Discretized light cone quantization and the coherent state basis

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We suggest the use of a coherent state basis to calculate the Hamiltonian matrix elements in the discretized light cone quantization method of bound state calculations as a possible way of avoiding the vanishing energy denominators and the resulting true infrared singularities. As an example, we obtain the light cone Schrödinger equation for positronium using the coherent state basis and show the absence of the ''Coulomb singularity.''

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I. INTRODUCTION

In the past few years, discretized light cone quantization (DLCQ) has attracted a lot of attention as a practical computational method to obtain the mass spectrum and wave functions of relativistic bound states [1]. The DLCQ method is based on diagonalization of the light cone Hamiltonian in a truncated Fock basis. The bound state, for example, positronium is an eigenstate of the light cone Hamiltonian

$$H_{\rm LC}|\psi\rangle = M^2|\psi\rangle. \tag{1}$$

Projecting this equation onto various Fock states $\langle e,\overline{e}|$, $\langle e,\overline{e},\gamma|, \ldots$ one gets an infinite number of coupled integral eigenvalue equations. Discretizing the momentum space and putting a cutoff on the number of Fock states as well as on large values of transverse momenta, this infinite set of equations reduces to a finite one and the problem of solving the eigenvalue equation basically reduces to one of diagonalizing a finite dimensional matrix. The initial success of DLCQ method has been established in $(1+1)$-dimensional models, where the mass spectrum and wave functions have been obtained successfully in Yukawa theory, QED, the massless Schwinger model, ϕ^4 theory and $1+1$ QCD [1–4]. In $3+1$ dimension the method has been developed [5] and applied to light cone QED to obtain the mass spectrum and wave functions of positronium [6]. Krautgärtner *et al.* have obtained the light cone Tamm-Dancoff equation for the charge zero sector under certain model assumptions and have solved the eigenvalue problem for various approximations to this equation. The successive approximations they have considered are the Tamm-Dancoff equation, the light cone Schrödinger equation and the Coulomb Schrödinger equation. They have developed a numerical procedure to solve these equations in a truncated and discretized Fock basis. In all the three cases, they have found good agreement with previous results. All of the equations considered in their paper suffer from the usual integrable singularity of equal time formulation, which has been dealt with by a mathematical artifact called the ''Coulomb trick.'' In the present work, we show that the Coulomb singularity does not appear in the light cone Schrödinger equation if one uses a coherent state basis to calculate the matrix elements. In a previous work [8], we had suggested using the coherent state basis as an alternative to the Fock basis for calculating the Hamiltonian matrix elements in the light cone bound state calculations. Now we will present such a calculation and show that the coherent state formalism leads to the elimination of the Coulomb singularity in light cone Schrödinger equation for positronium. More generally speaking, the calculation of matrix elements in a coherent state basis provides a cutoff on the $k^+=0$, $\mathbf{k}_\perp=0$ mode in a natural way and it can provide a way of avoiding the vanishing energy denominators in light cone bound state calculations. It is well known that the infrared catastrophe encountered in the corrections to the fine structure of positronium is eliminated by the use of an artificial infrared cutoff in divergent contributions [9]. However, it was shown by Fulton and Karplus [10] that the region cut off by the infrared regulator does not really contribute to the relevant integrals if one takes into account the binding in intermediate states. The Fulton-Karplus approach is an extension of the Bethe-Salpeter equation [11] for systems in which an instantaneous interaction is responsible for binding. The equation describing the bound state of two oppositely charged fermions is corrected by replacing the free particle Green's functions that occur in certain low order interactions by corrected Green's functions, which take into account the instantaneous Coulomb interaction. However, the terms including Coulomb energy are explicitely omitted from higher order corrections. This special treatment of the low energy region eliminates the infrared (ir) catastrophe in the hyperfine structure of positronium. The use of the coherent state basis in DLCQ calculations is analogous to the Fulton-Karplus approach in conventional calculations as it takes into account the emission and absorption of soft photons in intermediate states. In the present work, we have obtained the light cone Schrödinger equation of Ref. [6] using the coherent state basis and have shown that our formalism provides a natural cutoff on small values of photon momenta and thus avoids the Coulomb singularity from the discretized version of the equations. The analysis can be carried out to the more general case of the Tamm-Dancoff equation in a similar manner. The method presented here is in no way an alternative to the ''Coulomb trick'' of Ref. [6]. The Coulomb counterterm of Ref. [6] is just a mathematical artifact to produce convergent results, whereas the extra contribution to the light cone Schrödinger equation obtained here is a result of binding in the intermediate states. We would like to emphasize that the cutoff obtained in this manner is not equivalent to the procedure of eliminating $k^+=0$, $\mathbf{k}_\perp=0$ state from the discretized equation by hand as done by Tang *et al.* [5]. As a matter of fact, the contribution of soft photons is taken care of *before* the discretization is carried out and therefore the infrared cutoff eliminates not just the $k^+=0$, $\mathbf{k}_\perp=0$, but all the points in the soft region.

We would like to emphasize that the the coherent state method is in no way complimentary to the Tamn-Dancoff method of Fock space truncation in spite of the fact that a coherent state is a superposition of an arbitrary number of soft photon states. In fact, we are *supplementing* the Tamn-Dancoff truncation of hard photons with a summation over an arbitrary number of soft photons thereby providing a picture that combines the long distance and the short distance behavior of the theory. We would also like to remind the reader that the true ir divergences as defined in Ref. [8] are related to the long distance behavior of the theory as opposed to the UV divergences and the spurios ir divergences related to the short distance behavior and hence there is no reason to believe that they should be dealt with in a similar fashion. Moreover, the elimination of the true ir divergences does not remove the physics associated with the zero modes as the spurious ir divergences (i.e., those associated with $k^+=0$, $k_\perp \neq 0$ still remain).

Recently, there has been a lot of interest in the "zero mode" problem of light cone field theories [16–19]. It has been pointed out [18] that in light cone quatization, where the vacuum is simple in contrast to the complicated vacuum structure of equal time theories, the physics associated with the vacuum state is manifested in the zero mode and therefore, on first sight, the removal of true ir divergences attempted here may appear to be an undesirable feature. However, the removal of true ir divergences does in no way eliminate the physics associated with the zero modes as these divergences correspond to the *global* and not to the *proper* zero modes which play a role in spontanenous symmetry breaking. The proper zero modes, in our terminology, are related to the "spurious" ir divergences and the coherent state formalism developed in Ref. [8] does not remove these divergences. It has been pointed out by Kalloniatis *et al.* [18] that, at present, "there is no definite prescription for dealing with the global zero modes and in the discussion of vacuum problem, these are usually ignored without giving rise to any inconsistencies as far as spontaneous symmetry breaking (SSB) is concerned."

We would also like to point out in the very beginning the difference between our approach and the standard DLCQ. First, we are using a coherent state basis in place of the standard Fock basis. Second, we start from the outset in the continuum and discretize only after the true IR divergences are removed from the light cone Schrödinger equation. This is in contrast to the standard DLCQ technique. The formalism is established in a finite space with discrete momenta and sums right from the beginning [5].

The plan of the paper is as follows: In Sec. II we will set up our notation and will give a brief summary of the DLCQ method of solving the Hamiltonian eigenvalue problem. We will closely follow the notation of Ref. [6] and will show briefly how the light cone Schrödinger equation is obtained as the nonrelativistic limit of the light cone Tamm-Dancoff equation. In Sec. III we will obtain the modified light cone Schrödinger equation by using the coherent state basis to expand the Hamiltonian eigenstates and will then show how by using this basis the Coulomb singularity in the modified light cone Schrödinger equation is cancelled automatically up to $O(\alpha)$. Section IV extends the analysis of Sec. III to $O(\alpha^2)$ first in a Fock basis containing two hard photons and then in the corresponding coherent state basis. Here we show how the true infrared divergences arise in $O(\alpha^2)$ in the binding energy of positronium in the light cone Hamiltonian formalism and how the coherent state contributions exactly cancel these divergences. Section V contains a summary of the paper and emphasizes the equivalence of coherent state method and the Fulton-Karplus method of dealing with the infrared divergences in the equal time formulation. The Appendix contains the derivation of the modified light cone Schrödinger equation.

II. PRELIMINARIES

We will follow the notation of Ref. [6]. The light cone QED Hamiltonian is given by

$$H_{\mathrm{LC}}=P^+P^- - P_\perp^2, \tag{2}$$

where

$$P^- = H_0+V_1+V_2+V_3 \tag{3}$$

is the light cone energy operator, H_0 being the free part and V_1, V_2, and V_3 being the interaction parts. The expressions for these can be found in Ref. [7]. In the DLCQ method, the mass spectrum of a bound state is obtained by diagonalizing the discretized form of the Hamiltonian eigenvalue equation

$$H_{\mathrm{LC}}|\Psi\rangle = M^2|\Psi\rangle, \tag{4}$$

where M^2 is the mass-squared operator and $|\Psi\rangle$ is the bound state. $|\Psi\rangle$, in our case positronium, can be expanded in the Fock basis. Restricting only to the first two Fock states one can write

$$|\Psi\rangle = \sum_i \psi^i_{e^+e^-}(x,\mathbf{k}_\perp)|e^+,e^-\rangle_i + \sum_i \psi^i_{e^+e^-\gamma}(x,\mathbf{k}_\perp,q)|e^+,e^-,\gamma\rangle_i, \tag{5}$$

where the sum over i denotes the sum over spins and integration over all possible momentum configurations. Following the conventions of Ref. [6], the transverse momentum of the bound state $\mathbf{P}_\perp$ is chosen to be zero and the momenta of electron, positron and photon are parametrized as

$$k_e=\left(xP^+,\mathbf{k}_\perp,\frac{\mathbf{k}_\perp^2+m^2}{xP^+}\right),$$

$$k_{\bar{e}}=\left((1-x)P^+,-\mathbf{k}_\perp,\frac{\mathbf{k}_\perp^2+m^2}{(1-x)P^+}\right),$$

$$q=\left(yP^+,\mathbf{q}_\perp,\frac{q_\perp^2}{yP^+}\right). \tag{6}$$

Substituting the expansion in Eq. (5) into Eq. (4), one gets a set of coupled equations:

$$H_{00}\psi_{e^+e^-}+H_{01}\psi_{e^+e^-\gamma}=M^2\psi_{e^+e^-},$$

$$H_{10}\psi_{e^+e^-}+H_{11}\psi_{e^+e^-\gamma}=M^2\psi_{e^+e^-\gamma}, \tag{7}$$

where $H_{ij}=\langle i|H|j\rangle$, the states $|e^+,e^-\rangle$ and $|e^+,e^-,\gamma\rangle$ being denoted by $|0\rangle$ and $|1\rangle$, respectively. The above equations can be rewritten in terms of an effective Hamiltonian acting only in the space of $|e^+,e^-\rangle$ states:

$$H_{\text{eff}}\psi_{e^+e^-}=M^2\psi_{e^+e^-}, \tag{8}$$

where

$$H_{\text{eff}}=H_{00}+H_{01}\frac{1}{M^2-H_{11}}H_{10}. \tag{9}$$

Writing H_{eff} in terms of creation and annihilation operators and calculating the matrix elements of H between Fock states, Eq. (8) together with Eq. (9) gives the light cone Tamm-Dancoff equation [6]

$$\left[\frac{m^2+\mathbf{k}_\perp^2}{x(1-x)}-M^2\right]\psi(x,\mathbf{k}_\perp,s_e,s_{\bar{e}})+\sum_{s'_e,s'_{\bar{e}}}{}'\int_D dx'\langle x,\mathbf{k}_\perp,s_e,s_{\bar{e}}|V_{eff}(\omega)|x',\mathbf{k}'_\perp,s'_e,s'_{\bar{e}}\rangle\psi(x',\mathbf{k}'_\perp,s'_e,s'_{\bar{e}})=0\,, \tag{10}$$

where $\psi(x,\mathbf{k}_\perp,s_e,s_{\bar{e}})$ refers to $\psi_{e^+e^-}$ and V_{eff} is the effective interaction Hamiltonian defined by

$$V_{\text{eff}}=H_{01}\frac{1}{M^2-H_{11}}H_{10}. \tag{11}$$

Taking the nonrelativistic limit $[\mathbf{k}_\perp^2\ll m^2,(x-1/2)^2\ll 1]$ in the nondiagonal term of Eq. (10), one obtains the light cone Schrödinger equation [6]

$$\left[\frac{m^2+\mathbf{k}_\perp^2}{x(1-x)}\right]\psi(x,\mathbf{k}_\perp)-\frac{\alpha}{2\pi^2}\int_D dx'\,d^2\mathbf{k}'_\perp\,\frac{8m^2\psi(x',\mathbf{k}'_\perp)}{4m^2(x-x')^2+(\mathbf{k}_\perp-\mathbf{k}'_\perp)^2}=M^2\psi(x,\mathbf{k}_\perp). \tag{12}$$

The matrix elements of V_{eff} can be evaluated in standard manner. In Ref. [6], Eq. (12) has been discretized and solved numerically using the Gauss integration method under the following model assumptions: (1) Only those diagrams corresponding to instantaneous exchanges in V_{eff} are included which satisfy the gauge cutoff condition [5], i.e., a diagram involving instantaneous exchange is included only when a real dynamical parton with the same space like momentum and the same Fock space configuration is allowed by ultraviolet cutoff and Fock space truncation. (2) In the $(e^+e^-\gamma)$ space, the interaction corresponding to the exchange of an instantaneous photon is set equal to zero. (3) In the matrix elements of V_{eff}, the eigenvalue $M^2=\omega$ has been set equal to the symmetrized mass:

$$\omega=\omega^*=\frac{m^2+\mathbf{k}_\perp^2}{2x(1-x)}+\frac{m^2+\mathbf{k}'^2_\perp}{2x'(1-x')}. \tag{13}$$

Discretizing the momentum space and using the Gauss quadrature method to convert the integral in Eq. (12) into a finite sum, this equation is converted into a finite dimensional matrix equation and the problem of obtaining the mass spectrum is reduced to finding the eigenvalues of this matrix. Note that the finite dimensionality of the Hamiltonian matrix is a consequence of Fock space truncation and the ultraviolet cutoff. One can notice that the discretized version of Eq. (12) has a singularity at the point $x=x'$, $\mathbf{k}_\perp=\mathbf{k}'_\perp$ coming from the second term on the right-hand side (RHS). In Ref. [6], this singularity has been avoided by using the *Coulomb trick*, which is a mathematical artifact. In our opinion, however, the singularity has a physical origin, since expanding the bound state in the Fock basis, one assumes that the intermediate states are free particle states that do not interact with each other. Now, if one calculates the Hamiltonian matrix elements in the coherent state basis the Coulomb singularity does not appear as the the contribution of small momentum region in Eq. (12) is cancelled by a coherent state contribution which actually is a result of binding between constituent particles in the intermediate state. In other words, the coherent state formalism puts a cutoff on small values of the energy denominator in Eq. (12) in a natural manner. We will show this in the next section.

III. LIGHT CONE SCHRÖDINGER EQUATION AND THE COHERENT STATE BASIS

We will now obtain the light cone Schrödinger equation from the Hamiltonian eigenvalue equation by expanding the eigenstate in a coherent state basis. Coherent states can be defined in the usual manner by [12,8]

$$|n{:}\text{coh}\rangle=\exp\left[-e\sum_{\lambda=1,2}\int\frac{dk^+}{\sqrt{2k^+}}\int\frac{d^2\mathbf{k}_\perp}{(2\pi)^{3/2}}\sum_i\,[f_i(k,\lambda{:}p_i)a^\dagger(k,\lambda)-f_i^*(k,\lambda{:}p_i)a(k,\lambda)]\right]|n\rangle, \tag{14}$$

where

$$f_i(k,\lambda,:p_i)=s_i\frac{p_{i\mu}\epsilon_\lambda^\mu(k)}{p_i\cdot k}\,\theta(\Delta P^+-p_i\cdot k),\tag{15}$$

and the sum is over all the fermions in the Fock state $|n\rangle$. p_i is momentum of the ith fermion and s_i is -1 or $+1$ for positrons and electrons, respectively. P^+ is the total longitudinal momentum of the bound state $|\Psi\rangle$ and Δ is a small parameter with the dimension of light cone energy. Expanding $|\Psi\rangle$ in the basis defined by Eq. (14) and truncating in a manner so that only those states which are obtained by applying the asymptotic operator to the Fock states $|e^+(k_{\bar{e}}),e^-(k_e)\rangle$ and $|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\rangle$ are allowed, one can write

$$|\Psi\rangle=\sum_i\ \psi^i_{e^+e^-}(x,\mathbf{k}_\perp)|e^+(k_{\bar{e}}),e^-(k_e)\mathrm{:coh}\rangle_i+\sum_i\ \psi^i_{e^+e^-\gamma}(x,\mathbf{k}_\perp,q)|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\mathrm{:coh}\rangle_i\,,\tag{16}$$

where again the sum over i means a summation over the spins and integration over all possible momentum configurations of constituent fermions and the hard photon in the coherent state. The expressions for the coherent states in Eq. (16) follow from the general expression in Eq. (14):

$$\begin{aligned}|k_{\bar{e}},k_e\rangle_{\mathrm{coh}}&=|e^+(k_{\bar{e}}),e^-(k_e)\mathrm{:coh}\rangle\\&=\exp\left[-e\sum_{\lambda=1,2}\int\frac{dk^+}{\sqrt{2k^+}}\int\frac{d^2k_\perp}{(2\pi)^{3/2}}[f(k,\lambda:k_e,k_{\bar{e}})a^\dagger(k,\lambda)-f^*(k,\lambda:k_e,k_{\bar{e}})a(k,\lambda)]\right]|e^+(k_{\bar{e}}),e^-(k_e)\rangle,\end{aligned}\tag{17}$$

and

$$\begin{aligned}|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\mathrm{:coh}\rangle=\exp\Bigg[&-e\sum_{\lambda=1,2}\int\frac{dk^+}{\sqrt{2k^+}}\int\frac{d^2k_\perp}{(2\pi)^{3/2}}[f(k,\lambda:k_e,k_{\bar{e}})a^\dagger(k,\lambda)\\&-f^*(k,\lambda:k_e,k_{\bar{e}})a(k,\lambda)]\Bigg]|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\rangle,\end{aligned}\tag{18}$$

where

$$f(k,\lambda:k_e,k_{\bar{e}})=\left(\frac{k_{e\mu}\epsilon_\lambda^\mu(k)}{k_e\cdot k}\,\theta(\Delta P^+-k_e\cdot k)-\frac{k_{\bar{e}\mu}\epsilon_\lambda^\mu(k)}{k_{\bar{e}}\cdot k}\ \theta(\Delta P^+-k_{\bar{e}}\cdot k)\right).\tag{19}$$

These coherent states satisfy the properties

$$a(k,\lambda)|e^+,e^-,\gamma\mathrm{:coh}\rangle=-\frac{e}{(2\pi)^{3/2}}\ \frac{f(k,\lambda:k_e,k_{\bar{e}})}{\sqrt{2k^+}}|e^+,e^-,\gamma\mathrm{:coh}\rangle+\delta^3(k-q)|e^+,e^-\mathrm{:coh}\rangle,\tag{20}$$

and

$${}_f\langle e^+,e^-,\gamma\mathrm{:coh}|e^+,e^-\mathrm{:coh}\rangle_i=-\delta^{(3)}(k_e^i-k_e^f)\,\delta^{(3)}(k_{\bar{e}}^i-k_{\bar{e}}^f)\frac{e}{(2\pi)^{3/2}}\ \frac{f(k,\lambda:k_e^i,k_{\bar{e}}^i)}{\sqrt{2k^+}},\tag{21}$$

where the sum is over all the fermions. Substituting the expansion of Eq. (16) in Eq. (4), one gets the equation

$$\begin{aligned}&\sum_i\ \psi^i_{e^+e^-}(x,\mathbf{k}_\perp)(H_0+V)|e^+(k_{\bar{e}}),e^-(k_e)\mathrm{:\ coh}\rangle_i+\sum_i\ \psi^i_{e^+e^-\gamma}(x,\mathbf{k}_\perp,q)(H_0+V)|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\mathrm{:coh}\rangle_i\\&=\omega\left[\sum_i\ \psi^i_{e^+e^-}(x,\mathbf{k}_\perp)|e^+(k_{\bar{e}}),e^-(k_e)\mathrm{:coh}\rangle_i+\sum_i\ \psi^i_{e^+e^-\gamma}(x,\mathbf{k}_\perp,q)|e^+(k_{\bar{e}}),e^-(k_e),\gamma(q)\mathrm{:coh}\rangle_i\right],\end{aligned}\tag{22}$$

where $\omega=M^2$. Taking product of Eq. (22) with ${}_j\langle e^+,e^-\mathrm{:coh}|$ and ${}_j\langle e^+,e^-,\gamma\mathrm{:coh}|$, respectively, we arrive at the following set of equations:

$$\begin{aligned}&\sum_i\ \psi^i_{e^+e^-j}\langle e^+,e^-\mathrm{:\ coh}|(H_0+V)|e^+,e^-\mathrm{:coh}\rangle_i+\sum_i\ \psi^i_{e^+e^-\gamma j}\langle e^+,e^-\mathrm{:\ coh}|(H_0+V)|e^+,e^-\gamma\mathrm{:coh}\rangle_i\\&=\omega\left[\sum_i\ \psi^i_{e^+e^-j}\langle e^+,e^-\mathrm{:coh}|e^+,e^-\mathrm{:coh}\rangle_i+\sum_i\ \psi^i_{e^+e^-\gamma j}\langle e^+,e^-\mathrm{:coh}|e^+,e^-,\gamma\mathrm{:\ coh}\rangle_i\right],\end{aligned}\tag{23}$$

and

$$\sum_i \psi^i_{e^+e^-j}\langle e^+,e^-,\gamma\text{: coh}|(H_0+V)|e^+,e^-\text{:coh}\rangle_i+\sum_i \psi^i_{e^+e^-\gamma j}\langle e^+,e^-,\gamma\text{:coh}|(H_0+V)|e^+,e^-,\gamma\text{:coh}\rangle_i$$
$$=\omega\left[\sum_i \psi^i_{e^+e^-j}\langle e^+,e^-,\gamma\text{:coh}|e^+,e^-\text{:coh}\rangle_i+\sum_i \psi^i_{e^+e^-\gamma j}\langle e^+,e^-,\gamma\text{:coh}|e^+,e^-,\gamma\text{:coh}\rangle_i\right]. \tag{24}$$

Eliminating $\psi^i_{e^+e^-\gamma}$'s from Eqs. (23) and (24), one arrives at the following set of coupled integral equations for $\psi^i{}_{e^+e^-}$'s:

$$\sum_i \psi^i{}_{e^+e^-}(x,\mathbf{k}_\perp)_j\langle e^+,e^-\text{:coh}|(H_0+V)|e^+,e^-\text{:coh}\rangle_i-\sum_i\sum_k\sum_l \psi^k{}_{e^+e^-j}\langle e^+,e^-\text{:coh}|(H_0+V-\omega^*)|e^+,e^-,\gamma\text{:coh}\rangle_i$$
$$\times{}_i\left\langle e^+,e^-,\gamma\text{:coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma\text{:coh}\right\rangle_{ll}\langle e^+,e^-,\gamma\text{:coh}|(H_0+V-\omega^*)|e^+,e^-,\text{:coh}\rangle_k$$
$$=\omega\sum_i \psi^i_{e^+e^-j}\langle e^+,e^-\text{:coh}|e^+,e^-\text{:coh}\rangle_i\,, \tag{25}$$

where we have set $\omega=\omega^*$ and have also ignored the instantaneous interaction as assumed in the previous section. One must notice that this equation is different from the effective equation obtained in the previous section as neither ${}_j\langle e^+,e^-\text{:coh}|H_0|e^+,e^-,\gamma\text{:coh}\rangle_i$ nor ${}_j\langle e^+,e^-\text{:coh}|V|e^+,e^-\text{:coh}\rangle_i$ is zero in the coherent state basis. This follows from the properties of the coherent states, Eqs. (20) and (21). For example, the matrix element ${}_j\langle e^+,e^-\text{:coh}|P^+P^-|e^+,e^-\text{:coh}\rangle_i$ is not equal to just $(m^2+\mathbf{k}_\perp{}^2)/x(1-x)$, but has an additional contribution which is given by

$$\langle e^+(k'_{\bar e}),e^-(k'_e)\text{:coh}|P^+P^-|e^+(k_{\bar e}),e^-(k_e)\text{:coh}\rangle$$
$$=\delta^{(3)}(k_e-k'_e)\,\delta^{(3)}(k_{\bar e}-k'_{\bar e})\left\{\frac{m^2+\mathbf{k}_\perp{}^2}{x(1-x)}+\frac{\alpha}{2\pi^2}\int\frac{dy}{2y}\int d^2q_\perp\left[q_\perp^2\left(\frac{1+y}{y}\right)+\frac{m^2+k_\perp^2}{x(1-x)}(1+y)\right]\right.$$
$$\times\left[\left(-\frac{k_e^2}{k_e\cdot q}+\frac{2k_e^+}{q^+}\frac{1}{k_e\cdot q}\right)\theta(\Delta P^+-k_e\cdot q)+\left(-\frac{k_{\bar e}^2}{(k_{\bar e}\cdot q)^2}+\frac{2k_{\bar e}^+}{q^+}\frac{1}{k_{\bar e}\cdot q}\right)\theta(\Delta P^+-k_{\bar e}\cdot q)\right.$$
$$\left.\left.+\left(-\frac{2k_e\cdot k_{\bar e}}{(k_e\cdot q)(k_{\bar e}\cdot q)}+\frac{2k_e^+}{q^+}\frac{1}{k_e\cdot q}+\frac{2k_{\bar e}^+}{q^+}\frac{1}{k_{\bar e}\cdot q}\right)\theta(\Delta P^+-k_e\cdot q)\,\theta(\Delta P^+-k_{\bar e}\cdot q)\right]\right\}. \tag{26}$$

All the matrix elements in Eq. (25) can be calculated in a straightforward manner. Details of this calculation are given in the Appendix. The procedure is the standard one followed in Ref. [8] wherein we write the light cone Hamiltonian in terms of creation and annihilation operators and calculate the matrix elements between the coherent states by making use of the properties of coherent states. The equation so obtained is the *modified* light cone Schrödinger equation

$$\left[\frac{m^2+\mathbf{k}_\perp^2}{x(1-x)}\right]\psi(x,\mathbf{k}_\perp)-\frac{\alpha}{2\pi^2}\int_D dx'\,d^2\mathbf{k}'_\perp\,\frac{8m^2\psi(x',\mathbf{k}'_\perp)}{4m^2(x-x')^2+(\mathbf{k}_\perp-\mathbf{k}'_\perp)^2}+\frac{\alpha}{2\pi^2}\int_{D_c}dx'\,d^2\mathbf{k}'_\perp\,\frac{8m^2\psi(x',\mathbf{k}'_\perp)}{4m^2(x-x')^2+(\mathbf{k}_\perp-\mathbf{k}'_\perp)^2}$$
$$=M^2\psi(x,\mathbf{k}_\perp), \tag{27}$$

where D_c is defined in the Appendix and is actually a small region around the point at which the energy denominator vanishes:

$$D_c{:}\,4m^2(x-x')^2+(\mathbf{k}_\perp-\mathbf{k}'_\perp)^2<\overline{R}^{\,2}. \tag{28}$$

Equation (27) is different from Eq. (12) due to the presence of the third term on the LHS, which is a coherent state contribution. This term, on discretization, cancels the Coulomb singularity in the second term. Thus, we have shown that to $O(e^2)$, the coherent state formalism removes the Coulomb singularity in a natural manner. One may raise the point that the Coulomb singularity in the light cone Schrödinger equation under consideration is an integrable singularity and hence the relevance of the above discussion is not really clear. However, the analysis of this section can be carried over to higher orders too. It is well known that the hyperfine structure (hfs) of positronium has infrared (ir) divergences in $O(\alpha^2)$. In DLCQ method of bound state calculations this divergence appears in $O(\alpha^2)$ matrix elements when the number of basis states is increased to allow states with 2 hard photons. In the next section, we will extend our analysis to the case when the Fock space truncation allows $|e^+,e^-,\gamma,\gamma\rangle$ states also, in addition to the previously included $|e^+,e^-\rangle$ and $|e^+,e^-,\gamma\rangle$ states. We will not give the complete light cone Schrödinger equation in this approxima-

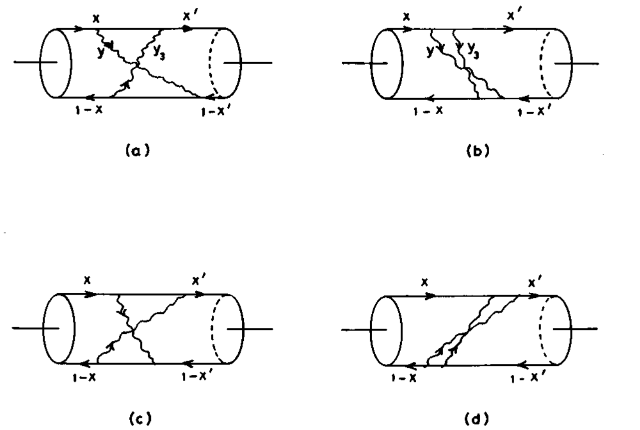

FIG. 1. Nontrivial $O(\alpha^2)$ graphs corresponding to matrix elements in Eq. (33).

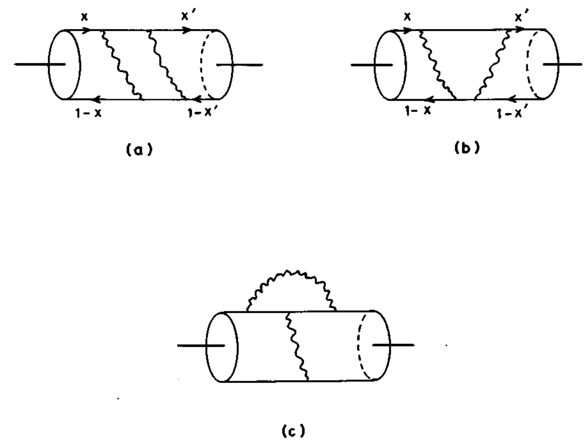

FIG. 2. Additional $O(\alpha^2)$ graphs corresponding to matrix element in Eq. (33).

tion. However, we will calculate the $O(\alpha^2)$ ir divergent terms and will then show that the additional coherent state contributions in the same order cancel these terms exactly, thus indicating the possibility of obtaining in a systematic manner Tamm-Dancoff equations which are free of true ir divergences.

IV. COHERENT STATE BASIS AND THE TRUE ir DIVERGENCES IN $O(\alpha^2)$ CORRECTIONS TO THE BINDING ENERGY OF POSITRONIUM

In this section, we will find the effective Hamiltonian in the truncated coherent state basis

$$\{|e^+,e^-:\text{coh}\rangle,|e^+,e^-,\gamma:\text{coh}\rangle,|e^+,e^-,\gamma,\gamma:\text{coh}\rangle\}$$

and will then show, with a model calculation, how the $O(\alpha^2)$ ir divergences are expected to cancel in the light cone Schrödinger equation. First, consider the Fock space expansion of the bound state:

$$|\Psi\rangle=\sum_i \psi^i_{e^+e^-}|e^+,e^-\rangle_i+\sum_i \psi^i_{e^+e^-\gamma}|e^+,e^-,\gamma\rangle_i + \sum_i \psi^i_{e^+e^-\gamma\gamma}|e^+,e^-,\gamma,\gamma\rangle_i . \tag{29}$$

Following the same steps as in Sec. II, we arrive at the following effective Hamiltonian in the space of $|e^+,e^-\rangle$ states [6]:

$$H_{\text{eff}}=H_{00}+H_{01}\frac{1}{\omega-H_{11}-H_{12}[1/(\omega-H_{22})]H_{21}}\left[H_{10}+H_{12}\frac{1}{\omega-H_{22}}H_{20}\right] + H_{02}\frac{1}{\omega-H_{22}}\left\{H_{20}+H_{21}\frac{1}{\omega-H_{11}-H_{12}[1/(\omega-H_{22})]H_{21}}\left[H_{10}+H_{12}\frac{1}{\omega-H_{22}}H_{20}\right]\right\}. \tag{30}$$

Omitting all the interactions that change the parton number by 2, the effective Hamiltonian reduces to

$$H_{\text{eff}}=H_{00}+H_{01}\frac{1}{\omega-H_{11}-H_{12}[1/(\omega-H_{22})]H_{21}}H_{10}, \tag{31}$$

and the light cone Tamm-Dancoff equation is given by

$$\left[H_{00}+H_{01}\frac{1}{\omega-H_{11}-H_{12}[1/\omega-H_{22})}\right]H_{21}H_{10}]\psi^i_{e^+e^-}=M^2\psi^i_{e^+e^-} . \tag{32}$$

$O(\alpha^2)$ contributions to Eq. (32) come from matrix elements

$$M=H_{01}\frac{1}{\omega-H_{11}}H_{12}\frac{1}{\omega-H_{22}}H_{21}\frac{1}{\omega-H_{11}}H_{10} . \tag{33}$$

The diagrams corresponding to these matrix elements are given in Figs. 1 and 2. (These are essentially the diagrams in Fig. 13 of Ref. [6].) For example, the term corresponding to Fig. 1(a) here can be shown to be equal to

$$M_{(1)}=\frac{\alpha^2}{4\pi^2}\int\frac{dq^+}{2q^+}\int d^2q_\perp\frac{1}{(2q_3^+)}\frac{1}{\sqrt{2k_e^+}}\frac{1}{\sqrt{2k_e'^+}}\frac{1}{\sqrt{2\bar{k}_e^+}}\frac{1}{\sqrt{2\bar{k}_e'^+}}\frac{1}{2(k_e'^+-q^+)}\frac{1}{2(\bar{k}_e^+-q^+)}d_{\sigma\mu}(k_e-k_e'+q)d_{\nu\rho}(q)$$
$$\times\frac{[\bar{v}(\bar{k}_e-q)\gamma^\sigma v(\bar{k}_e')][\bar{u}(k_e')\gamma^\rho u(k_e'-q)][\bar{v}(\bar{k}_e)\gamma^\nu v(\bar{k}_e-q)][\bar{u}(k_e'-q)\gamma^\mu u(k_e)]}{D_1D_2D_3},\tag{34}$$

where

$$q_3=k_e-k_e'+q,$$
$$D_1=\omega^*-(k_e'-q)^--q_3^--\bar{k}_e^-,$$
$$D_2=\omega^*-(k_e'-q)^--(\bar{k}_e-q)^--q^--q_3^-,$$
$$D_3=\omega^*-k_e'^--q_3^--(\bar{k}_e'-q)^-.\tag{35}$$

Using the parametrization in Eq. (6) and calculating the numerator in Eq. (34) by using the Dirac matrix elements for helicity spinors of Ref. [13] the gauge-independent $O(m^4)$ term of $M_{(1)}$ is calculated to be

$$M_{(1)}'=\frac{\alpha^2}{8\pi^2}\int\frac{dy}{2y}\frac{1}{(2y_3)}\int d^2q_\perp\frac{1}{\sqrt{x'}}\frac{1}{\sqrt{1-x'}}\frac{1}{1-x'-y}\frac{1}{x'-y}\left[\frac{1}{(1-x)(1-x-y)x(x'-y)}+\frac{1}{(1-x-y)(1-x')x(x'-y)}\right.$$
$$\left.+\frac{1}{(1-x)(x-y)x'(x'-y)}+\frac{1}{(1-x-y)(1-x')x'(x'-y)}\right]\frac{m^4}{D_1D_2D_3}.\tag{36}$$

In the nonrelativistic approxmation $[(x-1/2)^2\ll 1, k_\perp^2\ll m^2]$, [6] Eq. (36) simplifies to

$$M_{(1)}'=\frac{\alpha^2}{\pi^2}(8m^4)\int\frac{dy}{2y}\frac{1}{2y_3}\int\frac{d^2q_\perp}{D_1D_2D_3}.\tag{37}$$

At $\mathbf{k}_\perp=\mathbf{k}_\perp'$, $x=x'$, D_1 and D_3 can be shown to be equal to $-k_e\cdot q_3/(k_e^+-q_3^+)$ and $-(\bar{k}_e\cdot q_3)/(\bar{k}_e^+-q_3^+)$, respectively and hence $M_{(1)}'$ will have true ir singularity as defined in Ref. [8] at such points. Simplifying the denominator in Eq. (37) one obtains

$$M_{(1)}'=\frac{\alpha^2}{\pi^2}(8m^4)\int\frac{dy}{2y}\frac{1}{2y_3}\int d^2q_\perp\frac{1}{[\omega^*-k_e^--\bar{k}_e^--q_3^-]D_1D_3}+\frac{\alpha^2}{\pi^2}(8m^4)\int\frac{dy}{2y}\frac{1}{2y_3}\int d^2q_\perp\frac{q^-}{[\omega^*-k_e^--\bar{k}_e^--q_3^-]^2D_1D_3}.\tag{38}$$

A similar calculation for Fig. 1(b) will give the $O(m^4\alpha^2)$ contribution to the sum of Figs. 1(a) and 1(b) as

$$M=\frac{\alpha^2}{\pi^2}(4m^4)\int\frac{dy}{2y}\frac{1}{2y_3}\int d^2q_\perp\frac{1}{[\omega^*-k_e^--\bar{k}_e^--q_3^-]D_1D_3}+\frac{\alpha^2}{\pi}m^4\int\frac{dy}{yy_3}\int d^2q_\perp\frac{q^-}{[\omega^*-k_e'^--\bar{k}_e^--q_3^-]^2(k_e\cdot q)(\bar{k}_e\cdot q)}.\tag{39}$$

[One can readily check using Eq. (2) that at $\mathbf{k}_\perp=\mathbf{k}_\perp'$, D_1 and D_3 are proportional to $k_e\cdot q_3$ and $\bar{k}_e'\cdot q_3$, respectively]. Thus there will be singularities in the discretized version of light cone Tamm-Dancoff equation in $O(\alpha^2)$ at $\mathbf{k}_\perp=\mathbf{k}_\perp'$, $x=x'$ whenever $k_e\cdot q_3$ or $\bar{k}_e'\cdot q_3$ approaches zero. These singularities correspond to true ir divergence of the theory (and not the spurious divergences [8]). Such singularities and similar ones coming from other terms are the DLCQ analogue of $O(\alpha^2)$ ir divergences in hfs of positronium. It was shown by Fulton and Karplus [10] that $O(\alpha^2)$ ir divergences in the hfs of positronium are cancelled when one takes into account the binding between the constituents of the intermediate states. This was done by adding Coulomb corrections to the single- and two-photon exchange diagrams. In what follows, we will show a similar cancellation of true ir divergences can be achieved in the DLCQ approach also by using a coherent state basis. We claim our method to be analogous to the Fulton-Karplus method of obtaining "modified" bound state equations (i.e., equations which are free of ir divergences) in the Hamiltonian formalism because it takes into account the emission and absorption of soft photons. In the coherent state basis defined in Sec. III, the $O(\alpha^2)$ matrix elements of the Hamiltonian will have additional terms corresponding to diagrams in Figs. 3 and 4 (in addition to the diagrams in Fig. 13 of Ref. [6] where dotted lines represent the exchange of a coherent state photon). In the following text we will show the $O(m^4)$ term of Fig. 3(a) actually cancels the true ir divergences in Eq. (4), which is actually the

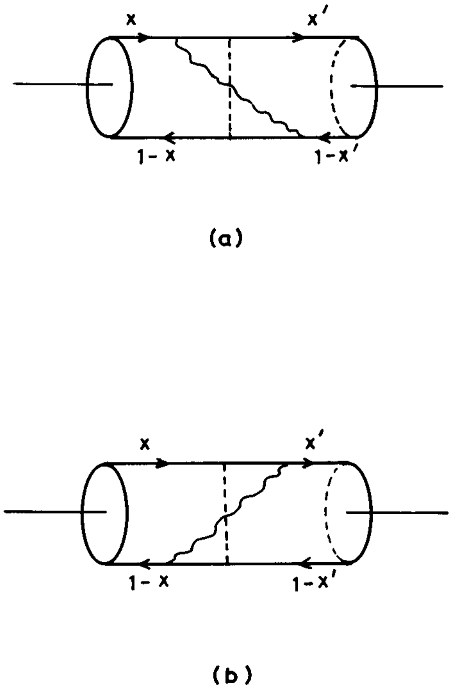

FIG. 3. $O(\alpha^2)$ graphs in coherent state basis which cancel the true ir divergences in diagrams of Figs. 1(a) and 1(b).

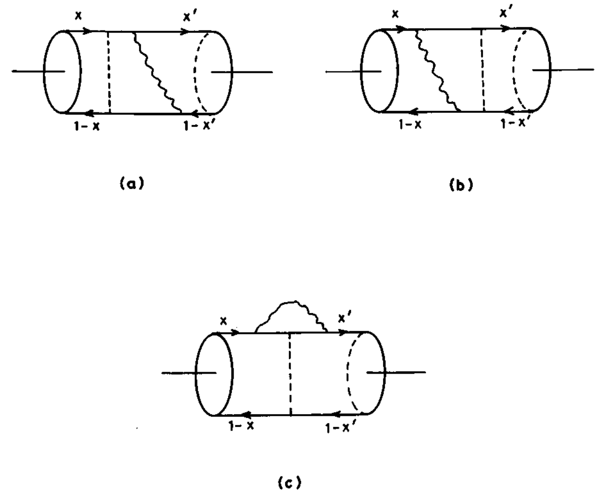

FIG. 4. $O(\alpha^2)$ graphs in coherent state basis which cancel the true IR divergences in diagrams of Figs. 2(a), 2(b), and 2(c).

$O(m^4)$ term in the sum of Figs. 1(a) and 1(b). One can show in an identical manner that the true ir divergences in the sum of Figs. 1(c) and 1(d) are cancelled by Fig. 3(b). The diagrams in Fig. 2 are much easier to deal with as they contain subdiagrams involving a single-photon exchange which, in turn, arises from the term $V_{01}[1/(\omega-H_{11})]V_{10}$ in the effective Hamiltonian. Now we have already seen in Sec. III that the true ir divergences in $V_{01}[1/(\omega-H_{11})]V_{10}$ are cancelled exactly by the $O(\alpha)$ contributions V_{00} and $(H_0-\omega)_{00}$ [see Eqs. (A8)–(A10)]. In our pictorial representation, the later two contributions correspond to a diagram in which a single-photon exchange has been replaced by a dotted line. Now corresponding to every diagram involving a (nonoverlapping) photon exchange, there exists another diagram in which the single-photon exchange is replaced by a dotted line and the sum of these digrams does not have true IR divergences. Hence, the true ir divergences in the diagrams of Fig. 2 cancel with the coherent state contributions shown in Fig. 4. Thus, the the diagrams in Fig. 1 are the only non-trivial diagrams in $O(\alpha^2)$ in the present context and we will now show that the true ir divergences in these given by Eq. (4) are cancelled exactly by the coherent state contributions represented by Figs. 3(a) and 3(b). To begin with, we will obtain the form of the effective Hamiltonian in the truncated coherent state basis:

$$\{|e^+,e^-:\text{coh}\rangle,|e^+,e^-,\gamma:\text{coh}\rangle,|e^+,e^-,\gamma,\gamma:\text{coh}\rangle\}.$$

Expanding the bound state in the above basis, we write

$$|\Psi\rangle=\sum_i \psi^i_{e^+e^-}|e^+,e^-:\text{coh}\rangle_i+\sum_i \psi^i_{e^+e^-\gamma}|e^+,e^-,\gamma:\text{coh}\rangle_i$$

$$+\sum_i \psi^i_{e^+e^-\gamma\gamma}|e^+,e^-,\gamma,\gamma:\text{coh}\rangle_i, \tag{40}$$

and following the procedure of Ref. [6], we arrive at the following effective Hamiltonian in the space of $|e^+,e^-:\text{coh}\rangle$ states:

$$H_{\text{eff}}=H_{00}+(H_{01}-\omega C_{01})\frac{1}{\omega-H_{11}-(H_{12}-\omega C_{12})[1/(\omega-H_{22})](H_{21}-\omega C_{21})}(H_{10}-\omega C_{10}), \tag{41}$$

where again, we have ignored the interactions that change the particle number by 2. C_{01}, C_{12}, and C_{21} appear due to nonorthogonality of the coherent states:

$$C_{01}=\langle e^+,e^-:\text{coh}|e^+,e^-,\gamma:\text{coh}\rangle,$$

$$C_{12}=\langle e^+,e^-,\gamma:\text{coh}|e^+,e^-,\gamma,\gamma:\text{coh}\rangle,$$

$$C_{21}=\langle e^+,e^-,\gamma,\gamma:\text{coh}|e^+,e^-,\gamma:\text{coh}\rangle. \tag{42}$$

It is now clear that there will be an additional $O(\alpha^2)$ contribution to H_{eff} apart from those in Eq. (33). In particular, the matrix element considered in the previous section,

$$M_2 = H_{01}\frac{1}{\omega - H_{11}}H_{10}, \tag{43}$$

will have an additional contribution which we had ignored in $O(\alpha)$ and which arises because H_{11} has an $O(\alpha)$ term too. In the following, we will calculate this contribution. M_2 can be rewritten as

$$M_2 = \sum_{i=1,2}\int d^3p_i d^3\overline{p}_i d^3k_i \langle k_e', k_{\bar e}': \mathrm{coh}|V|p_2, \overline{p}_2, k_2:\mathrm{coh}\rangle \left\langle p_2, \overline{p}_2, k_2:\mathrm{coh}\left|\frac{1}{\omega - H}\right| p_1, \overline{p}_1, k_1:\mathrm{coh}\right\rangle$$
$$\times \langle p_1, \overline{p}_1, k_1:\mathrm{coh}|V|k_e, k_{\bar e}:\mathrm{coh}\rangle. \tag{44}$$

This expression has an $O(\alpha^2)$ contribution because the second matrix element on the RHS involves $\langle e^+, e^-, \gamma: \mathrm{coh}|P^+P^-|e^+, e^-, \gamma:\mathrm{coh}\rangle$, which has an $O(\alpha)$ contribution in the coherent state basis:

$$\langle p_2, \overline{p}_2, k_2:\mathrm{coh}|P^+P^-|p_1, \overline{p}_1, k_1:\mathrm{coh}\rangle = \delta^{(3)}(p_1 - p_2)\,\delta^{(3)}(\overline{p}_1 - \overline{p}_2)\,\delta^{(3)}(k_1 - k_2)$$
$$\times\Big[(p_1^- + \overline{p}_1^- + k_1^-)P^+ + \frac{2\alpha}{2\pi}\int\frac{dq^+}{2q^+}\int d^2q_\perp[q^-(p_1^+ + \overline{p}_1^+ + k_1^+) + q^+(p_1^- + \overline{p}_1^- + k_1^-)$$
$$+ q_\perp^2] f^*(q,\lambda:p_2,\overline{p}_2) f(q,\lambda:p_1,\overline{p}_1). \tag{45}$$

Substituting Eq. (45) in Eq. (44), one gets

$$M_2 = M_2^{(1)} + M_2^{(2)}, \tag{46}$$

where $M_2^{(1)}$ and $M_2^{(2)}$ are the $O(\alpha)$ and $O(\alpha^2)$ terms in M_2 given by

$$M_2^{(1)} = \left\langle k_e', k_{\bar e}'\left| V\frac{1}{\omega - H_0}V\right| k_e, k_{\bar e}\right\rangle \tag{47}$$

and

$$M_2^{(2)} = M_2^{(2a)} + M_2^{(2b)}, \tag{48}$$

where

$$M_2^{(2a)} = \frac{\alpha^2}{4\pi^2}\frac{1}{\sqrt{2k_e^+}}\frac{1}{\sqrt{2k_e'^+}}\frac{1}{\sqrt{2k_{\bar e}^+}}\frac{1}{\sqrt{2k_{\bar e}'^+}}\,\delta^3(k_e + k_{\bar e} - k_e' - k_{\bar e}')\int\frac{dq^+}{2q^+}\int d^2q_\perp(q^- + \cdots)$$
$$\times f^*(q,\lambda:k_e,k_{\bar e}) f(q,\lambda:k_e,k_{\bar e})\frac{\overline{v}(k_{\bar e})\gamma^\nu v(k_{\bar e}')\overline{u}(k_e')\gamma^\mu u(k_e)}{[\omega^* - k_e'^- - k_{\bar e}^- - q_3^-]^2}\,d_{\nu\mu}(q_3), \tag{49}$$

$$M_2^{(2b)} = \frac{\alpha^2}{4\pi^2}\frac{1}{\sqrt{2k_e^+}}\frac{1}{\sqrt{2k_e'^+}}\frac{1}{\sqrt{2k_{\bar e}^+}}\frac{1}{\sqrt{2k_{\bar e}'^+}}\,\delta^3(k_e + k_{\bar e} - k_e' - k_{\bar e}')\int\frac{dq^+}{2q^+}\int d^2q_\perp(q^- + \cdots)$$
$$\times f^*(q,\lambda:k_e,k_{\bar e}) f(q,\lambda:k_e,k_{\bar e})\frac{\overline{u}(k_e')\gamma^\nu u(k_e)\overline{v}(k_{\bar e})\gamma^\mu v(k_{\bar e}')}{[\omega^* - k_e'^- - k_{\bar e}^- - q_3^-]^2}\,d_{\nu\mu}(q_3), \tag{50}$$

where $\cdots$ denotes terms that can be ignored at $k_\perp = k_\perp'$, $x = x'$ in the ir limit. $M_{2a}^{(2)}$ and $M_{2b}^{(2)}$ correspond to diagrams in Figs. 3(a) and 3(b), respectively. Substituting for $f(q,\lambda:k_e,k_{\bar e})$ and $f^*(q,\lambda:k_e,k_{\bar e})$, taking the nonrelativistic limit and simplifying, one finds that Eq. (49) has a term proportional to m^4 which exactly cancels the ir divergent part, of the second term in Eq. (4):

$$M_2' = -\frac{\alpha^2}{4\pi^2}4m^4\delta^3(k_e + k_{\bar e} - k_e' - k_{\bar e}')\int\frac{dy}{y}\int\frac{d^2q_\perp}{y_3}\,\frac{q^-\theta(\Delta P^+ - k_e\cdot q)\,\theta(\Delta P^+ - k_{\bar e}'\cdot q)}{(k_e'\cdot q)(k_{\bar e}\cdot q)[\omega^* - k_e'^- - k_{\bar e}^- - (k_e - k_e' + q)^-]^2}. \tag{51}$$

Similarly, Fig. 3(b) cancels the true ir divergences in the diagram shown in Figs. 1(c) and 1(d). As argued earlier, the true ir divergences in Figs. 3(a), 3(b), and 3(c) will be canceled by coherent state contributions in Figs. 4(a), 4(b), and 4(c), respectively, because these are basically $O(\alpha)$ divergences and we have already shown in Sec. III that to that order true ir divergences are absent in the coherent state basis. Thus the $O(m^4)$ term of $O(\alpha^2)$ true ir divergences is cancelled exactly by the corresponding term in the sum of all extra contributions to the Hamiltonian eigenvalue equation in the coherent state basis. This calculation indicates the possibility of the coherent state basis being a useful way of eliminating the vanishing enegry denominators and the resulting true ir divergences from the light cone bound state calculations.

V. SUMMARY

We have applied the coherent state formalism developed in an earlier work [8] for the continuum light cone QED to a bound state calculation in DLCQ. We have shown that the true ir singularities in the discretized light cone Schrödinger equation for positronium which are usually dealt with by adding Coulomb counterterms by hand are absent up to $O(\alpha)$ in the modified light cone Schrödinger equation which is the light cone Schrödinger equation obtained by calculating the Hamiltonian matrix elements between the appropriately chosen coherent states instead of using the usual Fock basis. We have also presented an $O(\alpha^2)$ calculation to show how the present formalism may be extended to remove the ir divergence in hfs of positronium. We claim that our method is essentially equivalent to a treatment of ir divergence in equal time bound state calculations by Fulton and Karplus where they introduce an instantaneous interaction between the constituents of the bound state. The reason for this claim is the following: Both the introduction of an instantaneous interaction and the use of coherent states are based on the observation that the constituents of intermediate states (which are to be taken as basis states) are actually not free states, and therefore one must take into account the binding between them. The binding can be incorporated by the introduction of an instantaneous interaction or equivalently by the exchange of soft photons between the constituents of the bound state. Now the (true) ir divergences of the theory are related to the long distance behavior of the interaction in contrast to the UV divergences and the spurious ir divergences which are related to short distance behavior. Since the ir divergence are associated with the classical limit [14], a classical treatment of the true ir divergences (based on the coherent state formalism) may be useful to seperate them from the other kind of divergences. An alternative method to remove the true ir divergences would be to add an instantaneous interaction $V_{\rm inst}$ to the free Hamiltonian and calculate the matrix elements between the Fock states. The coherent state formalism developed here will then be useful in deriving the form of $V_{\rm inst}$. In the case of QCD, one may use the form of artificial confining potential as proposed by Wilson and collaborators [15] to find the appropriate form of coherent states for infrared cancellations.

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APPENDIX

In this appendix, we will show how the modified light cone Schrödinger equation (27) is derived from Eq. (25). For notational convenience, we will rewrite Eq. (25) as

$$\sum_i \psi^i_{e^+e^-} M^d_{ji} - \sum_k \psi^k_{e^+e^-} M^{\rm ND}_{jk} = \omega \sum_i \psi^i_{e^+e^-\,j}\langle e^+,e^-:{\rm coh}|e^+,e^-:{\rm coh}\rangle_i\,, \tag{A1}$$

where we define

$$M^d_{ji} = {}_j\langle e^+,e^-:{\rm coh}|(H_0+V)|e^+,e^-:{\rm coh}\rangle_i \tag{A2}$$

and

$$M^{\rm ND}_{jk} = {}_j\langle e^+,e^-:{\rm coh}|(H_0+V-\omega^*)|e^+,e^-,\gamma:{\rm coh}\rangle_i \,{}_i\left\langle e^+,e^-,\gamma:{\rm coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma:{\rm coh}\right\rangle_{ll} \times\langle e^+,e^-,\gamma:{\rm coh}|(H_0+V-\omega^*)|e^+,e^-:{\rm coh}\rangle_k\,. \tag{A3}$$

The first two terms in M^d_{ij} can be calculated by substituting H_0 and V in terms of creation and annihilation operators and using the fact that $|e^+,e^-:{\rm coh}\rangle_i$ and $|e^+,e^-:{\rm coh}\rangle_j$ are eigenstates of the annihilation operator:

$$
{}_j\langle e^+,e^-:\text{coh}|H_0|e^+,e^-:\text{coh}\rangle_i=\delta^3(k_e-k_e')\,\delta^3(k_{\bar e}-k_{\bar e}')\left[(k_e^+ + k_{\bar e}^+)(k_e^- + k_{\bar e}^-)+\frac{\alpha}{2\pi^2}\int dq^+\int d^2q_\perp \times[(k_e^- + k_{\bar e}^-)q^+ + (k_e^+ + k_{\bar e}^+)q^- + q_\perp^2]f(q,\lambda:k_e,k_{\bar e})f^*(q,\lambda:k_e,k_{\bar e})\right]. \tag{A4}
$$

Substituting for $f(q,\lambda:k_e,k_{\bar e})$ and keeping only the cross terms in $f(q,\lambda:k_e,k_{\bar e})f^*(q,\lambda:k_e,k_{\bar e})$ (i.e., terms in which the soft photon is not emiited and absorbed by the same fermion), this expression simplifies to

$$
{}_j\langle e^+,e^-:\text{coh}|H_0|e^+,e^-:\text{coh}\rangle_i=\delta^3(k_e-k_e')\,\delta^3(k_{\bar e}-k_{\bar e}')\Bigg\{(k_e^+ + k_{\bar e}^+)(k_e^- + k_{\bar e}^-)+\frac{\alpha}{\pi^2}\int\frac{dy}{y}\int d^2q_\perp\Bigg[\left(\frac{k_e\cdot k_{\bar e}}{k_e\cdot q}-\frac{k_e^+}{q^+}\frac{k_{\bar e}\cdot q}{k_e\cdot q}-\frac{k_e^+}{q^+}\right)\times\theta(\Delta P^+ - k_e\cdot q)+\left(\frac{k_e\cdot k_{\bar e}}{k_{\bar e}\cdot q}-\frac{k_{\bar e}^+}{q^+}-\frac{k_{\bar e}^+}{q^+}\frac{k_e\cdot q}{k_{\bar e}\cdot q}\right)\theta(\Delta P^+ - k_{\bar e}\cdot q)\Bigg]\Bigg\}. \tag{A5}
$$

Similarly,

$$
{}_j\langle e^+,e^-:\text{coh}|V|e^+,e^-:\text{coh}\rangle_i=\delta^3(k_e-k_e')\,\delta^3(k_{\bar e}-k_{\bar e}')\Bigg\{-\frac{\alpha}{2\pi^2}4\int\frac{dy}{y}\int d^2q_\perp\Bigg[\left(\frac{k_e\cdot k_{\bar e}}{k_e\cdot q}-\frac{k_e^+}{q^+}\frac{k_{\bar e}\cdot q}{k_e\cdot q}-\frac{k_e^+}{q^+}\right)\theta(\Delta P^+ - k_e\cdot q)+\left(\frac{k_e\cdot k_{\bar e}}{k_{\bar e}\cdot q}-\frac{k_{\bar e}^+}{q^+}-\frac{k_{\bar e}^+}{q^+}\frac{k_e\cdot q}{k_{\bar e}\cdot q}\right)\theta(\Delta P^+ - k_{\bar e}\cdot q)\Bigg]\Bigg\}. \tag{A6}
$$

M_{ji}^{ND} is a sum of the terms

$$
M_{ji}^{\text{ND}}=M_1^{\text{ND}}+M_2^{\text{ND}}+M_3^{\text{ND}}+M_4^{\text{ND}}, \tag{A7}
$$

where

$$
M_1^{\text{ND}}=-\sum_i\sum_l {}_j\langle e^+,e^-:\text{coh}|(H_0-\omega^*)|e^+,e^-,\gamma:\text{coh}\rangle_i\,{}_i\left\langle e^+,e^-,\gamma:\text{coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma:\text{coh}\right\rangle_{ll}\times\langle e^+,e^-,\gamma:\text{coh}|V|e^+,e^-:\text{coh}\rangle_k, \tag{A8}
$$

$$
M_2^{\text{ND}}=-\sum_i\sum_l {}_j\langle e^+,e^-:\text{coh}|V|e^+,e^-,\gamma:\text{coh}\rangle_i\,{}_i\left\langle e^+e^-\gamma:\text{coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma:\text{coh}\right\rangle_{ll}\times\langle e^+,e^-,\gamma:\text{coh}|H_0-\omega^*|e^+,e^-:\text{coh}\rangle_k, \tag{A9}
$$

$$
M_3^{\text{ND}}=-\sum_i\sum_l {}_j\langle e^+,e^-:\text{coh}|(H_0-\omega^*)|e^+,e^-,\gamma:\text{coh}\rangle_i\,{}_i\left\langle e^+,e^-,\gamma:\text{coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma:\text{coh}\right\rangle_{ll}\times\langle e^+,e^-,\gamma:\text{coh}|H_0-\omega^*|e^+,e^-:\text{coh}\rangle_k, \tag{A10}
$$

$$
M_4^{\text{ND}}=-\sum_i\sum_l {}_j\langle e^+,e^-:\text{coh}|V|e^+,e^-,\gamma:\text{coh}\rangle_i\,{}_i\left\langle e^+,e^-,\gamma:\text{coh}\left|\frac{1}{H_0-\omega^*}\right|e^+,e^-,\gamma:\text{coh}\right\rangle_{ll}\times\langle e^+,e^-,\gamma:\text{coh}|V|e^+,e^-:\text{coh}\rangle_k. \tag{A11}
$$

Here, M_4^{ND} is the usual term corresponding to Fock representation which gives a nondiagonal term in the light cone Schrödinger equation (12) in $O(\alpha)$ and which has been calculated in $O(\alpha^2)$, Sec. IV. M_1^{ND}, M_2^{ND}, and M_3^{ND} can be calculated in the usual manner by substituting from Eqs. (A5) and (A6). We will give below the final expressions for the gauge independent parts of the cross terms in these:

$$
M_1^{\text{ND}}=M_2^{\text{ND}}=-\frac{\alpha}{2\pi^2 x}\int\frac{dy}{2y}\int d^2q_\perp\left[\frac{k_e\cdot k_{\bar e}}{k_e\cdot q}\theta(\Delta P^+ - k_e\cdot q)+\frac{k_e\cdot k_{\bar e}}{k_{\bar e}\cdot q}\theta(\Delta P^+ - k_{\bar e}\cdot q)\right], \tag{A12}
$$

$$
M_3^{\text{ND}}=\frac{\alpha}{2\pi^2}\int\frac{dy}{2y}\int d^2q_\perp q_\perp^2\left[\frac{2k_e\cdot k_{\bar e}}{(k_e\cdot q)(k_{\bar e}\cdot q)}\theta(\Delta P^+ - k_e\cdot q)\,\theta(\Delta P^+ - k_{\bar e}\cdot q)\right]. \tag{A13}
$$

Substituting from Eqs. (A5), (A6), (A12), and (A13) in Eq. (A1), we obtain

$$\left[\frac{m^2+\mathbf{k}_\perp{}^2}{x(1-x)}\right]\psi^i_{e^+e^-}-\sum_k \psi^k_{e^+e^-j}\left\langle e^+e^-:\text{coh}\left|V\frac{1}{\omega^*-H_0}V\right|e^+e^-:\text{coh}\right\rangle_i$$
$$+\frac{\alpha}{2\pi^2}4\int\frac{dy}{2y}\int d^2q_\perp\left[\left(\frac{m^2}{(k_e\cdot q)}+\frac{m^2}{(k_{\bar{e}}\cdot q)}\right)\theta(\Delta P^+-k_e\cdot q)\,\theta(\Delta P^+-k_{\bar{e}}\cdot q)\right]\psi^i_{e^+e^-}=M^2\psi^i_{e^+e^-}. \tag{A14}$$

The quantity in the square bracket in the third term on the LHS of Eq. (A14) is the coherent state contribution and it can be calculated by substituting for $k_e\cdot q$ and $k_{\bar{e}}\cdot q$:

$$k_e\cdot q=\frac{x}{2y}\left[\left(q_\perp-\frac{y}{x}\mathbf{k}_\perp\right)^2+\frac{m^2y^2}{x^2}\right],$$
$$k_{\bar{e}}\cdot q=\frac{1-x}{2y}\left[\left(q_\perp+\frac{y}{1-x}\mathbf{k}_\perp\right)^2+\frac{m^2y^2}{1-x^2}\right]. \tag{A15}$$

After appropriate changes of the integration variables and on taking the nonrelativistic limit, the coherent state contribution reduces to

$$I_C=\frac{\alpha}{2\pi^2}\int\frac{dy}{2y}\int d^2q_\perp\frac{8m^2}{4m^2y^2+q_\perp^2}\,\theta(\Delta P^+-(4m^2y^2+q_\perp^2)), \tag{A16}$$

which actually is an integral over the region defined in Eq. (28):

$$I_C=\frac{\alpha}{2\pi^2}\int_{D_c}\frac{dy}{2y}\int d^2q_\perp\frac{8m^2}{4m^2y^2+q_\perp^2}. \tag{A17}$$

Substituting I_C into Eq. (A14), we obtain Eq. (27).

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