Implications of anomaly constraints in the N_c-extended standard model

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We discuss some implications of anomaly cancellation in the standard model with (i) the color group extended to SU(N_c) and (ii) the leptonic sector extended to allow right-handed components for neutrinos. [S0556-2821(96)02611-2]

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I. INTRODUCTION

While the standard model [1] is quite successful in explaining known data, there are many questions which it leaves unanswered. One of the most basic is why the gauge group is $G_{\rm SM}=\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y$. There have been several appealing ideas which might answer, or help to answer, this question, such as grand unification and the more ambitious efforts to derive the standard model from a fundamental theory of all known interactions, including gravity [2]. A somewhat complementary approach is to consider how the standard model gauge group might be viewed as one of a sequence of gauge groups (which, in general, are still products of factor groups). In particular, it has proved quite useful to consider the number of colors as a parameter, and study the $N_c\to\infty$ limit of the quantum chromodynamics (QCD) sector of the theory [3–6], since this enables one to carry out analytic nonperturbative calculations (and, indeed, to obtain a soluble model in $d=2$ spacetime dimensions) [7]. Many of these discussions naturally concentrated on using the $1/N_c$ expansion to elucidate the properties of hadrons. When one includes electroweak interactions, however, one is led to address some additional questions. One of these concerns anomalies.

The freedom from anomalies is a necessary property of an acceptable quantum field theory. In $d=4$ dimensions, there are three types of possible anomalies in quantum field theories, including (i) triangle anomalies in gauged currents [8,9] which, if present, would spoil current conservation and hence renormalizability; and (ii) the global SU(2) anomaly resulting from the nontrivial homotopy group $\pi_4[\mathrm{SU}(2)]=Z_2$ [10] which, if present, would render the path integral ill-defined. Furthermore, (iii) if one includes gravitational effects on a semiclassical, even if not fully quantum level, one is motivated to require the absence of mixed gauge-gravitational anomalies [11] resulting from triangle diagrams involving two energy momentum tensor (graviton) vertices and a $\mathrm{U}(1)_Y$ gauge vertex, since this anomaly, if present, would also spoil conservation of the hypercharge current as well as precluding the construction of a generally covariant theory. As is well known, in the standard model (SM), all of these anomalies vanish [1,12,9], and for anomalies of types (i) and (ii), this vanishing occurs in a manner which intimately connects the quark and lepton sectors. Furthermore, in the standard model (with no right-handed neutrinos) the cancellation of the anomalies of type (i) implies the quantization of the fermion electric charges [13]; this also holds in an extension of the standard model where right-handed neutrinos are included but are assumed to have zero hypercharge [13]. Note that the gauge-gravitational anomaly vanishes separately for quark and lepton sectors.

The issue of anomalies in the N_c-extended standard model has recently been addressed explicitly by Chow and Yan [14]. These authors note that the anomaly cancellation conditions can be satisfied for arbitrary (odd) N_c, and the solution leads to unique, quantized, values of the electric charges of the up-type and down-type quarks, q_u and q_d. The present author had carried out a similar analysis for a different type of generalization of the standard model, namely, one in which the color group is extended to SU(N_c) and the leptonic sector is extended to include right-handed neutrino fields.

In this paper, we shall discuss the results of this analysis. These results present an interesting contrast to those in the N_c-extended standard model (with no right-handed neutrinos). Both types of generalizations of the SM (excluding or including right-handed neutrinos) are of interest. The generalization without any right-handed neutrinos may provide a more economical way of getting small neutrino masses (via dimension-five operators [15]), while the generalization with right-handed neutrinos is motivated in part by the fact that these make possible Dirac and right-handed Majorana mass terms for neutrinos at the renormalizable, dimension-four level, which naturally yield small observable neutrino masses via the seesaw mechanism [16,17], given that the natural scale for the mass coefficients of the right-handed Majorana neutrino bilinears is much larger than the electroweak symmetry breaking (EWSB) scale. In the usual extension of the standard model, the right-handed neutrino fields are electroweak singlets, a property which is crucial for the existence of the right-handed Majorana mass term. However, when one considers the N_c-extended standard model with right-handed neutrino fields from the perspective of determining the constraints on the fermion hypercharges Y_f which follow from the requirement of cancellation of anomalies, the hypercharge of the right-handed neutrinos (like the hypercharges of the other fields) naturally becomes a variable, not necessarily equal to zero [18]. If the hypercharge, and hence electric charge, of the right-handed neutrinos is nonzero, then the nature of the theory changes in a funda-

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mental way. Indeed, the term ''neutrino'' becomes a misnomer; we shall retain it here only to avoid proliferation of terms (it is no worse than the accepted term ''heavy lepton''). Clearly, if $Y_{\nu_R} \neq 0$, then the right-handed Majorana bilinear $\nu_{iR}^T C \nu_{jR}$ is forbidden by gauge invariance (where i,j denote generation indices, and C denotes the Dirac charge conjugation matrix). Given that Dirac mass terms for the neutrinos would be present in this type of theory, it would be natural for all of the fermions of a given generation to have comparable masses [19]. This class of models is of interest from an abstract field-theoretic viewpoint, because it serves as a theoretical laboratory in which to investigate the properties that follow from anomaly cancellation in a chiral gauge theory constituting a generalization of the standard model with N_c colors, constructed such that all left-handed Weyl components have right-handed components of the same electric charge.

In most of our discussion, we shall not need to make any explicit assumption concerning the still-unknown origin of electroweak symmetry breaking. At appropriate points, we shall comment on how various formulas would apply in the N_c-extended minimal supersymmetric standard model (MSSM) as well as the N_c-extended standard model itself (in both cases, including right-handed components for all matter fermions). As regards anomalies in the context of the MSSM, recall that in addition to the usual Higgs H_d, one must introduce another, H_u, with opposite hypercharge, both in order to be able to give the up-type quarks masses while maintaining a holomorphic superpotential, and in order to avoid anomalies in gauged currents which would be caused by the higgsino $\widetilde{H}_d$ if it were not accompanied by a $\widetilde{H}_u$. [The addition of a single $\widetilde{H}_d$ to the (even) number of matter fermion SU(2) doublets would also cause a global SU(2) anomaly.] All of this works in the same way regardless of the charges of the matter fermions, provided that the latter satisfy the anomaly cancellation condition by themselves. Moreover, as regards the neutralino sector, electric charge conservation by itself would allow mixing of neutrinos and neutralinos (the neutral higgsinos and superpartners of the gauge fields $\widetilde{A}^0$ and $\widetilde{B}$) if and only if $q_\nu = 0$. However, the R parity commonly invoked in the MSSM to prevent disastrously rapid proton decay also prevents mixing among the neutralinos and neutrinos even in the conventional case where $q_\nu = 0$, so there would be no change concerning this mixing even if $q_\nu \neq 0$. Finally, considering alternative ideas for electroweak symmetry breaking, one could envision embedding the $G'_{\rm SM}$ theory in a larger one in which this symmetry breaking is dynamical.

II. ANOMALY CONSTRAINTS AND THEIR IMPLICATIONS

A. General

Consider, then, the generalization

$$G_{\rm SM} \to G'_{\rm SM} = {\rm SU}(N_c) \times {\rm SU}(2) \times {\rm U}(1)_Y \tag{2.1}$$

with the fermion fields consisting of the usual $N_{\rm gen} = 3$ generations, each containing the following representations of $G'_{\rm SM}$:

$$Q_{iL} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L : \quad (N_c, 2, Y_{Q_L}), \tag{2.2}$$

$$u_{iR}: \quad (N_c, 1, Y_{u_R}), \tag{2.3}$$

$$d_{iR}: \quad (N_c, 1, Y_{d_R}), \tag{2.4}$$

$$\mathcal{L}_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L : \quad (1, 2, Y_{\mathcal{L}_L}), \tag{2.5}$$

$$\nu_{jR}: \quad (1, 1, Y_{\nu_R}), \tag{2.6}$$

$$e_{iR}: \quad (1, 1, Y_{e_R}), \tag{2.7}$$

where the index i denotes generation, $i = 1, \ldots, N_{\rm gen} = 3$, with $u_1 = u$, $u_2 = c$, $u_3 = t$, $d_1 = d$, $d_2 = s$, $d_3 = b$, etc. Thus, as usual, all generations have the same gauge quantum numbers. (In some formulas, we shall leave $N_{\rm gen}$ arbitrary for generality.) Because the SU(2) representations are the same in $G'_{\rm SM}$ as they were in $G_{\rm SM}$, the usual relations $Q = T_3 + Y/2$, $Y_{Q_L} = q_u + q_d$, $q_u = q_d + 1$, $q_\nu = q_e + 1$, and $Y_{f_R} = 2 q_{f_R}$ continue to hold, independent of the specific values of the fermion electric charges (where we have used the vectorial nature of the electric charge coupling, $q_{f_L} = q_{f_R} = q_f$ for all fermions f); and just as in the standard model itself, these relations imply

$$Y_{u_R} = Y_{Q_L} + 1, \quad Y_{d_R} = Y_{Q_L} - 1 \tag{2.8}$$

and

$$Y_{\nu_R} = Y_{\mathcal{L}_L} + 1, \quad Y_{e_R} = Y_{\mathcal{L}_L} - 1. \tag{2.9}$$

Before imposing the anomaly cancellation conditions, there are thus only two independent electric charges among the fermions; we may take these to be q_d and q_e. For $N_c = 3$ and $q_\nu = 0$, one may, *a priori*, have $j = 1, \ldots, N_s$ electroweak-singlet right-handed neutrinos ν_{jR}, where N_s need not be equal to $N_{\rm gen}$. However, in the general solution to the anomaly cancellation conditions for $N_c \neq 3$ (see below) the electric charges of all of the fermions will differ from their $N_c = 3$ values. In particular, since q_ν will not, in general, be equal to zero, the number N_s of electroweak-singlet right-handed neutrinos ν_{jR} must be equal to the number $N_{\rm gen}$ of left-handed lepton doublets in order to construct renormalizable, dimension-four neutrino mass terms, which in turn is necessary in this case to avoid massless, charged, unconfined fermions in the theory. Given that $N_s = N_{\rm gen}$, the number $N_{\rm gen.}$ enters in a trivial way as a prefactor in all of the expression for the anomalies of types (i) and (iii), i.e., these cancel separately for each generation of fermions. Accordingly, we shall often suppress the generational index in the notation henceforth.

The hypercharge relations (2.8) and (2.9) guarantee that, independent of the specific values of the fermion charges, one can write $G'_{\rm SM}$-invariant Yukawa couplings

$$-\mathcal{L}_{Yuk}=\sum_{i,j}\left[(Y_{ij}^{(d)}\overline{Q_{iL}}d_{jR}+Y_{ij}^{(\ell)}\overline{\mathcal{L}_{iL}}e_{jR})H_d+(Y_{ij}^{(u)}\overline{Q_{iL}}u_{jR}+Y_{ij}^{(\nu)}\overline{\mathcal{L}_{iL}}\nu_{jR})H_u\right]+\text{H.c.}, \tag{2.10}$$

where in a context in which one uses a single standard-model Higgs field, ϕ, with $I_\phi=1/2$, $Y_\phi=1$, then $H_d=\phi$ and $H_u=i\sigma_2\phi^*$ as usual, and in the minimal supersymmetric standard model (MSSM), H_d and H_u correspond to the scalar components of the two oppositely charged Higgs chiral superfields. [In Eq. (2.10), no confusion should result between the symbols $\mathcal{L}_{Yuk}$ for the Lagrangian terms and $\mathcal{L}_{iL}$ for the lepton doublets.] The vacuum expectation value(s) (VEV's) of the Higgs fields then yield fermion mass terms. We denote these VEV's as $\langle\phi\rangle=2^{-1/2}v$ in the SM, with $v=2^{-1/4}G_F^{-1/2}$, and $\langle H_{u,d}\rangle=2^{-1/2}v_{u,d}$ in the MSSM, with $\tan\beta=v_u/v_d$ and $v=\sqrt{v_u^2+v_d^2}$. In a scenario without Higgs, in which the electroweak symmetry breaking is dynamical, the fermion mass terms are envisioned to arise from four-fermion operators (the origin of which is explained with further theoretical inputs). In all three cases, this can be done just as in the respective $N_c=3$ model with conventional fermion charge assignments. It is also straightforward to see that in either a nonsupersymmetric model with the single Higgs ϕ, or the MSSM, or a model with dynamical electroweak symmetry breaking, the breaking pattern

$$G'_{\rm SM}\to SU(N_c)\times U(1)_{\rm em} \tag{2.11}$$

can be arranged, just as for the $N_c=3$ case with conventional fermion charges.

B. Anomalies in gauged currents

We proceed to analyze the constraints from the cancellation of the three types of anomalies. Among the triangle anomalies of type (i), the $SU(N_c)^3$ and $SU(N_c)^2U(1)_Y$ anomalies vanish automatically (as for $N_c=3$) because of the vectorial nature of the color and electromagnetic couplings. The condition for the vanishing of the $SU(2)^2U(1)_Y$ anomaly is

$$N_cY_{Q_L}+Y_{\mathcal{L}_L}=0, \tag{2.12}$$

i.e.,

$$N_c(2q_d+1)+(2q_e+1)=0. \tag{2.13}$$

The $U(1)_Y^3$ anomaly vanishes if and only if

$$N_c(2Y_{Q_L}^3-Y_{u_R}^3-Y_{d_R}^3)+(2Y_{\mathcal{L}_L}^3-Y_{\nu_R}^3-Y_{e_R}^3)=0. \tag{2.14}$$

Expressing this in terms of q_d and q_e yields the same condition as Eq. (2.13). Solving (2.13) for q_d yields

$$q_d=-\frac{1}{2}\left(1+\frac{1}{N_c}(2q_e+1)\right) \tag{2.15}$$

and hence

$$q_u=\frac{1}{2}\left(1-\frac{1}{N_c}(2q_e+1)\right) \tag{2.16}$$

or equivalently, taking q_d as the independent variable,

$$q_e=-\tfrac{1}{2}\left(1+N_c(2q_d+1)\right) \tag{2.17}$$

and thus

$$q_\nu=\tfrac{1}{2}\left(1-N_c(2q_d+1)\right) \tag{2.18}$$

C. Global SU(2) anomaly

The constraint from the global SU(2) anomaly is well known [10]: the number N_d of SU(2) doublets must be even:

$$N_d=(1+N_c)N_{\rm gen.}\ \text{is even.} \tag{2.19}$$

For odd $N_{\rm gen.}$, this implies that N_c is odd. For a nontrivial color group, this means $N_c=2s+1$, $s\geqslant 1$. Note that one gets a qualitatively different result in the hypothetical case in which $N_{\rm gen.}$ is even; here, there is no restriction on whether N_c is even or odd. From a theoretical point of view, one could perhaps regard it as satisfying that the physical value $N_{\rm gen.}=3$ is odd and hence is such as to yield a constraint on N_c [20]. Of course, a world with even N_c would be very different from our physical world, since baryons would be bosons.

D. Mixed gauge-gravitational anomalies

Finally, the anomalies of type (iii) do not add any further constraint; the mixed gauge-gravitational anomaly involving $SU(N_c)$ and SU(2) gauge vertices vanish identically since $Tr(T_a)=0$ where T_a is the generator of a nonabelian group, and the anomaly involving a $U(1)_Y$ vertex is proportional to

$$N_c(2Y_{Q_L}-Y_{u_R}-Y_{d_R})+(2Y_{\mathcal{L}_L}-Y_{\nu_R}-Y_{e_R})=0, \tag{2.20}$$

where the expression vanishes because of the vectorial nature of the electromagnetic coupling. Indeed, the two separate terms in parentheses each vanish individually: $2Y_{Q_L}-Y_{u_R}-Y_{d_R}=0$ and $2Y_{\mathcal{L}_L}-Y_{\nu_R}-Y_{e_R}=0$, so that this anomaly does not connect quark and lepton sectors, unlike (2.12), (2.14) and the global SU(2) anomaly. Hence, the only constraint on the fermion charges is provided by the condition that the anomalies of type (i) vanish.

E. Discussion

Our results show that the SM has a consistent generalization to the gauge group $G'_{\rm SM}$ in Eq. (2.1) with fermion charges given by (2.2)–(2.7). We find the one-parameter family of solutions given in (2.13) to the condition of zero anomalies in gauged currents. Since the values of q_d and q_e for which (2.13) is satisfied are, in general, real, and are not restricted to the rational numbers, it follows that in this generalization of the standard model, the anomaly cancellation conditions do not imply the quantization of electric charge (and hence, hypercharge). We note that this is qualitatively different from the type of generalization studied in Ref. [14], in which one extends $G_{\rm SM}\to G'_{\rm SM}$ but keeps the

TABLE I. Possibilities for quark charges.

Case	q_d	(q_u,q_d)	Y_{Q_L}	$Y_{\mathcal{L}_L}$
$C1_q$	>0	$(+,+)$	>1	$<-N_c$
$C2_q$	$-1<q_d<0$	$(+,-)$	$-1<Y_{Q_L}<1$	$-N_c<Y_{\mathcal{L}_L}<N_c$
$C2_{q,\mathrm{sym}}$	$-1/2$	$(1/2,-1/2)$	0	0
$C3_q$	<-1	$(-,-)$	<-1	$>N_c$
$C4_q$	0	(1,0)	1	$-N_c$
$C5_q$	-1	$(0,-1)$	-1	N_c

fermion content precisely as in the standard model, with no electroweak-singlet right-handed neutrinos. In that case, one must keep $q_\nu=0$ in order to avoid a massless, charged, unconfined fermion, and hence the lepton charges must be kept at their $N_c=3$ values while the quark charges are allowed to vary. Hence, the one-parameter family of solutions (2.13) reduces to a unique solution

$$q_d=q_u-1=\frac{1}{2}\left(-1+\frac{1}{N_c}\right) \tag{2.21}$$

and the anomaly cancellation conditions [specifically, of type (i)] do imply charge quantization, as was noted in Ref. [14]. In passing, we observe that for our type of generalization, although the generic situation for the solutions of Eq. (2.13) is that Y_{Q_L} and $Y_{\mathcal{L}_L}$ are real numbers, it is true that this equation implies that if either is rational, so is the other.

III. CLASSIFICATION OF SOLUTIONS FOR QUARK CHARGES

There is another important difference in the properties of the two types of N_c-extended standard model in which one includes or excludes right-handed neutrinos. In the case where one excludes them, Eq. (2.21) shows that (given a nontrivial color group) q_d is always negative, and q_u is always positive, and both decrease monotonically as functions of N_c [from $(q_u,q_d)=(2/3,-1/3)$ at $N_c=3$ to $(1/2,-1/2)$ in the limit as $N_c\to\infty$]. The situation is qualitatively different in the N_c-extended standard model with right-handed neutrinos; here, there are a number of different cases (denoted Cn_q) describing the up and down quark charges, of which three are generic and two are borderline. (We also list a certain special subcase because of its symmetry.) Regarding q_e as the independent variable in the solution of Eq. (2.13), these are

$$C1_q: \quad q_d>0 \ \ (\Rightarrow q_u>0), \tag{3.1}$$

i.e., $Y_{Q_L}>1$, which occurs if and only if $Y_{\mathcal{L}_L}<-N_c$, that is,

$$q_e<-\left(\frac{N_c+1}{2}\right), \tag{3.2}$$

$$C2_q: \quad q_u>0, \ \ q_d<0, \tag{3.3}$$

or equivalently, $-1<q_d<0$, which occurs if and only if

$$-\left(\frac{N_c+1}{2}\right)<q_e<\left(\frac{N_c-1}{2}\right) \tag{3.4}$$

and

$$C3_q: \quad q_u<0 \ \ (\Rightarrow q_d<0), \tag{3.5}$$

or equivalently, $q_d<-1$, which occurs if and only if

$$q_e>\left(\frac{N_c-1}{2}\right). \tag{3.6}$$

A symmetric special charge within case $C2_q$ is

$$C2_{q,\mathrm{sym}}: \quad q_u=-q_d=\tfrac{1}{2}\Leftrightarrow q_\nu=-q_e=\tfrac{1}{2}. \tag{3.7}$$

Finally, there are two special cases which are borderline between $C1_q$ and $C2_2$, and $C2_q$ and $C3_q$, respectively, and in which q_u or q_d is electrically neutral:

$$C4_q: \quad q_d=0, \ \ q_u=1\Leftrightarrow q_e=-\left(\frac{N_c+1}{2}\right) \tag{3.8}$$

and

$$C5_q: \quad q_u=0, \ \ q_d=-1\Leftrightarrow q_e=\left(\frac{N_c-1}{2}\right). \tag{3.9}$$

These cases are summarized in Table I.

From Eq. (2.12), it is clear that q_u and q_d are monotonically increasing (decreasing) functions of N_c if $q_e<-1/2$ ($q_e>-1/2$). In the borderline case $q_e=-1/2$, q_u and q_d are independent of N_c (and equal to the respective values in $C2_{q,\mathrm{sym}}$), so that the anomalies of type (i) cancel separately in the quark and lepton sectors. For $N_c=3$, the explicit conditions on q_e for the five cases are (1) $q_e<-2$; (2) $-2<q_e<1$; (3) $q_e>1$; (4) $q_e=-2$; and (5) $q_e=1$. As these results show, even for $N_c=3$, in the standard model with right-handed components for all fields, the cancellation of anomalies does not imply that any field, and in particular, any leptonic field, must have zero electric charge.

IV. CLASSIFICATION OF SOLUTIONS FOR LEPTON CHARGES

The corresponding possible cases for leptonic (ℓ) electric charges are as follows, taking q_d as the independent variable in Eq. (2.13):

$$C1_\ell: \quad q_e>0 \ \ (\Rightarrow q_\nu>0), \tag{4.1}$$

if and only if

TABLE II. Possibilities for lepton charges.

Case	q_e	(q_ν, q_e)	$Y_{\mathcal{L}_L}$	Y_{Q_L}
$C1_\ell$	>0	$(+,+)$	>1	$<-1/N_c$
$C2_\ell$	$-1<q_e<0$	$(+,-)$	$-1<Y_{\mathcal{L}_L}<1$	$-1/N_c<Y_{Q_L}<1/N_c$
$C2_{\ell,\mathrm{sym}}$	$-1/2$	$(1/2,-1/2)$	0	0
$C3_\ell$	<-1	$(-,-)$	<-1	$>1/N_c$
$C4_\ell$	0	(1,0)	1	$-1/N_c$
$C5_\ell$	-1	$(0,-1)$	-1	$1/N_c$

$$q_d<-\frac{1}{2}\left(1+\frac{1}{N_c}\right), \tag{4.2}$$

$$C2_\ell: \quad q_\nu>0, \;\; q_e<0, \tag{4.3}$$

if and only if

$$-\frac{1}{2}\left(1+\frac{1}{N_c}\right)<q_d<-\frac{1}{2}\left(1-\frac{1}{N_c}\right), \tag{4.4}$$

$$C3_\ell: \quad q_\nu<0 \;\; (\Rightarrow q_e<0), \tag{4.5}$$

if and only if

$$q_d>-\frac{1}{2}\left(1-\frac{1}{N_c}\right). \tag{4.6}$$

The symmetric subcase $C2_{\ell,\mathrm{sym}}$ is identical to $C2_{q,\mathrm{sym}}$ in Eq. (3.7). The two special cases which are borderline between $C1_\ell$ and $C2_\ell$, and between $C2_\ell$ and $C3_\ell$ are, respectively,

$$C4_\ell: \quad q_e=0, \;\; q_\nu=1 \Leftrightarrow q_d=-\frac{1}{2}\left(1+\frac{1}{N_c}\right) \tag{4.7}$$

and

$$C5_\ell: \quad q_\nu=0, \;\; q_e=-1 \Leftrightarrow q_d=-\frac{1}{2}\left(1-\frac{1}{N_c}\right). \tag{4.8}$$

These are summarized in Table II.

Several comments are in order. First, note that q_e and q_ν are monotonically increasing (decreasing) functions of N_c if $q_d<-1/2$ ($q_d>-1/2$). The special case $q_d=-1/2$ has been discussed above. Second, observe that, even if we include right-handed neutrinos, so that q_ν need not be zero in general, there is, for a given N_c, a solution of (2.13) where it is zero, namely case $C5_\ell$.

Of the various cases of lepton charges, two would yield a world similar to our own, in the sense that there would be neutral leptons with masses which are naturally much less than the electroweak symmetry breaking scale v. The closest would be case $C5_\ell$, where the neutrino has zero charge. As will be discussed further below, case $C4_\ell$, with $q_e=0$ would also be reminiscent of our world. The lightness of the masses of the observed electron-type leptons in this case would follow from a seesaw mechanism completely analogous to that for the neutrinos in the physical world; in this case, since $Y_{e_R}=0$, there would be gauge-invariant right-handed Majorana mass terms of the form $\Sigma_{i,j=1}^{N_{\mathrm{gen.}}} m_{R,ij} e_{iR}^T C e_{jR}+\mathrm{H.c.}$ in addition to the usual Dirac neutrino mass terms resulting from Eq. (2.10). By the usual argument, since the right-handed electron Majorana mass terms are electroweak singlets, the mass coefficients $m_{R,ij}$ are naturally much larger than the electroweak scale. Diagonalizing the combined Dirac-Majorana neutrino mass matrix would yield two sets of mass eigenvalues and corresponding (generically Majorana) mass eigenstates, the observed, light electron-type leptons having masses $\sim m_D^2/m_R \ll m_D$ and the heavy ones having masses $\sim m_R$ (where m_D denotes a generic Dirac mass, and we suppress generational indices).

Although the generic situation in our generalization of the standard model is that the electric charges of all the fundamental fermions are nonzero, there are evidently four special cases in which one type of fermion has zero charge, viz., $C4_q$ ($q_d=0$), $C5_q$ ($q_u=0$), $C4_\ell$ ($q_e=0$), and $C5_\ell$ ($q_\nu=0$). In each of the two leptonic cases containing a neutral lepton, one may define a new model in which one excludes the right-handed Weyl component for all generational copies of this lepton, viz., e_{iR} for case $C4_\ell$, and ν_{iR} for case $C5_\ell$, where $i=1,\ldots,N_{\mathrm{gen.}}$. Performing the excision of the ν_{iR} in case $C5_\ell$ and putting $N_c=3$ just yields the standard model. Performing the analogous excision of the e_{iR} fields in case $C4_\ell$ yields a model in which the electron-type leptons are naturally light, for the same reason that the neutrinos are naturally light in the standard model, namely that (a) there are no four-dimension Yukawa terms contributing to the masses of electron-type leptons; and (b) higher-dimension operators (which one would take account of when one views the model as a low-energy effective field theory) give naturally small masses. Indeed, the argument for the lightness of the neutral lepton in these reduced models, $C4_\ell$ with no e_{iR} fields, and $C5_\ell$ with no ν_{iR} fields, could be regarded as more economical than the seesaw mechanism, since the same result is achieved with a smaller field content (albeit by making reference to higher-dimension, nonrenormalizable operators). In passing, we note that in cases $C4_q$ and $C5_q$ where the conditions that there be no $\mathrm{SU}(2)^2\mathrm{U}(1)_Y$ or $\mathrm{U}(1)_Y^3$ anomalies, and no mixed gauge-gravitational or global SU(2) anomalies, by themselves, would allow one to define reduced models without d_{iR} and u_{iR} components, respectively (analogously to the removal of e_{iR} and ν_{iR} in the leptonic cases $C4_\ell$ and $C5_\ell$), this is, of course, forbidden because it would produce $\mathrm{SU}(N_c)^3$ and $\mathrm{SU}(N_c)^2\mathrm{U}(1)_Y$ anomalies, as well as rendering the color group chiral and thereby contradicting the observed absence of parity and charge conjugation violation in strong interactions.

An important observation concerns a connection between the values of the lepton charges and the perturbative nature of the electroweak sector. In the standard model, the observed electroweak decays and reactions are perturbatively calculable. However, in the generalized N_c-extended standard model that we consider here, this is no longer guaranteed to be the case, even if the SU(2) and $U(1)_Y$ gauge couplings g and g', and hence also the electromagnetic coupling, $e=gg'/\sqrt{g^2+g'^2}=g\sin\theta_W$, are small. Because the left-handed fermions have fixed, finite values of weak $T_3=\pm 1/2$, the SU(2) gauge interactions are still perturbative, as in the usual standard model. However, for a given value of N_c, the solution to the anomaly condition (2.12) allows arbitrarily large values of the magnitudes of fermion hypercharges and equivalently, electric charges, as is clear from the explicit solutions (2.15)–(2.18). If $|q_d|\gg 1$ (which, for a fixed value of N_c, implies $|q_e|\gg 1$), then even though the gauge coupling g' is small, the hypercharge interactions would involve strong coupling, since $|g'Y_f|\gg 1$ for each matter fermion f; similarly, even though g and hence e are also small, the electromagnetic interactions would also involve strong coupling, since $|eq_f|\gg 1$ for each matter fermion f. Thus, nothing in the general N_c-extended standard model (with right-handed components for all fermions) guarantees that hypercharge and electromagnetic interactions are perturbative, as observed in nature. This perturbativity is natural (provided that the g and g' are small) only if one has a criterion for restricting the fermion charges to values which are not $\gg 1$ in magnitude. Of course, this is automatic in an approach using grand unification; here we inquire what conditions make it natural without invoking grand unification. There are only two cases where one can naturally guarantee that the fermion charges are not $\gg 1$ in magnitude (and these both yield worlds reminiscent of our own), namely $C4_\ell$ and $C5_\ell$, where $q_e=0$ or $q_\nu=0$, and the electron-type leptons and neutrinos, respectively, are naturally very light compared to the electroweak scale. In these cases, as Eqs. (4.7) and (4.8) show, the quark charges cannot be large in magnitude. Of course, any set of charges in which $|q_e|$ (and hence $|q_\nu|$) are bounded above by a number of order unity implies by (2.13) that $|q_d|$ and $|q_u|$ are also bounded above in magnitude by $O(1)$, but one would lack a specific reason for choosing such a value of q_e or q_ν. We thus are led to conclude that, in the context of the general N_c-extended standard model, the condition that there be neutral (electron- or neutrino-type) leptons which are much lighter than the electroweak scale provides a natural way to get fermion charges which are not $\gg 1$ in magnitude and hence to get perturbative hypercharge and electromagnetic interactions, given that the electroweak gauge couplings are small. Note that this is true both in cases $C4_\ell$ and $C5_\ell$ themselves and in the reduced models in which one excludes the right-handed components of the respective neutral leptons, e_{iR} in $C4_\ell$ and ν_{iR} in $C5_\ell$, since in either case, albeit for different reasons (seesaw mechanism or higher-dimension operators), one has naturally light neutral leptons.

V. CONDITIONS FOR FINITENESS OF ELECTROWEAK EFFECTS AS $N_c\to\infty$

We have already noted that the anomaly conditions can be solved for fermion hypercharges and equivalently electric charges of arbitrarily large magnitude. Obviously, one condition for hypercharge and electromagnetic interactions to be finite is that one choose finite values of fermion charges to solve Eq. (2.13). It is also of interest to consider this from the viewpoint of the large-N_c limit. From Eqs. (2.17) and (2.18) it is clear that if Y_{Q_L} is nonzero, then q_e and q_ν will diverge, like $(-1/2)Y_{Q_L}N_c$, as $N_c\to\infty$. A necessary condition for the lepton charges to remain finite in this limit is that

$$\lim_{N_c\to\infty} q_d=-\tfrac{1}{2}, \tag{5.1}$$

i.e., $\lim_{N_c\to\infty}Y_{Q_L}=0$. However, this is not a sufficient condition; for example, if, as a function of N_c, q_d behaves as

$$q_d\to\frac{-1+aN_c^{-\alpha}}{2} \tag{5.2}$$

for large N_c (where $a\neq 0$), then, from Eq. (2.17),

$$q_e=-\tfrac{1}{2}(1+aN_c^{1-\alpha}) \tag{5.3}$$

which is finite as $N_c\to\infty$ if and only if $\alpha\geq 1$. In contrast, as is clear from (2.15), for any fixed (finite) value of q_e, q_d has a finite limit, namely $q_d=-1/2$, as $N_c\to\infty$.

However, this is still not sufficient for electroweak effects to remain finite in the limit $N_c\to\infty$. It will be recalled that in the large-N_c limit, one holds

$$g_s^2N_c=\text{const}, \tag{5.4}$$

where g_s denotes the SU(N_c) gauge coupling [3–6]. As has been noted in Ref. [14], to avoid a breakdown of large-N_c relations such as that for the $\pi^0\to\gamma\gamma$ amplitude while retaining nonzero electroweak interactions as $N_c\to\infty$, one sets

$$g^2N_c=\text{const} \tag{5.5}$$

and

$$(g')^2N_c=\text{const} \tag{5.6}$$

in this limit, where g and g' denote the SU(2) and $U(1)_Y$ gauge couplings, and the constants in Eqs. (5.4), (5.5), and (5.6) are, of course, different. It is easily seen that this is true for our generalization with right-handed neutrino fields and variable lepton charges, just as it was true of the generalization considered in Ref. [14] without any ν_{iR} fields and with fixed, conventional lepton charges. Hence also, the electromagnetic coupling $e=gg'/\sqrt{g^2+g'^2}$ satisfies the same scaling property

$$e^2N_c=\text{const} \tag{5.7}$$

as $N_c\to\infty$.

VI. RELATIONS BETWEEN QUARK AND LEPTON CHARGE CLASSES

It is of interest to work out the relationships between the various cases describing the possible quark and leptons charges. We thus consider a value of q_d lying in a given class, $C1_q$–$C5_q$, and determine to which class the corresponding lepton charges determined by Eq. (2.13) belong. First, as one can see from Table I, the condition that the quark charges fall in class $C1_q$ implies that the lepton charges fall in class $C3_{\ell}$. We symbolize this as

$$q_d \in C1_q \Rightarrow q_e \in C3_{\ell}. \tag{6.1}$$

The converse does not, in general, hold. The other implications are listed below (and again, the converses do not, in general, hold, except for $C2_{q,\mathrm{sym}}$):

$$q_d \in C2_{q,\mathrm{sym}} \Leftrightarrow q_e \in C2_{\ell,\mathrm{sym}}, \tag{6.2}$$

$$q_d \in C3_q \text{ or } C5_q \Rightarrow q_e \in C1_{\ell}, \tag{6.3}$$

$$q_d \in C4_q \Rightarrow q_e \in C3_{\ell}. \tag{6.4}$$

The condition that $q_d \in C2_q$ can be met for certain values of q_e in each of the leptonic charge classes. The implications following from a given leptonic charge class are

$$q_e \in C1_{\ell} \Rightarrow q_d \in C2_q, \; C3_q, \text{ or } C5_q, \tag{6.5}$$

$$q_e \in C2_{\ell} \Rightarrow q_d \in C2_q, \tag{6.6}$$

$$q_e \in C3_{\ell} \Rightarrow q_d \in C1_q, \; C2_q, \text{ or } C4_q, \tag{6.7}$$

$$q_e \in C4_{\ell} \text{ or } C5_{\ell} \Rightarrow q_d \in C2_q. \tag{6.8}$$

VII. A RELATION CONNECTING CERTAIN PAIRS OF SOLUTIONS

Two respective solutions S and S' of (2.13) with q_d (and a resultant q_e) and q'_d (and a resultant q'_e) have a certain simple relation if the corresponding hypercharges satisfy

$$Y_{Q_L} = -Y'_{Q_L} \tag{7.1}$$

or equivalently, by Eq. (2.13),

$$Y_{\mathcal{L}L} = -Y'_{\mathcal{L}_L}. \tag{7.2}$$

In terms of the fermion charges, these equivalent conditions read

$$q_d + q'_d + 1 = 0, \tag{7.3}$$

i.e.,

$$q_e + q'_e + 1 = 0. \tag{7.4}$$

To see the relation, we recall that the constraint (2.12) implies that all of the hypercharges for the two cases can be expressed in terms of any one, say Y_{Q_L} and Y'_{Q_L}, respectively; further, one can use condition (7.1) to express all hypercharges in terms of Y_{Q_L}. Then the fields for the original solution S are

$$Q_L: \quad (N_c,2,Y_{Q_L}), \tag{7.5}$$

$$u_R: \quad (N_c,1,1+Y_{Q_L}), \tag{7.6}$$

$$d_R: \quad (N_c,1,-1+Y_{Q_L}), \tag{7.7}$$

$$\mathcal{L}_L: \quad (1,2,-N_c Y_{Q_L}), \tag{7.8}$$

$$\nu_R: \quad (1,1,1-N_c Y_{Q_L}), \tag{7.9}$$

$$e_R: \quad (1,1,-1-N_c Y_{Q_L}). \tag{7.10}$$

Now, expressing the field content of solution S' in terms of the charge-conjugates fields,

$$(Q_R^c)': \quad (N_c^*,2,Y_{Q_L}), \tag{7.11}$$

$$(u_L^c)': \quad (N_c^*,1,-1+Y_{Q_L}), \tag{7.12}$$

$$(d_L^c)': \quad (N_c^*,1,1+Y_{Q_L}), \tag{7.13}$$

$$(\mathcal{L}_R^c)': \quad (1,2,-N_c Y_{Q_L}), \tag{7.14}$$

$$(\nu_L^c)': \quad (1,1,-1-N_c Y_{Q_L}), \tag{7.15}$$

$$(e_L^c)': \quad (1,1,1-N_c Y_{Q_L}), \tag{7.16}$$

where our notational convention is $\psi_R^c \equiv [(\psi_L)^c]_R$, $\psi_L^c \equiv [(\psi_R)^c]_L$. Evidently, there is a one-to-one correspondence between fields (7.5)–(7.10) of solution S and fields (7.11)–(7.16) of solution S' according to which $L \leftrightarrow R$, $N_c \to N_c^*$ [i.e., fundamental representation is replaced by conjugate fundamental representation of the SU(N_c) color group], and $(u_L^c)' \to d_R$, $(d_L^c)' \to u_R$, $(\nu_L^c)' \to e_R$ and $(e_L^c)' \to \nu_R$, etc. [Here we use the fact that the representations of SU(2) are (pseudo)real.] In particular, the leptonic fields $\mathcal{L}_L$, ν_R, and e_R for solution S transform according to precisely the same representations of $\mathrm{SU}(N_c)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ as the lepton fields $(\mathcal{L}_R^c)'$, $(e_L^c)'$, and $(\nu_L^c)'$ of solution S', respectively. We note that the special cases describing the quark charges (and their corresponding lepton charges) in $C4_q$ and $C5_q$ satisfy condition (7.1) [and the equivalent equation (7.2) for the leptons], so that $(C4_q, C5_q)$ form such a pair (S,S') of solutions. Similarly, the lepton charges (and their corresponding quark charges) in $C4_{\ell}$ and $C5_{\ell}$ satisfy condition (7.2) [and the equivalent equation (7.1) for the quarks], so that $(C4_{\ell}, C5_{\ell})$ form another such pair (S,S'). There is a (continuous) infinity of other pairs of solutions forming such pairs with hypercharges which are equal and opposite. Finally, the symmetric case $C2_{q,\mathrm{sym}} = C2_{\ell,\mathrm{sym}}$ with $Y_{Q_L} = 0 = Y_{\mathcal{L}_L}$ also satisfies condition (7.1) and thus forms a pair with itself $(S, S' = S)$.

VIII. SOME PROPERTIES OF VARIOUS CASES

We next comment on some properties of various classes of solutions of Eq. (2.13). In this discussion, we consider both fixed, finite N_c and the limit $N_c \to \infty$. The hadronic spectrum of the theory would contain various meson and glueball states, the latter being, in general, mixed with $\bar{q}q$ mesons of the same quantum numbers to form physical mass eigenstates. Independent of the specific values of fermion charges, the $\bar{q}q$ meson charges would always be 0 or ± 1 (the latter because $q_u = q_d + 1$). Other aspects of the spectrum would depend on the nature of electroweak symmetry breaking, such as superpartners in the MSSM; we shall not discuss these here. There are some interesting general results which one can derive concerning baryons, and we proceed to these.

A. Baryons

We consider baryons composed of r up-type and N_c-r down-type quarks [21] and denote their electric charge as $q[B(r,N_c-r)]$. (For considerations of electric charge, one can suppress the flavor dependence of the quark constituents; thus, for example, by down-type quarks, we include d, s, and b.) The electric charge of the baryon(s) $B(r,N_c-r)$ is

$$q[B(r,N_c-r)]=r+N_cq_d=r-\tfrac{1}{2}(N_c+2q_e+1). \qquad (8.1)$$

In the special case where all of the up-type quarks and down-type are u and d, respectively, two baryons which are related by a strong-isospin rotation $u\leftrightarrow d$ are $B(r,N_c-r)$ and $B(N_c-r,r)$. The charge difference between these is

$$q[B(r,N_c-r)]-q[B(N_c-r,r)]=2r-N_c. \qquad (8.2)$$

Now we assume that $m_u,m_d\ll\Lambda_{\rm QCD}$ [where $\Lambda_{\rm QCD}$ is the scale characterizing the SU(N_c) color interactions], as in the physical world. A strong-isospin mirror pair which constitutes a kind of generalization of the proton and neutron is the (light u, d quark, spin 1/2) pair

$$\mathcal{P}=B\left(\frac{N_c+1}{2},\frac{N_c-1}{2}\right) \qquad (8.3)$$

and

$$\mathcal{N}=B\left(\frac{N_c-1}{2},\frac{N_c+1}{2}\right). \qquad (8.4)$$

For $N_c=3$, $\mathcal{P}=p$, $\mathcal{N}=n$. From Eq. (8.1), it follows that

$$q_{\mathcal{P}}=-q_e \qquad (8.5)$$

and

$$q_{\mathcal{N}}=-q_\nu \qquad (8.6)$$

(and furthermore $q_{\mathcal{P}}=q_{\mathcal{N}}+1$). Some further general results on baryon charges are the following. First, if and only if q_d and q_u have the same sign, as they do for cases $C1_q$ and $C3_q$, then all baryons also have the same sign of electric charge. Second, as is clear from Eqs. (8.5) and (8.6), the proton and neutron $\mathcal{P}$ and $\mathcal{N}$ have the same sign of electric charge if and only if q_e and q_ν have the same sign, which holds in cases $C1_{\ell}$ and $C3_{\ell}$. Thus also, sgn$(q_{\mathcal{P}})=-{\rm sgn}(q_{\mathcal{N}})\Leftrightarrow{\rm sgn}(q_e)=-{\rm sgn}(q_\nu)\Leftrightarrow q_e\in C2_{\ell}$. The special case $C5_{\ell}$, with $q_\nu=0$ yields proton and neutron charges the same as in our world, while in the special case $C4_{\ell}$, with $q_e=0$, one would have $q_{\mathcal{P}}=0$ and $q_{\mathcal{N}}=-1$. In case $C2_{q,{\rm sym}}$, $q_{\mathcal{P}}=-q_{\mathcal{N}}=1/2$. In the cases $C1_{\ell}$ and $C3_{\ell}$ in which $\mathcal{P}$ and $\mathcal{N}$ have the same sign of electric charge, the resultant Coulomb repulsion would increase the energy (i.e., decrease the binding energy) of a nucleus, as compared to the cases where these nucleons have opposite signs of electric charge or one is neutral. Consequently, in these cases the Coulomb interaction could destabilize certain nuclei which are stable in the physical world.

B. Atoms

As a consequence of the charge relation, Eq. (8.5), for all cases of fermion charges [satisfying (2.13)] except the case $C4_{\ell}$ $(q_e=0)$, there will exist a Coulomb bound state of the proton $\mathcal{P}$ and electron, which is the N_c-extended generalization of the hydrogen atom. Furthermore, for all cases of fermion charges [satisfying Eq. (2.13)] except $C5_{\ell}$ $(q_\nu=0)$, there will exist a second neutral Coulomb bound state, which has no analogue in the usual $N_c=3$ standard model, namely, $(\mathcal{N}\nu_{r=1})$, where $\nu_{r=1}$ denotes the lightest neutrino mass eigenstate. Since the standard model conserves baryon number perturbatively [22], it follows that, if $q_e\neq 0$, so that the generalized H atom, $(\mathcal{P}e)$, exists, this bound state is stable. Even for cases other than $C5_{\ell}$, where the $(\mathcal{N}\nu_{r=1})$ atom exists, it would decay weakly, as a consequence of the decay $\mathcal{N}\to\mathcal{P}+e+\overline{\nu_e}$ (this really means the decays $\mathcal{N}\to\mathcal{P}+e+\overline{\nu_r}$, involving all mass eigenstates ν_r in the weak eigenstate $\nu_e=\Sigma_{r=1}^3U_{1r}\nu_r$ which are kinematically allowed to occur in the final state). For nonzero q_e and q_ν, no leptons would be generically expected to be very light compared with other fermions. However, depending on the fermion mass spectrum, there could exist other stable Coulomb bound states. For example, assuming the usual electron-type lepton mass spectrum, if $m(\nu_{r=2})<2m_e+m(\nu_{r=1})$, then the state $(\mathcal{N}\nu_{r=2})$ could also be stable. Henceforth, among the possible $(\mathcal{N}\nu_r)$ states, we shall only consider $(\mathcal{N}\nu_{r=1})$ and shall suppress the $r=1$ subscript in the notation.

In the following discussion of the H and $(\mathcal{N}\nu)$ atoms, we shall implicitly assume, respectively, that $q_e\neq 0$ and $q_\nu\neq 0$, so that these atoms exist. The condition that the H atom is a nonrelativistic bound state is

$$(\mathcal{P}e)\ \ {\rm nonrel.}\Leftrightarrow|q_e|\alpha\ll 1. \qquad (8.7)$$

Given that $q_\nu=q_e+1$, this is effectively the same condition for the $(\mathcal{N}\nu)$ bound state:

$$(\mathcal{N}\nu)\ \ {\rm nonrel.}|q_\nu|\alpha\ll 1. \qquad (8.8)$$

Note that if this condition is not met, i.e., if $|q_e|\alpha\gtrsim 1$, then the electromagnetic interaction between $\mathcal{P}$ and e would involve strong coupling. Assuming that these states are nonrelativistic, the binding energy of the ground state of the H atom is given, to lowest order, by

$$E_{(\mathcal{P}e)}=-\frac{(q_e\alpha)^2m_{\rm red.}}{2}, \qquad (8.9)$$

where $\alpha=e^2/(4\pi)$ and the reduced mass is $m_{\rm red.}=m_{\mathcal{P}}m_e/(m_{\mathcal{P}}+m_e)$. The Bohr radius for this ground state of the H atom would be

$$a_0=\frac{1}{q_e^2\alpha m_{\rm red.}}. \qquad (8.10)$$

Formulas (8.9) and (8.10) apply to the $(\mathcal{N}\nu)$ bound state with the obvious replacements $\mathcal{P}\to\mathcal{N}$ and $e\to\nu$.

For a fixed value of q_e and hence q_ν, such that the respective bound state $(\mathcal{P}e)$ or $(\mathcal{N}\nu)$ exists, assuming that the lepton masses are fixed, the magnitudes of the respective binding energies decrease as N_c increases. To see this, note first that since $m_{\mathcal{P}}, m_{\mathcal{N}} \sim N_c$ in this limit [6], the respective reduced mass $m_{\rm red.} \to m_e$ or m_ν and hence is finite. Then from Eq. (5.7), it follows that

$$E_{(\mathcal{P}e)}, E_{(\mathcal{N}\nu)} \sim N_c^{-2} \tag{8.11}$$

as N_c gets large.

If both q_e and q_ν are nonzero, then atoms with nuclei having atomic number $A \geqslant 2$ would exihibit a qualitatively new feature not present in our world: the leptons bound to the nucleus would be of two different types. A generic atom would be of the form

$$(\text{nucl}(N_{\mathcal{P}}\mathcal{P}, N_{\mathcal{N}}\mathcal{N}); N_{\mathcal{P}}e, N_{\mathcal{N}}\nu). \tag{8.12}$$

Whereas the characteristic size of atoms and molecules in the physical world is set by the Bohr radius, these atoms would have charged lepton clouds characterized by different sizes, reflecting their different masses and charges. These higher-A nuclei and atoms would undergo weak decays via e^- or e^+ emission or e^- capture, as in the physical world, and, in addition, ν or $\bar{\nu}$ emission or ν capture.

C. Possible lepton-lepton Coulomb bound state

There could also occur a stable purely leptonic Coulombic bound state, $(e\nu_{r=1})$. A necessary condition for this would be that q_e and q_ν are opposite in sign, i.e., that the lepton charges fall in case $C2_{\ell}$. However, unlike the $(\mathcal{P}e)$ and $(\mathcal{N}\nu_{r=1})$ atoms, this leptonic state (and other possible ones, e.g., for $r=2$) would, in general, have a nonzero charge, namely, $q[(e\nu)] = Y_{\mathcal{L}_L}$. In the physical world, one knows of many such Coulombic bound states with net charge, such as the negative hydrogen ion $(pee) = H^-$.

D. Lepton masses for the case $q_\nu = -q_e = 1/2$

Clearly, $q[(e\nu_r)] = 0$ only for the symmetric subcase $C2_{\ell,\rm sym} = C2_{q,\rm sym}$ in Eq. (3.7), where $q_\nu = -q_e = 1/2$ so that $Y_{\mathcal{L}_L} = 0$. In this case, the lepton mass spectrum exhibits some unusual features, which depend, moreover, on whether $N_{\rm gen.}$ is even or odd. We note first that, in addition to the lepton Yukawa interactions in (2.10), there would two more such terms, so that the total leptonic Yukawa part of the Lagrangian would be

$$\begin{aligned} -\mathcal{L}_{Yuk, C2_{\ell,\rm sym}} = \sum_{i,j=1}^{N_{\rm gen.}} &[(Y_{ij}^{(e)}\overline{\mathcal{L}_{ibL}}e_{jR} + Y_{ij}^{(\nu 2)}\epsilon_{ab}\mathcal{L}_{iL}^{Ta}C\nu_{jL}^c)H_d^b \\ &+ (Y_{ij}^{(\nu)}\overline{\mathcal{L}_{ibL}}\nu_{jR} + Y_{ij}^{(e2)}\epsilon_{ab}\mathcal{L}_{iL}^{Ta}Ce_{jL}^c)H_u^b] \\ &+ \text{H.c.}, \end{aligned} \tag{8.13}$$

where i,j are generation indices and a,b are SU(2) indices, and the notation applies to either the standard model or the MSSM, as discussed before, following Eq. (2.10). The VEV's of the Higgs fields would give rise to the mass terms

$$\begin{aligned} -\mathcal{L}_{Yuk,\rm mass} = \sum_{i,j=1}^{N_{\rm gen.}} &[M_{ij}^{(e)}\overline{e_{iL}}e_{jR} + M_{ij}^{(\nu 2)}\overline{\nu_{jR}}\nu_{iL} + M_{ij}^{(\nu)}\overline{\nu_{iL}}\nu_{jR} \\ &- M_{ij}^{(e2)}\overline{e_{jR}}e_{iL}] + \text{H.c.}, \end{aligned} \tag{8.14}$$

where

$$M^{(e)} = 2^{-1/2}Y^{(e)}v_d, \quad M^{(\nu 2)} = 2^{-1/2}Y^{(\nu 2)}v_d, \tag{8.15}$$

$$M^{(\nu)} = 2^{-1/2}Y^{(\nu)}v_u, \quad M^{(e2)} = 2^{-1/2}Y^{(e2)}v_u, \tag{8.16}$$

and we have used $\nu_{iL}^T C\nu_{jL}^c = \overline{\nu_{jR}}\nu_{iL}$ and $e_{iL}^T Ce_{jL}^c = \overline{e_{jR}}e_{iL}$. (In a theory without Higgs fields, these mass terms would arise, as discussed before, from certain multifermion operators.) Furthermore, there would be the electroweak-singlet bare leptonic mass terms

$$-\mathcal{L}_{\rm bare} = \sum_{i,j=1}^{N_{\rm gen.}} [M_{ij}^{(L)}\epsilon_{ab}\mathcal{L}_{iL}^{Ta}C\mathcal{L}_{jL}^b + M_{ij}^{(R)}e_{iR}^T C\nu_{jR}] + \text{H.c.} \tag{8.17}$$

(Note that $M_{1,ij}$ is automatically antisymmetric; $M_{1,ij} = -M_{1,ji}$ and that $\nu_{iR}^T Ce_{jR} = e_{jR}^T C\nu_{iR}$.) The mass coefficients multiplying these electroweak-singlet terms would be naturally much larger than the electroweak symmetry breaking scale v. Without having to analyze the full set of mass terms in Eqs. (8.14) and (8.17) in detail, one can thus immediately conclude that the SU(2)-singlet lepton fields will pick up masses which are naturally much larger than the EWSB scale. Indeed, (whether $N_{\rm gen.}$ is even or odd) one can always rewrite the SU(2)-singlet leptons as four-component Dirac fields; explicitly, these are

$$\psi_i = \begin{pmatrix} \nu_{iR} \\ e_{iL}^c \end{pmatrix}, \quad i = 1, \ldots, N_{\rm gen.} \tag{8.18}$$

with charge $q_\psi = 1/2$ [here the spinor refers to Dirac, not SU(2), space, and we use a representation in which γ_5 is diagonal]. These form bare mass terms $\Sigma_{i=1}^{N_{\rm gen.}} m_{Di}\bar{\psi}_i\psi_i$ with masses $m_{D,i}$ naturally $\gg v$.

In the hypothetical situation in which $N_{\rm gen.}$ is even, one could form $(N_{\rm gen.}/2)$ Dirac SU(2) doublets in a similar manner, combining $\mathcal{L}_{1L}$ and $\mathcal{L}_{2R}^c$ into the first doublet, say, $\mathcal{L}_{3L}$ and $\mathcal{L}_{4R}^c$ into the second one, and so forth for the others. Here we use the fact that the representations of SU(2) are (pseudo)real. Again, these SU(2) doublets would form electroweak-singlet Dirac bare mass terms with masses which are naturally much larger than the EWSB scale. Note that the only gauge interaction of these Dirac doublets, namely that involving the SU(2) gauge fields, is vectorial. This rewriting of the doublets as Dirac fields coupling in a vectorial manner is similar to the method that we used earlier in lattice gauge theory studies [23]. The matter fermions in the effective field theory at and below the electroweak scale then consist only of the quarks. In this hypothetical case, since $N_{\rm gen.}$ is even, there are $N_{\rm gen.}N_c$ doublets of quarks, which is even whether or not N_c is even or odd, so this effective field theory is free of any global SU(2) anomaly. [Recall that for the present case, $C2_{q,\rm sym} = C2_{\ell,\rm sym}$, the anomalies of type (i) cancel individually for the quark and lepton sectors.]

For the case where $N_{\rm gen.}$ is odd, one can form $(N_{\rm gen.}-1)/2$ Dirac SU(2) doublets which naturally gain large masses, as above. To see what happens to the one remaining leptonic chiral SU(2) doublet, it is sufficient to deal with a simple $N_{\rm gen.}=1$ example. Using the fact that the masses multiplying the electroweak-singlet bare mass terms are naturally much larger than the electroweak scale, while the masses resulting from the Yukawa couplings are $\lesssim v$, and diagonalizing the mass matrix, one finds two large eigenvalues $\pm M_R$, and two very small eigenvalues, $\pm m_1 m_2/M_R$ where M_R, m_1, and m_2 denote the coefficients of $e_R^T C \nu_R$, $\overline{\nu_L}\nu_R$, and $\overline{e_L}e_R$, respectively. The two large eigenvalues correspond, up to very small admixtures, to the SU(2)-singlet states already discussed above. The masses gained by the components in the remaining SU(2)-doublet are naturally much less than the electroweak scale, because of a kind of seesaw mechanism. The effective field theory (EFT) at and below the electroweak scale would consist of a remainder of $N_{d,\rm EFT}=(N_{\rm gen.}N_c+1)$ SU(2) doublets. Since $N_{\rm gen.}$ and hence N_c are odd, $N_{d,\rm EFT}$ is even, so again there is no global SU(2) anomaly in this sector. The sector containing high-mass leptons was rewritten in vectorial form, and hence is obviously free of any anomalies.

IX. QUESTION OF GRAND UNIFICATION

Finally, we address the issue of grand unification of the N_c-extended standard model with right-handed neutrinos. Although in the standard model (with no right-handed neutrinos) and its N_c extension, one already gets charge quantization without grand unification, the latter does provide an appealingly simple (if not unique [24–26]) way to obtain gauge coupling unification. Here we shall prove a strong negative result. Our proof will essentially consist of a counting argument and will not make use of the specific fermion charge assignments obtained as solutions to Eq. (2.13). Our proof will apply both for the case of odd $N_{\rm gen.}$ and for the hypothetical case of even $N_{\rm gen.}$. Again, we shall not need to make any explicit assumption concerning the nature of electroweak symmetry breaking. Clearly, however, if one discusses grand unification at all, it is natural to assume that physics remains perturbative from the electroweak scale up to the scale of the grand unified theory (GUT) (so that the unification of gauge couplings is not an accident) and hence work within the framework of a supersymmetric theory. This is also motivated by the fact that supersymmetry can protect the Higgs sector against large radiative corrections and, if the μ problem can be solved, can thereby account for the gauge hierarchy in a GUT [27]. We follow the standard rules of grand unification: first, in order have light fermions, one must use a group with complex representations. Second, in order to have natural cancellation of anomalies in gauged currents, one restricts the choices of a group to those which are ''safe'' (i.e., have identically zero triangle anomaly for all fermion representations) [28]. Note that at the GUT level, there are no mixed gauge-gravitational anomalies, since those necessarily involve a U(1) gauge group, and also no global anomaly involving the GUT group itself, since $\pi_4[{\rm SU}(N)]=\emptyset$ for $N\geqslant 3$, $\pi_4[{\rm SO}(N)]=\emptyset$ for $N\geqslant 6$, and $\pi_4(E_6)=\emptyset$ [29,10,30] [of course, restriction (2.19) still holds]. Now although E_6 has complex representations and is safe, it has a fixed rank of 6, and hence cannot be used for general N_c. The natural choice of GUT gauge group is thus $\mathrm{SO}(4k+2)$ with $k\geqslant 2$. Now, as in the original discussion [31], one must satisfy an inequality on ranks: for our case, in order for $\mathrm{SU}(N_c)\times\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ to be embedded in a GUT group G, it is necessary that

$$\mathrm{rank}(G)\geqslant N_c+1. \tag{9.1}$$

Using the standard result

$$\mathrm{rank}[\mathrm{SO}(2n)]=n, \tag{9.2}$$

setting $2n=4k+2$, and substituting (9.1), we obtain the inequality

$$2k\geqslant N_c. \tag{9.3}$$

If $N_{\rm gen.}$ is odd, and hence, by (2.19), N_c is odd, this becomes

$$2k\geqslant N_c+1 \quad \text{for } N_c \text{ odd}. \tag{9.4}$$

We are thus led to consider the special orthogonal group $G=\mathrm{SO}(2N_c+4)$ (corresponding to the algebra D_{N_c+2}, in the Cartan notation) with rank N_c+2. Since N_c+2 is odd, $G=\mathrm{SO}(2N_c+4)$ has a complex spinor representation of dimension 2^{N_c+1}. Now we would like to fit all of the fermions of each generation in a Weyl field transforming according to the spinor representation of this group. Note that in order for this to be possible, it is necessary that

$$\sum_f Y_f=0 \tag{9.5}$$

for each generation, since Y is now a generator of a simple (non-Abelian) group, and hence its trace must be zero. This requirement is met automatically as a consequence of the vectorial nature of the electromagnetic coupling; the left-hand side of Eq. (9.5) is, indeed, identical to that of Eq. (2.20) discussed before. Now there are

$$N_f=4(N_c+1)N_{\rm gen.} \tag{9.6}$$

Weyl matter fermion fields in the theory [32]. Requiring that these fit precisely in $N_{\rm gen.}$ copies of the spinor representation then yields the condition

$$2^{N_c+1}=4(N_c+1). \tag{9.7}$$

But this has a solution only for $N_c=3$. This is a very interesting result, and perhaps gives us a deeper understanding of why $N_c=3$ in our world.

We would reach the same conclusion even if $N_{\rm gen.}$ were even. In this case, (2.19) allows N_c to be either even or odd. If N_c is odd, the same reasoning as before applies directly. If N_c is even, then (9.3) can be satisfied as an equality, so that the group would be $\mathrm{SO}(2N_c+2)$ (with minimal rank N_c+1) corresponding to D_{N_c+1}, rather than $\mathrm{SO}(2N_c+4)$. Now since N_c is even, the dimension of the spinor representation of $\mathrm{SO}(2N_c+2)$ is 2^{N_c+1}. Hence, we are led to the same condition, (9.7) as before, and the same conclusion follows.

X. CONCLUDING REMARKS

In summary, we have explored the implications of the cancellation of anomalies for the N_c-extended standard model with right-handed components for all fermions. We have shown that anomaly cancellation does not imply the quantization of the fermion charges and have discussed some interesting properties of various classes of solutions for these charges. In particular, we have related the condition that there be neutral leptons with masses much less than the electroweak scale to the feature that the fermion charges are not $\gg 1$ in magnitude and hence that the electroweak interactions are perturbative. Finally, we have proved that the unification of the $SU(N_c)\times SU(2)\times U(1)_Y$ theory in $SO(2N_c+4)$ (with the usual assignment of the fermions to the spinor representation) can only be carried out for $N_c=3$. The world which we analyze here is, of course, a generalization of our physical one, but we believe that, as with the original $1/N_c$ expansion in QCD, by thinking about the standard model in a more general context, one may gain a deeper understanding of its features.

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[1] S. L. Glashow, Nucl. Phys. **B22**, 579 (1961); S. Weinberg, Phys. Rev. Lett. **27**, 1264 (1967); A. Salam, in *Elementary Particle Theory*, Proceedings of the 8th Nobel Symposium, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968); S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D **2**, 1285 (1970).

[2] Indeed, as is well known, the effort to derive known gauge interactions from a deeper theory involving gravity has a long history dating back at least to the works of Kaluza and Klein in 1921 and 1926.

[3] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974); **B75**, 461 (1974).

[4] There is a considerable literature on large-N_c methods. An early review is Ref. [5]. For a discussion of the $1/N_c$ expansion applied to 4D QCD, see Ref. [6].

[5] S. Coleman, in *Pointlike Structure Inside and Outside Hadrons*, Lectures at the 1979 Erice Summer School, edited by A. Zichichi (Plenum, New York, 1982), p. 11.

[6] E. Witten, Nucl. Phys. **B160**, 57 (1979); **B223**, 422 (1983); **B223**, 433 (1983).

[7] In principle, one could attempt an analogous generalization in the electroweak sector, considering one or both of the factor groups in this sector as members of an infinite sequence of groups, like the generalization $SU(3)\rightarrow SU(N_c)$ in the QCD sector. However, first, there is less motivation for this, since a primary reason for the study of the large-N_c limit in QCD was to have an analytic method for nonperturbative calculations, but perturbation theory is adequate to calculate the observed electroweak decays and reactions. Second, such a generalization of the electroweak sector is not straightforward; for example, if one extends the weak isospin gauge group from SU(2) to $SU(N_{wk})$, $N_{wk}\geqslant 3$, one encounters difficulties with anomalies.

[8] S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).

[9] D. J. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[10] E. Witten, Phys. Lett. **117B**, 324 (1982).

[11] R. Delbourgo and A. Salam, Phys. Lett. **40B**, 381 (1972); T. Eguchi and P. Freund, Phys. Rev. Lett. **37**, 1251 (1976); L. Alvaré-Gaumé and E. Witten, Nucl. Phys. **B234**, 269 (1983).

[12] C. Bouchiat, J. Iliopoulos, and P. Meyer, Phys. Lett. **38B**, 519 (1972).

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[14] C.-K. Chow and T.-M. Yan, Phys. Rev. D **53**, 5105 (1996).

[15] Such a dimension-five (gauge-invariant) operator is

$$\mathcal{O}=\frac{1}{M_X}\sum_{i,j} h_{ij}(\epsilon_{ak}\epsilon_{bm}+\epsilon_{am}\epsilon_{bk})[\mathcal{L}^{Ta}_{iL}C\mathcal{L}^{b}_{jL}]\phi^k\phi^m+\text{H.c.}, \tag{10.1}$$

where i,j are generation indices, a,b,k,m are SU(2) indices, and ϕ denotes the standard-model Higgs field or the H_d Higgs field in a supersymmetric extension of the standard model. The term arising from the VEV's of the Higgs is a (left-handed, Majorana) neutrino mass term, which is naturally small if M_X is much larger than the electroweak symmetry breaking scale.

[16] M. Gell-Mann, R. Slansky, and P. Ramond, *Supergravity* (North-Holland, Amsterdam, 1979), p. 315; T. Yanagida, *Proceedings of the Workshop on Unified Theory and Baryon Number in the Universe* (KEK, Japan, 1979).

[17] The conventional seesaw mechanism and the dimension-five operator in [15] both presume that the theory has the requisite fundamental Higgs fields. If, instead, one assumes that electroweak symmetry breaking occurs dynamically, without fundamental Higgs fields, then Dirac neutrino masses, like the masses of other fermions, would arise from multifermion operators; however, if neutral ν_{iR} or e_{iR} fields were present, as in the respective cases $C5_{\not{}}$ and $C4_{\not{}}$ [Eqs. (4.8) and (4.7)], they would, of course, still produce right-handed Majorana bare mass terms.

[18] One could consider the N_c-extended standard model with right-handed neutrinos and analyze the constraints on the quark hypercharges while fixing the hypercharges of the leptonic fields to be equal to their conventional values. From the viewpoint of anomalies alone, this would be unmotivated, since all of the fermion hypercharges enter, *a priori*, on an equal footing in the anomaly cancellation conditions. Of course, if one imposes the additional requirement beyond anomaly cancellation that there exist (gauge-invariant) right-handed Majorana mass terms, this implies that the leptonic hypercharges are automatically fixed to their conventional val-

ues. We shall not at the outset impose such a requirement here, since our purpose is to explore the consequences of anomaly cancellation by itself.

[19] As is discussed later in the text, there are two exceptions to this generic $Y_{\nu_R} \neq 0$ situation: (i) case $C4_{\not\ell}$ in Eq. (4.7) in which the electron charge $q_e = 0$, so that the electron-type leptons are naturally light; and (ii) case $C2_{q,\mathrm{sym}} = C2_{\ell,\mathrm{sym}}$ in which $q_\nu = -q_e = 1/2$ (see text for details).

[20] For completeness, on a purely empirical level, one must mention the possibility of further generations with electroweak nonsinglet neutrinos with masses $m_\nu > m_Z/2$ which would not be counted in the LEP-SLC determination of N_ν.

[21] Here, we use the conventional approach of describing a baryon as a color-singlet bound state of N_c (valence) quarks; another useful description, which is motivated by the large-N_c limit, is to describe a baryon as a soliton [6].

[22] Recall that the nonperturbative violation of B (and L) in the standard model, via electroweak instantons [G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976)] is negligibly small at zero temperature.

[23] I-H. Lee and R. Shrock, Phys. Rev. Lett. **59**, 14 (1987); Phys. Lett. B **196**, 82 (1987); Nucl. Phys. **B305**, 305 (1988); R. Shrock, Nucl. Phys. **B4** (Proc. Suppl.) 373 (1988); S. Aoki, I-H. Lee, and R. Shrock, Phys. Lett. B **207**, 471 (1988); **219**, 335 (1989).

[24] It may be recalled that string theory provides an alternate way of achieving gauge coupling unification (either without or with a simple gauge group) via the relation $g_a^2 k_a = \mathrm{const}$, $a = 1,2,3$ for the $U(1)_Y$, SU(2), and SU(3) factor groups, where k_a denotes the level of the Kac-Moody worldsheet algebra [25]. One has $k_2 = k_3 = 1$ naturally (e.g., to explain the absence of exotic representations of fermions); one can also argue for $k_1 = 5/3$. However, in string theory the rank of the gauge group is bounded above as $\mathrm{rank}(G) \leq 22$ [26], thereby precluding the discussion of G'_{SM} with arbitrary N_c, in a string context.

[25] P. Ginsparg, Phys. Lett. B **197**, 139 (1987); V. Kaplunovsky, Nucl. Phys. **B307**, 145 (1988); Errata: hep-th/9205070 (unpublished).

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[27] Of course, in this discussion, we do not consider the well-known problems faced by even supersymmetric GUT's, such as doublet-triplet splitting in the Higgs chiral superfields.

[28] H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 429 (1972).

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[30] C. Q. Geng, R. E. Marshak, Z. Y. Zhao, and S. Okubo, Phys. Rev. D **36**, 1953 (1987); S. Okubo, C. G. Geng, R. E. Marshak, and Z. Y. Zhao, *ibid.* **36**, 3268 (1987); S. Elitzur and V. P. Nair, Nucl. Phys. **B243**, 205 (1984).

[31] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438 (1974).

[32] For the the special cases $C4_{\not\ell}$ and $C5_{\not\ell}$ which allow removal of the e_{iR} or ν_{iR} fields, respectively, the low-energy theory has $N_f = 4N_c + 3$ Weyl matter fermion fields for each generation. Thus N_f is odd for both the cases N_c odd and N_c even. Since the spinor representations of the respective groups $SO(2N_c+4)$ and $SO(2N_c+2)$ for these cases are both even (both $= 2^{N_c+1}$), it is not possible to embed these fermion fields, by themselves, precisely in this spinor representation. The absence of a GUT for the case without ν_{iR}'s has been noted in Ref. [14]. For both case $C4_{\not\ell}$ and $C5_{\not\ell}$ one knows that it is necessary to add back the G'_{SM}-singlet fields e_{iR} or ν_{iR}, respectively, to be able even to consider fitting the fermions into the respective spinor representation. One is thus led back to the case where N_f is given by Eq. (9.6).