Hard dense loops in a cold non-Abelian plasma

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Classical transport theory is used to study the response of a non-Abelian plasma at zero temperature and high chemical potential to weak color electromagnetic fields. In this article the parallelism between the transport phenomena occurring in a non-Abelian plasma at high temperature and high density is stressed. In particular, it is shown that at high densities it is also possible to relate the transport equations to the zero-curvature condition of a Chern-Simons theory in three dimensions, even when quarks are not considered ultrarelativistic. The induced color current in the cold plasma can be expressed as an average over angles, which represent the directions of the velocity vectors of quarks having Fermi energy. From this color current it is possible to compute n-point gluonic amplitudes, with arbitrary n. It is argued that these amplitudes are the same as the ones computed in the high chemical potential limit of QCD, which are then called hard dense loops. The agreement between the two different formalisms is checked by computing the polarization tensor of QED due to finite density effects in the high density limit. [S0556-2821(96)05610-X]

I. INTRODUCTION

Quantum chromodynamics (QCD) undergoes a phase transition at high temperature and/or high density [1]. Above a critical temperature and critical chemical potential, quarks and gluons are no longer confined. Those extreme regimes are expected to be found in nature in some cosmological and astrophysical settings (e.g., in the interior of neutron stars) or in heavy ion collisions. These are the regimes of QCD that I am going to discuss here.

As is well known now, naive perturbative analysis of high temperature QCD fails completely. This was realized when physical quantities, such as the gluonic damping rate, were found to be gauge dependent when computed following the standard rules of quantum field theories at finite temperature. The connection between expanding in loops and expanding in the gauge coupling constant g is not valid in this regime. As was realized by Braaten and Pisarski [2], as well as by Frenkel and Taylor [3], there are one-loop corrections, the hard thermal loops (HTL's), which are as important as tree amplitudes, and they have to be resummed and included consistently in all computations to nontrivial order in g. Those hard thermal loops arise only for *soft* external momenta ($\sim gT$) and *hard* internal loop momentum ($\sim T$), where $g \ll 1$, and T denotes the plasma temperature. There is an infinite set of HTL's which possess very interesting properties, such as obeying QED-like Ward identities and being gauge independent [3,4].

The hard thermal loop resummation techniques that were proposed in [2] were successful in providing one-loop gauge independent physical answers. Much related work has been done since their discovery (see Refs. [5,6] for a review and references), but here I shall only review different approaches to HTL's.

Taylor and Wong [7] were able to construct an effective action for hard thermal loops just by solving the gauge invariance condition imposed on that effective action. Efraty and Nair [8,9] have related that condition to the zero-

curvature equation of a Chern-Simons theory in (2+1)-dimensional space at zero temperature, providing therefore a nonthermal framework to study a thermal effect. This identification has been used by Jackiw and Nair [10] to derive a non-Abelian generalization of the Kubo formula.

Other derivations of the effective action of hard thermal loops can be found in the literature. Blaizot and Iancu [11] could extract the hard thermal loop effective action by studying the truncated Schwinger-Dyson hierarchy of equations, after performing a consistent expansion in the gauge coupling constant and obtaining quantum kinetic equations for the QCD induced color current. Jackiw, Liu, and Lucchesi [12] have also shown that HTL's can be derived from the Cornwall-Jackiw-Tomboulis composite effective action [13], after requiring its stationarity.

A different derivation of the effective action of HTL's has been given in [14], which does not make use of quantum field theory, as opposed to all previous approaches. Hard thermal effects are exclusively due to thermal fluctuations, and that is why a classical formalism to describe them was developed in [14]. Just by writing the classical transport equations for non-Abelian particles [15], and using an approximation scheme that respects the non-Abelian gauge symmetry of the transport equations, the effective action of the infinite set of HTL's of QCD could be found.

A similar situation may be expected to arise for QCD at high density and zero temperature [16]. Actually, HTL's were also studied when a chemical potential was included for quarks, and it was concluded that the only effect of the chemical potential was in modifying the Debye mass by a term proportional to the chemical potential [17,18]. Therefore, at very high density or chemical potential μ and zero temperature, one also may expect that naive one-loop computations are incomplete, as one-loop diagrams with *soft* ($\sim g\mu$) external momenta, and quarks running inside the loop with Fermi energy, are comparable to tree amplitudes, and therefore they would have to be resummed. I shall call

those diagrams hard dense loops (HDL's) in analogy to the thermal case.

The purpose of this article is to give a derivation of the hard dense loops of gluonic amplitudes of QCD by using the classical transport formalism. It is expected that quantum field computations of QCD at high chemical potential will reproduce the transport results, exactly as happens in the high temperature case, although I do not attempt here to check complete agreement between these two different formalisms.

Although only gluonic amplitudes are studied in this article, it should be expected that hard dense loops also arise for Feynman diagrams with external quark legs. I do not take into consideration those diagrams in the present article, and refer to a recent publication [19] which considers the one-loop self-energy of the electron for QED at finite density for those kinds of HDL's.

I have structured this article as follows. In Sec. II the classical transport theory for a non-Abelian plasma is briefly reviewed. In Sec. III A it is studied how a cold quark plasma initially in equilibrium reacts to weak external color electromagnetic fields, and the transport equation that the induced color current density obeys are written. The parallelism between the high temperature and high density cases is clearly established, and I also comment on the connection between the transport equation and the Chern-Simons eikonal. In Sec. III B the transport equation in momentum space is solved, using recent results of Ref. [20]. In Sec. III C the polarization tensor in the quark plasma is given, and it is prescribed how to derive higher-order point amplitudes. As a check of the agreement between the transport approach and the high chemical potential limit of QCD, the polarization tensor of QED due to finite density effects is computed in Sec. IV, and from it the HDL corresponding to that graph is extracted, which coincides with the one of QCD, up to some color factors. Complete agreement is found between the two different computations. On this result, I base my expectation that complete agreement for higher n-point gluonic amplitudes should also be true. Finally, I present my conclusions.

II. CLASSICAL TRANSPORT THEORY FOR A NON-ABELIAN PLASMA

The classical transport theory for the QCD plasma was developed in [15] and further studied in [14], and here I will briefly review it. Consider a particle bearing a non-Abelian SU(N) color charge $Q^a, a=1,\ldots,N^2-1$, traversing a world line $\hat{x}^\alpha(\tau)$. The Wong equations [21] describe the dynamical evolution of the variables x^μ, p^μ, and Q^a (the effect of spin is neglected here):

$$m\frac{d\hat{x}^\mu(\tau)}{d\tau}=\hat{p}^\mu(\tau), \qquad (2.1a)$$

$$m\frac{d\hat{p}^\mu(\tau)}{d\tau}=g\hat{Q}^a(\tau)F_a^{\mu\nu}(\hat{x})\hat{p}_\nu(\tau), \qquad (2.1b)$$

$$m\frac{d\hat{Q}^a(\tau)}{d\tau}=-gf^{abc}\hat{p}^\mu(\tau)A_\mu^b(\hat{x})\hat{Q}^c(\tau), \qquad (2.1c)$$

where f_{abc} are the structure constants of the group, $F_a^{\mu\nu}$ denotes the field strength, g is the coupling constant, and I set $c=\hbar=k_B=1$ henceforth.

The main difference between the equations of electromagnetism and the Wong equations, apart from their intrinsic non-Abelian structure, comes from the fact that color charges precess in color space, and therefore they are dynamical variables. Equation (2.1c) guarantees that the color current associated with each colored particle,

$$j_\mu^a(x)=g\int d\tau\hat{p}_\mu(\tau)\hat{Q}^a(\tau)\delta^{(4)}(x-\hat{x}(\tau)), \qquad (2.2)$$

where $(\hat{x}^\mu,\hat{p}^\mu,\hat{Q}^a)$ are solutions of the equations of motion, is covariantly conserved,

$$(D_\mu j^\mu)_a(x)=\partial_\mu j_a^\mu(x)+gf_{abc}A_\mu^b(x)j^{\mu c}(x)=0, \quad (2.3)$$

therefore preserving the consistency of the theory.

The usual (x,p) phase space is thus enlarged to (x,p,Q) by including color degrees of freedom for colored particles. Physical constraints are enforced by inserting δ functions in the phase-space volume element $dxdPdQ$. The momentum measure

$$dP=\frac{d^4p}{(2\pi)^3}2\theta(p_0)\delta(p^2-m^2) \qquad (2.4)$$

guarantees positivity of the energy and on-shell evolution. The color charge measure enforces the conservation of the group invariants, e.g., for SU(3),

$$dQ=d^8Q\,\delta(Q_aQ^a-q_2)\delta(d_{abc}Q^aQ^bQ^c-q_3), \quad (2.5)$$

where the constants q_2 and q_3 fix the values of the Casimirs and d_{abc} are the totally symmetric group constants. The color charges which now span the phase space are dependent variables. These can be formally related to a set of independent phase-space Darboux variables [14]. For the sake of simplicity, we will use the standard color charges.

The one-particle distribution function $f(x,p,Q)$ denotes the probability for finding the particle in the state (x,p,Q). In the collisionless case, it evolves in time via a transport equation $df/d\tau=0$. Using the equations of motion (2.1), it becomes the Boltzmann equation [15]

$$p^\mu\left[\frac{\partial}{\partial x^\mu}-gQ_aF_{\mu\nu}^a\frac{\partial}{\partial p_\nu}-gf_{abc}A_\mu^bQ^c\frac{\partial}{\partial Q_a}\right]f(x,p,Q)=0. \qquad (2.6)$$

A complete, self-consistent set of non-Abelian Vlasov equations for the distribution function and the mean color field is obtained by augmenting the Boltzmann equation with the Yang-Mills equations:

$$[D_\nu F^{\nu\mu}]^a(x)=J^{\mu a}(x)=\sum_{\text{species}}\sum_{\text{helicities}}j^{\mu a}(x), \quad (2.7)$$

where the color current $j^{\mu a}(x)$ for each particle species is computed from the corresponding distribution function as

$$j^{\mu a}(x) = g \int dP dQ p^\mu Q^a f(x,p,Q). \qquad (2.8)$$

Notice that if the particle's trajectory in phase space were known exactly, then Eq. (2.8) could be expressed as in Eq. (2.2). Furthermore, the color current (2.8) is covariantly conserved, as can be shown by using the Boltzmann equation [14].

The Wong equations (2.1) are invariant under the finite gauge transformations (in matrix notation)

$$\overline{x}^\mu = x^\mu, \quad \overline{p}^\mu = p^\mu, \quad \overline{Q} = UQU^{-1},$$

$$\overline{A}_\mu = UA_\mu U^{-1} - \frac{1}{g} U \frac{\partial}{\partial x_\mu} U^{-1}, \qquad (2.9)$$

where $U = U(x)$ is a group element.

It can be shown [14] that the Boltzmann equation (2.6) is invariant under the above gauge transformation if the distribution function behaves as a scalar,

$$\overline{f}(\overline{x},\overline{p},\overline{Q}) = f(x,p,Q). \qquad (2.10)$$

To check this statement it is important to note that under a gauge transformation the derivatives appearing in the Boltzmann equation (2.6) transform as

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \overline{x}^\mu} - 2\mathrm{Tr}\left\{\left[\left[\left(\frac{\partial}{\partial \overline{x}^\mu}U\right)U^{-1},\overline{Q}\right]\frac{\partial}{\partial \overline{Q}}\right]\right\},$$

$$\frac{\partial}{\partial p^\mu} = \frac{\partial}{\partial \overline{p}^\mu}, \quad \frac{\partial}{\partial Q} = U^{-1}\frac{\partial}{\partial \overline{Q}}U, \qquad (2.11)$$

that is, they are not gauge invariant by themselves. Only the specific combination of the spatial and color derivatives that appears in (2.6) is gauge invariant.

The color current (2.8) transforms under (2.9) as a gauge covariant vector:

$$\overline{j}^\mu(\overline{x}) = g \int dP dQ p^\mu UQU^{-1} f(x,p,Q) = Uj^\mu(x)U^{-1}. \qquad (2.12)$$

This is due to the gauge invariance of the phase-space measure and to the transformation properties of f.

III. INDUCED COLOR CURRENT IN A COLD QUARK PLASMA

A. Transport equation for the color current

In this section the soft disturbances of a completely degenerate quark plasma are studied. The quark plasma at zero temperature and finite density is initially in equilibrium. In the absence of a net color field, and assuming isotropy and color neutrality, the equilibrium distribution function for this system is, up to a normalization constant,

$$f^{(0)}(p) = \theta(\mu - p_0), \qquad (3.1)$$

where θ is the step function, and μ is the chemical potential. That is, all particle states are occupied with occupancy number 1 up to the Fermi energy $p_0 = \mu$.

Now I study how the quark plasma reacts to weak external color electromagnetic fields. I consider that the chemical potential is large, $\mu \gg \mu_c$, where μ_c is the critical chemical potential, while the color fields are *soft* or of the scale order $F_{\mu\nu} \sim g\mu^2$, where g is the coupling constant which is assumed to be small. The distribution function can be expanded in powers of g as

$$f = f^{(0)} + gf^{(1)} + g^2 f^{(2)} + \cdots. \qquad (3.2)$$

The Boltzmann equation (2.6) for $f^{(1)}$ reduces to

$$p^\mu\left(\frac{\partial}{\partial x^\mu} - gf^{abc}A_\mu^b Q_c \frac{\partial}{\partial Q^a}\right)f^{(1)}(x,p,Q)$$

$$= p^\mu Q_a F_{\mu\nu}^a \frac{\partial}{\partial p_\nu}f^{(0)}(p_0). \qquad (3.3)$$

Notice that a complete linearization of the equation in A_μ^a would break the gauge invariance of the transport equation, which is preserved in this approximation. But notice as well that this approximation tells us that $f^{(1)}$ also carries a g dependence.

The equation that the color current density $J_a^\mu(x,p)$ obeys can be obtained by multiplying (3.3) by p^μ and Q_a and then integrating over the color charges. For quarks in the fundamental representation

$$\int dQ Q_a Q_b = \frac{1}{2}\delta_{ab}. \qquad (3.4)$$

Taking also into account

$$\frac{d}{dp_0}\theta(\mu - p_0) = -\delta(\mu - p_0), \qquad (3.5)$$

one finally gets, after summing over helicities,

$$[p \cdot DJ^\mu(x,p)]^a = -g^2 p^\mu p^\nu F_{\nu 0}^a(x)\delta(\mu - p_0). \qquad (3.6)$$

The total induced current density can be obtained after summing over the different contributions due to different quark flavors. In order to simplify the notation I will not write the quark flavor index in the color current, chemical potentials, masses, etc., and it should be understood that a sum over quark flavors is to be taken in all final formulas.

From Eq. (3.6) we see that only quarks which are on the Fermi surface contribute to the induced color current in the plasma. Quarks which are inside the Fermi sea are blocked from reacting to the presence of external fields due to the Pauli exclusion principle. Furthermore, the plasma only responds to the presence of external color electric fields, and that is why only those get a screening mass in this approach.

In order to solve (3.6), one first divides the equation by p_0, and then integrates it over p_0 and $|\mathbf{p}|$, using the momentum measure dP (2.4). Due to the δ function in the momentum measure and the δ function in the right-hand side (RHS) of Eq. (3.6), $J_a^\mu(x,p)$ is only nonvanishing when $p_0 = \sqrt{|\mathbf{p}|^2 + m^2} = \mu$. Then one can write Eq. (3.6) as

$$[v \cdot D \, \mathscr{J}^\mu(x,v)]^a = -M^2 v^\mu v^\rho F_{\rho 0}^a(x), \qquad (3.7)$$

with

$$M^2 = g^2 \frac{\mu p_F}{2\pi^2}, \qquad (3.8)$$

where $p_F = \sqrt{\mu^2 - m^2}$ is the Fermi momentum and $v^\mu = (1, (p_F/\mu)\mathbf{v})$, and I have defined

$$\mathscr{J}_a^\mu(x,v) = \int \frac{|\mathbf{p}|^2 d|\mathbf{p}| dp_0}{2\pi^2} 2\theta(p_0)\delta(p^2 - m^2) \quad J_a^\mu(x,p), \qquad (3.9)$$

so that the total color current is obtained by integrating over all directions of the unit vector $\mathbf{v}$,

$$J_a^\mu(x) = \int \frac{d\Omega}{4\pi} \, \mathscr{J}_a^\mu(x,v). \qquad (3.10)$$

Equation (3.7) has the same structure as the transport equation that is obeyed by the induced color density quantity $\mathscr{J}_a^{\mu(T)}(x,V)$ in a *softly* perturbed hot quark-gluon plasma [14]:

$$[V \cdot D \, \mathscr{J}^{\mu(T)}(x,V)]^a = -m_D^2 V^\mu V^\rho F_{\rho 0}^a(x), \qquad (3.11)$$

where $m_D^2 = g^2 T^2 (N + N_f/2)/3$ is the Debye mass squared for an SU(N) non-Abelian group, with N_f flavors of quarks, and V^μ is a lightlike four-vector.

Equations (3.7) and (3.11) exhibit a similar structure, although they correspond to two different physical situations. In the case of a hot quark-gluon plasma the contribution to the induced color current comes from *both* quarks and gluons which are thermalized at a temperature T. This is reflected in the coefficient of the RHS of (3.11), the Debye mass squared. For very high temperatures, we would expect that quarks in the plasma move very fast, and it is a good approximation to consider that their velocities are ultrarelativistic, as long as $T \gg m$, where m is the quark mass, which is then neglected. Gluons travel at light velocities, however, without any approximation. These are the reasons why the vector V^μ in Eq. (3.11) is a lightlike four-vector. In the zero-temperature quark plasma that I am considering here there are no *real* gluons, and that is why they do not contribute to the induced color current. Furthermore, only quarks with Fermi energy contribute to that current. For very large chemical potential, it can be a good approximation to neglect quark masses if $\mu \gg m$. If quark masses are neglected, then $p_F = \mu$, and Eq. (3.7) coincides with Eq. (3.11), except for the the factors m_D^2 and M^2. I have decided, however, to keep corrections due to quark masses, which is justified if $\mu > m \gg g\mu$. Therefore v^μ is not a lightlike vector in this more general situation where massive quarks are considered.

Before solving Eq. (3.7) I would like to comment on the connection between this transport problem and the Chern-Simons eikonal. Efraty and Nair [8,9] have shown that the gauge invariance condition for the generating functional of hard thermal loops, Γ_{HTL}, of QCD can be formally related to the zero-curvature condition of a Chern-Simons theory in a (2+1)-dimensional space-time at zero temperature. Further-

more, it has been shown that this condition can be obtained from the transport equation (3.11) after assuming $J_a^{\mu(T)}(x) = -\delta\Gamma_{\text{HTL}}[A]/\delta A_\mu^a(x)$. The same identifications can be done for the cold quark plasma, after assuming that the induced color current can be obtained from a generating functional. The same steps that were necessary to show that identification for the thermal problem can be repeated here with the only main difference that v^μ is not a lightlike vector when quark masses are not neglected.

For completeness I present that identification below. One first defines a new current density

$$\widetilde{\mathscr{J}}^{\mu a}(x,v) = \mathscr{J}^{\mu a}(x,v) + M^2 v^\mu A_0^a(x). \qquad (3.12)$$

Then $\widetilde{\mathscr{J}}^{\mu a}$ obeys the equation

$$[v \cdot D \, \widetilde{\mathscr{J}}^\mu(x,v)]^a = M^2 v^\mu \frac{\partial}{\partial x^0}[v \cdot A^a(x)]. \qquad (3.13)$$

Now one assumes that $\widetilde{\mathscr{J}}_a^\mu$ can be derived from a generating functional as

$$\widetilde{\mathscr{J}}_a^\mu(x,v) = \frac{\delta W(A,v)}{\delta A_\mu^a(x)}. \qquad (3.14)$$

Equation (3.13) then implies that $W(A,v)$ depends only on $A_+^a \equiv (1/\sqrt{2f}) \, v \cdot A^a$, with $f = p_F/\mu$, i.e., $W(A,v) = W(A_+)$, and $\widetilde{\mathscr{J}}_a^\mu = [\delta W(A_+)/\delta A_+^a]v^\mu$. If I now define

$$x_+ = \sqrt{\frac{f}{2}}\left(x_0 + \frac{1}{f}\,\mathbf{v} \cdot \mathbf{x}\right), \quad \partial_+ = \frac{1}{\sqrt{2f}}\,v \cdot \partial,$$

$$\qquad (3.15)$$

$$x_- = \sqrt{\frac{f}{2}}\left(x_0 - \frac{1}{f}\,\mathbf{v} \cdot \mathbf{x}\right), \quad \partial_- = \frac{1}{\sqrt{2f}}\,\overline{v} \cdot \partial,$$

with $\overline{v} \equiv (1, -(p_F/\mu)\mathbf{v})$, and

$$F^a = \frac{\delta W(A_+)}{\delta A_{+a}} - \frac{1}{2} M^2 A_+^a, \qquad (3.16)$$

then, after a Wick rotation to Euclidean space, $x_+ \to z$, $x_- \to \overline{z}$, $A_+ \to A_z$, and the identification $F^a = \sqrt{2/f}(1/M^2)a_{\overline{z}}^a$, Eq. (3.13) is translated into [in matrix notation, and using anti-Hermitian generators in the fundamental representation of SU(N)]

$$\partial_z a_{\overline{z}} - \partial_{\overline{z}} A_z + g[A_z, a_{\overline{z}}] = 0, \qquad (3.17)$$

which corresponds to the zero-curvature condition of a Chern-Simons theory in a (2+1) Euclidean dimensional space-time, in the gauge $A_0 = 0$.

Solutions of Eq. (3.17) are provided by the Chern-Simons eikonal and were studied long ago [22]. The generating functional of the induced color current in a cold quark plasma can then be expressed in terms of the Chern-Simons eikonal, exactly as in the case of the hot quark-gluon plasma.

In spite of the very interesting connection between these, in principle, unrelated problems, I will not pursue the Chern-Simons approach to the transport equations in the remaining part of this article.

B. Solution to the transport equation

Equation (3.7) can be solved after inverting the $v \cdot D$ operator and imposing proper boundary conditions. Then the solution of (3.7) is expressed in terms of link operators [7,11]. In this subsection, I prefer to solve the transport equation by going to momentum space exactly as has been done for Eq. (3.11) in Ref. [20]. Writing

$$\mathcal{J}_a^{\mu}(k,v) = \int \frac{d^4x}{(2\pi)^4} e^{ik \cdot x} \mathcal{J}_a^{\mu}(x,v), \qquad (3.18)$$

Eq. (3.7) becomes in momentum space

$$v \cdot k \mathcal{J}_a^{\mu}(k,v) + igf_{abc} \int \frac{d^4q}{(2\pi)^4} v \cdot A^b(k-q) \mathcal{J}^{\mu c}(q,v)$$

$$= -M^2 v^{\mu} \left[v \cdot k A_0^a(k) - k_0 v \cdot A^a(k) \right.$$

$$\left. + igf_{abc} \int \frac{d^4q}{(2\pi)^4} v \cdot A^b(k-q) A_0^c(q) \right]. \qquad (3.19)$$

Now, after assuming that $\mathcal{J}_a^{\mu}(k,v)$ can be expressed as an infinite power series in the gauge field $A_{\mu}^a(k)$, Eq. (3.19) can be solved iteratively for each order in the power series. Retarded boundary conditions are imposed by the prescription $p_0 \to p_0 + i\epsilon$, with $\epsilon \to 0^+$, which should be understood in all following formulas.

The first-order solution is

$$\mathcal{J}_a^{\mu(1)}(k,v) = M^2 v^{\mu} \left(k_0 \frac{v \cdot A_a(k)}{v \cdot k} - A_a^0(k) \right). \qquad (3.20)$$

Inserting (3.20) in (3.19) allows solving for the second-order term in the series, which reads

$$\mathcal{J}_a^{\mu(2)}(k,v) = -igM^2 f_{abc}$$

$$\times \int \frac{d^4q}{(2\pi)^4} v^{\mu} q_0 \frac{v \cdot A^b(k-q) v \cdot A^c(q)}{(v \cdot k)(v \cdot q)}. \qquad (3.21)$$

The nth-order term ($n > 2$) can be expressed as a function of the $(n-1)$th one as

$$\mathcal{J}_a^{\mu(n)}(k,v) = -igf_{abc} \int \frac{d^4q}{(2\pi)^4} \frac{v \cdot A^b(k-q)}{v \cdot k} \mathcal{J}_c^{\mu(n-1)}(q,v). \qquad (3.22)$$

The complete expression of the induced color current in the cold quark plasma is thus given by

$$J_a^{\mu}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \sum_{n=1}^{\infty} J_a^{\mu(n)}(k)$$

$$= \int \frac{d\Omega}{4\pi} \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \sum_{n=1}^{\infty} \mathcal{J}_a^{\mu(n)}(x,v). \qquad (3.23)$$

The color current is then expressed as an average over the three-dimensional unit vector $\mathbf{v}$, which represents the directions of the velocity vectors of quarks with Fermi energy.

It should be possible to construct a generating functional Γ_{HDL} that generates this current by solving

$$J_a^{\mu}(x) = -\frac{\delta \Gamma_{\text{HDL}}[A]}{\delta A_{\mu}^a(x)}. \qquad (3.24)$$

It is first required to check that $J_a^{\mu}(x)$ obeys integrability conditions, as has been done in Ref. [7] for the massless and thermal case.

C. Polarization tensor

The polarization tensor $\Pi_{ab}^{\mu\nu}$ can be computed from (3.20) by using the relation

$$J_a^{\mu(1)}(k) = \Pi_{ab}^{\mu\nu}(k) A_{\nu}^b(k). \qquad (3.25)$$

It reads:

$$\Pi_{ab}^{\mu\nu}(k) = M^2 \left(-g^{\mu 0} g^{\nu 0} + k_0 \int \frac{d\Omega}{4\pi} \frac{v^{\mu} v^{\nu}}{v \cdot k} \right) \delta_{ab}, \qquad (3.26)$$

where we recall that $v^{\mu} = (1,(p_F/\mu)\mathbf{v})$. To avoid the poles in the above integrand, retarded boundary conditions are imposed, i.e., $k_0 \to k_0 + i\epsilon$.

The polarization tensor (3.26) obeys the Ward identity

$$k_{\mu} \Pi_{ab}^{\mu\nu} = 0, \qquad (3.27)$$

and it is also gauge independent due to the gauge invariance of the transport formalism.

The real part of the polarization tensor is

$$\text{Re}\Pi_{ab}^{00}(k_0,\mathbf{k}) = \delta_{ab} \Pi_l(k_0,\mathbf{k}), \qquad (3.28a)$$

$$\text{Re}\Pi_{ab}^{0i}(k_0,\mathbf{k}) = \delta_{ab} k_0 \frac{k^i}{|\mathbf{k}|^2} \Pi_l(k_0,\mathbf{k}), \qquad (3.28b)$$

$$\text{Re}\Pi_{ab}^{ij}(k_0,\mathbf{k}) = \delta_{ab} \left[\left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2} \right) \Pi_t(k_0,\mathbf{k}) \right.$$

$$\left. + \frac{k^i k^j}{|\mathbf{k}|^2} \frac{k_0^2}{|\mathbf{k}|^2} \Pi_l(k_0,\mathbf{k}) \right], \qquad (3.28c)$$

where

$$\Pi_l(k_0,\mathbf{k}) = M^2 \left(\frac{\mu}{p_F} \frac{k_0}{2|\mathbf{k}|} \ln \left| \frac{\mu k_0 + p_F |\mathbf{k}|}{\mu k_0 - p_F |\mathbf{k}|} \right| - 1 \right), \qquad (3.29a)$$

$$\Pi_t(k_0,\mathbf{k}) = -M^2 \frac{k_0^2}{|\mathbf{k}|^2}\left[1 + \frac{1}{2}\left(\frac{p_F}{\mu} \frac{|\mathbf{k}|}{k_0} \right.\right.$$

$$\left.\left. - \frac{\mu}{p_F} \frac{k_0}{|\mathbf{k}|} \right) \ln\left| \frac{\mu k_0 + p_F |\mathbf{k}|}{\mu k_0 - p_F |\mathbf{k}|} \right| \right]. \qquad (3.29\text{b})$$

The imaginary part of the polarization tensor describes damping in the quark plasma, explicitly:

$$\text{Im}\Pi_{ab}^{\mu\nu}(k_0,\mathbf{k}) = -\delta_{ab}M^2\pi k_0 \int \frac{d\Omega}{4\pi} v^\mu v^\nu \delta\left(k_0 - \frac{p_F}{\mu}\mathbf{k}\cdot\mathbf{v} \right). \qquad (3.30)$$

My result for the polarization tensor agrees with the one computed in Ref. [23] using classical transport theory for a quark-gluon plasma at finite temperature and baryon density in the zero-temperature limit.

Notice that if quark masses are neglected, then $p_F = \mu$, and then (3.26) reduces to the same polarization tensor as the one found in a hot quark-gluon plasma [15,23,14], with a different screening mass.

Higher order n-point functions can be found as

$$\delta^{(4)}\left(k - \sum_{i=1}^{i=n-1} q_i \right)\Gamma_{ab_1\cdots b_{n-1}}^{\mu\nu_1\cdots\nu_{n-1}}(k,q_1,\ldots,q_{n-1})$$

$$= \left.\frac{\delta J_a^\mu(k)}{\delta A_{\nu_1}^{b_1}(q_1)\cdots\delta A_{\nu_{n-1}}^{b_{n-1}}(q_{n-1})}\right|_{A_\rho^c = 0}, \qquad (3.31)$$

where the δ function on the RHS of (3.31) accounts for conservation of momentum, and permutations of indices and momentum have to be taken.

Due to the fact that the induced color current can be expressed as an infinite power series in the gauge field A_μ^a, it is obvious that there are n-point gluonic amplitudes, with arbitrary n, from 2 to ∞. All these gluonic amplitudes describe the polarizability properties of the non-Abelian plasma.

IV. THE POLARIZATION TENSOR OF QED AT FINITE DENSITY

In this section the computation of the polarization tensor of QED due to the presence of a background density of electrons is presented, and from it the HDL corresponding to that amplitude in the massless case is extracted. This computation has been previously performed by several authors in Euclidean space-time; see Refs. [24–27], and also [28] and [29] for the nonrelativistic case. Here I review how this computation is performed in Minkowski space-time when retarded boundary conditions are imposed. I find it more appropriate to carry out this computation in Minkowski space-time to obtain the correct analytic properties of the retarded polarization tensor.

In the presence of a finite density of electrons the usual definition of creation and annihilation operators for particles and holes, or antiparticles, of the vacuum has to be modified [29]. The ground state $|\Phi_0\rangle$ of the system is constituted by the Fermi sea, that is, by electrons occupying all particle states according to the Pauli exclusion principle up to the Fermi energy $p_0 = \mu$. Then one defines a creation operator

$b_\alpha^\dagger(p)$ that acting on the ground state creates one particle with energy $p_0 > \mu$, and the creation operator $d_\alpha^\dagger(-p)$, for $p_0 < \mu$, which creates a hole, which may be interpreted as the absence of one electron in the Fermi sea. The corresponding particle and hole destruction operators annihilate the ground state

$$b_\alpha(p)|\Phi_0\rangle = d_\alpha(p)|\Phi_0\rangle = 0, \qquad (4.1)$$

since the ground state does not contain electrons with energies above the Fermi energy or holes inside the Fermi sea.

The usual averages of products of creation and annihilation operators that are required to compute Feynman amplitudes are then modified from their vacuum values, explicitly,

$$\langle\Phi_0|b_\alpha(p)b_{\alpha'}^\dagger(p)|\Phi_0\rangle = \theta(p_0 - \mu)\delta_{\alpha,\alpha'}. \qquad (4.2)$$

Then one can compute the Feynman propagator to get [29,5]

$$iS_F(x,y) = \langle\Phi_0|\mathscr{T}\psi(x)\bar{\psi}(y)|\Phi_0\rangle$$

$$= \theta(x^0 - y^0)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}[(\gamma\cdot p + m)$$

$$\times \theta(p_0 - \mu)e^{-ip(x-y)}] - \theta(y^0 - x^0)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p_0}$$

$$\times [(\gamma\cdot p + m)\theta(\mu - p_0)e^{-ip(x-y)}$$

$$+ (\gamma\cdot p - m)e^{ip(x-y)}] \qquad (4.3)$$

where $p_0 = \sqrt{|\mathbf{p}|^2 + m^2}$. Notice that in the limit $\mu\to 0$, (4.3) agrees with the Feynman propagator in the vacuum.

In order to compute the polarization tensor of QED in the presence of a background density of electrons, the propagator (4.3) is needed. Retarded boundary conditions are taken into account by introducing convergence factors $e^{\epsilon x_0}$, $e^{-\epsilon y_0}$, with $\epsilon\to 0^+$, as needed when going from configuration to momentum space (see Refs. [9] and [10] for explicit details). Taking that into account, one finds for the chemical potential dependent part of the retarded polarization tensor in the massless case

$$\Pi^{\mu\nu}(k) = g^2 \int \frac{d^3q}{(2\pi)^3}\frac{1}{2p_0}\frac{1}{2q_0}\left[[\theta(\mu - q_0) \right.$$

$$- \theta(\mu - p_0)]\frac{T^{\mu\nu}(p,q)}{p_0 - q_0 - k_0 - i\epsilon}$$

$$- \theta(\mu - p_0)\frac{T^{\mu\nu}(p,q')}{p_0 + q_0 - k_0 - i\epsilon}$$

$$\left. - \theta(\mu - q_0)\frac{T^{\mu\nu}(p',q)}{p_0 + q_0 + k_0 + i\epsilon} \right] \qquad (4.4)$$

where $p = (p_0,\mathbf{p})$, $p' = (p_0,-\mathbf{p})$, $q = (q_0,\mathbf{q})$, $q' = (q_0,-\mathbf{q})$, with $p_0 = |\mathbf{p}|$, $q_0 = |\mathbf{q}|$, and $\mathbf{p} = \mathbf{q} + \mathbf{k}$. We have further defined

$$T^{\mu\nu}(p,q) \equiv \text{Tr}[\gamma^\mu\gamma\cdot p\,\gamma^\nu\gamma\cdot q]. \qquad (4.5)$$

In the high chemical potential limit, one considers $k_0, |\mathbf{k}| \ll \mu$. Then one can make the approximations [3,5]

$$T^{\mu\nu}(p,q) \sim 8q_0^2 V^\mu V^\nu, \qquad (4.6)$$

$$T^{\mu\nu}(p,q') \sim T^{\mu\nu}(p',q) \sim 4q_0^2 (V^\mu \bar{V}^\nu + \bar{V}^\mu V^\nu - 2g^{\mu\nu}), \qquad (4.7)$$

$$p_0 - q_0 - k_0 \sim -k \cdot V, \qquad (4.8)$$

$$p_0 + q_0 \pm k_0 \sim 2q_0, \qquad (4.9)$$

$$\theta(\mu - q_0) - \theta(\mu - p_0) \sim \mathbf{V} \cdot \mathbf{k} \delta(\mu - q_0), \qquad (4.10)$$

where $V = (1, \mathbf{q}/q_0)$ and $\bar{V} = (1, -\mathbf{q}/q_0)$, to finally get

$$\Pi^{\mu\nu}(k) = -g^2 \int \frac{d^3q}{(2\pi)^3} \left(\bar{V}^\mu V^\nu + V^\mu V^\nu - 2k_0 \frac{V^\mu V^\nu}{V \cdot k + i\epsilon} \right)$$
$$\times \delta(\mu - q_0), \qquad (4.11)$$

which after performing the integral over the modulus of $|\mathbf{q}|$ reduces to the same polarization tensor (3.26) for the Abelian and massless case.

V. CONCLUSIONS

In this paper classical transport theory has been used to study the response of a completely degenerate non-Abelian plasma at zero temperature to weak color electromagnetic fields. An approximation scheme for the classical transport equations that respects their non-Abelian gauge symmetry has been used, and from it the induced color current in the non-Abelian plasma has been obtained.

The study of the transport phenomena occurring at high density in a quark plasma that I have carried out here is quite similar to the one that was done in [14] for a QCD plasma at high temperature, and I direct the reader to consult that reference. Throughout this paper the parallelism that these two different problems exhibit has been stressed, as well as their differences.

At both high temperature and high density, the color constituents of the non-Abelian plasma are not confined, and that is why one can model color degrees of freedom classically.

The transport equations obeyed by the induced color current for a non-Abelian plasma at high temperature and at high density present a similar structure. If quark masses are neglected, then the two transport equations are essentially the same, except for the following. At high temperature T, both quarks and gluons which are thermalized at the temperature T contribute to the induced color current in the plasma. At high chemical potential, only quarks on the Fermi surface contribute to that current, as there are no *real* gluons at zero temperature. Then, in that situation, the only difference between the two transport equations and their solutions comes from the screening masses squared, which are $m_D^2 = g^2 T^2 (N + N_f/2)/3$ and $M^2 = g^2 N_f \mu^2 / 2\pi^2$, respectively, for an SU(N) theory with N_f flavors of quarks (and assuming that for all quark flavors $\mu_f = \mu$).

I have decided to keep corrections due to quark masses in the present study. The solution to the transport equation can be expressed in terms of an average of a three-dimensional unit vector $\mathbf{v}$, which represents the directions of the velocity vectors of quarks which have Fermi energy.

It has been also shown that the solution of the transport equation can be expressed in terms of the Chern-Simons eikonal, even for massive quarks. Even though one cannot define natural light-cone coordinates when quarks are not considered ultrarelativistic, we have seen that it is still possible to relate the transport equation to the zero-curvature condition of a Chern-Simons theory in $(2+1)$ Euclidean space-time. The transport equation only depends on the projections of the vector gauge fields over the four vectors v^μ and $\bar{v}^\mu$ defined in Sec. III A, and this is enough to define a two-dimensional plane where the zero-curvature condition of a Chern-Simons theory can be recognized, after proper identifications.

I have presented a systematic way of computing n-point gluonic amplitudes using the solution of the classical transport equation, and I claim that these results should agree with the high density limit of a quantum field theory approach. Complete agreement is found for the polarization tensor ($n=2$) in the massless case, and complete agreement should be expected for higher n-point functions, although I have not checked this statement, nor am I aware of the computation of those amplitudes in the literature.

Therefore, it must be claimed that a Braaten-Pisarski resummation procedure should also be used in QCD with high chemical potential, as was generally expected, although a more detailed quantum field theoretical study should be carried out to see exactly how this resummation is implemented. I think that it is interesting to have this alternative computation of HDL's, specially due to the simplicity and transparent physical interpretation of the classical transport formalism.

Finally, one should mention that although the response theories of the high temperature limit (at zero chemical potential) and the high chemical potential limit (at zero temperature) of QCD seem almost identical, and one could naively conclude that in both situations static color electric fields are exponentially damped (at least at leading order), this is not so. Kapusta and Toimela [30] have pointed out that the static potential between two static charges in a plasma at zero temperature becomes oscillatory and vanishes as a power at large distances due to the existence of a sharp Fermi surface. This effect was known as Friedel oscillation in nonrelativistic quantum theory [29].

A further study of the high density limit of QCD should be carried out, and I postpone that study to a further publication.

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