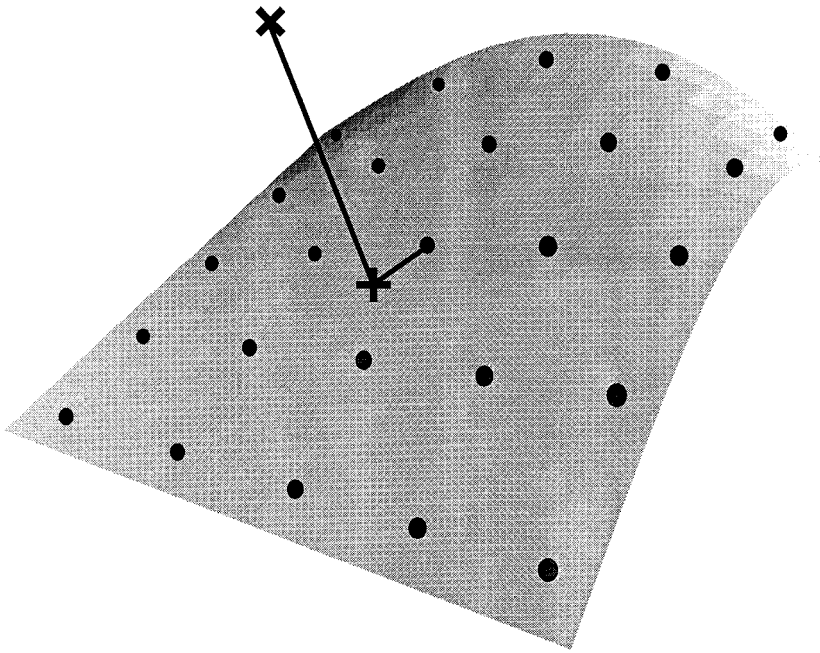


FIG. 1. A schematic depiction of the manifold formed by the continuous template family, here represented as a two-dimensional surface lying within a three-dimensional space. The discrete template family, shown by the dots, resides within this manifold. The \times indicates the location of an actual signal, which because it is an exact solution to the Einstein equations does not lie within the manifold. The $+$ marks the spot in the manifold which is closest to (has the highest inner product with) the signal. In general, this location falls in between the actual discrete templates.

The matched filtering strategy is to compute a cross correlation between the interferometer output and a template wave form, weighted inversely by the noise spectrum of the detector. The signal-to-noise ratio is defined as the value of the cross correlation of the template with a particular stretch of data divided by the rms value of the cross correlation of the template with pure detector noise. If the signal-to-noise ratio exceeds a certain threshold, which is set primarily to control the rate of false alarms due to fluctuations of the noise, a detection is registered. If the functional form of the template is identical to that of the signal, the mean signal-to-noise ratio in the presence of a signal is the highest possible for any linear data-processing technique, which is why matched filtering is also known as optimal filtering.

In practice, however, the template wave forms will differ somewhat from the signals. True gravitational-wave signals from inspiraling binaries will be exact solutions to the Einstein equations for two bodies of non-negligible mass, while the templates used to search for these signals will be, at best, finite-order approximations to the exact solutions. Also, true signals will be characterized by many parameters (e.g., the masses of the two objects, their spins, the eccentricity, and orientation of the orbit, . . .), some of which might be neglected in construction of the search templates. Thus, the true signals will lie somewhat outside the submanifold formed by the search templates in the full manifold of all possible detector outputs (see Fig. 1).

Apostolatos [11] has defined the “fitting factor” FF to quantitatively describe the closeness of the true signals to the template manifold in terms of the reduction of the signal-to-noise ratio due to cross correlating a signal lying outside the manifold with all the templates lying inside the manifold. If the fitting factor of a template family is unity, the signal lies in the template manifold. If the fitting factor is less than unity, the signal lies outside the manifold, and the fitting factor represents the cross correlation between the signal and the template nearest it in the template manifold.

Even if the signal were to lie within the template mani-

fold, it would not in general correspond to any of the actual templates used to search the data. The parameters describing the search templates (masses, spins, etc.) can vary continuously throughout a finite range of values. The set of templates characterized by the continuously varying parameters is of course infinite, so the interferometer output must be cross correlated with a finite subset of the templates whose parameter values vary in discrete steps from one template to the next. This subset (the “discrete template family”) has measure zero on the manifold of the full set of possible templates (the “continuous template family”), so the template which most closely matches a signal will generally lie in between members of the discrete template family (again, see Fig. 1). The mismatch between the signal and the nearest of the discrete templates will cause some reduction in the signal-to-noise ratio and therefore in the observed event rate, as some signals which would lie above the threshold if cross correlated with a perfectly matched filter are driven below the threshold by the mismatch. Thus, the spacing between members of the discrete template family must be chosen so as to render acceptable the loss in event rate, without requiring a prohibitive amount of computing power to numerically perform the cross correlations of the data stream with all of the discrete templates.

The high computational demands of a laser interferometric detector may in fact make it desirable to perform a *hierarchical search*. In a hierarchical search, each stretch of data is first filtered by a set of templates which rather sparsely populates the manifold, and the stretches which fail to exceed a relatively low signal-to-noise threshold are discarded. The surviving stretches of data are filtered by a larger set of templates which more densely populates the manifold, and are subjected to a higher threshold. The sparseness of the first-pass template set ensures that most of the data need only be filtered by a small number of templates, while the high threshold of the final pass reduces the false alarm rate to an acceptable level.

Theoretical foundations for choosing the discrete set of

templates from the continuous family were laid by Sathyaprakash and Dhurandhar for the case of white noise in Ref. [12], and for (colored) power-recycling interferometer noise in Ref. [13]. Both papers used a simplified (so-called “Newtonian”) version of the wave form which can be characterized by a single parameter, the binary’s “chirp mass” \mathcal{M} . Recently, Sathyaprakash [14] began consideration of an improved, “post-Newtonian” set of templates characterized by two mass parameters. He found that, by a judicious choice of the two parameters, the spacing between templates can be made constant in both dimensions of the intrinsic parameter space. Sathyaprakash’s parameters also make it obvious (by producing a very large spacing in one of the dimensions) that a two-parameter set of templates can be constructed which, if it does not populate the manifold too densely, need not be much more numerous than the one-parameter set of templates used in Refs. [12,13].

In this paper I shall recast the Sathyaprakash-Dhurandhar (SD) formalism in geometric language which, I believe, simplifies and clarifies the key ideas. I shall also generalize the SD formalism to an arbitrary spectrum of detector noise and to a set of template shapes characterized by more than one parameter. This is necessary because, as Apostolatos [11] has shown, no one-parameter set of templates can be used to filter a post-Newtonian signal without causing an unacceptably large loss of signal-to-noise ratio.

In one respect, my analysis will be more specialized than that of the SD formalism. My geometric analysis requires that the templates of the discrete set be spaced very finely in order that certain analytical approximations may be made, while the numerical methods of Sathyaprakash and Dhurandhar are valid even for a large spacing between templates (as would be the case in the early stages of a hierarchical search). The small spacing approximation is justified on the grounds that at some point, even in a hierarchical search, the data must be filtered by many closely spaced templates in order to detect a reasonable fraction (of order unity) of the binary inspirals occurring in the Universe within range of the LIGO-VIRGO network.

The rest of this paper is organized as follows. In Sec. II, I develop my generalized, geometric variant of the SD formalism. I then apply this formalism to the general problem of detection of gravitational waves from inspiraling binaries, and develop general formulas for choosing a discrete template family from a given continuous template family. In Sec. III, I detail an example of the use of my formalism, choosing discrete templates from a continuous template family which describes nonspinning, circularized binaries to post¹-Newtonian order in the evolution of the wave form’s phase. I also estimate the computing power required for a single-pass (nonhierarchical) search using this discrete template family, and compare to the previous work of Sathyaprakash [14]. Finally, in Sec. IV, I summarize my results and suggest future directions for research on the choice of discrete search templates.

II. THEORY OF MISMATCHED FILTERING

In this section, a geometric, multiparameter variant of the SD formalism is developed. Unless otherwise stated, the following conventions and definitions are assumed.

Following Cutler and Flanagan [15], we define the inner product between two functions of time $a(t)$ and $b(t)$ (which may be templates or interferometer outputs) as

$$\begin{aligned} \langle a|b \rangle &\equiv 2 \int_0^\infty df \frac{\tilde{a}^*(f)\tilde{b}(f) + \tilde{a}(f)\tilde{b}^*(f)}{S_h(f)} \\ &= 4 \operatorname{Re} \left[\int_0^\infty df \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_h(f)} \right]. \end{aligned} \quad (2.1)$$

Here, $\tilde{a}(f)$ is the Fourier transform of $a(t)$,

$$\tilde{a}(f) \equiv \int_{-\infty}^\infty dt e^{i2\pi ft} a(t), \quad (2.2)$$

and $S_h(f)$ is the detector’s noise spectrum, defined below.

The interferometer output $o(t)$ consists of noise $n(t)$ plus a signal $\mathcal{A}s(t)$, where \mathcal{A} is a dimensionless, time-independent amplitude and $s(t)$ is normalized such that $\langle s|s \rangle = 1$. Thus, \mathcal{A} describes the strength of a signal and $s(t)$ describes its shape.

Wave form templates are denoted by $u(t; \boldsymbol{\mu}, \boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is the vector of “intrinsic” or “dynamical” parameters characterizing the template shape and $\boldsymbol{\mu}$ is the vector of “extrinsic” or “kinematical” parameters describing the offsets of the wave form’s end point. Examples of intrinsic parameters λ^i are the masses and spins of the two objects in a compact binary; examples of extrinsic parameters μ^i are the time of a compact binary’s final coalescence t_0 and the phase of the wave form at coalescence Φ_0 .

Templates are assumed to be normalized such that $\langle u(\boldsymbol{\mu}, \boldsymbol{\lambda}) | u(\boldsymbol{\mu}, \boldsymbol{\lambda}) \rangle = 1$ for all $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$.

Expectation values of various quantities over an infinite ensemble of realizations of the noise are denoted by $E[\]$.

The interferometer’s strain spectral noise density $S_h(f)$ is the one-sided spectral density, defined by

$$E[\tilde{n}(f_1)\tilde{n}^*(f_2)] = \frac{1}{2} \delta(f_1 - f_2) S_h(f_1) \quad (2.3)$$

for positive frequencies and undefined for negative frequencies. The noise is assumed to have a Gaussian probability distribution.

Newton’s gravitational constant G and the speed of light c are set equal to one.

A. Formalism

In developing our formalism, we begin by defining the signal-to-noise ratio. For any single template $u(t)$ of unit norm, the cross correlation with pure noise $\langle n|u \rangle$ is a random variable with mean zero and variance unity [cf. Sec. II B of Ref. [15], wherein it is shown that $E[\langle n|a \rangle \langle n|b \rangle] = \langle a|b \rangle$]. The signal-to-noise ratio of a given stretch of data $o(t)$, after filtering by $u(t)$, is defined to be

$$\rho \equiv \frac{\langle o|u \rangle}{\operatorname{rms}\langle n|u \rangle} = \langle o|u \rangle. \quad (2.4)$$

This ratio is the statistic which is compared to a predetermined threshold to decide if a signal is present.

If the template u is the same as the signal s , it optimally filters the signal, and the corresponding (mean) optimal signal-to-noise ratio is

$$E[\rho] = E[\langle n + Au | u \rangle] = \mathcal{A}. \quad (2.5)$$

If the template u used to filter the data is not exactly the same as the signal s , the mean signal-to-noise ratio is decreased somewhat from its optimal value:

$$E[\rho] = E[\langle n + As | u \rangle] = \mathcal{A} \langle s | u \rangle. \quad (2.6)$$

The inner product $\langle s | u \rangle$, which is bounded between zero and one, is the fraction of the optimal $E[\rho]$ retained in the mismatched filtering case, and as such is a logical measure of the effectiveness of the template u in searching for the signal shape s .

Now, suppose that we search for the signal with a family of templates specified by an extrinsic parameter vector $\boldsymbol{\mu}$ and an intrinsic parameter vector $\boldsymbol{\lambda}$. Let us denote the values of the parameters of the actual templates by $(\boldsymbol{\mu}_{(k)}, \boldsymbol{\lambda}_{(k)})$. For example, $\mu_{(k)}^1$ might be the value of the time t_0 of coalescence for the k th template in the family, and $\mu_{(k)}^2$ might be the phase of the k th template wave form at coalescence.

The search entails computing, via fast Fourier transforms (FFT's), all the inner products $\langle o | u(\boldsymbol{\mu}_{(k)}, \boldsymbol{\lambda}_{(k)}) \rangle$ for $k = 1, 2, \dots$. In these numerical computations, the key distinction between the extrinsic parameters $\boldsymbol{\mu}$ and the intrinsic parameters $\boldsymbol{\lambda}$ is this: One explores the whole range of values of $\boldsymbol{\mu}$ very quickly, automatically, and efficiently for a fixed value of $\boldsymbol{\lambda}$; but one must do these explorations separately for each of the $\boldsymbol{\lambda}_{(k)}$. In this sense, dealing with the extrinsic parameters is far easier and more automatic than dealing with the intrinsic ones [16].

As an example (for further detail see Sec. 16.2.2 of Schutz [17]), for a given stretch of data one explores *all* values of the time of coalescence ($t_0 \equiv \mu^1$) of a compact binary simultaneously (for fixed values of the other template parameters) via a single FFT. If we write the Fourier transform (for notational simplicity) as a continuous integral rather than a discrete sum, we get

$$\begin{aligned} & \langle o | u(\boldsymbol{\mu}_{(k)}, \boldsymbol{\lambda}_{(k)}) \rangle \\ &= \int_0^\infty df \frac{e^{i2\pi f t_0}}{S_h(f)} \tilde{o}^*(f) \tilde{u}(f; \text{other } \boldsymbol{\mu}_{(k)}, \boldsymbol{\lambda}_{(k)}). \end{aligned} \quad (2.7)$$

The discrete fast Fourier transform (FFT) yields the discrete analogue of the function of t_0 as shown above, an array of numbers containing the values of the Fourier transform for all the discrete values of t_0 .

Because, for fixed $\boldsymbol{\lambda}_{(k)}$, the extrinsic parameters $\boldsymbol{\mu}$ are dealt with so simply and quickly in the search, throughout this paper we shall focus primarily on a template family's intrinsic parameters $\boldsymbol{\lambda}$, which govern the shape of the template. Correspondingly, we shall adopt the following quantity as our measure of the effectiveness with which a particular template shape—i.e., a particular vector $\boldsymbol{\lambda}_{(k)}$ of the intrinsic parameters—matches the incoming signal:

$$\max_{\boldsymbol{\mu}} \langle s | u(\boldsymbol{\mu}, \boldsymbol{\lambda}_{(k)}) \rangle. \quad (2.8)$$

Here, the maximization is over all continuously varying values of the extrinsic parameters. Then, the logical measure of the effectiveness of the entire discrete family of templates in searching for the signal shape is

$$\max_k [\max_{\boldsymbol{\mu}} \langle s | u(\boldsymbol{\mu}, \boldsymbol{\lambda}_{(k)}) \rangle], \quad (2.9)$$

which is simply (2.8) maximized over all the discrete template shapes.

In order to focus on the issue of discretization of the template parameters rather than on the inadequacy of the continuous template family, let us assume that the signal shape s is identical to some template. The discussion of the preceding paragraphs suggests that in discussing the discretization of the template parameters we will want to make use of the *match* between two templates $\tilde{u}(f; \boldsymbol{\mu}, \boldsymbol{\lambda})$ and $\tilde{u}(f; \boldsymbol{\mu} + \Delta\boldsymbol{\mu}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda})$ which we will define as

$$M(\boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) \equiv \max_{\boldsymbol{\mu}, \Delta\boldsymbol{\mu}} \langle u(\boldsymbol{\mu}, \boldsymbol{\lambda}) | u(\boldsymbol{\mu} + \Delta\boldsymbol{\mu}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) \rangle. \quad (2.10)$$

This quantity, which is known in the theory of hypothesis testing as the *ambiguity function*, is the fraction of the optimal signal-to-noise ratio obtained by using a template with intrinsic parameters $\boldsymbol{\lambda}$ to filter a signal identical in shape to a template with intrinsic parameters $\boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}$.

Using the match (2.10) it is possible to quantify our intuitive notion of how “close” two template shapes are to each other. Since the match clearly has a maximum value of unity at $\Delta\boldsymbol{\lambda} = 0$, we can expand in a power series about $\Delta\boldsymbol{\lambda} = 0$ to obtain

$$M(\boldsymbol{\lambda}, \Delta\boldsymbol{\lambda}) \approx 1 + \frac{1}{2} \left(\frac{\partial^2 M}{\partial \Delta\lambda^i \partial \Delta\lambda^j} \right)_{\Delta\lambda^k=0} \Delta\lambda^i \Delta\lambda^j. \quad (2.11)$$

This suggests the definition of a metric

$$g_{ij}(\boldsymbol{\lambda}) = -\frac{1}{2} \left(\frac{\partial^2 M}{\partial \Delta\lambda^i \partial \Delta\lambda^j} \right)_{\Delta\lambda^k=0} \quad (2.12)$$

so that the *mismatch* $1 - M$ between two nearby templates is equal to the square of the proper distance between them:

$$1 - M = g_{ij} \Delta\lambda^i \Delta\lambda^j. \quad (2.13)$$

Having defined a metric on the intrinsic parameter space, we can now use it to calculate the spacing of the discrete template family required to retain a given fraction of the ideal event rate. Schematically, we can think of the templates as forming a lattice in the N -dimensional intrinsic parameter space whose unit cell is an N -dimensional hypercube with sides of proper length dl . The worst possible case (lowest $E[\rho]$) occurs if the point $\tilde{\boldsymbol{\lambda}}$ describing the signal is exactly in the middle of one of the hypercubes. If the templates are closely spaced, i.e., $dl \ll 1$, such a signal has a squared proper distance

$$g_{ij} \Delta\lambda^i \Delta\lambda^j = N(dl/2)^2 \quad (2.14)$$

from the templates at the corners of the hypercube.

We define the *minimal match* MM to be the match between the signal and the nearest templates in this worst possible case, i.e., the fraction of the optimal signal-to-noise ratio retained by a discrete template family when the signal falls exactly “in between” the nearest templates. This minimal match is the same quantity that Dhurandhar and Sathyaprakash in Ref. [13] denote as κ^{-1} ; but since it is the central quantity governing template spacing it deserves some recognition in the form of its own name. Our choice of name closely parallels the term “fitting factor” FF , which Apostolatos introduced in Ref. [11] to measure the similarity between actual signals and a continuous template family.

The minimal match, which is chosen by the experimenter based upon what he or she considers to be an acceptable loss of ideal event rate, will determine our choice of spacing of the discrete template parameters and therefore the number of discrete templates in the family. More specifically, the experimenter will choose some desired value of the minimal match MM ; and then will achieve this MM by selecting the templates to reside at the corners of hypercubes with edge dl given by

$$MM = 1 - N(dl/2)^2. \quad (2.15)$$

The number of templates in the resulting discrete template family will be the proper volume of parameter space divided by the proper volume per template dl^N : i.e.,

$$\mathcal{N} = \frac{\int d^N \lambda \sqrt{\det \|g_{ij}\|}}{(2\sqrt{(1-MM)/N})^N}. \quad (2.16)$$

B. Inspiring binaries detected by LIGO

The formalism above applies to the detection of any set of signals which have a functional form that depends on a set of parameters which varies continuously over some range. We now develop a more explicit formula for the metric, given an analytical approximation to the LIGO noise curve and a particular class of inspiraling binary signals.

We approximate the “initial” and “advanced” benchmark LIGO noise curves by the following analytical fit to Fig. 7 of Ref. [5]:

$$S_h(f) = \begin{cases} \frac{1}{5} S_0 \{ (f/f_0)^{-4} + 2[1 + (f/f_0)^2] \}, & f > f_s, \\ \infty, & f < f_s, \end{cases} \quad (2.17)$$

where f_0 is the “knee frequency” or frequency at which the interferometer is most sensitive (which is determined by the reflectivities of the mirrors and is set by the experimenters to the frequency where photon shot noise begins to dominate the spectrum) and S_0 is a constant whose value is not important for our purposes. This spectrum describes photon shot noise in the “standard recycling” configuration of the interferometer (second term) superposed on thermal noise in the suspension of the test masses (first term), and it approximates seismic noise by setting S_h infinite at frequencies below the “seismic-cutoff frequency” f_s .

Throughout the rest of this paper, the “first LIGO noise curve” will refer to (2.17) with $f_s = 40$ Hz and $f_0 = 200$

Hz, and the “advanced LIGO noise curve” will refer to (2.17) with $f_s = 10$ Hz and $f_0 = 70$ Hz. These numbers are chosen to closely fit Fig. 7 of Ref. [5] for the first LIGO interferometers and for the advanced LIGO benchmark. In this paper, when various quantities (such as the number of discrete templates) are given including a scaling with f_0 , this indicates how the quantity changes while f_0 is varied but f_s/f_0 is held fixed.

At this point it is useful to define the moments of the noise curve (2.17), following Poisson and Will [18], as

$$I(q) \equiv S_h(f_0) \int_{f_s/f_0}^{f_c/f_0} dx \frac{x^{-q/3}}{S_h(xf_0)} = \int_{f_s/f_0}^{f_c/f_0} dx \frac{5x^{-q/3}}{x^{-4} + 2(1+x^2)}, \quad (2.18a)$$

$$J(q) \equiv I(q)/I(7). \quad (2.18b)$$

The upper limit of integration f_c denotes the coalescence frequency or high-frequency cutoff of whatever template we are dealing with, which very roughly corresponds to the last stable circular orbit of a test particle in a nonspinning black hole’s Schwarzschild geometry.

For both first and advanced LIGO noise curves, the majority of inspiraling binary search templates will occupy regions of parameter space for which f_c is many times f_0 . Because we will always be dealing with $I(q)$ for $q > 0$, and because the noise term in the denominator of the integrand in Eq. (2.18a) increases as f^2 for $f \gg f_0$, we can simplify later calculations by approximating $f_c = \infty$ in the definition of the moments.

To illustrate the metric formalism, we shall use templates based on a somewhat simplified version of the post-Newtonian expansion. Since the inner product (2.1) has negligible contributions from frequencies at which the integrand oscillates rapidly, it is far more important to get the phase of $u \sim(f)$ right than it is to get the amplitude dependence. Therefore, we adopt templates based on the “restricted” post-Newtonian approximation in which one discards all multipolar components except the quadrupole, but keeps fairly accurate track of the quadrupole component’s phase (for more details see Secs. II C and III A of Ref. [15]). Applying the stationary phase approximation to that quadrupolar wave form, we obtain

$$\bar{u}(f; \lambda, \mu) = f^{-7/6} \exp \left\{ i \left[-\frac{\pi}{4} - \Phi_0 + 2\pi f t_0 + \Psi(f; \lambda) \right] \right\}, \quad (2.19)$$

up to a multiplicative constant which is set by the condition $\langle u|u \rangle = 1$ [19].

The function Ψ , describing the phase evolution in (2.19), is currently known to post²-Newtonian order for the case of two nonspinning point masses in a circular orbit about each other as

$$\begin{aligned} \Psi(f; M, \eta) = & \frac{3}{128} \eta^{-1} (\pi M f)^{-5/3} \left[1 + \frac{20}{9} \left(\frac{743}{336} + \frac{11}{4} \eta \right) \right. \\ & \times (\pi M f)^{2/3} - 16 \pi (\pi M f) + 10 \left(\frac{3058673}{1016064} \right. \\ & \left. \left. + \frac{5429}{1008} \eta + \frac{617}{144} \eta^2 \right) (\pi M f)^{4/3} \right] \quad (2.20) \end{aligned}$$

[cf. Eq. (3.6) of Ref. [18]]. Here, the mass parameters have been chosen to be M , the total mass of the system, and η , the ratio of the reduced mass to the total mass.

The actual amplitude \mathcal{A} of a wave form is proportional to $1/R$, where R is the distance to the source; and this lets us find the relation between minimal match and event rate which we will need in order to wisely choose the minimal match. Assuming that compact binaries are uniformly distributed throughout space on large distance scales, this means that the rate of events with a given set of intrinsic parameters and with an amplitude greater than \mathcal{A} is proportional to $1/\mathcal{A}^3$. Setting a signal-to-noise threshold ρ_0 is equivalent to setting a maximum distance $R_0 \propto 1/\mathcal{A}$ to which sources with a given set of intrinsic parameters can be detected. Thus, if we could search for signals with the entire continuous template family, we would expect the observed event rate to scale as $1/\rho_0^3$. This ideal event rate is an upper limit on what we can expect with a real, discrete template family.

We can obtain a lower bound on the observed event rate by considering what happens if all signals conspire to have parameters lying exactly in between the nearest search templates. In this case, all events will have reduced signal-to-noise ratios of MM times the optimal signal-to-noise ratio \mathcal{A} . This is naively equivalent to optimally filtering with a threshold of ρ_0/MM , so a pessimistic guess is

$$\text{event rate} \propto \left(\frac{MM}{\rho_0} \right)^3 \quad (2.21)$$

for a fixed rate of false alarms. Note that the fraction of false dismissals *due to template spacing* is $1 - MM^3$, independently of the threshold ρ_0 (assuming $\rho_0 \gg 1$). This is because the scaling of absolute event rate with \mathcal{A} is independent of \mathcal{A} , as discussed in the previous paragraph.

In real life, ρ_0 is affected by the total number of discrete templates and by the minimal match of the discrete template family. This can be seen by the fact that the signal-to-noise ratio,

$$\rho \equiv \max_k [\max_\mu \langle o | u(\boldsymbol{\mu}, \boldsymbol{\lambda}_{(k)}) \rangle], \quad (2.22)$$

is the maximum of a number of random variables. The covariance matrix of these variables will be determined by the minimal match, and will itself determine the probability distribution of ρ (in the absence of a signal) which is used to set the threshold ρ_0 in order to keep the false alarm rate below a certain level. However, since these effects are fairly small at high signal-to-noise ratios (such as those to be used by LIGO) and the issue of choosing thresholds is a problem worthy of its own paper [20], we will use (2.21) for the rest of this paper.

For this two-parameter template family, the formula for the match (2.10) can be simplified somewhat by explicitly performing the maximization over the extrinsic parameters $\boldsymbol{\mu}$ and $\Delta\boldsymbol{\mu}$. Since the integrand in the inner product (2.1) depends on $\boldsymbol{\mu}$ and $\Delta\boldsymbol{\mu}$ as $\exp\{i[2\pi f\Delta t_0 - \Delta\Phi_0]\}$, there is no dependence on $\boldsymbol{\mu}$ but only on $\Delta\boldsymbol{\mu}$. Maximizing over $\Delta\Phi_0$ is easy: instead of taking the real part of the integral in the inner product (2.1), we take the absolute value.

To maximize over Δt_0 we go back a step. Let us define $\lambda^0 \equiv t_0$, and consider the $(N+1)$ -dimensional space formed by λ^0 and λ^j . We expand the inner product between adjacent templates to quadratic order in $\Delta\lambda^\alpha$ to get a preliminary metric $\gamma_{\alpha\beta}$, where the Greek indices range from 0 to N (Latin indices range from 1 to N):

$$\gamma_{\alpha\beta}(\boldsymbol{\lambda}) = -\frac{1}{2} \left[\frac{\partial^2}{\partial \Delta\lambda^\alpha \partial \Delta\lambda^\beta} \left\{ \frac{\left| \int_0^\infty df \frac{f^{-7/3}}{S_h(f)} \exp\{i[2\pi f\Delta t_0 + \Delta\Psi(f; \lambda^j, \Delta\lambda^j)]\} \right|}{\int_0^\infty df \frac{f^{-7/3}}{S_h(f)}} \right\} \right] \Delta\lambda^\alpha = 0. \quad (2.23)$$

Here $\Delta\Psi \equiv \Psi(f; \lambda^j + \Delta\lambda^j) - \Psi(f; \lambda^j)$ [21].

We define the moment functional \mathcal{J} such that, for a function a ,

$$\mathcal{J}[a] = \frac{1}{I(7)} \int_{f_s/f_0}^{f_c/f_0} dx \frac{x^{-7/3}}{S_h(xf_0)} a(x), \quad (2.24)$$

and thus

$$\mathcal{J} \left[\sum_n a_n x^n \right] = \sum_n a_n J(7-3n). \quad (2.25)$$

We also define the quantities ψ_α such that

$$\psi_0 \equiv 2\pi f, \quad \psi_j \equiv \frac{\partial \Delta\hat{\Psi}}{\partial \Delta\lambda^j}, \quad (2.26)$$

where the derivative is evaluated at $\Delta\lambda^j = 0$ and $\hat{\Psi}$ is the part of Ψ in Eq. (2.19) that is frequency dependent (any non-frequency-dependent, additive parts of Ψ are removed when

we take the absolute value in the maximized inner product). Evaluation of the derivative in Eq. (2.23) then shows that, in the limit $f_c/f_0 \rightarrow \infty$,

$$\gamma_{\alpha\beta} = \frac{1}{2} (\mathcal{J}[\psi_\alpha \psi_\beta] - \mathcal{J}[\psi_\alpha] \mathcal{J}[\psi_\beta]). \quad (2.27)$$

Finally, we minimize $\gamma_{\alpha\beta} \Delta \lambda^\alpha \Delta \lambda^\beta$ with respect to Δt_0 (i.e., we project $\gamma_{\alpha\beta}$ onto the subspace orthogonal to t_0) and thereby obtain the following expression for the metric of our continuous template family:

$$g_{ij} = \gamma_{ij} - \frac{\gamma_{0i} \gamma_{0j}}{\gamma_{00}}. \quad (2.28)$$

By taking the square root of the determinant of this metric and plugging it into Eq. (2.16), we can compute the number of templates \mathcal{N} that we need in our discrete family as a function of our desired minimal match MM , or equivalently of the loss of ideal event rate.

III. EXAMPLE: CIRCULARIZED NONSPINNING BINARIES TO POST¹-NEWTONIAN ORDER

Although the phase of the inspiraling binary signal has recently been calculated to post²-Newtonian order [9], it is useful to calculate the number of templates that would be required in a Universe where the wave forms evolve only to post¹-Newtonian order and all binaries are composed of non-spinning objects in circular orbits. There are several reasons for this exercise.

(1) Spin effects are expected to be significant only for a small fraction of sources. Blanchet *et al.* [9] estimate very small spin-orbit and spin-spin phase modulation terms for typical neutron-star binaries based on models of evolution for currently known binary pulsars. Apostolatos [11] has shown that templates with no spin-related terms perform quite well when searching for “signals” with spin-induced phase modulation but no amplitude modulation, corresponding approximately to a favorable orientation of the spin axes. He has also shown that amplitude modulation due to spin-orbit precession can indeed be a significant effect, but only for large spins and certain unfavorable orientations of the axes. Therefore, consideration of a spinless family of templates is quite sufficient for a rough estimate of the computing power required to perform a search that will overlook a small fraction of binary inspiral events due to mismatched filtering.

(2) We assume circular orbits because gravitational radiation reaction circularizes most eccentric orbits on a time scale much less than the lifetime of a compact binary [22].

(3) The phase of the templates is truncated at post¹-Newtonian order for simplicity. Although Apostolatos has demonstrated in Ref. [11] that post¹-Newtonian templates will not have a large enough fitting factor to be useful, consideration of such a set is a first step toward obtaining an adequate set of templates—and it is a particularly important step since the metric coefficients will turn out to be constant over the template manifold.

A. Calculation of the two-dimensional metric

Having chosen as the continuous template family the set of post¹-Newtonian, circular, spinless binary wave forms, we must now choose the discrete templates from within this continuous family. The first step is to calculate the coefficients of the metric on the two-dimensional dynamical parameter space.

It is convenient to change the mass parametrization from the variables (M, η) to the Sathyaprakash variables [14]

$$\tau_1 = \frac{5}{256} \eta^{-1} M^{-5/3} (\pi f_0)^{-8/3}, \quad (3.1)$$

$$\tau_2 = \frac{5}{192} (\eta M)^{-1} \left(\frac{743}{336} + \frac{11}{4} \eta \right) (\pi f_0)^{-2}. \quad (3.2)$$

Note that τ_1 and τ_2 are simply the Newtonian and post¹-Newtonian contributions to the time it takes for the carrier gravitational wave frequency to evolve from f_0 to infinity. The advantage of these variables is that the metric coefficients in (τ_1, τ_2) coordinates are constant (in the limit $f_c \gg f_0$) for all templates. This is because the phase of the wave form $\tilde{u}(f)$ is linear in the Sathyaprakash variables, and so the integral in the definition of the match (2.10) depends only on the displacement $(\Delta \tau_1, \Delta \tau_2)$ between the templates, and not on the location (τ_1, τ_2) of the templates in the dynamical parameter space.

The dynamical parameter-dependent part of the templates’ phase is given by [Eq. (2.20) truncated to first post-Newtonian order and reexpressed in terms of (τ_1, τ_2) using Eqs. (3.1) and (3.2)]

$$\Psi(f; \tau_1, \tau_2) = \frac{6}{5} \pi f_0 (f/f_0)^{-5/3} \tau_1 + 2 \pi f_0 (f/f_0)^{-1} \tau_2, \quad (3.3)$$

and it is easy to read off ψ_1 and ψ_2 [Eq. (2.26)] as the coefficients of τ_1 and τ_2 [23]. By inserting these ψ_j into Eq. (2.25), the relevant moment functionals can be expressed in terms of the moments of the noise:

$$\begin{aligned} \mathcal{J}[\psi_0] &= 2 \pi f_0 J(4), & \mathcal{J}[\psi_1] &= 2 \pi f_0^{\frac{3}{5}} J(12), \\ \mathcal{J}[\psi_2] &= 2 \pi f_0 J(10), & \mathcal{J}[\psi_0^2] &= (2 \pi f_0)^2 J(1), \\ \mathcal{J}[\psi_0 \psi_1] &= (2 \pi f_0)^2 J(9), & \mathcal{J}[\psi_0 \psi_2] &= (2 \pi f_0)^2 J(7), \\ \mathcal{J}[\psi_1^2] &= (2 \pi f_0)^2 J(17), & \mathcal{J}[\psi_1 \psi_2] &= (2 \pi f_0)^2 J(15), \\ \mathcal{J}[\psi_2^2] &= (2 \pi f_0)^2 J(13). \end{aligned} \quad (3.4)$$

We can compute the needed moments of the noise by numerically evaluating the integrals (2.18b). By setting the upper limit of integration to infinity, i.e., by approximating f_c/f_0 as infinite for all templates under consideration, we find that the moments have the constant values given in Table I; and therefore the moment functionals (3.4) have the constant values given in Table II. Inserting these values into Eqs. (2.27) and (2.28) yields, for the coordinates $(\lambda^0 = t_0, \lambda^1 = \tau_1, \lambda^2 = \tau_2)$, the three-metric and two-metric

TABLE I. Numerical values of the moments of the noise in the limit $f_c/f_0 \rightarrow \infty$, for the noise curves of the first LIGO interferometers ($f_s/f_0 = 1/5$) and advanced LIGO interferometers ($f_s/f_0 = 1/7$).

Noise moment	First LIGO value	Advanced LIGO value
J(1)	1.27	1.26
J(4)	0.927	0.919
J(7)	1 (exact)	1 (exact)
J(9)	1.24	1.26
J(10)	1.44	1.49
J(12)	2.13	2.31
J(13)	2.69	3.03
J(15)	4.67	5.80
J(17)	8.88	12.7

$$\gamma_{\alpha\beta} = (2\pi f_0)^2 \begin{pmatrix} +0.208 & -0.220 & -0.168 \\ . & +0.784 & +0.481 \\ . & . & +0.309 \end{pmatrix} \text{ and } \quad (3.5)$$

$$g_{ij} = (2\pi f_0)^2 \begin{pmatrix} 0.552 & 0.304 \\ . & 0.173 \end{pmatrix} \quad (3.6)$$

for the first LIGO noise curve, and

$$\gamma_{\alpha\beta} = (2\pi f_0)^2 \begin{pmatrix} +0.209 & -0.257 & -0.183 \\ . & +1.320 & +0.712 \\ . & . & +0.407 \end{pmatrix} \text{ and } \quad (3.7)$$

$$g_{ij} = (2\pi f_0)^2 \begin{pmatrix} 1.01 & 0.486 \\ . & 0.246 \end{pmatrix} \quad (3.8)$$

for the advanced LIGO noise curve (where the dots denote terms obtained by symmetry).

We shall also estimate the errors in the metric coefficients due to the approximation $f_c/f_0 \rightarrow \infty$: The moment integrals defined in Eq. (2.18b) can be rewritten as

$$\int_{f_s/f_0}^{\infty} dx \frac{5x^{-q/3}}{x^{-4} + 2(1+x^2)} - \int_{f_c/f_0}^{\infty} dx \frac{5x^{-q/3}}{x^{-4} + 2(1+x^2)},$$

TABLE II. Numerical values of the moment functionals, under the same assumptions as in Table I.

Moment functional	First LIGO value	Advanced LIGO value
$\mathcal{A}[\psi_0]$	$(2\pi f_0)0.927$	$(2\pi f_0)0.919$
$\mathcal{A}[\psi_1]$	$(2\pi f_0)1.28$	$(2\pi f_0)1.38$
$\mathcal{A}[\psi_2]$	$(2\pi f_0)1.44$	$(2\pi f_0)1.49$
$\mathcal{A}[\psi_0^2]$	$(2\pi f_0)^2 1.27$	$(2\pi f_0)^2 1.26$
$\mathcal{A}[\psi_0\psi_1]$	$(2\pi f_0)^2 0.743$	$(2\pi f_0)^2 0.756$
$\mathcal{A}[\psi_0\psi_2]$	$(2\pi f_0)^2$ (exact)	$(2\pi f_0)^2$ (exact)
$\mathcal{A}[\psi_1^2]$	$(2\pi f_0)^2 3.20$	$(2\pi f_0)^2 4.56$
$\mathcal{A}[\psi_1\psi_2]$	$(2\pi f_0)^2 2.80$	$(2\pi f_0)^2 3.48$
$\mathcal{A}[\psi_2^2]$	$(2\pi f_0)^2 2.69$	$(2\pi f_0)^2 3.03$

where the first integral is the expression used in the above metric coefficients and the second is the correction due to finite f_c/f_0 . The second integral can be expanded to lowest order in f_0/f_c as

$$\frac{5}{2(1+q/3)} (f_0/f_c)^{1+q/3},$$

and from this the errors in the moments (and therefore in the metric coefficients) due to approximating f_c as infinite are estimated to be less than or of order ten percent for the first LIGO interferometers and one percent for the advanced LIGO interferometers over most of the relevant volume of parameter space. Since the two-parameter, post¹-Newtonian continuous template family is known to be inadequate for the task of searching for real binaries, these errors are small enough to justify our use of the $f_c \rightarrow \infty$ approximation in this exploratory analysis.

B. Number of search templates required

Since the metric coefficients are constant in this analysis, the formula for the required number of templates [Eq. (2.16)] reduces to

$$\mathcal{N} = \frac{\sqrt{\det \|g_{ij}\|}}{2(1-MM)} \int d\tau_1 d\tau_2. \quad (3.9)$$

The square root of the determinant of the metric is given by $(2\pi f_0)^2 0.108$ for the advanced LIGO noise curve and by $(2\pi f_0)^2 0.058$ for the initial LIGO noise curve, so once we have decided on the range of parameters we deem astrophysically reasonable we will have a formula for \mathcal{N} as a function of MM .

The most straightforward belief to cherish about neutron stars is that they all come with masses greater than a certain minimum M_{\min} , which might be set to $0.2M_{\odot}$ (based on the minimum mass that any neutron star can have [24]) or $1.0M_{\odot}$ (based on the observed masses of neutron stars in binary pulsar systems [25]). In terms of the variables (M, η) the constraint $M_1 > M_{\min}$ and $M_2 > M_{\min}$ is easily expressed as

$$\frac{1}{2} M(1 - \sqrt{1-4\eta}) > M_{\min},$$

but in terms of the Sathyaprakash variables [Eqs. (3.1) and (3.2)] the expression becomes rather unwieldy to write down. However, see Fig. 2 for a plot of the allowed region in (τ_1, τ_2) coordinates.

For this reason we have found it convenient to use a Monte Carlo integration routine [26] to evaluate the coordinate volume integral $\int d\tau_1 d\tau_2$. The Monte Carlo approach becomes especially attractive when evaluating the proper volume integral $\int d\tau_1 d\tau_2 \sqrt{\det \|g_{ij}\|}$ for cases where the integrand is allowed to vary—and in fact may itself have to be evaluated numerically, as will be the case for a post²-Newtonian set of templates. The integral has numerical values of 0.18 and 24 seconds² for initial and advanced LIGO interferometer parameters, respectively, assuming a M_{\min} of $0.2M_{\odot}$ and arbitrarily large M_{\max} . The integral can be shown (numerically) to scale roughly as $f_0^{-4.5}$ (independent of f_0) and as $M_{\min}^{-2.7}$ for M_{\min} ranging from 0.2 to 1.0

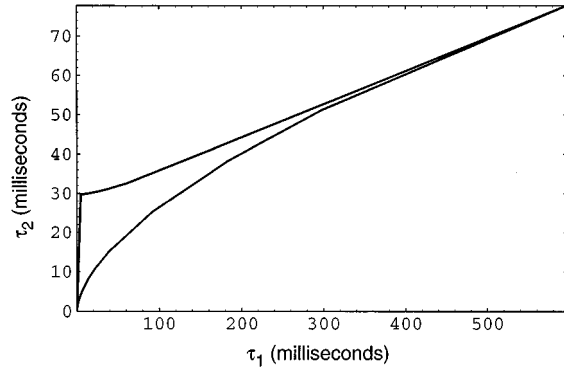


FIG. 2. The two-dimensional region of parameter space inhabited by binaries composed of objects more massive than $1M_\odot$. τ_1 and τ_2 are expressed (in milliseconds) for $f_0=200$ Hz, but the shape of the region does not change with f_0 . The upper boundary of the wedge is set by M_{\min} . The left-hand boundary is set by M_{\max} , but is essentially identical to the τ_2 axis for M_{\max} greater than a few solar masses. The region below the wedge corresponds to $\eta > 1/4$, which is *a priori* impossible.

solar masses (the dependence on M_{\max} is negligible for any value greater than a few solar masses).

Inserting the above numbers into Eq. (3.9), we find that

$$\mathcal{N} \approx 2.7 \times 10^5 \left(\frac{1-MM}{0.03} \right)^{-1} \left(\frac{M_{\min}}{0.2M_\odot} \right)^{-2.7} \left(\frac{f_0}{200 \text{ Hz}} \right)^{-2.5} \quad (3.10)$$

for the first LIGO noise curve, and

$$\mathcal{N} \approx 8.4 \times 10^6 \left(\frac{1-MM}{0.03} \right)^{-1} \left(\frac{M_{\min}}{0.2M_\odot} \right)^{-2.7} \left(\frac{f_0}{70 \text{ Hz}} \right)^{-2.5} \quad (3.11)$$

for the advanced LIGO noise curve. The fiducial value of MM has been chosen as 0.97 to correspond to an event rate of roughly 90 percent of the ideal event rate [cf. Eq. (2.21)]. In terms of the template-spacing-induced fractional loss \mathcal{L} of event rate, the number of templates required is

$$\mathcal{N} \approx 2.4 \times 10^5 \left(\frac{\mathcal{L}}{0.1} \right)^{-1} \left(\frac{M_{\min}}{0.2M_\odot} \right)^{-2.7} \left(\frac{f_0}{200 \text{ Hz}} \right)^{-2.5} \quad (3.12)$$

for the first LIGO noise curve, and

$$\mathcal{N} \approx 7.6 \times 10^6 \left(\frac{\mathcal{L}}{0.1} \right)^{-1} \left(\frac{M_{\min}}{0.2M_\odot} \right)^{-2.7} \left(\frac{f_0}{70 \text{ Hz}} \right)^{-2.5} \quad (3.13)$$

for the advanced LIGO noise curve. (For a minimum mass of $1.0M_\odot$, these numbers reduce to 3.1×10^3 and 9.9×10^4 , respectively.)

The thirtyfold increase in the number of templates between the initial LIGO and advanced LIGO noise curves is due to the lowering of the seismic frequency f_s . If f_s is lowered, a given signal evolves a greater number of cycles in the interferometer bandwidth. Therefore, the filtered signal-to-noise ratio is sensitive to smaller changes in the signal parameters, and the discrete template spacing must be tightened to compensate—resulting in a larger number of templates. A factor of 2 in the increase is due to the difference in

the shape of the noise curve (f_s/f_0 changes from 1/5 to 1/7); the remainder is due to the scaling with f_0 (or, equivalently, f_s) as f_s/f_0 is held fixed, which is shown in the previous four equations.

C. Template spacing

With the aid of the metric coefficients given in Eqs. (3.6) and (3.8), it is a simple task to select the locations of the templates and the spacing between them.

Because the metric coefficients form a constant 2×2 matrix, we can easily find the eigenvectors e_{x_1} and e_{x_2} of $\|g_{ij}\|$ and use them as axes to lay out a grid of templates. The numerical values are

$$\begin{aligned} e_{x_1} &= 0.874e_{\tau_1} + 0.485e_{\tau_2}, \\ e_{x_2} &= -0.485e_{\tau_1} + 0.874e_{\tau_2} \end{aligned} \quad (3.14)$$

for the first LIGO noise curve, and

$$\begin{aligned} e_{x_1} &= 0.899e_{\tau_1} + 0.437e_{\tau_2}, \\ e_{x_2} &= -0.437e_{\tau_1} + 0.899e_{\tau_2} \end{aligned} \quad (3.15)$$

for the advanced LIGO noise curve. The infinitesimal proper distance is given in terms of the eigencoordinates as $E_1(dx_1)^2 + E_2(dx_2)^2$, where E_1 and E_2 are the eigenvalues of the metric.

Therefore, we simply use Eq. (2.15) to obtain the template spacings

$$dx_j = \sqrt{\frac{2(1-MM)}{E_j}}, \quad j=1,2. \quad (3.16)$$

We find that the eigenvalues of the metrics (3.6) and (3.8) are $(2\pi f_0)^2$ times 0.721 and 0.004 27 (first LIGO), and $(2\pi f_0)^2$ times 1.25 and 0.009 84 (advanced LIGO). Therefore, the template spacings are given by

$$\begin{aligned} dx_1 &= 0.22 \text{ ms} \left(\frac{1-MM}{0.03} \right)^{1/2} \left(\frac{f_0}{200 \text{ Hz}} \right)^{-1}, \\ dx_2 &= 2.9 \text{ ms} \left(\frac{1-MM}{0.03} \right)^{1/2} \left(\frac{f_0}{200 \text{ Hz}} \right)^{-1} \end{aligned} \quad (3.17)$$

for the first LIGO noise curve, and by

$$\begin{aligned} dx_1 &= 0.49 \text{ ms} \left(\frac{1-MM}{0.03} \right)^{1/2} \left(\frac{f_0}{70 \text{ Hz}} \right)^{-1}, \\ dx_2 &= 5.6 \text{ ms} \left(\frac{1-MM}{0.03} \right)^{1/2} \left(\frac{f_0}{70 \text{ Hz}} \right)^{-1} \end{aligned} \quad (3.18)$$

for the advanced LIGO noise curve. Figure 3 shows the locations of some possible templates superposed on a contour plot of the match with the template in the center of the graph.

D. Computing power requirements

Drawing on the previous work of Schutz [17] concerning the mechanics of fast-Fourier-transforming the data, we can

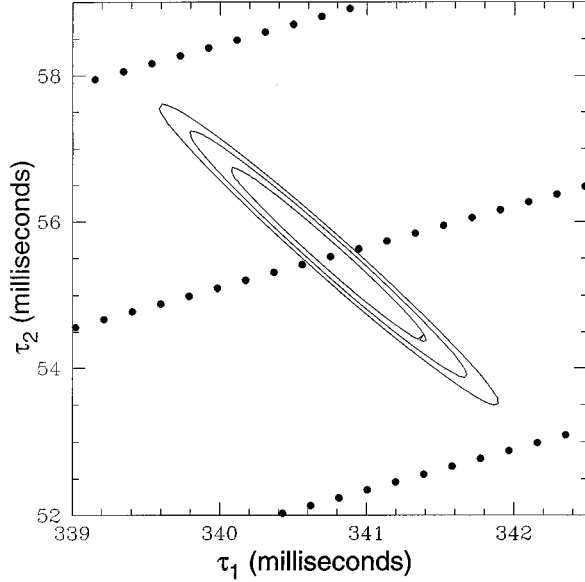


FIG. 3. Locations of various discrete templates for the first LIGO interferometers are shown by dots. The contours indicate the value of the match between a template located at (τ_1, τ_2) and the template located in the center of the figure. The contours are drawn at match values of 0.97, 0.98, and 0.99. Templates are placed along the principal axes of the elliptical contours. However, in order to make the ellipses visible the figure had to be stretched horizontally, distorting the contours and making the axes appear nonorthogonal.

estimate the CPU power required to process the interferometer output on-line through a single-pass (nonhierarchical) search involving \mathcal{N} templates.

Although the data will be sampled at a rather high rate (tens of kHz), frequencies above some upper limit $f_u \approx 4f_0$ can be thrown away (in Fourier transforming the data) with only negligible effects on the signal-to-noise ratio. This lowers the effective frequency of sampling to $2f_u$ (the factor of two is needed so that the Nyquist frequency is f_u), and thereby considerably reduces the task of performing the inner-product integrals. If the length of the array of numbers required to store a template is F and that required to store a given stretch of data is D , the number of floating-point operations required to process that data stretch through \mathcal{N} filters is

$$N_{\text{FO}} \approx D\mathcal{N}(16 + 3 \log_2 F) \quad (3.19)$$

[cf. Eq. (16.37) of Schutz [17], with the fractional overlap between data segments x chosen as roughly $1/15$].

Actually, F varies from filter to filter, but most of the search templates occupy regions of parameter space where the mass is very low—and thus the storage size of the template,

$$F \approx 2f_u \tau_1 (f_0/f_s)^{8/3}, \quad (3.20)$$

is very large [27]. The longest filter is the one computed for two stars of mass M_{\min} , so by inserting $\eta = 1/4$ and $M = 2M_{\min}$ into Eq. (3.1) and combining with Eq. (3.20), we find that we can make a somewhat pessimistic estimate of the computational cost by using

$$F \approx 2^{24} \left(\frac{M_{\min}}{0.2M_{\odot}} \right)^{-5/3} \left(\frac{f_s}{10 \text{ Hz}} \right)^{-8/3} \left(\frac{f_0}{70 \text{ Hz}} \right). \quad (3.21)$$

The required CPU power \mathcal{P} for an on-line search is obtained by dividing Eq. (3.19) by the total duration of the data set,

$$T_{\text{tot}} \approx D/(2f_u),$$

to find that

$$\mathcal{P} \approx 2\mathcal{N}f_u(16 + 3 \log_2 F). \quad (3.22)$$

Combining Eq. (3.22) with Eqs. (3.10) and (3.11) gives us

$$\mathcal{P} \approx 30 \text{ Gflops} \left(\frac{\mathcal{L}}{0.1} \right)^{-1} \left(\frac{M_{\min}}{0.2M_{\odot}} \right)^{-2.7} \left(\frac{f_0}{200 \text{ Hz}} \right)^{-1.5} \quad (3.23)$$

for an on-line search by the first LIGO interferometers, and

$$\mathcal{P} \approx 400 \text{ Gflops} \left(\frac{\mathcal{L}}{0.1} \right)^{-1} \left(\frac{M_{\min}}{0.2M_{\odot}} \right)^{-2.7} \left(\frac{f_0}{70 \text{ Hz}} \right)^{-1.5} \quad (3.24)$$

for the advanced LIGO interferometers.

Although the estimates in the paragraph above are not to be believed beyond a factor of order unity, the magnitude of the numbers shows that a hierarchical search strategy may be desirable to keep the computing power requirements at a reasonable level for nonsupercomputing facilities. That is, the data would first be filtered through a more widely spaced (low minimal match) set of templates with a relatively low signal-to-noise threshold, and only the segments which exceed this preliminary threshold would be analyzed with the finely spaced (high minimal match) templates.

The metric-based formalism of this paper only holds for the finely spaced set of templates used in the final stage of the hierarchical search; the template spacing used in the earlier stages of the search will have to be chosen using more complex methods such as those of Sathyaprakash and Dhurandhar [12,13].

E. Comparison with previous results

The only previous analysis of the problem of choosing the discrete search templates from the two-parameter, restricted post¹-Newtonian continuous template family is that of Sathyaprakash [14], in which he found that the entire volume of parameter space corresponding to $M_{\min} = 1M_{\odot}$ could be covered by a set of templates which vary only in $\tau_1 + \tau_2$ —thereby reducing the effective dimensionality of the mass parameter space to 1. This implied a value of \mathcal{N} similar to that obtained in the one-parameter (Newtonian template) analysis of Dhurandhar and Sathyaprakash in Ref. [13].

It is not possible to fairly compare my value for \mathcal{N} to the values given by Dhurandhar and Sathyaprakash in Table II of Ref. [13] due to our differing assumptions concerning the sources and the desirable level of the minimal match. Therefore, I will compare the assumptions.

Dhurandhar and Sathyaprakash typically consider a minimal match of 0.8 or 0.9 rather than 0.97. This would lead to a loss of 30–50% of the ideal event rate [cf. Eq. (2.21)]. If the current “best estimates” of inspiring binary event rates

[2,3] are correct, the ideal event rate for LIGO and VIRGO will not be more than about one hundred per year even when operating at the “advanced interferometer” noise levels, and the loss of up to half of these events would be unacceptable.

From Eqs. (3.23) and (3.24) it can be seen that the dependence of \mathcal{N} on M_{\min} is the most important factor influencing the computing power requirement. The two-parameter analysis of Sathyaprakash [14] uses a value for M_{\min} of $1M_{\odot}$, which is based on the statistics of (electromagnetically) known binary pulsars. However, because there is no known, firm physical mechanism that prevents neutron stars from forming with masses between 0.2 and $1M_{\odot}$, LIGO and VIRGO should use a discrete template family with $M_{\min}=0.2M_{\odot}$. After all, laser interferometer gravitational wave detectors are expected to bring us information about astronomical objects as yet unknown.

During the final stages of completion of this manuscript, a new paper by Balasubramanian, Sathyaprakash, and Dhurandhar appeared [28], applying differential geometry to the problem of detecting compact binary inspiral events and extracting source parameters from them. It applies the tools of differential geometry primarily to the problem of parameter measurement rather than that of signal detection, and so does not develop the geometrical formalism as far as is done in Sec. II of this paper. The metric constructed in Ref. [28] is identical to the information matrix which was suggested for use in the construction of a closely spaced, discrete template family in the authors’ previous work [13]. While this is quite useful for parameter measurement, it neglects maximization over kinematical parameters and thus is not very useful for the construction of search templates. Also, the assumptions about MM and M_{\min} are no different from those made in previous work up to and including Ref. [14], and so the result for \mathcal{N} is no different.

The main difference between the results of Ref. [28] and previous analyses by the same authors—and therefore the most important part of their paper as far as the detection problem is concerned—is the introduction of the possibility of choosing search templates to lie outside the manifold of the continuous template family. Using an *ad hoc* example, the authors show that such a placement of templates can result in a spacing roughly double that between discrete templates chosen from the manifold formed by the continuous template family. My analysis in this paper does not consider this possibility, but the formalism of Sec. II can easily be extended to investigate this problem in the future.

IV. CONCLUSIONS

A. Summary of results

This paper has presented a method for semianalytically calculating the number of templates required to detect gravity waves from inspiraling binaries with LIGO as a function of the fraction of event rate lost due to the discrete spacing of the templates in the binary parameter space. This method, based on differential geometry, emphasizes that ultimately a finer template spacing is required than has previously been taken as typical in the literature, in order to retain a reasonable fraction of the event rate. This paper details the first calculation of this kind that uses post-Newtonian templates and a noise curve which takes into account the coloration of

noise in the detector due to both standard recycling photon shot noise and thermal noise in the suspension of the test masses.

The result is that it is possible to search the data for binaries containing objects more massive than $0.2M_{\odot}$ thoroughly enough to lose only $\sim 10\%$ of the ideal event rate without requiring a quantum leap in computing technology. The computational cost of such a search, conducted on-line using a single pass through the data, is roughly 20 Gflops for the first LIGO interferometers (ca. 2000) and 270 Gflops for the advanced LIGO interferometers (some years later). Due to the nature of the approximations made in this paper, these numbers should not be believed to any greater precision than a factor of order unity; but they are certainly good order-of-magnitude estimates. These numbers are feasible (or very nearly feasible) even for a present-day supercomputing facility, but a hierarchical search strategy (using as its first stage a widely spaced set of templates similar to that analyzed by Sathyaprakash [14]) may be desirable to reduce the cost.

B. Future directions

There are several issues in this area which remain open for further investigation. Generally speaking, they fall into two categories characterized by pursuit of two different but complementary goals: obtaining an estimate of computing requirements to a precision better than the factor of unity achieved by the rough estimate in this paper, and reducing those computing requirements by using different algorithms to cover the templates’ parameter space.

(1) A thorough investigation of hierarchical search strategies is in order. How should the threshold and the minimal match of the first stage be set in order to minimize the CPU power required while keeping the false alarm and false dismissal rates at acceptable levels? How would non-Gaussian noise statistics affect the first stage threshold and minimal match? Would a hierarchical search benefit by using more than two stages? How is the threshold affected by the minimal match when the approximation of high signal-to-noise ratio can no longer be made?

(2) The formalism of this paper should be applied to choose discrete templates from a better continuous template family than the one considered here. The best two-parameter templates will be based on the highest post-Newtonian order computations that have been performed for circularized, spinless binaries, augmented perhaps by terms of still higher order from the theory of perturbations of Schwarzschild or Kerr spacetime. I plan to soon apply my geometric formalism to the post^{2.5}-Newtonian templates which are currently the best available.

(3) In order to detect some of the more interesting signals, it will be necessary to consider templates with more than two intrinsic parameters. Apostolatos has shown in Ref. [11] that when the objects comprising the binary are no longer constrained to have negligible spins, large volumes of the resulting five-dimensional parameter space are very poorly detected by spinless templates. It remains to be seen how many sources are expected by astrophysicists to populate those regions of parameter space, although current estimates [9] place the number at a small fraction of all compact binaries. It would also be wise to consider the effect of an

eccentricity parameter, since binaries formed by close capture (which may be common in globular clusters) would not be guaranteed to have circularized orbits.

(4) The fact that the wave form templates are close but not exact solutions to the general relativistic two-body problem (i.e., the fitting factor of even the best continuous template family will be slightly less than unity) will change this paper's estimates of computing power somewhat. Clearly, the spacing between templates must be tightened to some degree in order to compensate for the imperfection, but a more detailed analysis of this correction has not yet been done.

(5) The effect of nonquadrupolar harmonics of the gravitational wave on the construction of search templates should be considered. These harmonics have been ignored in previous analyses of detection and even of parameter measure-

ment, but they may have a noticeable effect when a very low loss of ideal event rate is desired.

(6) Finally, a systematic investigation of the optimal choice of search templates outside the continuous template family is in order. This problem has been addressed in a preliminary way in Ref. [28], but it deserves further scrutiny, especially since it relates to item (4) above.

ACKNOWLEDGMENTS

My thanks to Theodoros Apostolatos and Eanna Flanagan for helping me get started. I am most indebted to Kip Thorne for his guidance and his patience in reading the manuscript. This work was supported in part by the NSF graduate program and in part by NSF Grant No. PHY-9424337.

-
- [1] K. S. Thorne, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), p. 330.
 - [2] E. S. Phinney, *Astrophys. J.* **380**, L17 (1991).
 - [3] R. Narayan, T. Piran, and A. Shemi, *Astrophys. J.* **379**, L17 (1991).
 - [4] For example, A. V. Tutukov and L. R. Yungelson, *Mon. Not. R. Astron. Soc.* **260**, 675 (1993).
 - [5] A. Abramovici, W. E. Althouse, R. W. P. Drever, Y. Gürsel, S. Kawamura, F. J. Raab, D. Shoemaker, L. Sievers, R. E. Spero, K. S. Thorne, R. E. Vogt, R. Weiss, S. E. Whitcomb, and M. E. Zucker, *Science* **256**, 325 (1992).
 - [6] C. Bradaschia, E. Calloni, M. Cobal, R. Del Fabbro, A. Di Virgilio, A. Giazotto, L. E. Holloway, H. Kautzky, B. Michelozzi, V. Montelatici, D. Pascuello, and W. Velloso, in *Gravitation 1990*, Proceedings of the Banff Summer Institute, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).
 - [7] L. Blanchet, T. Damour, and B. R. Iyer, *Phys. Rev. D* **51**, 5360 (1995).
 - [8] C. M. Will and A. G. Wiseman (in preparation).
 - [9] L. Blanchet, T. Damour, B. R. Iyer, C. M. Will, and A. G. Wiseman, *Phys. Rev. Lett.* **74**, 3515 (1995).
 - [10] See, e.g., N. Wiener, *The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications* (Wiley, New York, 1949) or L. A. Wainstein and V. D. Zubakov, *Extraction of Signals from Noise* (Prentice-Hall, London, 1962).
 - [11] T. A. Apostolatos, *Phys. Rev. D* **52**, 605 (1995).
 - [12] B. S. Sathyaprakash and S. V. Dhurandhar, *Phys. Rev. D* **44**, 3819 (1991).
 - [13] S. V. Dhurandhar and B. S. Sathyaprakash, *Phys. Rev. D* **49**, 1707 (1994).
 - [14] B. S. Sathyaprakash, *Phys. Rev. D* **50**, R7111 (1994).
 - [15] C. Cutler and E. E. Flanagan, *Phys. Rev. D* **49**, 2658 (1994).
 - [16] Note the terms “intrinsic” and “extrinsic” are meant with regard to the shape of the wave form, not to the source itself. Consider, for example, the direction angles of a source on the celestial sphere. These parameters are clearly not intrinsic to the binary itself—it does not care how it is oriented with respect to the Earth. These parameters are not intrinsic to the wave form either, if the signal is of short duration—they simply change the overall phase of the signal, and can be lumped together with the coalescence phase. But they *are* intrinsic to the shape of the wave form if the signal lasts long enough to be significantly modulated by the Doppler effect of the interferometer's motion with respect to the solar system barycenter. In the former case (typical of stellar-mass objects detected by LIGO) the direction angles are very easily handled in the analysis; in the latter [typical of binaries detected by the Laser Interferometer Space Antenna (LISA)] they are intrinsic to the wave form and must be treated with great care, as is the case for long-term, all-sky pulsar searches.
 - [17] B. F. Schutz, in *The Detection of Gravitational Radiation*, edited by D. Blair (Cambridge University Press, Cambridge, England, 1989).
 - [18] E. Poisson and C. M. Will, *Phys. Rev. D* **52**, 848 (1995).
 - [19] Note that the Φ_0 used here is the coalescence phase of the wave form. In the restricted post-Newtonian approximation, this is twice the coalescence phase of the orbit ϕ_0 , which is used by Cutler and Flanagan in Ref. [15].
 - [20] S. V. Dhurandhar and S. D. Mohanty (in preparation); also L. S. Finn (in preparation).
 - [21] This $(N+1)$ -dimensional metric $\gamma_{\alpha\beta}$ is related to the Fisher information matrix, which was used by Dhurandhar and Sathyaprakash in the appendix of Ref. [13] to compute the template spacing analytically in the small-spacing limit. However, in taking the absolute value to maximize over $\Delta\Phi_0$, we have lost some dependence on the dynamical (intrinsic) template parameters, so our γ metric is not even the projection of the information matrix onto the subspace perpendicular to Φ_0 . This lets us tolerate a greater mismatch between the templates' dynamical parameters than was possible before maximization, thereby reducing the total number of templates required.
 - [22] P. C. Peters, *Phys. Rev.* **136**, B1224 (1964).
 - [23] The ψ_j given here are different from those given in Eq. (13a–e) of Ref. [14] because Sathyaprakash does not use t_0 and Φ_0 as kinematical parameters, but rather the time and phase at which the quadrupole part of the wave form reaches a frequency of f_0 .

- [24] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, New York, 1983).
- [25] J. H. Taylor, Rev. Mod. Phys. **66**, 711 (1994).
- [26] Numerical integrations were performed with the aid of Mathematica [S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer* (Addison-Wesley, Redwood City, CA, 1988)] and with a Monte Carlo code based on Numerical Recipes [W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, Cambridge, England, 1992)].
- [27] The storage size of the template is equal to the effective sampling frequency $2f_u$ times the duration of the template wave form. In the restricted post-Newtonian approximation, the duration of a binary chirp wave form template is equal to the time it takes for the gravitational wave frequency to sweep from the seismic frequency f_s up to infinity. Since $\tau_1 \propto f_0^{-8/3}$ is the dominant (Newtonian) contribution to the time it takes the gravitational wave frequency to sweep from f_0 up to infinity, the duration of a wave form template is roughly $\tau_1(f_0/f_s)^{8/3}$. Thus, we obtain Eq. (3.20).
- [28] R. Balasubramanian, B. S. Sathyaprakash, and S. V. Dhurandhar, Phys. Rev. D **53**, 3033 (1996).