Magnetic Polarizability of the nucleon

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We derive an expression for the magnetic polarizability of the nucleon, as related to sums of products of its electromagnetic transition moments involving the electric and magnetic dipoles and mean-square radii, as well as the electric quadrupole moment. Two sum rules emerge from the calculation.

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I. INTRODUCTION

Recently we have studied the third-order spin polarizabilities of the nucleon [1,2] (hereafter called papers I and II, respectively), which are the coefficients of the spin-dependent terms of the excited state part of the Compton amplitude that are of third order in the frequency of the incoming photon, expressing them in terms of the nucleon transition moments up to quadrupole order. Looking at the second-order terms of the amplitude [2] and following the same methodology, we have found here an expression for the magnetic polarizability β of the nucleon in terms of its static properties and its transition moment matrix elements up to electric quadrupole order. Also, two sum rules emerge from the calculation when one makes use of the low-energy theorems which are associated with the second-order terms of the amplitude [1].

A formula for β has been found by Maksimenko and Shul'ga [3] in terms of derivatives of the nucleon transition current matrix elements, by expanding the amplitude in terms of these quantities. A simpler formula for β has been derived by L'Vov [4] using a dispersion method. This simpler formula is the one that we use to make the expansion of β in terms of the transition moments, using the multipole expansion of the nucleon transition current matrix elements derived before [1].

This work corrects and extends a previous one [5] which contains an incorrect expression for β that was based on an erroneous definition for the nucleon magnetic moment transition matrix elements [4] and an incorrect procedure when extracting the nucleon contribution to the amplitude, by using identities for the current operator before expanding the amplitude in intermediate states.

The experimental situation of the polarizabilities is discussed in Ref. [4]. On the theoretical side a quark model calculation may come to mind and for that purpose expressions of the polarizabilities in terms of the nucleon transition moments can be of help. A calculation of the multipole transition moments to low-lying nucleon resonances could be attempted by using available experimental data allied with the determination of the nucleon transition matrix elements in the context of the relativized quark model [6–8]. On the assumption of low-lying resonances, we could then estimate the polarizabilities. In turn, the two sum rules could be used as a test for the low-lying dominance hypothesis.

In Sec. II we discuss the amplitude. In Sec. III we show that one of the sum rules appears from the analysis of the forward scattering amplitude, a procedure that, as we shall see, permits a derivation of the formula for β [4] in a different way. After expressing these two results in terms of the multipole transition moments in Sec. IV, we go to the general, nonforward, situation and derive the second sum rule already in terms of the transition moments. This is done in Sec. V and in Sec. VI we discuss the results.

II. SCATTERING AMPLITUDE

Using the same notation as before [1,2], we write the reduced scattering amplitude of light by the nucleon, in the transverse gauge $\varepsilon_0 = \varepsilon'_0 = 0, \boldsymbol{\varepsilon} \cdot \mathbf{k} = \boldsymbol{\varepsilon}' \cdot \mathbf{k}' = 0$, as

$$A = \varepsilon'^i T_{ij} \varepsilon^j = \varepsilon'^i \left(E_{ij} + U_{ij}\right) \varepsilon^j , \tag{2.1}$$

where ε^i and ε'^i are the polarization vectors of the incoming and outgoing photons with momenta $k^\mu = (\omega, \mathbf{k})$ and $k'^\mu = (\omega', \mathbf{k}')$, respectively. Following Low [9] we have separated out the contribution U_{ij} of the one-nucleon on-shell intermediate state. U_{ij} is called the unexcited part of T_{ij} and the rest, E_{ij}, is the excited part. U_{ij} is given by

$$U_{ij} = V^2 \left[\frac{\langle \mathbf{p}' | J_i | \mathbf{p} + \mathbf{k} \rangle \langle \mathbf{p} + \mathbf{k} | J_j | \mathbf{p} \rangle}{E\left(\mathbf{p}+\mathbf{k}\right) - E - \omega} \times \frac{m}{E\left(\mathbf{p}+\mathbf{k}\right)} + \text{c.t.} \right], \tag{2.2}$$

where c.t. stands for the crossed term ($i \leftrightarrow j$, $\mathbf{k} \leftrightarrow -\mathbf{k}'$, $\omega \leftrightarrow -\omega'$). V is the normalization volume and $\mathbf{p}$ ($\mathbf{p}'$) designates the incoming (outgoing) nucleon momentum with energy E (E'). $|\mathbf{p} + \mathbf{k}\rangle$ is the one-nucleon on-shell intermediate state with energy $E\left(\mathbf{p} + \mathbf{k}\right)$ and

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mass m, and a summation over the intermediate spin states is implied. An invariant normalization has been used and the nucleon current matrix element is then given by

$$\langle \mathbf{p}_2|J_\mu|\mathbf{p}_1\rangle$$

$$= \frac{e}{V}\bar{u}(\mathbf{p}_2)\left[F_1\left(q^2\right)\gamma_\mu + \frac{i\lambda}{2m}F_2\left(q^2\right)\sigma_{\mu\nu}q^\nu\right]u(\mathbf{p}_1), \quad (2.3)$$

with $q = p_2 - p_1$ and the normalization $\bar{u}u = 1$, and λ is the anomalous magnetic moment of the nucleon in units of $e/2m$. The calculation of $\varepsilon'^i\ U_{ij}\ \varepsilon^j$ is given in Appendix A for future reference, to the order ω^2 that we are interested in. The Breit frame, where

$$\mathbf{p}' = -\mathbf{p} = \frac{\mathbf{k}-\mathbf{k}'}{2},\quad E' = E,\quad \omega' = \omega, \quad (2.4)$$

is the frame where the requirement of time-reversal invariance achieves its simplest form as used by Pais [10] to find the minimal basis $B_{ij}^{(N)}$ in which E_{ij} is to be expanded. To order ω^2 we have [1]

$$\begin{aligned} E_{ij} &= \sum_N a_N(\mathbf{k},\mathbf{k}')B_{ij}^{(N)} \\ &= \left[a_1(0) + a_{1,1}\mathbf{k}\cdot\mathbf{k}' + a_{1,2}\omega^2\right]\delta_{ij} + i\omega a_{2,1}\epsilon_{ijm}\sigma^m \\ &\quad + a_3(0)\left(-k_ik_j' + \mathbf{k}\cdot\mathbf{k}'\delta_{ij}\right) + a_4(0)k_i'k_j \\ &\quad + a_5(0)\left(k_i'k_j' + k_ik_j\right). \end{aligned} \quad (2.5)$$

The expansion of the coefficients is in accordance with the even-crossing-symmetry property of E_{ij}, that is, invariance under the transformation $i \leftrightarrow j$, $\mathbf{k} \leftrightarrow -\mathbf{k}'$, $\omega \leftrightarrow -\omega' = -\omega$. The coefficients a_1, $a_{1,1}$, $a_{2,1}$, a_4, and a_5 are the ones that obey low-energy theorems, with values in the Breit frame given by Eqs. (I-2.5a)–(I-2.5e), which we repeat here for convenience:

$$a_1(0) = -\frac{e^2}{m}, \quad (2.6a)$$

$$a_{2,1} = \frac{2\mu-1}{2m^2}e^2, \quad (2.6b)$$

$$a_{1,1} = \frac{e^2}{4m^3}, \quad (2.6c)$$

$$a_4(0) = 0, \quad (2.6d)$$

$$a_5(0) = \frac{\mu(1-\mu)}{4m^3}e^2, \quad (2.6e)$$

where $\mu = 1+\lambda$ is the magnetic moment of the nucleon in units of $e/2m$. Because of the transversality condition, a_4 and a_5 will not be present in the amplitude but, as we shall see, their values will give one of the sum rules. Also, as discussed before [1], one needs the value of a_5 to disentangle the known part of the coefficient $a_{1,2}$ whose unknown part α_1 represents a first contribution to the electric polarizability of the nucleon [1]:

$$a_{1,2} = \alpha_1 + \frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3} + \frac{2\mu^2-1}{4m^2}\right). \quad (2.7)$$

Here, $\langle r_1^2\rangle = 6F_1'$ with $F_1' = [dF_1(t)dt]_{t=0}$ and [11]

$$\alpha_1 = 2\sum_n \frac{|\langle n,\mathbf{0}|d_z|\mathbf{0}\rangle|^2}{M_n - m}, \quad (2.8)$$

where the sum extends to all but the one-nucleon on-shell intermediate state, and $\langle n,\mathbf{0}|d_i|\mathbf{0}\rangle$ is the electric dipole transition matrix element between the nucleon and the excited state n, with mass M_n, at rest. Our main purpose here is to obtain an expression for $a_3(0)$ which, as we shall discuss in Sec. III, is what is usually called the magnetic polarizability β of the nucleon:

$$a_3(0) = \beta. \quad (2.9)$$

According to Eq. (II-3.8) we can write the explicit expression of the excited state part, in the Breit frame where $\mathbf{p}' = -\mathbf{p}$, as

$$E_{ij} = V^2\sum_n\left[\frac{\langle -\mathbf{p}|J_i|n,\mathbf{p}+\mathbf{k}\rangle\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle}{E_n(\mathbf{p}+\mathbf{k}) - E - \omega} \times\frac{M_n}{E_n(\mathbf{p}+\mathbf{k})} + \text{c.t.}\right] + V\langle -\mathbf{p}|\rho_{ij}|\mathbf{p}\rangle, \quad (2.10)$$

where the sum stands for all but the nucleon intermediate state itself, E_n is the energy of the intermediate n state, and ρ_{ij}, which is symmetric in i and j, is related to the time-space current commutation relation as

$$[J_0(x), J_i(y)]\,\delta(x_0 - y_0) = i\frac{\partial}{\partial x_j}\left[\delta^4(x-y)\,\rho_{ij}(x)\right]. \quad (2.11)$$

It is easy to see that the last term of Eq. (2.10) can contain only even powers of ω. In fact, being symmetric in i,j its expansion in terms of the basis elements $B_{ij}^{(N)}$ defined in (2.5) can contain only the symmetric ones, $N = 1$ and N =3–5, to order ω^2. Next we extract from E_{ij} another piece that is even in ω by multiplying the first term inside the sum of Eq. (2.10) by $E_n(\mathbf{p}+\mathbf{k}) - E$ and the second by $E_n(\mathbf{p}-\mathbf{k}') - E'$ to obtain [1]

$$E_{ij} = \omega\Gamma_{ij} + \Delta_{ij} + V\langle -\mathbf{p}|\rho_{ij}|\mathbf{p}\rangle, \quad (2.12)$$

where, in the Breit frame where $E' = E$,

$$\Gamma_{ij} = V^2\sum_n\left[\frac{\langle -\mathbf{p}|J_i|n,\mathbf{p}+\mathbf{k}\rangle\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle}{[E_n(\mathbf{p}+\mathbf{k}) - E]\,[E_n(\mathbf{p}+\mathbf{k}) - E - \omega]} \times\frac{M_n}{E_n(\mathbf{p}+\mathbf{k})} - \text{c.t.}\right] \quad (2.13)$$

and

$$\Delta_{ij} = V^2 \sum_n \left[\frac{\langle -\mathbf{p}|J_i|n, \mathbf{p}+\mathbf{k}\rangle\langle n, \mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle}{E_n(\mathbf{p}+\mathbf{k}) - E} \times \frac{M_n}{E_n(\mathbf{p}+\mathbf{k})} + \text{c.t.} \right] . \tag{2.14}$$

In the Breit frame Γ_{ij} is then odd under crossing while Δ_{ij} is even under the less restricted transformation, $i \leftrightarrow j$, $\mathbf{k} \leftrightarrow -\mathbf{k}'$. As a consequence of this last fact it follows that the expansion of Δ_{ij} in terms of the basis elements $B_{ij}^{(N)}$ defined in Eq. (2.5) can contain only those for N=1 and N=3–5. Therefore Δ_{ij} is even in ω.

III. FORMULA FOR β AND A FIRST SUM RULE

Before we study E_{ij} in the general situation we consider the forward scattering amplitude $\mathbf{k}' = \mathbf{k}$ and show how we can get a closed expression for β [4], by making use of the second-order low-energy theorems. A first sum rule is also obtained. In the forward direction, the excited state tensor amplitude (2.5) becomes, for $i = j = 1$,

$$E_{11}(\mathbf{k},\mathbf{k}) = a_1(0) + [a_{1,1} + a_{1,2} + a_3(0)]\,\omega^2 + [-a_3(0) + a_4(0) + 2a_5(0)]\,k_x^2 . \tag{3.1}$$

Therefore

$$\left(\frac{\partial}{\partial k_z^2} E_{11}(\mathbf{k},\mathbf{k})\right)_0 = a_{1,1} + a_{1,2} + a_3(0) , \tag{3.2}$$

where the zero indicates $\mathbf{k}$=$\mathbf{0}$. From (2.12) and (2.4),

$$E_{11}(\mathbf{k},\mathbf{k}) = \omega\Gamma_{11}(\mathbf{k},\mathbf{k}) + \Delta_{11}(\mathbf{k},\mathbf{k}) + V\langle\mathbf{0}|\rho_{11}|\mathbf{0}\rangle . \tag{3.3}$$

The first-order term of Γ_{ij}, which is the only one that will contribute to the left-hand side of Eq. (3.2), is [1]

$$\Gamma_{ij}^{(1)} = \alpha_1\omega\delta_{ij} . \tag{3.4}$$

Therefore (3.2) can be written

$$\alpha_1 + \left(\frac{\partial}{\partial k_z^2}\Delta_{11}(\mathbf{k},\mathbf{k})\right)_0 = a_{1,1} + a_{1,2} + a_3(0) . \tag{3.5}$$

Using now the low-energy result (2.6c) and Eq. (2.7) with (2.9) we get

$$\beta = \beta_1 + \beta_2 \tag{3.6}$$

where

$$\beta_1 = \left(\frac{\partial}{\partial k_z^2}\Delta_{11}(\mathbf{k},\mathbf{k})\right)_0 \tag{3.7}$$

and

$$\beta_2 = -\frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3} + \frac{\mu^2}{2m^2}\right) . \tag{3.8}$$

From Eqs. (2.14) and (2.4) we can write

$$\beta_1 = 2V^2 \left(\frac{\partial}{\partial k_z^2} \sum_n \frac{|\langle n,\mathbf{k}|J_x|\mathbf{0}\rangle|^2}{E_n(\mathbf{k}) - m}\frac{M_n}{E_n(\mathbf{k})}\right)_0 , \tag{3.9}$$

and introducing $\langle r_E^2\rangle = \langle r_1^2\rangle + 3\lambda/2m^2$

$$\beta_2 = -\frac{e^2}{m}\left(\frac{\langle r_E^2\rangle}{3} + \frac{\lambda^2+\lambda+1}{2m^2}\right) . \tag{3.10}$$

The formula for β agrees then with the one obtained by L'Vov [4] using a dispersion method. Notice now that from (3.1) we also have the relation

$$\left(\frac{\partial}{\partial k_x^2}E_{11}(\mathbf{k},\mathbf{k})\right)_0 = a_{1,1} + a_{1,2} + a_4(0) + 2a_5(0) . \tag{3.11}$$

From (3.3) and (3.4), and making use of the low-energy results (2.6) and of (2.7) we then obtain

$$\left(\frac{\partial}{\partial k_x^2}\Delta_{11}(\mathbf{k},\mathbf{k})\right)_0 = \frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3} + \frac{\mu}{2m^2}\right) . \tag{3.12}$$

With (2.14) this gives the sum rule

$$2V^2 \left(\frac{\partial}{\partial k_x^2} \sum_n \frac{|\langle n,\mathbf{k}|J_x|\mathbf{0}\rangle|^2}{E_n(\mathbf{k}) - m}\frac{M_n}{E_n(\mathbf{k})}\right)_0 = \frac{e^2}{m}\left(\frac{\langle r_E^2\rangle}{3} + \frac{1}{2m^2}\right) . \tag{3.13}$$

Using now the decomposition of the transition current in multipoles [1] we shall have both β_1 in Eq. (3.9) and the sum rule (3.13) expressed in terms of the nucleon multipole transition moments. We shall do so in the next section.

We mention now that, as we have three low-energy theorems associated to second-order terms, given by Eqs. (2.6b)–(2.6d), we might expect another two sum rules. However, as the study of the general situation will show, there is only one more sum rule associated with the second order low-energy theorems.

Before closing this section we shall justify Eq. (2.9). The electric and magnetic dipole polarizabilities α and β are usually defined as the coefficients of the second-order spin-independent terms of the scattering amplitude after extracting the Born term contribution, through the relation

$$A = A_B + \alpha\omega\omega'\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + \beta(\mathbf{k}'\times\boldsymbol{\varepsilon}')\cdot(\mathbf{k}\times\boldsymbol{\varepsilon}) , \tag{3.14}$$

plus order ω^3. A_B is the Born amplitude, which is to be calculated with a point nucleon with charge e and anomalous moment λ. Its expression is given in Eq. (A4) of Appendix A, in the Breit frame. On the other hand, Eq. (A3) gives the expression of the amplitude calculated from (2.1). One then sees that this amplitude is related to the Born amplitude by the relation

$$A = A_B + \left(a_{1,2} - \frac{\lambda^2 + 2\lambda}{4m^3} \right) \omega^2 \varepsilon \cdot \varepsilon' + a_3(0) \left(\mathbf{k}' \times \varepsilon' \right) \cdot \left(\mathbf{k} \times \varepsilon \right) . \quad (3.15)$$

From here and Eq. (3.13) we conclude that

$$\alpha = a_{1,2} - \frac{\lambda^2 + 2\lambda}{4m^3} e^2 \quad (3.16)$$

and

$$\beta = a_3(0) , \quad (3.17)$$

which is Eq. (2.9). Also, by using (2.7), we see that α in (3.16) is given by

$$\alpha = \alpha_1 + \alpha_2 , \quad (3.18)$$

where α_1 is given in (2.8) and

$$\alpha_2 = \frac{e^2}{m} \left(\frac{\langle r_E^2 \rangle}{3} + \frac{\lambda^2 + 1}{4m^2} \right) , \quad (3.19)$$

which agrees with the result derived by L'Vov [4] using a dispersion approach.

IV. MULTIPOLE EXPANSION OF β AND OF THE FIRST SUM RULE

We shall now express β_1 in Eq. (3.9) and the sum rule (3.13) in terms of the multipole transition moments. For that purpose we need the multipole expansion of the nucleon transition current matrix elements. This was discussed in Ref. [1] and given for the excited state at rest. When the nucleon is at rest, which is what we need in Eqs. (3.9) and (3.13), the result, to order ω^2, is ($\epsilon_{123} = +1$)

$$\begin{aligned} V\langle n, \mathbf{k} | J_j | \mathbf{0} \rangle = {} & i \left(E_n \left(\mathbf{k} \right) - m \right) [\langle n, \mathbf{0} | d_j | \mathbf{0} \rangle \\ & - \frac{i}{2} k^a \langle n, \mathbf{0} | \bar{Q}_{ja} | \mathbf{0} \rangle - \frac{1}{6} k^a k^b \langle n, \mathbf{0} | \bar{O}_{jab} | \mathbf{0} \rangle \\ & + i \epsilon_{ja}{}^{r} k^a \Big[\langle n, \mathbf{0} | m_r | \mathbf{0} \rangle \\ & - \frac{i}{3} k^b \langle n, \mathbf{0} | L_{rb} | \mathbf{0} \Big] + O(\omega^3). \end{aligned} \quad (4.1)$$

On the right-hand side, we first have the rest state transition matrix of the electric dipole. The second matrix element contains those of the (traceless) quadrupole moment, and of the charge mean-square radius [1],

$$\langle n, \mathbf{0} | \bar{Q}_{ja} | \mathbf{0} \rangle = \langle n, \mathbf{0} | Q_{ja} | \mathbf{0} \rangle + \frac{1}{3} \delta_{ja} \langle n, \mathbf{0} | C | \mathbf{0} \rangle , \quad (4.2)$$

where $\langle n, \mathbf{0} | C | \mathbf{0} \rangle = - \langle n, \mathbf{0} | Q^i{}_i | \mathbf{0} \rangle$ is the charge mean-square radius transition matrix element. The third is

$$\langle n, \mathbf{0} | \bar{O}_{jab} | \mathbf{0} \rangle = \langle n, \mathbf{0} | O_{jab} | \mathbf{0} \rangle + \frac{1}{5} [\langle n, \mathbf{0} | S_j | \mathbf{0} \rangle \delta_{ab} + \text{c.p.}] , \quad (4.3)$$

where $\langle n, \mathbf{0} | S_j | \mathbf{0} \rangle = - \langle n, \mathbf{0} | \bar{O}_{ja}{}^{a} | \mathbf{0} \rangle$ and where c.p. stands for the cyclic permutation of the indices j, a, b. The first term on the right of Eq. (4.3) is the matrix element of the (traceless) octopole moment and the second is the matrix element of the electric dipole mean-square radius [see below Eq. (4.4)]. Next, we have in (4.1) the matrix element of the magnetic dipole moment, and the last term contains the matrix element of the magnetic quadrupole and of the magnetic dipole mean-square radius:

$$\langle n, \mathbf{0} | L_{rb} | \mathbf{0} \rangle = \langle n, \mathbf{0} | H_{rb} | \mathbf{0} \rangle + \epsilon_{rb}{}^{m} \langle n, \mathbf{0} | I_m | \mathbf{0} \rangle . \quad (4.4)$$

A little explanation is probably in order [1]. In Eq. (4.2), the matrix elements on the right-hand side contain, respectively, the rest-state matrix elements of the usual traceless electric quadrupole operator and of the charge mean-square operator $c = \int J_0 r^2 dV$, plus contributions of the moving electric dipole represented by a k^a derivative of $\langle n, \mathbf{k} | d_j | \mathbf{0} \rangle$ at $\mathbf{k} = \mathbf{0}$. Likewise, the two matrix elements on the right of Eq. (4.3) correspond to the usual traceless electric octopole moment operator and to the electric dipole mean-square radius operator, $s_j = \int J_0 r^2 x_j dV$, plus contributions of the moving dipole and quadrupole. The magnetic moment transition matrix element in (4.1) contains the rest-state matrix element of the magnetic dipole operator, $\mu_i = \frac{1}{2} \int (\mathbf{r} \times \mathbf{J})_i \, dV$, plus a contribution of the moving dipole moment [1], which we repeat here:

$$\langle n, \mathbf{0} | m_i | \mathbf{0} \rangle = \langle n, \mathbf{0} | \mu_i | \mathbf{0} \rangle - \frac{1}{2} \left(M_n - m \right) \times \epsilon_i{}^{ab} \left(\frac{\partial}{\partial k^a} \langle n, \mathbf{k} | d_b | \mathbf{0} \rangle \right)_0 , \quad (4.5)$$

where again the zero stands for $\mathbf{k}$=$\mathbf{0}$. The last term of (4.5) was missing in the definition used in Ref. [5]. Finally, on the right of Eq. (4.4), the first matrix element refers to the usual traceless magnetic quadrupole operator and the second to the magnetic dipole mean-square radius $i_m = \int \left(\mathbf{r} \times \mathbf{J} \right)_m r^2 dV$, plus contributions of the moving magnetic dipole moment. All these generalized multipole transition moments can be expressed in terms of the transition current matrix element [1], as we can see from (4.1):

$$V \langle n, \mathbf{0} | J_j | \mathbf{0} \rangle = i (M_n - m) \langle n, \mathbf{0} | d_j | \mathbf{0} \rangle , \quad (4.6)$$

$$\frac{V}{2i} \left(\frac{\partial}{\partial \mathbf{k}} \times \langle n, \mathbf{k} | \mathbf{J} | \mathbf{0} \rangle \right)_0 = \langle n, \mathbf{0} | \mathbf{m} | \mathbf{0} \rangle , \quad (4.7)$$

and

$$V \left[\left(\frac{\partial}{\partial k^a} \langle n, \mathbf{k} | J_j | \mathbf{0} \rangle \right)_0 + (j, a) \right] = \left(M_n - m \right) \langle n, \mathbf{0} | \bar{Q}_{ja} | \mathbf{0} \rangle , \quad (4.8)$$

where (j, a) stands for the previous term with j and a interchanged. By subtracting and adding the trace, we obtain the matrix element of the quadrupole moment and mean-square radius. Next we have

$$-V\left\{\left[\left(\frac{\partial^2}{\partial k^a \partial k^b}\langle n,\mathbf{k}|J_j|\mathbf{0}\rangle\right)_0 - \delta_{ab}\frac{\langle n,\mathbf{0}|J_j|\mathbf{0}\rangle}{M_n\left(M_n-m\right)}\right] +\text{c.p.}\right\} = \left(M_n-m\right)\langle n,\mathbf{0}|\bar{O}_{jab}|\mathbf{0}\rangle \quad (4.9)$$

and

$$V\left[\epsilon_r{}^{bj}\left(\frac{\partial^2}{\partial k^a k^b}\langle n,\mathbf{k}|J_j|\mathbf{0}\rangle\right)_0 + \epsilon_{ra}{}^j\frac{\langle n,\mathbf{0}|J_j|\mathbf{0}\rangle}{M_n\left(M_n-m\right)}\right] = \langle n,\mathbf{0}|L_{ra}|\mathbf{0}\rangle\,. \quad (4.10)$$

From the equation of continuity we have

$$\left[E_n(\mathbf{k})-m\right]\langle n,\mathbf{k}|J_0|\mathbf{0}\rangle = -k^j\langle n,\mathbf{k}|J_j|\mathbf{0}\rangle\,, \quad (4.11)$$

and hence the electric multipoles can also be calculated from the charge density matrix element:

$$V\left(\frac{\partial}{\partial k^i}\langle n,\mathbf{k}|J_0|\mathbf{0}\rangle\right)_0 = -i\langle n,\mathbf{0}|d_i|\mathbf{0}\rangle\,, \quad (4.12)$$

$$V\left(\frac{\partial}{\partial k^i\partial k^j}\langle n,\mathbf{k}|J_0|\mathbf{0}\rangle\right)_0 = -\langle n,\mathbf{0}|\bar{Q}_{ij}|\mathbf{0}\rangle, \quad (4.13)$$

$$V\left(\frac{\partial^3}{\partial k^i\partial k^j\partial k^m}\langle n,\mathbf{k}|J_0|\mathbf{0}\rangle\right)_0 = \langle n,\mathbf{0}|\bar{O}_{ijm}|\mathbf{0}\rangle\,. \quad (4.14)$$

We go back now to Eq. (4.1). We could substitute this equation directly in Eqs. (3.9) and (3.13) to reach our aim but it is a somewhat simpler procedure to substitute Eq. (4.1) and its companion

$$\begin{aligned} V\langle\mathbf{0}|J_i|n,\mathbf{k}\rangle = & -i\left(E_n(\mathbf{k})-m\right)\left[\langle\mathbf{0}|d_i|n,\mathbf{0}\rangle + \frac{i}{2}k^a\langle\mathbf{0}|\bar{Q}_{ia}|n,\mathbf{0}\rangle - \frac{1}{6}k^ak^b\langle\mathbf{0}|\bar{0}_{iab}|n,\mathbf{0}\rangle\right] \\ & -i\epsilon_{ia}{}^s k^a\left[\langle\mathbf{0}|m_s|n,\mathbf{0}\rangle + \frac{i}{3}k^b\langle\mathbf{0}|L_{sb}|n,\mathbf{0}\rangle\right] \end{aligned} \quad (4.15)$$

directly in Eq. (2.14) for $\mathbf{k}'=\mathbf{k}$ and afterwards use Eqs. (3.7) and (3.12). Also, this procedure is more akin to the one that we shall have to use in the nonforward situation in Sec. V. With the indication

$$\langle n,\mathbf{0}|X|\mathbf{0}\rangle = (X)_{nN}\,, \quad (4.16)$$

where N stands for the implicit nucleon rest state, and paying attention that there are no cross products between matrix elements of multipoles with opposite parities, we obtain, to order ω^2,

$$\begin{aligned} \Delta_{ij}(\mathbf{k},\mathbf{k}) = & \sum_n \frac{M_n}{E_n(\mathbf{k})}\left[E_n(\mathbf{k})-m\right]\left[(d_i)_{Nn}(d_j)_{nN}+(i,j)\right] \\ & -\frac{1}{6}k^ak^b\sum_n\left(M_n-m\right)\left[(d_i)_{nN}\left(\bar{O}_{jab}\right)_{Nn}+\left(\bar{O}_{jab}\right)_{Nn}(d_i)_{nN}+(i,j)\right] \\ & +\frac{1}{4}k^ak^b\sum_n\left(M_n-m\right)\left[\left(\bar{Q}_{ia}\right)_{Nn}\left(\bar{Q}_{jb}\right)_{nN}+(i,j)\right] \\ & -\frac{i}{3}k^ak^b\sum_n\left\{\epsilon_{ja}{}^r\left[(d_i)_{Nn}(L_{rb})_{nN}-(L_{rb})_{Nn}(d_i)_{nN}\right]+(i,j)\right\} \\ & +\frac{i}{2}k^ak^b\sum_n\left\{\epsilon_{ja}{}^r\left[\left(\bar{Q}_{ia}\right)_{Nn}(m_r)_{nN}-(m_r)_{Nn}\left(\bar{Q}_{ia}\right)_{nN}\right]\right\} \\ & +\epsilon_{ia}{}^r\epsilon_{jb}{}^s k^ak^b\sum_n\frac{(m_r)_{Nn}(m_s)_{nN}+(r,s)}{M_n-m}, \end{aligned} \quad (4.17)$$

where (i,j) stands for the previous term with i and j interchanged. Next we notice that, on reducing $\left(\bar{O}_{jab}\right)_{nN}$ according to (4.3), the octopole can give no contribution since $(d_i)_{Nn}(O_{jab})_{nN} = 0$. In fact, as d_i has spin parity $J^p = 1^-$ and O_{jab} is a 3^-object, the state n in the first matrix element can be nonzero only if it is a $\frac{1}{2}^-$ or $\frac{3}{2}^-$state and in the second it can only be a $\frac{5}{2}^-$or $\frac{7}{2}^-$ state. Also, on reducing $\left(\bar{Q}_{ia}\right)_{Nn}$ according to (4.2), the crossed quadrupole-charge mean-square radius term can give no contribution, $(Q_{ia})_{Nn}(C)_{nN} = 0$, since Q_{ia} is a 2^+ object and C is a 0^+ object. We introduce now quantities which will be present in the right of Eq. (4.17) together with others that will appear when we shall study the amplitude in the nonforward direction. First we define

$$\sum_n E_{vn}\left[\langle\mathbf{0}|d_i|n,\mathbf{0}\rangle\langle n,\mathbf{0}|d_j|\mathbf{0}\rangle + (i,j)\right] = b_v\delta_{ij},$$

$$v = 1,\,2\,, \quad (4.18)$$

where

$$E_{1n} = \frac{m}{2M_n^2}, \quad E_{2n} = \frac{1}{M_n}. \tag{4.18a}$$

The case $v = 1$ will be present here and $v = 2$ will appear in Sec. V. Then we define, for future reference in Sec. V,

$$\frac{1}{4}\sum_n \frac{M_n - m}{mM_n}\left[\langle 0|d_i|n,0\rangle\langle n,0|d_j|0\rangle - (i,j)\right]$$
$$= -ih\epsilon_{ijm}\sigma^m . \tag{4.19}$$

Next, in the order that will appear in (4.17),

$$\frac{1}{30}\sum_n (M_n - m)\left[\langle 0|d_i|n,0\rangle\langle n,0|S_j|0\rangle\right.$$
$$\left.+\langle 0|S_j|n,0\rangle\langle n,0|d_i|0\rangle\right] = c\delta_{ij} . \tag{4.20}$$

A contribution $c'\epsilon_{ijm}\sigma^m$ is excluded by the fact that under time reversal (T) σ^m changes sign but the left-hand side of (4.20) does not, since $\langle 0|d_i|n,0\rangle$ goes into $\langle n,0|d_i|0\rangle$ and similarly for S_j. Next we define

$$\frac{1}{24}\sum_n F_{vn}(M_n - m)\left[\langle 0|Q_{ia}|n,0\rangle\langle n,0|Q_{jb}|0\rangle\right.$$
$$\left.+\langle 0|Q_{jb}|n,0\rangle\langle n,0|Q_{ia}|0\rangle\right]$$
$$= d_v\left(\delta_{ij}\delta_{ab} + \delta_{ib}\delta_{ja} - \frac{2}{3}\delta_{ia}\delta_{jb}\right), \tag{4.21}$$

which is symmetric and traceless in i and a, as is Q_{ia}, where

$$F_{1n} = 1, \quad F_{2n} = \frac{M_n}{m}. \tag{4.21a}$$

A term $\delta_{ij}\epsilon_{abm}\sigma^m$, properly symmetrized and made traceless, is excluded by T invariance. Then we define

$$\frac{1}{36}\sum_n F_{vn}(M_n - m)\left|\langle 0|C|n,0\rangle\right|^2 = e_v \tag{4.22}$$

and

$$\frac{i}{3}\sum_n\left[\langle 0|d_i|n,0\rangle\langle n,0|I_j|0\rangle\right.$$
$$\left.-\langle 0|I_j|n,0\rangle\langle 0|d_i|0\rangle\right] = f\delta_{ij}. \tag{4.23}$$

Here a $f'\epsilon_{ijm}\sigma^m$ contribution is excluded by T invariance, since under time reversal $\langle n,0|I_j|0\rangle$ goes into $-\langle 0|I_j|n,0\rangle$ with a minus sign not present for d_i. Finally, we define

$$\sum_n F_{vn}\frac{\langle 0|m_i|n,0\rangle\langle n,0|m_j|0\rangle + (i,j)}{M_n - m} = g_v\delta_{ij}. \tag{4.24}$$

The quantities involving

$$(d_i)_{Nn}(H_{rb})_{nN} + (H_{rb})_{Nn}(d_i)_{nN}$$

and

$$(\bar{Q}_{ib})_{Nn}(m_r)_{nN} - (M_r)_{Nn}(\bar{Q}_{ib})_{nN}$$

in (4.17) will be both equal to zero because the corresponding sums do not change sign under time reversal but both would have to have a term of the form $\delta_{ir}\sigma_b$ on their right-hand side, by parity invariance. We then obtain, to order ω^2,

$$\begin{aligned}\Delta_{ij}(\mathbf{k},\mathbf{k}) = &\sum_n (M_n - m)\left[(d_i)_{Nn}(d_j)_{nN} + (i,j)\right]\\ &+\omega^2(b_1 - 2c + 6d_1 + 2f + g_1)\delta_{ij}\\ &+k_ik_j(-4c + 2d_1 + 2e_1 - 2f - g_1) .\end{aligned} \tag{4.25}$$

Taking $i = j = 1$ and substituting in Eq. (3.7) we get

$$\beta_1 = b_1 - 2c + 6d_1 + 2f + g_1 . \tag{4.26}$$

This gives the desired expression for the nonstatic piece of β in terms of the nucleon multipole transition matrix elements. It includes those up to electric quadrupole order. Explicitly we have, from (4.18)–(4.24),

$$\begin{aligned}\beta_1 = &\sum_n \frac{m}{M_n^2}\left|\langle n,0|d_z|0\rangle\right|^2\\ &-\frac{2}{15}\sum_n (M_n - m)\,\mathrm{Re}\left[\langle 0|d_z|n,0\rangle\langle n,0|S_z|0\rangle\right]\\ &+\frac{3}{8}\sum_n (M_n - m)\left|\langle 0|Q_{zz}|n,0\rangle\right|^2\\ &-\frac{4}{3}\sum_n (M_n - m)\,\mathrm{Im}\left[\langle 0|d_z|n,0\rangle\langle n,0|I_z|0\rangle\right]\\ &+2\sum_n \frac{\left|\langle n,0|m_z|0\rangle\right|^2}{M_n - m},\end{aligned} \tag{4.27}$$

where Re and Im stand for real and imaginary part of, respectively. Likewise, the sum rule (3.12) can now be expressed in terms of the multipole moments as

$$\frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3} + \frac{\mu}{2m^2}\right) = b_1 - 6c + 8d_1 + 2e_1 . \tag{4.28}$$

A second sum rule will be derived in the next section.

V. MULTIPOLE EXPANSION OF THE AMPLITUDE IN THE GENERAL SITUATION

In this section we shall study the second-order terms of E_{ij} in the general, nonforward, situation. From Eq. (2.12) we see that one of them comes from the first-order term of Γ_{ij} and the others come from the last two terms which, as discussed before, are even in ω. We shall start by expressing the last one in terms of the current matrix elements by making use of Eq. (2.11). We first take this equation between $\langle -\mathbf{p}|$ and $|\mathbf{p}\rangle$, multiply by $\exp(i\mathbf{Q}\cdot\mathbf{r})$, take $y=0$, and integrate over $\mathbf{r}$. Then we introduce a complete set of intermediate states on the left-hand side and make an integration by parts on its right-hand side. Separating out the one-nucleon on-shell intermediate state we obtain, in the Breit frame,

$$\begin{aligned} VQ^j\langle -\mathbf{p}|\rho_{ij}|\mathbf{p}\rangle = {} & V^2\frac{m}{E(\mathbf{p}+\mathbf{k})}\left[\langle -\mathbf{p}|J_i|\mathbf{p}+\mathbf{Q}\rangle\langle \mathbf{p}+\mathbf{Q}|J_0|\mathbf{p}\rangle - \langle -\mathbf{p}|J_0|-\mathbf{p}-\mathbf{Q}\rangle\langle -\mathbf{p}-\mathbf{Q}|J_i|\mathbf{p}\rangle\right] \\ & -V^2Q^j\sum_n\frac{M_n}{E_n(\mathbf{p}+\mathbf{Q})\left[E_n(\mathbf{p}+\mathbf{Q})-E\right]} \\ & \times\left[\langle -\mathbf{p}|J_i|n,\mathbf{p}+\mathbf{Q}\rangle\langle n,\mathbf{p}+\mathbf{Q}|J_j|\mathbf{p}\rangle + \langle -\mathbf{p}|J_j|n,-\mathbf{p}-\mathbf{Q}\rangle\langle n,-\mathbf{p}-\mathbf{Q}|J_i|\mathbf{p}\rangle\right], \end{aligned} \tag{5.1}$$

where we have used the equation of continuity to write the excited state part in a convenient way. To obtain $\langle -\mathbf{p}|\rho_{ij}|\mathbf{p}\rangle$ to order p^2, that is, to order ω^2 by Eq. (2.4), we have to calculate the right-hand side of (5.1) to order Qp^2. To this order the unexcited term is equal to

$$-\frac{e^2}{m}Q^i + \frac{e^2}{m}\left(4F_1' + \frac{2\mu^2-1}{2m^2}\right)p^2Q^i - \frac{e^2\mu(\mu-1)}{m^3}\mathbf{p}\cdot\mathbf{Q}p^i\,. \tag{5.2}$$

Therefore, Eq. (5.1) gives us

$$V\langle -\mathbf{p}|\rho_{ij}|\mathbf{p}\rangle = u_{ij} + e_{ij} \tag{5.3}$$

where, to order ω^2,

$$u_{ij} = -\frac{e^2}{m}\delta_{ij} + \frac{e^2}{m}\left(4F_1' + \frac{2\mu^2-1}{2m^2}\right)p^2\delta_{ij} - \frac{e^2\mu(\mu-1)}{m^3}p_ip_j \tag{5.4}$$

and

$$e_{ij} = -V^2\sum_n\frac{M_n}{E_n(\mathbf{p})}\frac{\langle -\mathbf{p}|J_i|n,\mathbf{p}\rangle\langle n,\mathbf{p}|J_j|\mathbf{p}\rangle + \langle -\mathbf{p}|J_j|n,-\mathbf{p}\rangle\langle n,-\mathbf{p}|J_i|\mathbf{p}\rangle}{E_n(\mathbf{p})-E}, \tag{5.5}$$

which is to be calculated to order p^2, where $\mathbf{p}$ is given by (2.4). Equation (5.3) is now to be substituted in Eq. (2.12). To order ω^2 we shall have

$$E_{ij}^{(2)} = \omega\Gamma_{ij}^{(1)} + \Delta_{ij}^{(2)} + e_{ij}^{(2)} + u_{ij}^{(2)}\,. \tag{5.6}$$

We shall need now the multipole expansion of the current matrix elements present in Eqs. (2.14) and (5.5). As $\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle$ in Eq. (2.14) depends on two variables we shall first reduce one of the states to rest by means of a Lorentz transformation, dealing thereafter with a single variable. In Ref. [1] we brought the excited state to rest but here we want it to be so for the nucleon state, to maintain close contact with the nucleon rest-state results derived in the previous section. We could bring the nucleon state to rest directly but to avoid a rotation matrix for the excited state we shall do so in two steps: First we bring the excited state to rest as before [1], by means of a Lorentz transformation L, with velocity

$$\mathbf{V} = \frac{\mathbf{p}+\mathbf{k}}{E_n(\mathbf{p}+\mathbf{k})}, \tag{5.7}$$

and obtain, to the order ω^2 that we are interested in,

$$\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle = \langle n,\mathbf{0}|J_j + V_j(\tfrac{1}{2}\mathbf{V}\cdot\mathbf{J}+J_0)|\mathbf{q}\rangle D_{\mathbf{p}} \tag{5.8}$$

where

$$\mathbf{q} = \mathbf{p} - m\mathbf{V} + O(\omega^3) \tag{5.9}$$

is the transformed of $\mathbf{p}$ by L and $D_{\mathbf{p}}$ is the rotation matrix corresponding to the rotation of the nucleon momentum $\mathbf{p}$,

$$D_{\mathbf{p}} = 1 - i\frac{\boldsymbol{\sigma}\cdot(\mathbf{V}\times\mathbf{p})}{4m} + O(\omega^3)\,. \tag{5.10}$$

Now we perform a second Lorentz transformation with velocity

$$\mathbf{V}' = \frac{\mathbf{q}}{E(\mathbf{q})} \tag{5.11}$$

to bring the nucleon state $|\mathbf{q}\rangle$ to rest. This gives for the two matrix elements on the right of Eq. (5.8), to order ω^2,

$$\langle n,\mathbf{0}|J_j|\mathbf{q}\rangle = \langle n,\mathbf{K}|J_j + V_j'(\tfrac{1}{2}\mathbf{V}'\cdot\mathbf{J}+J_0)|\mathbf{0}\rangle \tag{5.12}$$

and

$$\langle n,\mathbf{0}|J_0|\mathbf{q}\rangle = \langle n,\mathbf{K}|J_0 + \mathbf{V}'\cdot\mathbf{J}|\mathbf{0}\rangle\,, \tag{5.13}$$

where $\mathbf{K} = -M_n\mathbf{V}'$ is the new momentum of the excited state, to order ω^2. On account of (5.11) and with the help of (5.7) and (5.9), it can be written

$$\mathbf{K} = \mathbf{k} + \left(1 - \frac{M_n}{m}\right)\mathbf{p}\,, \tag{5.14a}$$

plus terms of order ω^3, or by (2.4)

$$\mathbf{K}=\frac{1}{2}\left(1-\frac{M_n}{m}\right)\mathbf{k}'+\frac{1}{2}\left(1+\frac{M_n}{m}\right)\mathbf{k}\,. \tag{5.14b}$$

Now we substitute (5.12) and (5.13) in (5.8) and notice that on account of (5.9) and (5.11), to order ω^3,

$$\mathbf{V}'=\frac{\mathbf{p}}{m}-\mathbf{V}\,. \tag{5.15}$$

After using the relation

$$\langle n,\mathbf{K}|J_0|\mathbf{0}\rangle=-K^a\frac{\langle n,\mathbf{K}|J_a|\mathbf{0}\rangle}{E_n(\mathbf{K})-m}\,, \tag{5.16}$$

which follows from the equation of continuity, we get

$$\begin{aligned}\langle n,\mathbf{p}+\mathbf{k}|J_j|\mathbf{p}\rangle&=\langle n,\mathbf{K}|J_j|\mathbf{0}\rangle\left(1-i\frac{\boldsymbol{\sigma}\cdot(\mathbf{V}\times\mathbf{p})}{4m}\right)\\&\quad+\langle n,\mathbf{0}|J^a|\mathbf{0}\rangle\left(\frac{p_ap_j}{2m^2}\right.\\&\quad\left.-\frac{k_jp_a-k_ap_j}{2mM_n}-\frac{k_ap_j}{m(M_n-m)}\right).\end{aligned} \tag{5.17}$$

Likewise

$$\begin{aligned}&\langle-\mathbf{p}|J_i|n,\mathbf{p}+\mathbf{k}\rangle\\&=\left[1-i\frac{\boldsymbol{\sigma}\cdot(\mathbf{V}\ \times\mathbf{p})}{4m}\right]\langle\mathbf{0}|J_i|n,\mathbf{K}'\rangle\\&\quad+\langle\mathbf{0}|J^a|n,\mathbf{0}\rangle\left(\frac{p_ap_i}{2m^2}+\frac{k_i'p_a-k_a'p_i}{2mM_n}+\frac{k_a'p_i}{m(M_n-m)}\right),\end{aligned} \tag{5.18}$$

where

$$\mathbf{K}'=\mathbf{k}+\left(1+\frac{M_n}{m}\right)\mathbf{p}\,, \tag{5.19a}$$

or, by (2.4),

$$\mathbf{K}'=\frac{1}{2}\left(1+\frac{M_n}{m}\right)\mathbf{k}'+\frac{1}{2}\left(1-\frac{M_n}{m}\right)\mathbf{k}\,. \tag{5.19b}$$

Notice that when $\mathbf{k}\leftrightarrow-\mathbf{k}'$ we have $\mathbf{K}\leftrightarrow-\mathbf{K}'$, and, as $\mathbf{p}+\mathbf{k}=\frac{1}{2}(\mathbf{k}'+\mathbf{k})$, $\mathbf{V}\to-\mathbf{V}$. For the matrix elements that appear in (5.5) we shall have

$$\begin{aligned}\langle-\mathbf{p}|J_i|n,\mathbf{p}\rangle&=\langle\mathbf{0}|J_i|n,\mathbf{K}_0\rangle+\langle\mathbf{0}|J^a|n,\mathbf{0}\rangle\\&\quad\times\frac{p_ap_i(M_n+3m)}{2m^2(M_n-m)}\,,\end{aligned} \tag{5.20}$$

where

$$\mathbf{K}_0'=\left(1+\frac{M_n}{m}\right)\mathbf{p}\,, \tag{5.21}$$

and

$$\langle n,\mathbf{p}|J_j|\mathbf{p}\rangle=\langle n,\mathbf{K}_0|J_j|0\rangle+\langle n,0|J^a|\mathbf{0}\rangle\frac{p_ap_j}{2m^2}\,, \tag{5.22}$$

where

$$\mathbf{K}_0=\left(1-\frac{M_n}{m}\right)\mathbf{p}\,. \tag{5.23}$$

We use now these results together with (3.4), (4.1), and (4.15) to calculate (5.6). The calculation is rather long but straightforward, along the same line of thought that we used in the previous section. We quote here only the final results, giving the details in Appendix B. Calculating the right-hand side of (5.6) in terms of the quantities defined in Eqs. (4.18)–(4.24) and comparing with (2.5) we reobtain the zeroth-order low-energy result (2.6a) and get the following five relations from the second-order terms.

$$\begin{aligned}a_{1,1}=&-\frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3}+\frac{\mu^2+\mu-1}{4m^2}\right)+b_1-b_2-4c+7d_1\\&+5d_2+e_1-e_2+f+\frac{1}{2}(g_1-g_2)+h,\end{aligned} \tag{5.24}$$

$$a_{1,2}=\alpha_1+\frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3}+\frac{2\mu^2-1}{4m^2}\right), \tag{5.25}$$

and, as $a_3(0)=\beta$,

$$\begin{aligned}\beta=&\frac{\mu(1-\mu)e^2}{4m^3}+b_2+2c-d_1-5d_2-e_1\\&+e_2+f+\frac{1}{2}(g_1+g_2)-h\,.\end{aligned} \tag{5.26}$$

Then

$$\begin{aligned}a_4(0)=&\frac{\mu(\mu-1)e^2}{4m^3}+b_2-2c+d_1-5d_2+e_1\\&+e_2-f+\frac{1}{2}(g_2-g_1)-h\ ,\end{aligned} \tag{5.27}$$

and

$$a_5(0)=\frac{\mu(1-\mu)e^2}{4m^3}\,. \tag{5.28}$$

Equations (5.25) and (5.28) agree with the results (2.6e) and (2.7) obtained [1] by the gauge method. Equation (5.26) is an expression for β in term of the static properties of the nucleon and its multipole transition moments up to electric quadrupole order. Finally Eqs. (5.24) and (5.28) will give us two sum rules when we make use of the low-energy results (2.6c) and (2.6d). Adding (5.24) and (5.26) and using (2.6c) the expression for β can be rewritten as

$$\begin{aligned}\beta=&-\frac{e^2}{m}\left(\frac{\langle r_E^2\rangle}{3}+\frac{\lambda^2+\lambda+1}{2m^2}\right)+b_1\\&-2c+6d_1+2f+g_1\,.\end{aligned} \tag{5.29}$$

This agrees with (3.6), as given by (3.10) and (4.26). On the other hand, by adding (5.24) and (5.27) we obtain, after using (2.6c) and (2.6d),

$$\frac{e^2}{m}\left(\frac{\langle r_1^2\rangle}{3}+\frac{\mu}{2m^2}\right)=b_1-6c+8d_1+2e_1\,, \tag{5.30}$$

which agrees with (4.28). We then recover the expression for β and the sum rule obtained from the forward-direction analysis. The additional sum rule is, from (5.27) and (2.6d),

$$\frac{\mu(1-\mu)e^2}{4m^3} = b_2 - 2c + d_1 - 5d_2 + e_1 + e_2 - f + \frac{1}{2}(g_2 - g_1) - h. \tag{5.31}$$

VI. DISCUSSION

By a direct analysis of the excited state part of the nucleon Compton scattering amplitude to the second order in the frequency of the incoming photon we have founded an expression for the magnetic polarizability β of the nucleon, related to sums of products of its transition moments up to quadrupole order. For that purpose we have used the multipole expansion of the transition current matrix elements between the nucleon and its excitations derived before [1], in the formula for β derived by L'Vov [4]. This formula, which was obtained using a dispersion method, is here reobtained by a method that immediately leads to a first sum rule, when one makes use of the low-energy theorems which are of second order in the frequency of the incoming photon. As the formula for β, this sum rule is also related to the nucleon transition current and, therefore, by using the multipole expansion it could also be put in terms of the nucleon transition multipoles. All this analysis has been made in the forward-scattering situation. Next we moved to the general, nonforward, case and obtain a second sum rule. The closed expression for the magnetic polarizability can be of help for a quark model calculation. This could be achieved by using the nucleon transition amplitudes derived in the context of the relativized quark model [6–8], on the assumption of dominance of the low-lying resonances. In turn, the two sum rules could be used to test this assumption. A further test could be provided by the other three sum rules previously obtained [1].

The whole procedure developed here for the nucleon is quite general and can be applied to hadrons of arbitrary spin.

APPENDIX A

The calculation of $\varepsilon'^i U_{ij}\varepsilon^j$ is made somewhat easier by employing the equivalent expression

$$\langle \mathbf{p}_2|J_\mu|\mathbf{p}_1\rangle = \frac{e}{V}\bar{u}(\mathbf{p}_2)\left[(F_1 + \lambda F_2)\gamma_\mu - \frac{\lambda}{2m}F_2(p_2+p_1)_\mu\right]u(\mathbf{p}_1) \tag{A1}$$

for the nucleon current matrix element. A straightforward calculation gives, in the Breit frame, to second order, and with $\mathbf{h} = \mathbf{k}\times\boldsymbol{\varepsilon}$,

$$\begin{aligned}\varepsilon'^i U_{ij}\varepsilon^j = {} & \frac{ie^2\mu}{2m^2\omega}[\mathbf{k}'\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\mathbf{h}' - \mathbf{k}\cdot\boldsymbol{\varepsilon}'\boldsymbol{\sigma}\cdot\mathbf{h} - \mu\boldsymbol{\sigma}\cdot(\mathbf{h}'\times\mathbf{h})] \\ & + \frac{e^2\cos\theta}{4m^3}[\mathbf{k}'\cdot\boldsymbol{\varepsilon}\mathbf{k}\cdot\boldsymbol{\varepsilon}'(2\lambda+\lambda^2) \\ & - \mu^2\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'\mathbf{k}\cdot\mathbf{k}'],\end{aligned} \tag{A2}$$

where $\cos\theta = \mathbf{k}\cdot\mathbf{k}'\omega^{-2}$. Adding the excited part $\varepsilon'^i E_{ij}\varepsilon^j$ with the low-energy results (2.6a)–(2.6c), we get, for the scattering amplitude,

$$\begin{aligned}A = {} & \varepsilon'^i U_{ij}\varepsilon^j - \frac{e^2}{m}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + i\omega\frac{2\mu-1}{2m^2}e^2\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}'\times\boldsymbol{\varepsilon}) \\ & + \frac{e^2}{4m^3}\mathbf{k}\cdot\mathbf{k}'\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + a_{1,2}\omega^2\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + a_3(0)\,\mathbf{h}'\cdot\mathbf{h}.\end{aligned} \tag{A3}$$

The Born term is given by ($\underline{a} = \gamma^\mu a_\mu$)

$$\begin{aligned}A_B = {} & e^2\bar{u}(-\mathbf{p})\left[\underline{\varepsilon}' - \frac{\lambda}{2m}i\underline{\varepsilon}'\underline{k}'\right]\frac{m}{(p+k)+m^2} \\ & \times\left[\underline{\varepsilon} + \frac{\lambda}{2m}i\underline{\varepsilon}\,\underline{k}\right]u(\mathbf{p}) + \text{c.t.}.\end{aligned}$$

A rather long but direct calculation gives, to second order,

$$\begin{aligned}A_B = e^2\Bigg(& -\frac{1}{m}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + i\omega\frac{2\mu-1}{2m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{\varepsilon}'\times\boldsymbol{\varepsilon}) + \frac{i\mu}{2m^2\omega}[\mathbf{k}'\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot\mathbf{h}' - \mathbf{k}\cdot\boldsymbol{\varepsilon}'\boldsymbol{\sigma}\cdot\mathbf{h} - \mu\boldsymbol{\sigma}\cdot(\mathbf{h}'\times\mathbf{h})] \\ & + \frac{1}{4m^3}\{\mathbf{k}\cdot\mathbf{k}'\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' + \cos\theta[\mathbf{k}'\cdot\boldsymbol{\varepsilon}k\cdot\boldsymbol{\varepsilon}'(\lambda^2+2\lambda) - \mu^2\mathbf{k}\cdot\mathbf{k}'\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'] + (\lambda^2+2\lambda)\,\omega^2\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'\}\Bigg).\end{aligned} \tag{A4}$$

Comparing (A3) and (A4) we find Eq. (3.15).

APPENDIX B

Using (5.20) and (5.22) in (5.1) and recalling that ρ_{ij}, and therefore $e_{ij}^{(2)}$, is symmetric we can write

$$\begin{aligned}e_{ij}^{(2)} = {} & -V^2\sum_n \frac{M_n}{2E_n(\mathbf{p})}\left[\frac{\langle 0|J_i|n,\mathbf{K}_0'\rangle\langle n,\mathbf{K}_0|J_j|0\rangle + \langle 0|J_j|n,-\mathbf{K}_0\rangle\langle n,-\mathbf{K}_0'|J_i|0\rangle}{E_n(\mathbf{p})-E} + \text{c.t.}\right] \\ & -\frac{1}{2}V^2\sum_n\frac{1}{M_n-m}\left\{[(J_i)_{Nn}(J^a)_{nN} + (J^a)_{Nn}(J_i)_{nN}]\left[-\frac{p_ap_j}{2m^2} + \frac{p_ap_j(M_n+3m)}{2m^2(M_n-m)}\right] + \text{c.t.}\right\}.\end{aligned} \tag{B1}$$

Using (4.1) and (4.15) in (B1) and using (5.17) and (5.18) in (2.14) we obtain, to order ω^2,

$$\begin{aligned}
\triangle_{ij}^{(2)} + e_{ij}^{(2)} = &\sum_n \left[-\frac{\mathbf{k}\cdot\mathbf{k}'}{M_n}\left(1-\frac{m}{2M_n}\right)+\frac{K'^2+K^2-K_0^2-K_0'^2}{2M_n}\right]\left[(d_i)_{Nn}(d_j)_{nN}+(i,j)\right] \\
&-\frac{1}{6}\sum_n (M_n-m)\left\{\left(K^aK^b-\frac{K_0^aK_0^b+K_0'^aK_0'^b}{2}\right)\left[(d_i)_{Nn}\left(\bar{O}_{jab}\right)_{nN}+\left(\bar{O}_{jab}\right)_{Nn}(d_i)_{nN}\right]+\text{c.t.}\right\} \\
&-\frac{i}{3}\sum_n \left\{\left(K^aK^b-\frac{K_0^aK_0^b+K_0'^aK_0'^b}{2}\right)\epsilon_{ja}{}^{r}\left[(d_i)_{Nn}(L_{rb})_{nN}-(L_{rb})_{Nn}(d_i)_{nN}\right]+\text{c.t.}\right\} \\
&+\frac{1}{4}\sum_n (M_n-m)\left(K'^aK^b-K_0'^aK_0^b\right)\left[\left(\bar{Q}_{ia}\right)_{Nn}\left(\bar{Q}_{jb}\right)_{nN}+\left(\bar{Q}_{jb}\right)_{Nn}\left(\bar{Q}_{ia}\right)_{nN}\right] \\
&+\frac{1}{2}\sum_n \left\{\left(K'^aK^b-K_0'^aK_0^b\right)\varepsilon_{jb}{}^{r}\left[\left(\bar{Q}_{ia}\right)_{Nn}(m_r)_{nN}-(m_r)_{Nn}\left(\bar{Q}_{ia}\right)_{nN}\right]+\text{c.t.}\right\} \\
&+\varepsilon_{ia}{}^{r}\varepsilon_{jb}{}^{s}\sum_n\left(K'^aK^b-K_0'^aK_0^b\right)\frac{(m_r)_{Nn}(m_s)_{nN}+(r,s)}{M_n-m} \\
&+i\sum_n (M_n-m)\left\{\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}\left[(d_i)_{Nn}(d_j)_{nN}+(i,j)\right]+\left[(d_i)_{Nn}(d_j)_{nN}+(i,j)\right]\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{k})}{8mM_n}\right\} \\
&-\sum_n (M_n-m)\left\{\left[(d_i)_{Nn}(d^a)_{nN}+(d^a)_{Nn}(d_i)_{nN}\right]\left[\frac{k_jp_a-k_ap_j}{2mM_n}+\frac{k_ap_j+p_ak_j}{2m(M_n-m)}\right]+\text{c.t.}\right\}. \qquad \text{(B2)}
\end{aligned}$$

From (5.14b), (5.19b), (5.21), and (5.23) we get the relations

$$K'^2+K^2-K_0'^2-K_0^2=2\mathbf{k}\cdot\mathbf{k}'\ , \qquad \text{(B3a)}$$

$$K^aK^b-\frac{1}{2}\left(K_0^aK_0^b+K_0'^aK_0'^b\right)=\frac{1}{2}\left[k_a'k_b+k_ak_b'+\frac{M_n}{m}\left(k_ak_b-k_a'k_b'\right)\right]\ , \qquad \text{(B3b)}$$

and

$$K'^aK^b-K_0'^aK_0'^b=\frac{1}{2}\left(1+\frac{M_n}{m}\right)k_a'k_b+\frac{1}{2}\left(1-\frac{M_n}{m}\right)k_ak_b'\,. \qquad \text{(B3c)}$$

Taking these relations in (B2) we get

$$\begin{aligned}
\triangle_{ij}^{(2)} + e_{ij}^{(2)} = &\left[b_1-b_2-4c+7d_1+5d_2+e_1-e_2+f+\frac{1}{2}(g_1-g_2)+h\right]\mathbf{k}\cdot\mathbf{k}'\delta_{ij} \\
&+\left[b_2-2c+d_1-5d_2+e_1+e_2-f-\frac{1}{2}(g_1-g_2)-h\right]k_i'k_j \\
&+\left[-b_2-2c+d_1+5d_2+e_1-e_2-f-\frac{1}{2}(f_1+f_2)+h\right]\left(k_ik_j'-\mathbf{k}\cdot\mathbf{k}'\delta_{ij}\right). \qquad \text{(B4)}
\end{aligned}$$

Substituting this result in (5.6) together with (3.4) and (5.4), and comparing with (2.5) we reobtain the zeroth-order low-energy result (2.6a) and obtain Eqs. (5.24)–(5.28).

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[1] Silvestre Ragusa, Phys. Rev. D **47**, 3757 (1993). Equation (n) of this and of Ref. [2] will be indicated by Eq. (I-n) and Eq. (II-n), respectively.

[2] Silvestre Ragusa, Phys. Rev. D **49**, 3157 (1994).

[3] N.V. Maksimenko and S.G. Shul'ga, Sov. J. Nucl. Phys. **52**, 335 (1990).

[4] A.I. L'Vov, Int. J. Mod. Phys. A **8**, 5267 (1993). This paper contains a very complete list of references of previous works on the dipole polarizabilities.

[5] S. Ragusa, Phys. Rev. D **11**, 1536 (1975).

[6] M. Warns, H. Schroder, W. Pfeil, and H. Rollnik, Z. Phys. C **45**, 613 (1990); **45**, 627 (1990); M. Warns, W. Pfeil, and H. Rollnick, Phys. Rev. D **42**, 2215 (1990).

[7] Zhenping Li and Frank E. Close, Phys Rev. D **42**, 2207 (1990).

[8] Simon Capstick, Phys. Rev. D **46**, 2864 (1992).

[9] F.E. Low, Phys. Rev. **96**, 1428 (1954).

[10] A. Pais, Nuovo Cimento A **53**, 433 (1968).

[11] A.M. Baldin, Nucl. Phys. **18**, 310 (1960).