

Winding number correlation functions and cosmic string formation

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We develop winding number correlation functions that allow us to assess the role of field fluctuations on vortex formation in an Abelian gauge theory. We compute the behavior of these correlation functions in simple circumstances and show how fluctuations are important in the vicinity of the phase transition. We further show that, in our approximation, the emerging population of long or infinite string is produced by the classical dynamics of the fields alone, being essentially unaffected by field fluctuations.

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I. INTRODUCTION

The production of topological defects seems to be an inescapable consequence of phase transitions in field theories with a topologically nontrivial vacuum manifold [1,2]. This is observed experimentally in numerous condensed matter systems from superfluid helium [3] and superconductors to ordinary liquid crystals [4] and is expected to occur in the early universe for most theories of grand unification. Defects produced at such energy scales are strong candidates to generate the observed large scale structure of the universe, competing with fluctuations arising from inflationary models to match the observations [5].

In spite of their universality, the problem of predicting the details of networks of defects emerging from a phase transition is still unsolved, being necessarily entangled with the more general question of how to determine the dynamical evolution of fields throughout the transition. Our qualitative picture of defect production is given by the well-known Kibble mechanism [1]. With the $U(1)$ theory of a complex scalar field in mind, it essentially assumes that the phase transition proceeds by the formation of domains, which are characterized by some random value of the field phase, which later coalesce to complete the transition. Such domains, at the time of coalescence, can, in turn, be characterized by their average radial length scale ξ , which in most applications is taken to be the correlation length as computed for a massive scalar field in thermal equilibrium [1]. As domains coalesce phase gradients must be minimized and integer windings can be formed. Because of its simplicity this picture has great appeal and is amenable to a straightforward approximate numerical implementation by the so-called Vachaspati-Vilenkin algorithm [6], which in turn constitutes the usual starting point for numerical network evolutions.

As it stands the Kibble mechanism fails to answer any questions regarding the role of fluctuations in the fields. If important enough, such fluctuations will necessarily alter the number and distribution of defects produced by the dynamics of domain coalescence and could thus distort seriously the predictions solely based upon it.

A change in initial string density and especially in the amount of long strings versus that in small closed loops, for example, necessarily affects the transient regime that characterizes a string network prior to scaling. Moreover, it is precisely the point at which thermal fluctuations cease to be able to change string configurations appreciably that determines the emerging defect network. This problem has been attracting great enthusiasm, leading to a number of different approaches [7-9].

Our aim in this paper is to develop quantities and associated criteria for evaluating the role of such fluctuations on cosmic string distributions.

We focus our analysis on cosmic strings in the Abelian Higgs model and consider correlation functions of the winding number (defined as the number of quantized units of magnetic flux) in several circumstances. Even though the present study is motivated by cosmological questions the analogy with superconducting systems is almost complete because of the role of the gauge fields in both models. As such our results should apply equally well there, where contact with experiment can prove a good test on the ideas developed herein. In principle the interpolation between both pictures can be obtained essentially by replacing cosmic strings with vortex lines and discarding some specific aspects of relativistic self-energies in Sec. III. In practice we have yet to do so.

The outline of the paper is as follows. We start, in Sec. II, by describing the qualitative aspects of the classical dynamics of the fields during the phase transition and determine under which conditions the measurement of quantized units of magnetic flux becomes a reliable estimate of the number of strings present. We then proceed, in Sec. III, to compute the fluctuations in winding number in the simplest scenario, namely, that of a constant background scalar field with an equilibrium distribution, and under the assumption that the gauge field, too, displays a thermal spectrum. In Sec. IV we show that the computations of Sec. III can easily be extended to the calculation of correlation functions of the winding number threading through two different contours with relative specific geometries. We use these correlation functions to answer questions concerning the conditional probability of finding strings crossing a given surface in

space, subject to the condition it has crossed another, with a given separation and orientation relative to the former. In Sec. V, we return to the computation of winding number fluctuations, by including inhomogeneities in the scalar field. We show how a systematic expansion around a homogeneous background can be performed, generating as a zeroth order term the result of Sec. III, and compute the first order correction resulting from the presence of strings. By comparing their relative magnitude we can formulate well-defined criteria for the breakdown of such an expansion and consequently for the non-validity of our results computed about the homogeneous background. In Sec. VI we present our conclusions and compare our results with those of other recent approaches to the same problem.

II. PHASE TRANSITION FIELD DYNAMICS AND DEFECTS

We begin with the semiclassical analysis of the simplest theory that admits gauged strings, the Abelian Higgs model. We consider it in the symmetry broken phase so that the decomposition of the complex scalar field into modulus φ and phase α is possible as

$$\phi(x) = \varphi(x)e^{-i\alpha(x)} \quad (1)$$

for nonzero $\varphi(x)$. Thus the classical action, including the gauge-fixing term, can be written as

$$S[\varphi, \alpha, A_\mu] = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} [\partial_\mu \varphi \partial^\mu \varphi + \varphi^2 (eA_\mu - \partial_\mu \alpha)(eA^\mu - \partial^\mu \alpha)] - \frac{\lambda}{8} (\varphi^2 - \eta^2)^2 \right] + \frac{1}{2a} \partial_\mu A^\mu \partial_\nu A^\nu, \quad (2)$$

where the last term enforces gauge fixing explicitly. Our objective is to compute string formation in gauge theories, as a dynamical process occurring as a by-product of the symmetry breaking in the theory, by measuring units of the quantized magnetic flux crossing a surface bounded by the contour γ , given by

$$\langle N_\gamma \rangle = \frac{e}{2\pi} \int_\gamma dl^i \langle A_i \rangle. \quad (3)$$

The idea of using the magnetic flux as an indicator of the net winding number springs from the vacuum field configurations in the broken phase: namely,

$$\langle \varphi \rangle = \eta, \quad (4)$$

$$\left\langle A_\mu - \frac{\partial_\mu \alpha}{e} \right\rangle \equiv \langle A'_\mu \rangle = 0. \quad (5)$$

This guarantees that the gauge field indeed traces the gradient in phase of the scalar field. In a situation where the scalar modulus is constant, however, this becomes simply a gauge transformation, corresponding to the residual freedom in defining A_μ in the Lorentz gauge ($\partial_\mu A^\mu = 0$). However, it is the case of a nonconstant scalar modulus that is relevant for our purposes, corresponding to the presence of strings. In this case the phase can change around a closed contour in space by a nonzero multiple of 2π , implying the existence of a singularity in its gradient as the contour is shrunk to a point (at which there is a necessary restoration of symmetry). The gauge field at such a point, the core of the string, exhibits therefore its expectation value in the symmetric phase, i.e., $\langle A_\mu \rangle = 0$. The approximate solution of the static Euler-Lagrange equation interpolating between these two regimes is well known to be [10]

$$A_\theta = \frac{n}{er} - n\varphi K_1(e\varphi r) \quad (6)$$

for a homogeneous scalar modulus φ . At large distances from the core of the string ($r \gg \frac{1}{e\varphi}$), the modified Bessel function K_1 becomes exponentially decreasing and the vacuum of the spontaneously broken theory is rapidly approached. In this limit, applying the form (6) in (3) gives precisely n units of flux with exponentially small corrections.

As the phase transition proceeds and the fields become smooth we therefore expect A_μ to trace the overall differences in phase left over from the dynamics of the scalar field with increasing accuracy. To see in more qualitative detail how this happens we need to consider the actual Euler-Lagrange equations of the theory,

$$\partial_\mu \partial^\mu \varphi = \varphi \left[(eA_\mu - \partial_\mu \alpha)(eA^\mu - \partial^\mu \alpha) - \frac{\lambda}{2} (\varphi^2 - \eta^2) \right], \quad (7)$$

$$\partial^\mu \left(F_{\mu\nu} + \frac{1}{a} \partial_\nu A_\mu \right) = J_\nu, \quad (8)$$

and

$$\partial^\mu J_\mu = 0, \quad (9)$$

where

$$J_\mu = e\varphi^2 (eA_\mu - \partial_\mu \alpha). \quad (10)$$

The phase transition is triggered by an instability in the long wavelength modes of the scalar field,¹ whose amplitudes then start growing exponentially towards η . This

¹The details of the triggering mechanism depend crucially on the hierarchy of the couplings in the theory determining thereby the order of the transition, as seen from thermodynamic effective potential constructions.

is expected to happen independently in regions of space which are causally disconnected for the time scales involved. As a result a domain structure is expected to form, leading to an inhomogeneous scalar modulus on a scale of several domains. These inhomogeneous field configurations do not minimize energy and will therefore evolve in time so as to coarsen and smooth out.²

What is the behavior of the phase and vector potential

during this dynamical process? The vector potential is characterized initially, in the symmetric phase, by $\langle A_\mu \rangle = 0$. During the transition A_μ is sensitive to the change in expectation values of φ and α in quite a different manner. Whereas the change in the expectation value of φ gives a mass to A_μ , without changing its expectation value, the presence of a gradient in phase does act as a source for the latter. This can be seen by writing (8) as

$$\left[\partial^\mu \partial_\mu A_\nu - \left(1 - \frac{1}{a} \right) \partial_\nu \partial^\mu A_\mu \right] - e^2 \phi^2 A_\nu = J_\nu^{\text{ext}} - e\varphi^2 \partial_\nu \alpha \equiv J'_\nu, \quad (11)$$

where J_ν^{ext} is any external source, imposed on the system.

The general $\langle A_\mu \rangle$ can, in principle, be computed from the generating functional for the theory which in turn follows from expanding the action (2) around a field configuration that satisfies the Euler-Lagrange equations with boundary conditions characterizing the fields at some given time, as discussed above.

By introducing (11) in the action, we obtain

$$\begin{aligned} S[\varphi, \alpha, A_\mu] &= -\frac{1}{2} \int d^4x \left[(e\varphi^2 \partial_\mu \alpha - J_\mu^{\text{ext}}) A^\mu + \text{terms in } \varphi \text{ and } \alpha \right] \\ &\equiv \frac{1}{2} \int d^4x \left[J'_\mu A^\mu + \text{terms in } \varphi \text{ and } \alpha \right]. \end{aligned} \quad (12)$$

This can in turn be rewritten exclusively in terms of the currents, as usual, since

$$A_\mu(x) = \int d^4y G_{\mu\nu}(x, y, \varphi) J'^\nu(y), \quad (13)$$

where

$$\left[\partial^\mu \partial_\mu - \left(1 - \frac{1}{a} \right) \partial_\nu \partial^\mu - e^2 \varphi^2(x) \right] G_{\mu\nu}(x, y, \varphi) = -\delta_\nu(x - y). \quad (14)$$

In the absence of external sources A_μ then becomes

$$A_\mu(x) = \int d^4y G_{\mu\nu}(x, y, \varphi) e\varphi(y) \partial^\nu \alpha(y). \quad (15)$$

As the phase transition approaches completion the scalar field should become smoother, both in modulus and phase, leading to small derivative terms relative to the mass scale. This allows us to perform a small derivative expansion of the gauge propagator, leading to the result

$$A_\mu(x) \simeq \left(\frac{\varphi(x)^2}{\eta^2} \right) \frac{\partial_\mu \alpha(x)}{e} + \text{derivative terms in } \varphi \text{ and } \partial_\mu \alpha. \quad (16)$$

In the case of a homogeneous and static scalar modulus, this reduces to the vacuum in the spontaneously broken phase (5). Unlike its higher derivative corrections, the first term in (16) is gauge independent.

Having understood qualitatively from this semi-classical analysis how the gauge field reacts to a difference in phase, it is important to discuss how a gradient in phase itself can occur and how it will evolve. Departing from the symmetric phase, where $\langle A_\mu \rangle = 0$, Eq. (9) can be written as

$$\partial_\mu \partial^\mu \alpha + (\partial_\mu \ln \varphi^2) \partial^\mu \alpha \simeq 0. \quad (17)$$

Equation (17) is homogeneous in $\partial^\mu \alpha$, showing that a gradient in phase must exist initially for the phase to change in space-time. Simultaneously, given that $\langle A_\mu \rangle = 0$, the energy of a given configuration is minimized for a constant value of α . We therefore expect the field phase to be approximately constant inside each domain formed by the dynamics of the scalar modulus.³ Differences in

²This is explicitly observed in condensed matter systems, such as specific types of liquid crystals, which display a large relaxation time [4].

³By domains we mean the volume associated with the scale over which the phase typically changes. This is dependent on the actual dynamics of the fields and will correspond to quite different configurations in the context of a first or second order transition.

phase can then only occur when different domains come into causal contact, and then should evolve so as to be minimized. This picture corresponds to the well-known Kibble mechanism, in a gauge theory.⁴ In particular, according to (17) and under plane wave *Ansätze* for the fields the amplitudes of the small frequency modes of α will decrease exponentially, given an exponential growth in the amplitudes of φ . As domains coalesce and $\partial_\mu\varphi$ tends to zero the remaining gradients in phase are frozen in and the gauge field catches up with them, in the sense of (16).

Having this qualitative picture in mind for the description of the field dynamics during the phase transition we proceed, in the next sections, to compute the effect of fluctuations on the total winding number using the quantized magnetic flux (3) as a probe. These are given by

$$\langle TA_\mu(x)A_\nu(y) \rangle = G_{\mu\nu}(x, y, \varphi) + \langle J^\rho(z)G_{\rho\mu}(x, z, \varphi)J^\sigma(z')G_{\sigma\nu}(z', y, \varphi) \rangle, \quad (19)$$

where $G_{\mu\nu}$ is given by (14) and $J^\mu(x) = e\varphi(x)\partial^\mu\alpha(x)$. If we assume that the contours do not run over any strings,⁵ the dominant term will be

$$\langle TA_\mu(x)A_\nu(y) \rangle = G_{\mu\nu}(x, y, \varphi). \quad (20)$$

We stress that we are not just quantizing A_μ . Scalar-field radiative corrections about $\langle\varphi\rangle$ will be taken into account in the calculation of $G_{\mu\nu}$. The remaining uncertainty in defining the gauge propagator explicitly results from the unknown form of $\varphi(x)$. Having (16) in mind we will treat the situation of a quasihomogeneous background, expanding the propagator in the perturbations. The time dependence of φ remains a problem, however. We deal with it approximately by considering the fields in equilibrium and changing the corresponding temperature. In the next section we treat the simplest case, i.e., that of a homogeneous background where the fields display equilibrium distributions.

III. FLUCTUATIONS IN WINDING NUMBER IN THERMAL EQUILIBRIUM

In this section we specialize the previous discussion to the case of a homogeneous scalar modulus background and assume the fields to have equilibrium distributions. The first of these conditions will be lifted later, in Sec. V. Dropping the condition of equilibrium, however, would require the knowledge of the time dependence of the fields, in closed form, which is unknown in the necessary detail.⁶

⁴We see that the geodesic rule comes about naturally from dynamical considerations. Ambiguities [11] result from considering the *static* gauge theory in its spontaneously broken phase, where the phase decouples and loses physical meaning.

⁵This seems reasonable if the string density is not too high. See Sec. V.

⁶More is known for the theory of a simple scalar field numerically, e.g., [12], but this proves of little use in our present context.

$$\langle N_\gamma N_{\gamma'} \rangle = \int_\gamma \int_{\gamma'} dl^i dl'^j \langle A_i(x) A_j(y) \rangle. \quad (18)$$

In most of what follows we will be concerned solely with thermal fluctuations. Quantum fluctuations can be readily included but are known to be subleading. Their effect is to change the value of the vacuum expectation value of the φ field as well as the couplings in the theory. We will in general adopt the position of treating φ as providing a classical temperature-dependent background field $\langle\varphi\rangle$ in which the fields can be quantized. In general this background will include strings. We then need to know the form of the two-point function for the gauge fields, which in the absence of external sources, is given by

The degree of approximation actually involved in adopting a thermal distribution for the fields will necessarily depend on the dynamics of the phase transition, which in turn will be determined by the hierarchy of couplings in the theory. First, for a theory with strong gauge couplings (type I) one expects the system to depart significantly from equilibrium (by bubble nucleation) and our approximation will probably grossly misrepresent the actual fluctuations in the system. In the opposite limit, however (strong type II), it should hold as a good approximation more or less throughout the whole dynamics. The latter is the case of the most high temperature superconducting systems. Second, in any case, the late stages of the transition should be characterized by the approach to a thermal distribution and consequently the estimates below should describe the actual fluctuations in the system increasingly more accurately. This expectation is also consistent with the indications of when the magnetic flux becomes a good indicator of the number of strings crossing a given surface, as seen in the last section.

A strictly homogeneous scalar field is of course a necessary and sufficient condition for the existence of *no* strings. It then becomes necessary to be able to estimate how the values of the fluctuations computed below relate to those in more realistic inhomogeneous backgrounds. This issue will be addressed in Sec. V, where we will show that under well-defined circumstances the fluctuations computed here are the first term in a systematic expansion about a homogeneous background and constitute, in general, the dominant contribution.

In order to compute the thermal fluctuations in winding number, given by (18), we need to know the form of the time-ordered thermal correlation function of the A_μ . There are only two symmetric tensors in the spatial indices which can be made in the covariant and R_ζ gauges, namely, δ^{ij} , $k^i k^j$. The Ward or Becchi-Rouet-Stora (BRS) identities and the Lorentz structure at finite temperature then tell us that the most general form in covariant and R_ζ gauges is

$$iG^{ij}(t-t', \vec{x}-\vec{x}') = \langle TA^i(t, \vec{x})A^j(t', \vec{x}') \rangle \quad (21)$$

$$= \frac{i}{8\pi^3\beta} \sum_{n=-\infty}^{+\infty} \int d^3\vec{k} \exp\{-i[k_0(t-t') - \vec{k} \cdot (\vec{x}-\vec{x}')]\} iG^{ij}(k) \quad (22)$$

$$-G^{ij}(k) = \frac{1}{k^2 - e^2\eta^2 - \Pi_T} \left(\delta^{ij} - \frac{k^i k^j}{k^2} \right) + \left(\frac{1}{k^2 - e^2\eta^2 - \Pi_L} + f(k, M, \zeta) \right) \frac{k^i k^j}{k^2}, \quad (23)$$

where the energy $k_0 = 2\pi i n/\beta$. The f function varies with the gauge chosen while the other Π_T and Π_L terms correspond to the physical modes of the photon, two transverse (magnetic) and a single longitudinal (electric) mode.

From (18) and (21)–(23), taking the integrals over circular paths of radius L , we see that we require two sorts of integration,

$$\oint_L dx_i e^{-i\vec{k} \cdot \vec{x}}, \quad \oint_L dx_i e^{-i\vec{k} \cdot \vec{x}} k_i. \quad (24)$$

Since the latter is zero, we only pick up the term in the propagator with the δ_{ij} . This is not surprising as we are really looking at the magnetic field correlations, and the term with δ_{ij} is associated with magnetic fields and its self-energy Π_T contains information about the magnetic screening. Thus we have that

$$\langle N_\gamma^2 \rangle = \frac{e^2}{4\pi^2} \frac{i}{8\pi^3\beta} \sum_{n=-\infty}^{+\infty} \int d^3\vec{k} \frac{-i}{k^2 - e^2\eta^2 - \Pi_T} \left| \oint_L d\vec{x}_i e^{-i\vec{k} \cdot \vec{x}} \right|^2 n(\omega). \quad (25)$$

Note that this form is independent of the gauge chosen and the only source of gauge dependence is in the expression used for Π_T . Proceeding with (25), we use the leading term in the high temperature expansion of the one-loop self-energy. In this we are working to leading order in the resummation scheme of Braaten and Pisarski [13] which is necessary at high temperatures. Performing the energy sum leads to two terms. There is a pole contribution which corresponds to propagation of a “plasmon.” The pole is found at $k_0 = \pm\omega(k)$ where

$$0 = \omega(k)^2 - e^2\eta(T)^2 - \Pi_T(\omega(k), k) \quad (26)$$

$$\Pi_T(k_0, k) = \frac{3M_P^2}{2} \left\{ \frac{k_0^2}{k^2} + \left(1 - \frac{k_0^2}{k^2} \right) \frac{k_0}{2k} \ln \left[\frac{k_0 - k}{k_0 + k} \right] \right\},$$

where we have worked with respect to the vacuum

$$\left. \frac{\partial V_{\text{eff}}}{\partial \phi} \right|_{\phi=\eta(T)} = 0. \quad (27)$$

The usual $O(e^2)$ result is $\eta(T)^2 = \eta^2 \left(1 - \frac{T^2}{T_c^2} \right)$, which vanishes at the critical temperature T_c . The value of M_P is

$$M_P^2 = \frac{e^2 T^2}{3}, \quad (28)$$

for a U(1) theory with a single charged scalar or fermion. While the position of the pole for general three-momentum is given by a nontrivial dispersion relation, to within about 10% this is approximately expressible as $k_0^2 = k^2 + m^2$ where

$$m^2 = e^2\eta(T)^2 + M_P^2. \quad (29)$$

The difference in the final results, between choosing this approximate dispersion relation or the true one, has been studied numerically in some of the cases below and is found to be negligible (less than 1%).

The real complication is encountered with the cut term, running across the zero energy point, seen in the imaginary part of Π_T for $|k_0| < k$. This corresponds to the physical process of Landau damping allowed in a heat bath. This must be treated numerically as in this case we are not able to use any sum rules of [14], which can be seen in the logarithmic dependence on temperature. Equivalently, trying to look at possible $O(T)$ terms by including only the $n = 0$ term in the energy sum (e.g., see the Appendix of [15]) leads to divergent integrals and to the breakdown of the calculation.

Doing the energy sum gives

$$\langle N_\gamma^2 \rangle = \frac{e^2}{32\pi^5} \int d^3\vec{k} \left| \oint_L d\vec{x}_i e^{-i\vec{k} \cdot \vec{x}} \right|^2 \left(\frac{n(\omega)}{\omega} + (\text{cut terms}) \right) + (T = 0 \text{ terms}). \quad (30)$$

On computing the integral in (25) we find

$$\langle N_\gamma^2 \rangle = \frac{e^2}{4\pi^2} L^2 \int k_{\parallel} dk_{\parallel} dk_z J_1^2(k_{\parallel} L) \left(\frac{n(\omega)}{\omega} + (\text{cut terms}) \right) + (T = 0 \text{ terms}), \quad (31)$$

where k_{\parallel} is the momentum component on the plane defined by the contours and J_1 is the usual Bessel function of order 1. This last integral cannot be fully computed analytically. For large enough loops ($k_{\parallel}L < 1$) we can take the large argument form of the Bessel function. Looking only at the pole contribution and using the good approximation that $\omega^2 = k^2 + M_P^2$ leads to

$$\langle N_{\gamma}^2 \rangle \simeq \frac{e^2}{4\pi^2} \frac{2LT}{3} \ln \left(\frac{T}{m} \right). \quad (32)$$

This estimate is indeed confirmed by computing the integral (31), for the pole term, numerically. Figures 1 and 2 show the result of a numerical integration looking at just the pole term with the approximate dispersion relation $\omega^2 = k^2 + m^2$ on the dependence of $\langle N_{\gamma}^2 \rangle$ on LT and $\frac{M}{T}$, respectively. The $\frac{1}{3}$ coefficient results from the best fit, given the functional form (32). Using the exact dispersion relation for the pole contribution makes only a very slight difference to the results. The cut term shows a similar sensitivity on $\eta(T)$ which acts as an infrared cutoff. For $L\eta(T) < 1$ the cut contribution to N_{γ}^2 rises as $L \ln L$. However, for $L\eta(T) \sim 1$ and larger, a similar qualitative behavior to (32) is found, with magnetic screening mass⁷ $m_{\text{mag}} = e\eta(T)$ replacing the plasmon mass in the formula above to give

$$\langle N_{\gamma}^2 \rangle \propto \frac{e^2}{4\pi^2} LT \ln \left(\frac{T}{\eta(T)} \right), \quad (33)$$

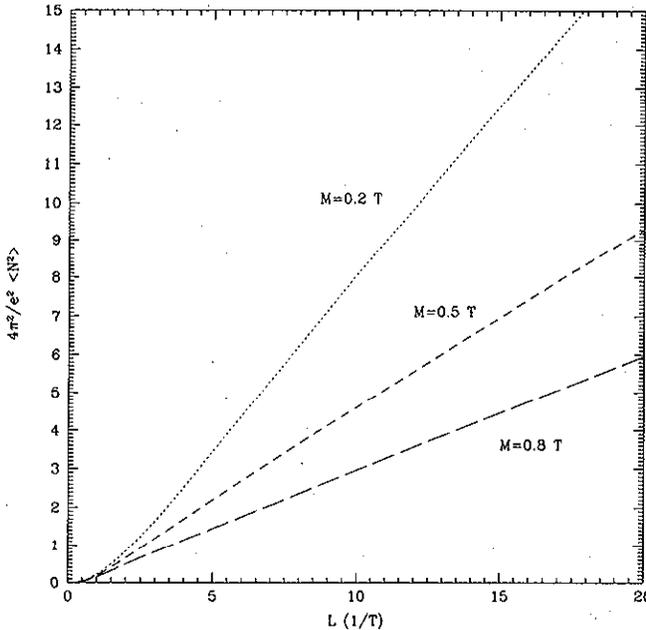


FIG. 1. The dependence of the winding number fluctuations on the contour radius L from the pole contribution, for $m = 0.2T$, $0.5T$, and $0.8T$.

⁷The magnetic screening mass for Abelian theories only has a contribution from the scalar field shift.

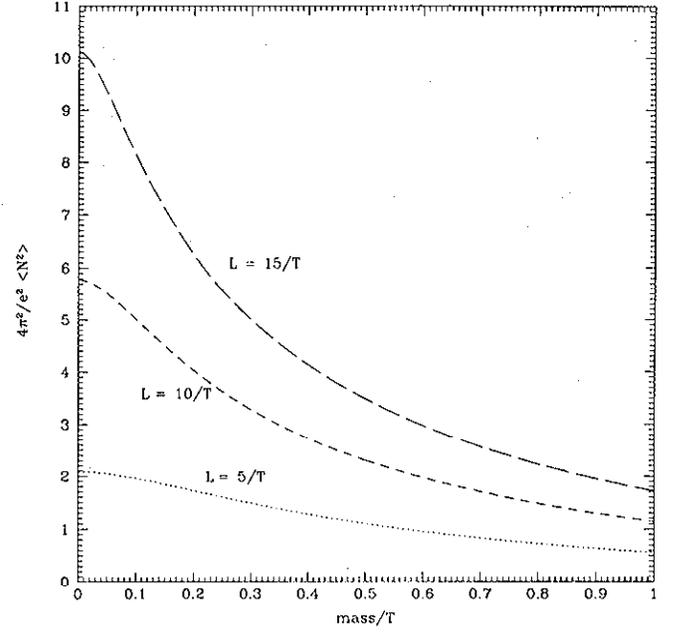


FIG. 2. The dependence of the winding number fluctuations on the gauge field mass from the pole contribution, for $L = 1/T$, $5/T$, and $15/T$.

with a coefficient of proportionality comparable (typically, a few times larger) to that of the pole in (32). Thus we see that the contributions from the pole and cut terms essentially generate the same form for the winding number fluctuations away from the critical temperature, when $\eta(T) \ll M_P$. As such the influence of the cut term, in these circumstances, is to change the overall coefficient in the functional form (32). In most of what follows we will therefore restrict ourselves to computing pole contributions.

Qualitatively the results above can be thought of as follows. The dependence of $\langle N_{\gamma}^2 \rangle$ on L can be understood in terms of a random walk in the field fluctuations along the contour perimeter. This is commensurate with the Kibble mechanism of domain formation, with random phases correlated on a scale $\xi = 1/e^2T$ (up to logarithms). This should not be surprising as thermal fluctuations ensure that fields are correlated on scales smaller or equal to ξ , and not on longer scales. The precise correlation function is therefore not relevant as a first guess and the random walk paradigm is a suitable starting point.

The quasilinear dependence on the temperature ensures that, as the system cools, fluctuations in winding number become less and less relevant. The logarithmic dependence on the field thermal mass, in turn, guarantees that as one approaches the critical temperature and the mean field mass $\eta(T)$ vanishes, the magnitude of winding number fluctuations diverges. This is certainly the instance when thermal fluctuations can significantly change any underlying expected winding number. The actual divergence of $\langle N_{\gamma}^2 \rangle$ at the critical temperature for the pole term, however, is an artifact of our best fit since, as can be seen in Fig. 2, as the mass vanishes (for $M_P \simeq 0$) the

fluctuations in winding number remain finite even though maximal. This is not the case for the cut contribution, as far as we could find numerically where the logarithmic behavior persists, with extraordinary accuracy, down to $m_{\text{mag}} = 10^{-4}$.

By themselves, however, the magnitude of fluctuations tell us little about how string configurations can be changed. This then takes us to the crucial issue of comparing the magnitude of fluctuations to the underlying expected winding number. The magnitude of relative fluctuations is given by

$$\frac{(\Delta N)^2}{\langle \bar{N} \rangle^2} = \frac{\langle N^2 \rangle}{\langle \bar{N} \rangle^2} - 1, \quad (34)$$

where $\langle \bar{N} \rangle$ is the total string number, without distinguishing strings from antistrings. In principle $\langle N^2 \rangle$ and $\langle \bar{N} \rangle^2$ must be computed self-consistently given a specific field background. To estimate the behavior of (34) in our specific setting, however, we can take our values for $\langle N^2 \rangle$ and borrow the value for $\langle \bar{N} \rangle^2$, obtained using the Vachaspati-Vilenkin algorithm, for the number of strings crossing a disk of area πL^2 . As required, this algorithm does not distinguish between strings and antistrings. Using this value in (34) should clearly yield an underestimate for the value of the relative fluctuations. The expected value of the winding number thus computed crossing a disk of radius L is [6]

$$\langle \bar{N} \rangle = \frac{1}{4} \frac{L^2}{\xi^2}, \quad (35)$$

where ξ is the average domain radius at coalescence. This can be written as an inverse of a mass scale $\xi = \frac{1}{m_\xi}$, whence

$$\frac{(\Delta N)^2}{\langle \bar{N} \rangle^2} \simeq \frac{8e^2}{3\pi^2} \frac{T}{m_\xi} \frac{1}{m_\xi L} \ln \left(\frac{T}{m} \right) - 1. \quad (36)$$

The identification of this scale with the mean domain radius at coalescence springs directly from its numerical implementation. Its computation from the field theory, however, implies a knowledge of the domain dynamics which is, in general, very poor [16]. The original proposal by Kibble [1] was to identify this scale with the inverse Ginzburg temperature which is a well-defined quantity in a field theory undergoing a second order phase transition.⁸ More recently, Kibble and Vilenkin [7] argued that a more suitable scale could be found by invoking the scaling of the string network emerging from the phase transition. They discuss carefully the different length scales in the problem and essentially conclude that individual strings can then be identified once this typical string separation scale becomes larger than the Ginzburg length. Before then fluctuations in the fields

are argued to be very large and coherent long-lived field configurations cannot persist. Looking at (36) we can indeed observe that fluctuations become large for temperatures larger than m_ξ and for length scales smaller than its inverse. At the critical temperature the relative fluctuations should diverge due to the cut contribution discussed above.

Our present analysis thus lends support to several of their qualitative arguments and clarifies the emergence of some scales from field theory propagators.

IV. STRING SPATIAL DISTRIBUTION AND OTHER CORRELATION FUNCTIONS

One interesting question to ask about string formation that is of the upmost importance for the subsequent evolution of a string network is what is the fraction of string in loops as opposed to long strings. We will see in this section that winding number correlation functions can give us some insights into this problem.

Since we expect to be dealing with a complicated background in which many string configurations will thread through our contour, these questions can only be answered statistically. Intuitively, then, the question of how a string is distributed in space assumes the form of a conditional probability question; i.e., knowing the amount of string that crosses a given surface, what is the amount of string that crosses a second surface with a given orientation relative to the former? Answers to these questions are naturally provided in terms of correlation functions and the natural way ahead, in the context of our present discussion, is to take our two-point function for the winding number (18) through two different contours. In so doing we expect that the correlations between net winding number threading through the two loops in a given geometrical configuration can also give us information about how the string is distributed relative to that geometry. The general case is too difficult, but there are two natural extensions of (25) that are solvable, namely, that for which the two circles remain concentric, but now have different radii l and $L > l$, and that of two coaxial circles of equal radius L separated by a distance h . These are shown in Figs. 3 and 4, respectively. Intuitively we would expect the first of these correlation functions to give us information about the population of strings with curvature radius greater than that of the longest contour whereas the second should provide a measure of straightness on the scale of separation between the contours.

We start with the former. In what follows we compute the contributions to the correlation functions due to the pole term only. As discussed above the cut contribution should not change any qualitative aspect of our results but merely the overall coefficient by a number of order 1. As before in order to compute the two-point function for the winding number we can show that only the integrals of the type I_1 survive, i.e., we have to compute

$$\begin{aligned} I_1 &= \int dl_i dl_j e^{ik \cdot (\vec{L} - \vec{l})} \delta_{ij} \\ &= lL \int_{-\pi}^{\pi} d\theta d\theta' e^{ik_{\parallel} (L \cos \theta' - l \cos \theta)} \cos(\theta' - \theta). \end{aligned} \quad (37)$$

⁸The corresponding, though in general not computable, natural scale in a first order transition is obviously the bubble radius at coalescence.

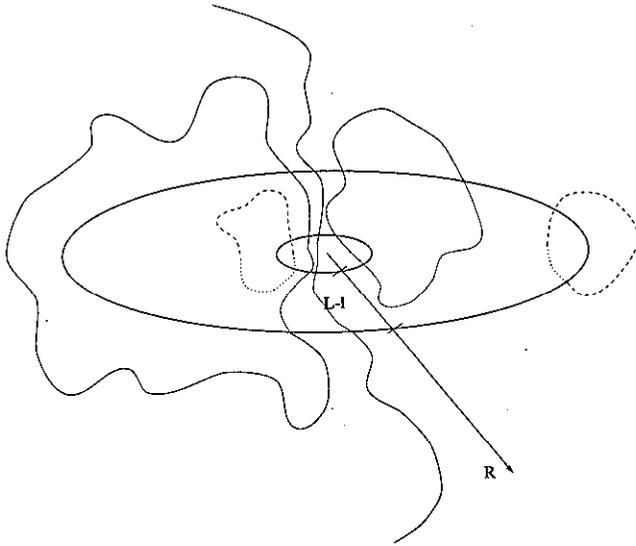


FIG. 3. A schematic view of the two concentric contours and of the strings that contribute to the winding number correlation function between them.

This can be shown to yield

$$I_1 = 4\pi^2 l L J_1(k_{\parallel} L) J_1(k_{\parallel} l). \quad (38)$$

We then see that in this case the correlation function is no longer, necessarily, positive definite but rather depends on the interplay between the radius of the two contours.

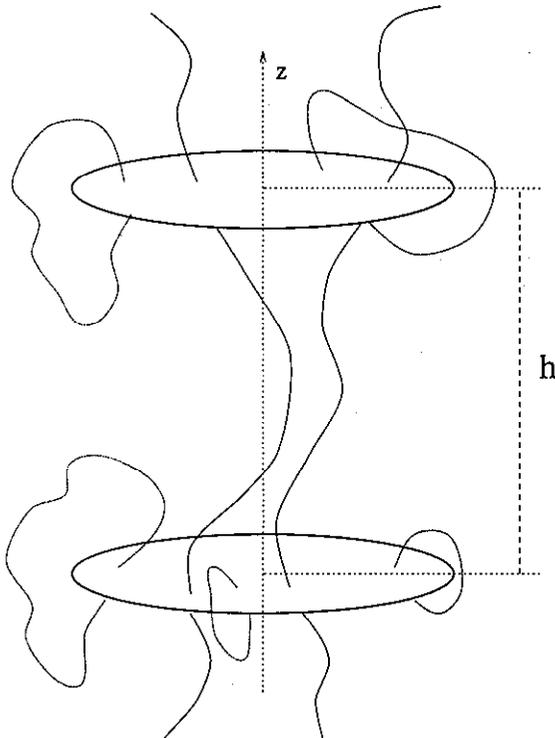


FIG. 4. The same as in Fig. 3 for the two coaxial circular contours.

The expression for the correlation function then becomes

$$\langle N_{\gamma} N_{\gamma'} \rangle = \frac{e^2}{4\pi^2} L l \int k_{\parallel} dk_{\parallel} dk_z \frac{J_1(k_{\parallel} L) J_1(k_{\parallel} l)}{\omega} n(\omega). \quad (39)$$

Figure 5 shows the result of the numerical integration of (39), having fixed the radius of one of the contours to be $l = \frac{1}{T}$, $\frac{5}{T}$, and $\frac{10}{T}$ and $m = 0.2T$. It shows that the winding number is most correlated when the radius of the two contours is comparable. Moreover, when the difference between the radii of the two contours becomes appreciable the correlation function exhibits the approximate behavior

$$\langle N_{\gamma} N_{\gamma'} \rangle \simeq \frac{e^2}{4\pi^2} L T 0.05 \frac{e^{-1.25m(L-l)}}{\sqrt{L-l}}. \quad (40)$$

Thus, for radial separations larger than essentially a few units of the inverse mass, the winding number through the two contours becomes essentially uncorrelated, showing that it should correspond to different strings. Strings, if produced by a thermal fluctuation, would therefore be expected to have a curvature radius smaller than that scale, corresponding necessarily to loops of that form.

Another possible generalization of (25) corresponds to translating the two contours, in the direction perpendicular to the plane they define, relative to each other so that they now define the two basis of a cylinder of height h ; see Fig. 4. In this case the form of the original integral still holds but the Fourier transform corresponding to the z direction ceases to be trivial. We then obtain

$$\langle N_{\gamma} N_{\gamma'} \rangle = \frac{e^2}{4\pi} L^2 \int_0^{\infty} k_{\parallel} dk_{\parallel} J_1^2(k_{\parallel} L) \int_0^{\infty} dk_z e^{ik_z h} \frac{n(\omega)}{\omega}, \quad (41)$$

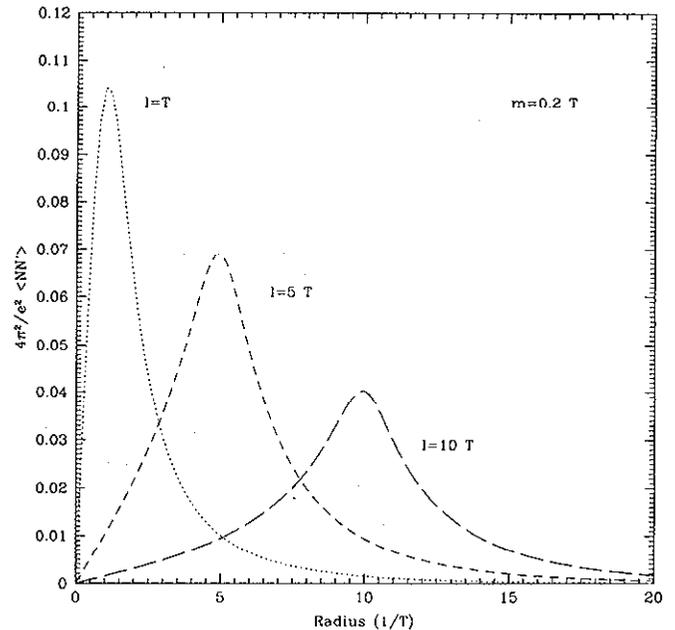


FIG. 5. The dependence of the winding number correlation function between the two concentric contours on one of the contour's radius L , for $m = 0.2T$ and $l = 1/T$, $5/T$, and $10/T$.

where h is the separation between the two contours along the vertical axis. This gives

$$\langle N_\gamma N_{\gamma'} \rangle = 2 \frac{e^2}{4\pi} L^2 \int_0^\infty k_\parallel dk_\parallel J_1^2(k_\parallel L) \times \int_0^\infty dk_z \cos(k_z h) \frac{n(\omega)}{\omega}. \quad (42)$$

Figure 6 shows the result of these integrations for several values of L and $m = 0.2T$. Again an approximate analytical behavior can be found for (42). The best fit on the mass and h dependences yields

$$\langle N_\gamma N_{\gamma'} \rangle \simeq \frac{e^2}{4\pi} 0.6LT e^{-1.25mh}. \quad (43)$$

Both correlation functions (39) and (42) above exhibit exponential falloffs with contour separation, determined by a correlation length multiple of the inverse gauge field mass. This behavior strictly holds for $m(L-l)$ and mh both larger than 1. Together with the result of the previous section about the magnitude of fluctuations, these allow us to characterize the temperature for which winding number fluctuations are relevant on a given length scale that can in turn be compared to the string's proper length. It is clear that, as the temperature drops, only the population of string loops smaller than $\xi = O(\frac{1}{m(T)})$ will be affected. Thus any long or infinite string emerging from the phase transition will be very little changed by fluctuations. This is an important result as it states that the formation of long strings is essentially a classical process occurring at the level of the Euler-Lagrange equations for the theory.

V. FLUCTUATIONS IN WINDING NUMBER IN THE PRESENCE OF STRINGS

In this section we investigate how a nonhomogeneous scalar modulus changes the fluctuations in winding number, as computed in Sec. III. This is an obvious requirement of any description of fluctuations taking place during a phase transition. Inhomogeneities occur naturally in the dynamics of domain growth and coalescence, corresponding to their interfaces as well as to topological defects.⁹ As the phase transition approaches completion

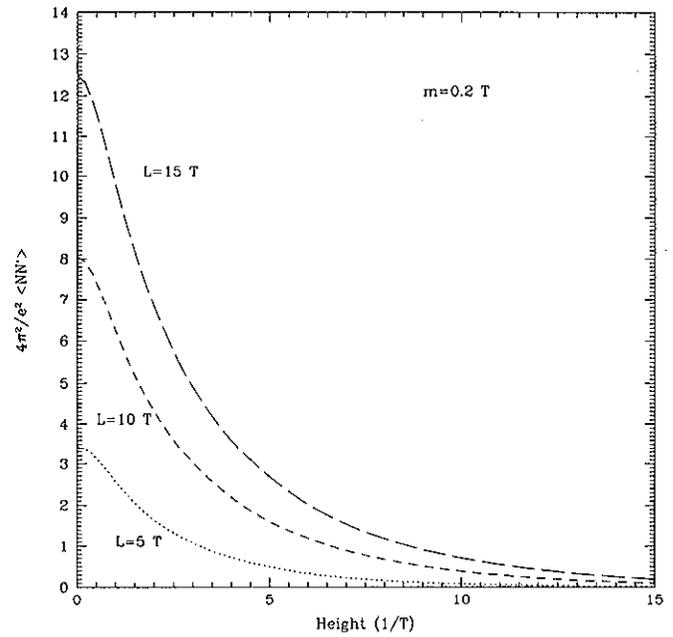


FIG. 6. The different curves exhibit the dependence of the winding number correlation function between the two circular coaxial contours on their separation h , for $m = 0.2T$ and $L = 5/T, 10/T, \text{ and } 15/T$.

we expect the domain structure to coarsen and smooth away and the only inhomogeneities left behind should correspond to topological defects. At all stages, thermal fluctuations in winding number will occur and a freeze-out in defect number will correspond to the instant in time when their role is rendered negligible, in some well-specified way.

In what follows we will attempt to give a general treatment of fluctuations in the presence of a weakly inhomogeneous scalar modulus by systematically expanding the gauge field propagator in a small perturbation around a homogeneous scalar field. We will later specialize to the case where strings constitute such inhomogeneities. As before we will only keep the pole term in the gauge field propagator.

Let us consider the case of a localized deviation from the otherwise homogeneous background so that we can write the scalar modulus as

$$\varphi(x) = \eta - f(x), \quad (44)$$

where η is the background value of the scalar field modulus and $f(x)$ the small perturbation. Then we can expand the gauge propagator around the homogeneous background field. We restrict ourselves to the relevant part of the transverse propagator, for the sake of clarity. Formally, we obtain

⁹In the early stages of this dynamics such a distinction probably fails to apply.

$$G_{\mu\nu}(x, y, M) = \langle x | \frac{-\delta_{\mu\nu}}{\square^2 + m^2 + M^2(z)} | y \rangle \simeq \langle x | \frac{-\delta_{\mu\nu}}{\square^2 + m^2} \left[1 - |z| M^2(z) |z| \frac{1}{\square^2 + m^2} + \frac{1}{2} |z| M^2(z) |z| \frac{1}{\square^2 + m^2} |z'| M^2(z') \langle z' | \frac{1}{\square^2 + m^2} + \text{higher order terms} \right] | y \rangle, \quad (45)$$

where $M^2(x) = e^2 f(x) [2\eta - f(x)]$. The first term in the expansion is just the gauge field propagator in the presence of a homogeneous, static scalar field in the broken phase. When used to compute winding number correlation functions it will generate the results of the previous two sections, when taken in thermal equilibrium. The first correction to that can be written as

$$G_{\mu\nu}^{(1)}(x, y) = \delta_{\mu\nu} 1 / (2\pi)^8 \int d^4 p d^4 k d^4 z d^4 z' G(p) e^{ip(z-x)} M^2(z, z') G(k) e^{ik(y-z')}, \quad (46)$$

where $M^2(z, z') = M^2(z')(2\pi)^4 \delta^4(z - z')$. This leads to

$$G_{\mu\nu}^{(1)}(x, y) = \delta_{\mu\nu} 1 / (2\pi)^4 \int d^4 p d^4 k G(p) e^{-ipx} M^2(k - p) G(k) e^{iky}, \quad (47)$$

where

$$M^2(k - p) = \int d^4 z e^{-iz(k-p)} M^2(z) \quad (48)$$

is the usual Fourier transform of $M^2(z)$. We see that the presence of inhomogeneities can be accounted for as the occurrence of sources, in the usual way. Their effect is naturally to change the four-momentum of photons from an in-going state to an out-going one. The order in the expansion (45) then corresponds to the number of such momentum transfers. To actually compute corrections to our previous results we have to specify the functional form of $M^2(x)$. As an illustration of the effect of inhomogeneities we take the simplest case of associating M^2 with one straight string at rest, placed in the center of the loop. That is,

$$M^2(x) = e^2 f(x) [2\eta - f(x)] \simeq 2e^2 \eta f(x), \quad (49)$$

where we specialize the form of $f(x)$ to correspond the field profiles of a string at distances larger than its width [17],

$$f(r) = k_s K_0(m_s r) \simeq k_s \sqrt{\frac{1}{2\pi m_s r}} e^{-m_s r}, \quad (50)$$

where K_0 is the modified Bessel function of order zero and k_s is a constant that for critically coupled strings can be shown to be $k_s = |n|\eta$, with n as the string's winding number. Here we assume strings to be the usual static solutions of the effective three-dimensional theory, obtained after the usual frequency summations at finite temperature. As such all mass scales exhibit their usual temperature dependence. The exponential regime corre-

sponds to the asymptotic form of the Bessel function for large arguments. In particular we see that, in this regime, the corrections to the homogeneous background propagator will be exponentially suppressed by the distance to the string.

It is straightforward to generalize the situation for one string to an arbitrary number of strings or other well localized sources of inhomogeneity with a definite profile. We simply write the scalar modulus as the superposition of all sources of inhomogeneity as

$$\varphi = \eta^{1-N} \prod_{i=1}^N (N - f_i) \simeq \eta - \sum_{i=1}^N f_i + \text{higher order terms in } f. \quad (51)$$

Let us then proceed to compute the change in fluctuations brought about by a single string, to first order. The form (50), corresponding to a single static cylindrically symmetric string placed in the center of our circular contours, actually allows us to compute its Fourier transform analytically; namely,

$$M^2(k - p) \simeq \int d^4 x e^{-i(p-q)x} 2m_e^2 K_0(m_s r) = 4\pi^2 m_e^2 \frac{\delta(k_0 - p_0) \delta(k_z - p_z)}{(k - p)_{\parallel}^2 + m_s^2}, \quad (52)$$

which is just the potential, in momentum space, for a massive cylindrically symmetric field, as it should be. Using this form in (47) yields the correction to the winding number correlation functions

$$\langle N_{\gamma}^2 \rangle^{(1)} = \frac{e^2}{8\pi^3} (Lm_e)^2 \int dp_{\parallel} dk_{\parallel} dk_z \frac{J_1(p_{\parallel} L) J_1(k_{\parallel} L)}{w_q w_k} \left(\frac{p_{\parallel}^2 + k_{\parallel}^2 + m_s^2}{\sqrt{(p_{\parallel}^2 + k_{\parallel}^2 + m_s^2)^2 - 4p_{\parallel}^2 k_{\parallel}^2}} - 1 \right) \times \left[\frac{1}{w_p + w_k} [n(w_p) + n(w_k)] - \frac{1}{w_k - w_p} [n(w_k) - n(w_p)] \right], \quad (53)$$

where we have assumed that both the in-going and out-going momenta display thermal distributions. This follows automatically since a static source causes no energy shift.

The integral (53) has to be computed numerically. Its dependence on L , together with that of the zeroth order result of Sec. II and their sum, are depicted in Fig. 7, for $\frac{m_e}{T} = 0.5$, $\frac{m_s}{T} = 0.6$. It shows that the effect of this correction is only to change winding number fluctuations significantly for small contour radius, smaller than or equal to the string's width. This can also be seen by fixing L and m_e and varying m_s . Indeed, the magnitude of the correction term becomes larger the smaller scalar masses we take, i.e., with greater string widths, signaling the breakdown of the weakly inhomogeneous scalar field expansion for the gauge propagator. In this regime the sum of the two terms yields a negative quantity, showing the clear breakdown of the expansion (45) since the full quantity is positive definite. For large contour radius the effect of the correction becomes increasingly more negligible and the negative shift probably becomes a genuine feature. This is an important feature as it seems to indicate that winding number fluctuations computed on a homogeneous background are an overestimate of their real values.

Another relevant issue is to know how this correction term behaves close to the critical temperature. This can be probed by assuming that both masses vanish approximately as we approach the critical temperature. In practice we know this to be untrue for the gauge field due to higher order effects, as discussed in Sec. III. We can distinguish two cases, namely, when the gauge coupling is larger or smaller than the scalar self-coupling. This differentiates between a type-I and a type-II theory, re-

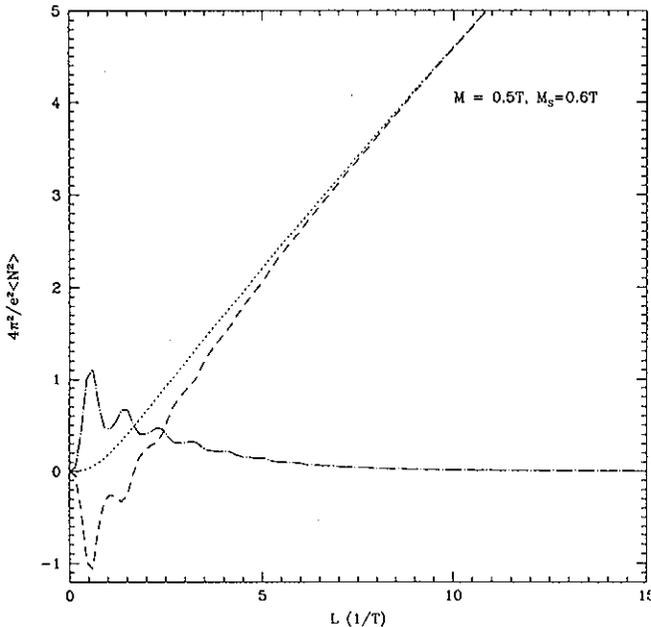


FIG. 7. The magnitude of the zeroth order term (dotted line), the absolute value of its first correction, and their sum as a function of contour radius and for $m_s = 0.4T$ and $m = 0.2T$.

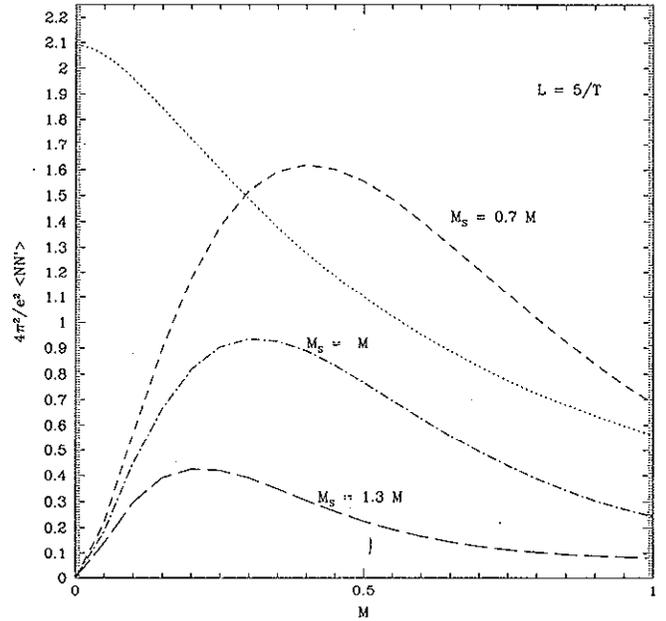


FIG. 8. The dependence of the zeroth order term in winding number fluctuations (dotted line) and its first order correction on the gauge field mass M , for $L = 5/T$ and in the cases $m_s = 0.7M$, M , and $1.2M$. The mass scale M is in units of temperature.

spectively. Figure 8 shows the dependence of the zeroth order term on the gauge mass as well as that of (54) for $m_s = 1.5m_e$, m_e , $0.5m_e$, for fixed $L = 5/T$. We see that the magnitude of the correction term grows with the gauge coupling, while still remaining smaller than the zeroth order term. This, however, does not hold for strong type-I theories for which we see that larger values of L will be required for the series expansion to hold. One important feature of the correction term, which holds regardless of the hierarchy of the couplings in the theory, is that it vanishes, at critical temperature, with the scalar mass. This should constitute a genuine behavior of all correction terms since they are proportional to increasing powers of $M(x)$. Thus as the scalar thermal mass goes to zero so does $M(x)$ and we recover the usual electromagnetic propagator. This allows us to conclude that the weakly inhomogeneous background expansion can always be trusted in a sufficiently small neighborhood of the critical temperature and that the effect of the presence of strings will manifest itself mostly at intermediate and low temperatures. These conclusions would radically change if we do not adopt a thermal equilibrium mass scale (vanishing at the critical temperature) in our defect configurations. In assuming this we hope to mimic an average defect configuration in the presence of a thermal plasma and thereby render the whole picture self-consistent.¹⁰

¹⁰In the context of a first order transition, when strong deviations from equilibrium are expected, this picture will probably, very probably, not hold.

VI. CONCLUSIONS

In this paper we have developed techniques to allow us to study the role of fluctuations in cosmic string formation in an Abelian gauge theory. Their application to the simplest case of quasihomogeneous scalar field backgrounds leads us to conclude that thermal fluctuations in winding number are strongest the closer the system is to the critical temperature and, further, that the relative winding number fluctuations diverge at that temperature, signaling the presence of no stable defects, in these circumstances. This effect persists even in the presence of inhomogeneities in the fields, such as strings themselves, if thermal equilibrium is used throughout. As the temperature drops the role of thermal fluctuations in changing winding number becomes less and less important. The energy scale that characterizes an approximate freeze-out in defect numbers is most naturally the inverse mean domain size at the time of coalescence, which in turn is associated with the average string separation. From the point of view of the field theory its determination necessarily implies a more detailed knowledge of the field dynamics during the transition than what is currently known.

On developing generalizations of our initial winding number correlation function we also conclude that string configurations can only be changed by thermal fluctuations on scales of the order of the thermal correlation length of the fields. As such the population of small string loops can be drastically modified but not that of long and infinite strings, which is the most relevant input for network numerical evolutions. A change in loop population is only likely, for the present cosmic string network implementations, to change the approach to the scaling regime and will therefore bring no change to the details of late time structure seeding in the canonical scenario.

An analogous winding number correlation function over the same contour can be defined using the phase of the Higgs field directly [9]. Using the gauge sector instead has great advantages in computational simplicity and reliability as the former is divergent as the symmetric phase is approached from below. This leads to infinities

and computability can only be maintained by adopting severe approximations. Both methods will necessarily measure fluctuations in the gauge and scalar fields, respectively, that do not necessarily correspond to strings. This is because, even though one of the fields can have the right configuration to generate a string, it will take the two at the same point in space to produce a Nielsen-Olesen vortex. One is then over counting strings as we will be computing the number of strings present at a given instant together with that of fluctuations that can potentially produce a string or decay away.¹¹

Our methods should have other applications beyond the grand unified theory (GUT) transitions. The system considered above has obvious analogies with a superconductor where the Ginzburg-Landau free energy is given precisely by a nonrelativistic analogue of the Abelian Higgs model. Our analysis for the winding number correlation functions generalizes to both models provided the mean field masses and self-energies get properly translated. This has yet to be done, but would possibly constitute the natural test on the approach developed above as comparison with experiment seems a feasible task. The next closest example of string configurations in a specific model concerns those in the electroweak standard model. Their study in the simplest approach consists in treating the model as an effective Abelian theory of a complex scalar field and the Z gauge boson. Even though the supplementary field excitations that are known render them unstable [19], our winding number correlation functions should find applicability here, too.

The eventual applicability of this approach to genuinely out-of-equilibrium and inhomogeneous scenarios hinges upon a better knowledge of the corresponding fields and their two-point correlation functions. A considerable amount of effort is being devoted to this problem [12] and it is conceivable that the full problem will be tractable in the near future.

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¹¹Other criteria exist to count the number of strings and their fluctuations, namely, those inspired in counting the density of zeros of the scalar field [18]. However, the same methods have yet to be applied to gauge theories.

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