Lessons from two-dimensional QCD ($N\to\infty$): Vacuum structure, asymptotic series, instantons, and all that

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We discuss two-dimensional QCD ($N_c\to\infty$) with fermions in the fundamental as well as adjoint representation. We find factorial growth $\sim(g^2N_c\pi)^{2k}(2k)!(-1)^{k-1}/(2\pi)^{2k}$ in the coefficients of the large order perturbative expansion. We argue that this behavior is related to classical solutions of the theory, instantons; thus it has nonperturbative origin. Phenomenologically such a growth is related to highly excited states in the spectrum. We also analyze the heavy-light quark system $Q\bar{q}$ within the operator product expansion (which turns out to be an asymptotic series). Some vacuum condensates $\langle\overline{q(x_\mu D_\mu)^{2n}q}\rangle\sim(x^2)^n n!$ which are responsible for this factorial growth are also discussed. We formulate some general puzzles which are not specific for two-dimensional physics, but are inevitable features of any asymptotic expansion. We resolve these apparent puzzles within two-dimensional QCD and we speculate that analogous puzzles might occur in real four-dimensional QCD as well. [S0556-2821(96)07210-4]

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I. INTRODUCTION

The problem of large-order behavior of perturbative theory attracted renewed attention recently. One of the motivating factors is a common wisdom that the corresponding asymptotic behavior is related somehow to very deep physics. This is the area where perturbative and nonperturbative physics strongly interfere. An understanding of this interplay may shed light on the nature of the nonperturbative vacuum structure in general and the origin of vacuum condensates in particular.

With these general remarks in mind we would like to analyze these problems in solvable two-dimensional QCD ($N\to\infty$) [1–7]. We would like to test assumptions, hypotheses, and interpretations, made in four-dimensional field theory, within toy two-dimensional QCD ($N\to\infty$), where we expect confinement and many other properties inherent to real QCD. Additionally, we extend the analysis to QCD with adjoint matter [8,9]. As is known, in this theory, the pair creation is not suppressed even in the large N_c limit, and thus this model can mimic an exponentially growing density of states with large mass $\rho(m)\sim\exp(m)$. In this case no exact solution is available, but we argue that general methods such as dispersion relations, duality and unitarity can provide all the information we need about spectrum for the calculation of large order behavior.

Why are we so conscious about the large order behavior? We see at least a few theoretical and phenomenological reasons for that. Let us start from the pure theoretical reasons. One may think that the crucial question in this case is whether the perturbative series is Borel summable or not.

Contrary to the common belief, we do not think that the issue of Borel summability (or its loss) is the fundamental point. In particular, let us mention an example of the principal chiral field theory at large N [10]. In this case, the explicit solution as well as the coefficients of the perturbative expansion can be calculated. These coefficients grow factorially with the order and the series is non-Borel summable, but nevertheless, the physical observables perfectly exist for any finite coupling constant. The exact result can be recovered by special prescription which uses a nontrivial procedure of analytical continuation (which might be a good example for other asymptotically free theories).

The second important theoretical issue can be formulated as follows: Because of dimensionality of the coupling constant in two-dimensional QCD the perturbative expansion $\sim\Sigma(g^2)^n c_n$ and the operator product expansion (OPE) for a correlation function $\sim\Sigma(g^2/Q^2)^n c_n$ are *one and the same* expansion. From this simple observation we learn that the OPE is an asymptotic series. Thus many interesting questions arise.

(a) What kind of vacuum condensates are responsible for such a behavior?

(b) Do we extract the actual condensates from the OPE or only effective ones?

(c) What kind of vacuum configurations are responsible for such $n!$ growth?

(d) Do these configurations saturate the vacuum condensates?

What is more important, sometimes these questions (and many others) can be answered. We expect, will argue, that the analogous phenomena might occur in real four-dimensional QCD, thus these questions have not only pure academic interest.

Phenomenologically, there are issues which are even more interesting and give much more freedom for speculation. First of all, let us recall that the reason for interest in the large order behavior is related to the factorial $n!\simeq n^n$ growth of the perturbative coefficients. This growth can be considered via the dispersion relation and is commonly interpreted as a reflection of the divergence of the multiparticle cross section with large number of particles and energy $\simeq n$ (see discussions in [11]). The naive interpretation would be the

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violation of unitarity.[1] We will show however that in two-dimensional QCD while we have a factorial growth of coefficients, this growth has nothing to do with multiparticle production since at large N the pair creation is suppressed by factor $1/N$. Rather, this growth is related to the highly excited two-particle meson states. Another phenomenological issue looks mysterious: vacuum condensates extracted from the spectrum might be quite different from the actual magnitude of condensates.

II. 't HOOFT MODEL

Let us start from the analysis of two-dimensional QCD with fermions in the fundamental representation—the 't Hooft model. It is completely solvable in the limit where the number of colors $N\to\infty$ [1]. The Bethe-Salpeter equation for mesonic bound states was solved in [1] yielding a spectrum whose states lie asymptotically on a single "Regge trajectory." We want to point out that many general questions in this model can be answered without solving an equation, but using such powerful methods as dispersion relations, duality and unitarity. In particular, in the weak coupling regime,

$$g^2N\sim\text{const},\quad N\to\infty,\quad m_q\gg g\sim\frac{1}{\sqrt{N}}\tag{1}$$

the chiral $\langle\bar{\psi}\psi\rangle$ and gluon condensates $\langle G_{\mu\nu}^2\rangle$ can be calculated *exactly*; see below. Additionally, few low-energy theorems can be tested and obtained result imply that there are no other states in addition to those found by 't Hooft. In other words, the dispersion and duality relations would indicate missing states.

Here, the entire spectrum is discrete and is classified by the integer n. The model we shall consider consists of quark in fundamental representation interacting via an SU(N) color gauge group. We follow the notation of Ref. [2] and present the 't Hooft equation [1] in the following form:

$$m_n^2\phi_n(x)=\frac{m_q^2}{x(1-x)}\phi_n(x)-m_0^2P\int dy\frac{\phi_n(y)}{(x-y)^2},\tag{2}$$

where symbol P notes as the principal value of the integral, and $0<x<1$ is the fraction of the total momentum of the bound state carried by quark q with mass m_q. The quantity $m_0^2\equiv g^2N/\pi$ is the basic mass scale in the theory and the index n classifies the ordering number of the bound states $|n,p\rangle$ with total momentum p_μ.

The same wave function can be expressed in terms of the following matrix element [6]:

$$\phi_n(x)=\sqrt{\frac{N}{\pi}}\int dy_+e^{-iy_+(1-2x)p_-}\langle 0|\overline{q}(-y)q(y)|n,p\rangle|_{y_-}=0.\tag{3}$$

Let us note that the matrix element on the right is written in the light cone gauge $A_-=0$; to restore the manifest gauge invariance one can insert the standard exponential factor $e^{ig\int A_-dy_+}$ into the formula (3).

Let us review some important properties of Eq. (2). The entire spectrum is discrete and classified by the integer number n. The wave functions $\phi_n(x)$ are orthogonal, complete, and obey the following boundary conditions:

$$\phi_n(x)\to[x(1-x)]^\beta,\quad x\to0,\quad x\to1,$$

$$\pi\beta\cot(\pi\beta)=1-\frac{m_q^2}{m_0^2}.\tag{4}$$

For large n the spectrum is linear

$$m_n^2\simeq\pi^2m_0^2n,\quad \phi_n(x)\simeq\sqrt{2}\sin(\pi nx)\tag{5}$$

and does not depend on mass of the quark. More importantly, in the chiral limit ($m_q\to0$) the lowest level (we call it π meson) tends to zero ($m_\pi^2\sim m_q$) and one could expect a nonzero magnitude for the chiral condensate. Thus we have come to the very important connection between spectrum and vacuum structure.

As the vacuum of the model is a very important issue for the following analysis, we would like to recall some results with an explanation of the general methods which have been used to derive them.

We define the chiral condensate in the current algebra terms as follows:

$$0=\lim_{p_\mu\to0}i\int d^2x\,e^{ipx}\partial_\mu\langle0|T\{\overline{q}\gamma_\mu\gamma_5q(x),\overline{q}\gamma_5q(0)\}|0\rangle$$
$$=2i\langle0|\overline{q}q|0\rangle+2m_q\langle0|T\{\overline{q}i\gamma_5q(x),\overline{q}i\gamma_5q(0)\}|0\rangle.\tag{6}$$

As we have already mentioned, the only states of 't Hooft's solution are the quark-antiquark bound states, thus they must saturate the dispersion relation. Upon inserting this complete set of mesons to the (6) one obtains

$$\langle0|\overline{q}q|0\rangle=-m_q\sum_n\frac{N}{\pi}\frac{f_n^2\pi^2m_0^2}{m_n^2},\tag{7}$$

where f_n is defined in terms of the following matrix elements:

$$\langle0|\overline{q}i\gamma_5q|n\rangle=\sqrt{\frac{N}{\pi}}\pi m_0f_n,\quad f_n\pi m_0=\frac{m_q}{2}\int_0^1\frac{\phi_n(x)}{x(1-x)}dx.\tag{8}$$

In the chiral limit the only state which can contribute to the formula (7) is the π meson. Its matrix element can be calculated exactly and we end up with the following expression for the chiral condensate in the $m_q\to0$ limit [12]:

$$\langle0|\overline{q}q|0\rangle=-N\frac{m_0}{\sqrt{12}},\quad m_0^2=\frac{g^2N}{\pi},$$

$$f_{n=0}=\frac{1}{\sqrt{3}},\quad m_\pi^2=m_q\frac{2m_0}{\sqrt{3}}.\tag{9}$$

[1] In the physical theory, the unitarity is preserved, of course. The physical question is the following: what can stop this growth?

The result is confirmed by numerical [13,14] and independent analytical calculations [15]. Moreover, the method has been generalized for the nonzero quark mass and the corresponding explicit formula for the chiral condensate $\langle \overline{q}q\rangle$ with arbitrary m_q has been obtained [16].

As was expected, we find that $\langle 0|\overline{q}q|0\rangle \sim N$. Besides that, as we have already noticed in [12], if we put $m_q=0$ from the very beginning, then $\langle 0|\overline{q}q|0\rangle=0$. This corresponds to the different regime when $m_q \ll g \sim 1/\sqrt{N}$, when nonplanar diagrams come into the game as we will discuss later. The last remark is the observation that the entire nonzero answer for the condensate comes from the infrared region of the integration in Eq. (8): $x\sim 0, x\sim 1$ which corresponds to the situation when one of the quarks carries all the momentum and the second one is at rest.

The sum (7) can be calculated exactly for arbitrary m_q [16]. The crucial point is that for arbitrary m_q the nonzero contribution comes from the highly excited states ($n \gg 1$) only. The properties of these states are well known,

$$f_n^2 \to 1, \quad m_n^2 \to \pi^2 m_0^2 n, \quad n \gg 1, \tag{10}$$

and thus the sum (7) can be explicitly evaluated with the result [16]

$$\langle 0|\overline{q}q|0\rangle = \frac{m_q N}{2\pi}\left\{ \ln(\pi\alpha) - 1 - \gamma_E + \left(1-\frac{1}{\alpha}\right)[I(\alpha) - \alpha I(\alpha) - \ln 4]\right\}, \tag{11}$$

where $\alpha = m_0^2/m_q^2, \gamma_E = 0.5772\ldots$ is Euler's constant and

$$I(\alpha) = \int_0^\infty \frac{dy}{y^2}\frac{1-(y/\sinh y\ \cosh y)}{[\alpha(y\ \coth y - 1)+1]}.$$

This result is exact for large N and arbitrary quark mass within the 't Hooft regime, i.e., $m_q \gg g \sim 1/\sqrt{N}$ (1). In the limit $\alpha \to \infty$, it reduces to Eq. (9) as it should.

The last condition ($m_q \gg g$) which has to be satisfied for the 't Hooft solution to be valid requires some additional explanation. Roughly speaking, nonplanar diagrams may contain a factor $\sim m_q^{-1}$ which at $m_q=0$ blows up and the theory changes completely. The concept of the proof that there exists a factor $\sim m_q^{-1}$ in nonplanar diagrams is the following.

Let us consider the correlation function for $p\to 0$

$$i\int d^2x\ e^{ipx}\langle 0|T\{\overline{q}q(x), \overline{q}q(0)\}|0\rangle = P(p^2). \tag{12}$$

The 't Hooft solution suggests that only planar graphs are taken into account and, consequently, the spectral density contains only contributions from one meson states for which $P_{\text{planar}} \sim N$. At the same time in the chiral limit, we can calculate the two-pion contribution exactly. This contribution is *not accounted for* in deriving (2). Of course, the two-pion contribution is suppressed by a factor $1/N$. However, it contains a term $\sim m_0^2/m_\pi^2$ which tends to infinity for $m_q \to 0$. The presence of the factor $\sim m_q^{-1}$ in nonplanar diagrams leads to the aforementioned constraint on m_q.

Now let us explicitly demonstrate the existence of the term $\sim m_q^{-1}$ for the two-pion contribution. In order to do so, we write down a dispersion relation for P,

$$P(0) = \frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{ds}{s}\text{Im}P(s), \tag{13}$$

where $\text{Im}P(s)$ is the physical spectral density. The $\pi\pi$ contribution is fixed uniquely by (9) because of the special role of pions [12]

$$\langle \pi\pi|\overline{q}q|0\rangle|_{p\to 0} = \frac{m_0\pi}{\sqrt{3}},$$

$$\frac{1}{\pi}\ \text{Im}P^{\pi\pi}(s) = \frac{m_0^2\pi^2}{6}\frac{1}{\sqrt{s(s-4m_\pi^2)}}, \tag{14}$$

$$P^{\pi\pi}(0) = \frac{m_0^2\pi^2}{6}\int_{4m_\pi^2}^\infty \frac{ds}{s\sqrt{s(s-4m_\pi^2)}} = \frac{m_0^2\pi^2}{12 m_\pi^2} \sim \frac{1}{m_q}.$$

It is clear that the only cause for a singular $\sim 1/m_q$ behavior is the finiteness of the pion matrix elements at zero momentum. At the same time this contribution does not contain the large factor N which accompanies a one meson contribution to the same correlator. To suppress these nonplanar diagrams we require $N \gg m_0^2/m_\pi^2$. Thus we expect that some kind of phase transition may occur in the region $m_q \sim g$, which would cause a complete restructuring of the theory.

The last subject we would like to discuss in this section is the strict Coleman theorem [17] which states that a continuous symmetry cannot be broken spontaneously in a two dimensional theory. As we discussed earlier [12] we expect that as in the SU($N\to\infty$) Thirring model (where the chiral symmetry is ''almost'' spontaneously broken [18]), the Berezinski-Kosterlitz-Thouless (BKT) effect [19] operates in regime (1). This fact also confirms the 't Hooft spectrum: states with opposite P parity are not degenerate in mass and there is an ''almost'' Goldstone boson with $m_\pi^2 \sim m_q + 1/N$.

To be more specific, one can show [12] that in two-dimensional QCD ($N\to\infty$) the behavior of the proper two-point correlation function is as follows:

$$\langle 0|T\{\overline{q_L}q_R(x), \overline{q_R}q_L(0)\}|0\rangle \sim x^{-1/N}. \tag{15}$$

Such a behavior together with cluster property as $x\to\infty$ implies the existence of the condensate at $N=\infty$ in a full agreement with our previous discussion. At the same time, for any finite but large N, the correlator falls off very slowly demonstrating the BKT behavior with no signs of contradiction to the Coleman theorem.

Having these general remarks on two-dimensional QCD (N) in mind, we turn to our main subject.

III. LARGE ORDER BEHAVIOR IN TWO-DIMENSIONAL QCD ($N=\infty$)

A. 't Hooft model

Let us consider the asymptotic limit $Q^2=-q^2\to\infty$ of the two-point correlation function [2,12]:

$$i\int dx\; e^{iqx}\langle 0|T\{\overline{qi}\gamma_5 q(x),\overline{qi}\gamma_5 q(0)\}|0\rangle = P(Q^2). \qquad (16)$$

It is clear that the large Q^2 behavior of $P(Q^2)$ is governed by the free, massless theory, where

$$P(Q^2\to\infty) = -\frac{N_c}{2\pi}\ln Q^2. \qquad (17)$$

At the same time the dispersion relations state that

$$P(Q^2) = \frac{N_c m_0^2 \pi^2}{\pi}\sum_{n=0,2,4,\ldots}\frac{f_n^2}{Q^2+m_n^2} \qquad (18)$$

and the sum is over states with even n because we are considering the pseudoscalar currents. Here residues f_n are defined as follows:

$$\langle 0|\overline{qi}\gamma_5 q|n\rangle = \sqrt{\frac{N_c m_0^2\pi^2}{\pi}}f_n, \quad n=0,2,4,\ldots . \qquad (19)$$

Bearing in mind that for large n, $f_n^2\to 1$ and $m_n^2\to m_0^2\pi^2 n$, we recover the asymptotic result (17). We can reverse arguments by saying that in order to reproduce $\ln Q^2$ dependence in the dispersion relation (17), the residues f_n^2 must approach the constant $\sim(m_{n+1}^2 - m_n^2)$ for large n.

Now consider a Q^{-2k} expansion for the correlator (18) in order to find the coefficients c_k of this series at large k

$$P(Q^2)\sim\sum_k c_{2k}g^{2k}\sim\sum_k c_{2k}\left(\frac{g^2N_c}{Q^2}\right)^k. \qquad (20)$$

As we mentioned earlier, in two dimensions the perturbative expansion $c_k(g^2N)^k$ and the $1/(Q^2)^k$ expansion coincide.

Now, if we knew f_n and m_n for arbitrary n we could calculate the sum (18) precisely, and thus, we would find the coefficients c_k from (20). Unfortunately, we do not know them. However, the key observation is as follows: in spite of the fact that we do not know an analytical expression for f_n and m_n for arbitrary n we still can calculate the leading behavior of c_k. The reason for that is related to the fact that the only asymptotics of residues $f_n=1$, $n\to\infty$ and masses $m_n^2=m_0^2\pi^2 n$, $n\to\infty$ are essential; the corrections to $f_n=1+O(1/n)$, $m_n^2=m_0^2\pi^2 n+O(1)$, $n\to\infty$ might change the preasymptotic behavior of c_k, $k\to\infty$, but cannot change the factorial behavior $(k)!$, found below. Using the asymptotic expressions for f_n and m_n we find that $P(Q^2)$ is expressed in terms of transcendental function $\Psi(z)=\Gamma'(z)/\Gamma(z)$ where $z=Q^2/m_0^2\pi^2$. However we can trust only in the leading terms of the corresponding formula

$$P(Q^2)-P(0) = \frac{N_c}{2\pi}\sum\left(\frac{1}{n+z}-\frac{1}{n}\right)$$
$$= -\frac{N_c}{2\pi}\left[\ln z+\gamma_E+\frac{1}{2z}-\sum_{k=1}^{\infty}\frac{B_{2k}}{2k}\frac{1}{z^{2k}}\right],$$
$$z=\frac{Q^2}{m_0^2\pi^2}, \quad B_{2k}=\frac{(2k)!2(-1)^{k-1}}{(2\pi)^{2k}}, \qquad (21)$$

where B_{2k} is asymptotic expression for the Bernoulli numbers.

A few comments are in order. First of all, we have explicitly demonstrated that the coefficients c_{2k} in the operator expansion $\Sigma c_{2k}/Q^{2k}$ are factorial divergent in high orders, $c_{2k}\sim(2k)!$, so the expansion is asymptotic in full agreement with the general arguments of Ref. [21].

Additionally we note that the only even powers g^2/Q^2 are essential in the expansion (generally speaking, arbitrary powers of g^2/Q^2 could contribute). Nonleading terms in f_n^2 and m_n^2 might contribute to the odd powers g^2/Q^2.

From the physical point of view this factorial behavior is related to highly excited states with excitation number $2n_0\sim Q^2/m_0^2$, and not with multiple production as one might naively expect. Indeed let us consider $Q^2/m_0^2\sim z\leqslant 2n_0$ in the expansion $\Sigma(1/2n+z)\sim\Sigma(2n/z)^k$. It is clear that the main contribution comes from $k\simeq 2n_0$ and $(2n_0)^{2n_0}\sim(2n_0)!$ exactly corresponds with the behavior found above. It is in agreement with phenomenological analysis [11], where it was assumed that the production of a highly excited resonance might be responsible for the large order behavior.

From the theoretical point of view we would expect that this behavior is related, somehow, to purely imaginary instantons [in order to provide the correct $(-1)^k$ behavior]. Any additional arguments in favor of this point will be discussed latter.

We would also like to point out that the numerical coefficient, which enters to the formula (21) is follows:

$$c_{2k}g^{2k}\sim\left(\frac{g^2N_c}{Q^2}\right)^{2k}. \qquad (22)$$

At the same time, the perturbative $2k$-loop graph gives a different contribution $\sim(g^2N_c/\pi Q^2)^{2k}$ with an extra factor $1/\pi$ per each coupling constant. This extra factor π must be taken very seriously as it is a large parameter. We definitely know (from exact solution), that the real scale of the problem is $m_0^2\pi^2$, and not $m_0^2=g^2N_c/\pi$ as one would naively expect from the perturbative theory. This means that vacuum condensates which are determined by nonperturbative physics come into the game. Even more, their contribution is much more important than pure perturbative diagrams. Let us note that the lowest vacuum condensates, found exactly in [12], exhibit this additional factor π. Thus the factorial growth is related, somehow, to the nonperturbative physics.

To further investigate the nonperturbative nature of the asymptotic series (in order to support the previous arguments) let us, instead of correlation function (16), consider the following difference of correlators:

$$i\int dx\; e^{iqx}\{\langle 0|T\{\overline{qi}\gamma_5 q(x),\overline{qi}\gamma_5 q(0)\}|0\rangle$$
$$-\langle 0|T\{\overline{q}q(x),\overline{q}q(0)\}|0\rangle\} = \Delta P(Q^2). \qquad (23)$$

One can argue that in the chiral limit $m_q\to 0$ the perturbative contribution to (23) is zero. At the same time dispersion relations lead to the same result: the coefficients of the OPE are factorially divergent. This growth is related not to some perturbative diagrams, but to nonperturbative physics. We will present more arguments for this point of view in the next section.

Finally we have explicitly demonstrated that the OPE is an asymptotic series. However, we cannot answer the important question of what kind of operators are responsible for such behavior. The reason for that is simple—too many operators contribute to the correlation function (16) and the corresponding classification problem is quite involved. In the following we will consider a special heavy-light quark system, where such an identification can be made. We find that some vacuum condensates exhibit a factorial growth. Exactly this fact is the source of such an asymptotic behavior.

B. QCD coupled to adjoint fermions. Instantons

Now we repeat the preceding analysis for the much more interesting model of QCD with adjoint Majorana fermions [20,8,9]:

$$S_{\rm adj}=\int d^2x\ {\rm Tr}\left[-\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi+m\bar{\Psi}\Psi\right]. \tag{24}$$

As is known, the most important difference with the 't Hooft model is that the bound states may contain, in general, *any* number of quanta. In other words, pair creation is not suppressed even in the large N limit. The problem becomes more complicated, but much more interesting, because the pair creation imitates some physical gluon effects.

We consider the following correlator analogous to (16):

$$i\int dx\ e^{iqx}\langle 0|T\left\{\frac{1}{N_c}{\rm Tr}\bar{\Psi}\Psi(x),\frac{1}{N_c}{\rm Tr}\bar{\Psi}\Psi(0)\right\}|0\rangle =P_2(Q^2), \tag{25}$$

where $\bar{\Psi}=\Psi^T\gamma_0$ and the label P_2 shows the number of partons in the external source $\bar{\Psi}\Psi(x)$; the factor $1/N_c$ is included in the definition of the external current in order to make the right hand side of the equation independent on N. In the large Q^2 limit the leading contribution to correlation functions is given as before by

$$P_2(Q^2\to\infty)=-\frac{2}{2\pi}\ln Q^2. \tag{26}$$

The additional factor 2 comes from two options in calculation of Tr and related to Z_2 symmetry mentioned in [9].

Now the problem arises. In 't Hooft model we definitely know that only two-particle bound states contribute to the corresponding correlation function. However, this is not true for the model under consideration and any states may contribute to P_2. The *key* observation is as follows: any pair creation (quantum loops which describe the virtual effects) is suppressed by a factor g^2N_c/Q^2 because of dimensionality of the coupling constant in two-dimensions (in a big contrast with real four-dimensional QCD).[2] Besides that, the quark mass term produces the analogous small factor m_q^2/Q^2 and can be neglected as well. Thus information about highly excited states (which provides the $\ln Q^2$ dependence) can be obtained exclusively from the analysis of the correlation function at large Q^2. In this case the analysis is very similar to 't Hooft case, considered in the previous section:

$$\langle 0|\frac{1}{N_c}\bar{\Psi}\Psi|n_1\rangle=\sqrt{\frac{(m_0^2\pi^2)}{\pi}}f_{n_1},\quad n_1\gg 1,\quad n_1\in 2Z,$$

$$m_{n_1}^2=m_0^2\pi^2n_1,\quad f_{n_1}^2=1,\quad m_0^2=\frac{2g^2N_c}{\pi}. \tag{27}$$

The only difference is the doubling of the strength of the interaction $g^2\to 2g^2$, [8] and the additional degeneracy Z_2, mentioned above. Now the formula for P_2 (26) can be easily recovered:

$$P_2(Q^2\to\infty)-P_2(0)=\frac{2}{\pi}\sum_{n_1=0,2,4\ldots}\frac{(m_0^2\pi^2)f_{n_1}^2}{m_{n_1}^2+Q^2} \to -\frac{2}{2\pi}\ln(Q^2). \tag{28}$$

As before, any correction to the asymptotic expression (7), such as $f_{n_1}^2=1+O(1/n_1)$, $m_{n_1}^2=m_0^2\pi^2n_1+O(m_q)$ will produce power corrections $\sim 1/Q^2$ and they are not interesting at the moment.

Apparently, the formulas (26)–(28) are very similar to Eqs. (17)–(19) which correspond to the 't Hooft model. However, there is a big difference in interpretation of these two cases: in the 't Hooft model we have exclusively two-parton states (two bits, in terminology of Refs. [8,20]). They saturate the dispersion relations.

In the case (26)–(28) we have much more states with arbitrary number of partons. As we explained in [22] the mixing between the different numbers of partons is not suppressed because of $\langle\bar{\Psi}\Psi\rangle$ condensation. Effectively, however, all these complex states contribute to the correlation function (28) in the same way as in the 't Hooft model. In this case the integer number n_1 from (27) should be interpreted as an excitation number of two bits in those states. The matrix element $f_{n_1}^2\simeq 1$ can be interpreted as a total probability to find two bits among the complete set of the mixed states. The total number of states is increasing the mass increases. Thus the probability to find two bits in the given state is decreasing correspondingly. However, the dispersion

[2]Naively, one could interpret such a result that the mixing between different number of partons is highly suppressed. Such a conclusion would be in contradiction with numerical results [20]. However, as we argued in recent papers [22], the puzzle can be resolved by introducing a nonzero value for vacuum condensate $\langle\bar{\Psi}\Psi\rangle$. Such a condensation does not break any continuous symmetries. Thus no Goldstone boson appears as a consequence of the condensation.

relations (28) tell us that the total probability $f_{n_1}^2$ with the given excitation number n_1 remains the same.

We can repeat the previous analysis, which led us to the formula (21) with small changes. Instead of factor B_{2k} in the expression (21) we will find

$$B_{2k} \Rightarrow 2^{2k} B_{2k}$$

for the theory with adjoint matter. This difference was mentioned above and is related to the doubling of the strength of the interaction $g^2 \to 2g^2$.

How could one interpret this result? First of all, let us recall that factorial behavior may occur for three different reasons: ultraviolet renormalons, infrared renormalons, and instantons. Clearly, the first two reasons cannot cause for such behavior in a field theory with the dimensional coupling constant g. Thus we expect that some kind of classical solution should be responsible for such behavior. If we accept the instanton hypothesis, then from the very general arguments, one would expect that the instanton contribution with action S to the large k-order coefficients c_k is given by [23]

$$c_k (g^2)^k \sim (k)! S^{-k} (g^2)^k. \tag{29}$$

In this case the factor 2, mentioned above, has the following interpretation: when we go from the QCD with fundamental matter to the theory with adjoint matter the instanton action is decreased by a factor of 2. In this case the factor S^{-k} from the formula (29) is exactly equal 2^k. It can be interpreted as the decreasing of instanton action by factor 2. Why is the instanton with the action one-half allowed in the theory with adjoint matter and forbidden in the theory with fundamental fermions? This question has yet to be answered.

Let us conclude this section by noting that from arguments given above we expect that some classical, pure imaginary solution (we call it instanton), is responsible for the factorial behavior found above.

IV. HEAVY-LIGHT QUARK SYSTEM IN TWO-DIMENSIONAL QCD

A. General remarks

As we mentioned in previous sections we are not able to identify the $n!$ behavior in the OPE (found from the spectrum) with some specific vacuum condensates. Such an identification can be done if one considers the heavy-light quark system. In this case the problem is reduced to the analysis of vacuum expectation value of the Wilson line $\langle W \rangle = \langle \overline{q(x)} P \exp(ig \int_0^x A_\mu dx_\mu) q(0) \rangle$. Indeed, if we consider as in [24,25] the correlation function $\langle T\{\overline{q}Q(x), \overline{Q}q(0)\}\rangle$, describing this system, we end up (in the limit $M_Q \to \infty$) with the object which is completely factorized (in accordance with HQET; see, e.g., review [26]) from the heavy quark:

$$\langle T\{\overline{q}Q(x), \overline{Q}q(0)\}\rangle \sim \left\langle \overline{q(x)} P \exp\left(ig \int_0^x A_\mu dx_\mu \right) q(0) \right\rangle + \text{perturb. part.} \tag{30}$$

By definition, $\langle W \rangle$ in this formula is understood as the Taylor expansion:

$$\langle W \rangle = \langle 0 | \overline{q(x)} P e^{ig \int_0^x A_\mu dx_\mu} q(0) | 0 \rangle = \sum_{n=0}^{n=\infty} \frac{1}{(2n)!} \langle \overline{q(x_\mu D_\mu)^{2n}} q \rangle. \tag{31}$$

All nontrivial, large distance physics of the system is hidden there. Together with perturbative contributions one should expect the following behavior for this correlator [25]:

$$\langle T\{\overline{q}Q(x), \overline{Q}q(0)\}\rangle \sim e^{-\Lambda \cdot x}. \tag{32}$$

The perturbative terms, proportional to $\sim (g^2 N/\pi)^n (x^2)^n$, contribute to (32) as nonperturbative ones due to the dimensional coupling constant g in two dimensions; thus they interfere with expansion (31). As was suggested in the same context by Shifman [21], one can get rid of the perturbative terms by considering a special combination of scalar and pseudoscalar correlation functions [analogous to (23) with the replacement of a light quark for a heavy quark]. The perturbative contribution vanishes in the chiral limit for the combination and we can study the pure nonperturbative physics.

Let us note that the general connection (based on dispersion relations) between spectrum and vacuum condensates was considered earlier [21]. It was proven that the OPE is asymptotic series. Besides that, for demonstration purposes, it was suggested a specific model for the spectral density (linear trajectory) and it was found that the vacuum condensates (31) get the form $\langle \overline{q(x_\mu D_\mu)^{2n}} q \rangle \sim (2n)!$.

In this section we essentially follow the steps from the paper [21] with the only difference being that we start from theory defined as two-dimensional QCD and derive the $Q\overline{q}$ spectrum from this Lagrangian. We find that we have not a linear spectrum but rather $E_n \sim \sqrt{n}$ in two-dimensional QCD for a heavy-light quark system. It gives a different behavior for the vacuum condensates $\langle \overline{q(x_\mu D_\mu)^{2n}} q \rangle \sim n!$. However, the main statement that the OPE is an asymptotic series remains the same.

Before going on, we would like to make the following remark. In two-dimensional QCD one can calculate the appropriate vacuum condensates in the chiral limit from first principles [28]. Such a calculation (which will be reviewed in the next section) leads to puzzling results. Roughly speaking, the results of direct computation do not agree with indirect calculations based on the dispersion relations and spectrum. We formulate this puzzle as well as its resolution in the next section. Anticipating the event, we would like to note here that the origin of the puzzle is the factorial divergent coefficients in the asymptotic series. If these expansions were convergent series, we would expect an exact coincidence of the results, based on these two methods. However, the analysis in field theory demonstrates that this is not the case.

B. Spectrum $\overline{Q}q$ system in two-dimensional QCD ($N = \infty$)

As we have discussed in previous sections, if we knew the spectrum of highly excited states we would calculate (via dispersion relations) the large order behavior for the corresponding correlation function. As we mentioned above, the heavy-light quark system is very special in this sense, because it allows us to identify the corresponding factorial behavior with specific vacuum condensates. This is the main motivation for the present section: find the spectrum for highly excited states. Let us note that the heavy-light quark system in this model was considered previously in Ref. [29], but in a quite different context.

As we discussed in Sec. II, the spectrum of highly excited states in two-dimensional QCD is linear (4). This is certainly true, but only for the finite parameter m_Q, *with* $n\to\infty$. We are now interested in a different limit, when $m_Q\to\infty$ first, and $n\gg1$ afterwards. These limits do not commute and we have to start our analysis from exact original Eq. (2).

In order to perform the limit $m_Q\to\infty$ in 't Hooft Eq. (2), let us make the following change of variables:

$$x=1-\frac{m_0}{m_Q}\alpha,\quad 0\leqslant\alpha\leqslant\frac{m_Q}{m_0}=\infty. \tag{33}$$

Additionally we would like to rescale the wave function and redefine the energy scale (from now on, the counting of the energy starts from m_Q) in the following way:

$$m_n^2=(m_Q+E_n)^2\simeq m_Q^2+2m_QE_n,$$
$$\phi_n(x)\to\sqrt{\frac{m_Q}{m_0}}\;\widetilde{\phi_n(\alpha)}\;. \tag{34}$$

Once all these changes have been made we arrive to the following equation, which replaces the 't Hooft equation, for the heavy-light quark system in the limit $m_Q\to\infty$:

$$2E_n\widetilde{\phi_n(\alpha)}=m_0\left[\alpha+\frac{1}{\alpha}\;\frac{m_q^2-m_0^2}{m_0^2}\right]\times\widetilde{\phi n(\alpha)}-m_0^2P\int_{0\infty}d\beta\frac{\widetilde{\phi_n(\beta)}}{(\alpha-\beta)^2}\;. \tag{35}$$

The new set of wave functions in terms of new variables is orthogonal and complete:

$$\Sigma_n\widetilde{\phi_n(\alpha)}\widetilde{\phi_n(\beta)}=\delta(\alpha-\beta),$$
$$\textstyle\int_0^\infty d\alpha\widetilde{\phi_n(\alpha)}\widetilde{\phi_m(\alpha)}=\delta_{nm}\,. \tag{36}$$

For the future analysis we need not only the wave functions, but also some physical matrix elements in terms of these wave functions. It is convenient to separate the common factor, related to m_Q, and define the matrix elements in the following way[3]:

$$\langle0|\overline{qi}\gamma_5Q|n\rangle=\sqrt{\frac{N}{\pi}}\sqrt{m_0m_Q}f_n,\quad f_n=\int_0^\infty d\alpha\;\widetilde{\phi_n(\alpha)}\;. \tag{37}$$

[3]We keep the same notation f_n for the corresponding matrix elements. For light quark system they are defined in a different way (8). We hope it will not confuse the reader.

Using the parity relation [2], which in our notations takes the form,

$$\int_0^\infty d\alpha\widetilde{\phi_n(\alpha)}=\frac{m_q}{m_0}\int_0^\infty d\alpha\frac{\widetilde{\phi_n(\alpha)}}{\alpha}, \tag{38}$$

one can show that the scalar matrix elements have the same expression in terms of wave functions as the pseudoscalar ones (37).

Having these results in mind, and using the standard techniques [1–6], one can calculate [32] the matrix elements f_n and energies E_n in the quasiclassical approximation for $n\gg1$:

$$E_n=2m_0\sqrt{\pi n}\left[1+O\left(\frac{\ln n}{n}\right)\right],\quad f_n^2=\sqrt{\frac{\pi}{n}}\left[1+O\left(\frac{\ln n}{n}\right)\right]. \tag{39}$$

This is the main result of this section. It will be used in what follows for the calculation of large order behavior and high dimensional condensates.

Let us conclude this section with a few remarks. First of all, as was expected, in the $m_Q\to\infty$ limit, Eqs. (35)–(39) do not depend on m_Q (after an appropriate rescaling) in accordance with HQET (see, e.g., review [26]).

Our second remark is the observation that the chiral limit $m_q=0$ is very peculiar. In particular, one cannot take the limit $m_q=0$ in the identity (38), because it clearly leads to a nonsensical result. The reason for that is very simple. The edge region $\alpha\sim m_q$ plays a very important role in making the theory (and this identity in particular) self-consistent. As a consequence, one cannot derive the boundary conditions on wave function from the truncated equation (35) where the limit $m_Q\to\infty$ already has been taken. In order to do so we need to come back to the original 't Hooft equation (2). We shall return to this point in the Conclusion. We believe that the situation in four-dimensional QCD is quite similar in that the information about edge behavior in the theory cannot be found from the truncated Lagrangian with the limit $m_Q\to\infty$ already taken.

C. High order condensates from duality and dispersion relations

The starting point, as usual, is the correlation function

$$P(Q^2)=i\int e^{iqx}dx\langle0|T\{\overline{qQ}(x),\overline{Q}q(0)\}|0\rangle$$
$$\sim i\int e^{iqx}dx\langle0|\overline{q(x)}Pe^{ig\int_0^xA_\mu dx_\mu}q(0)|0\rangle$$
$$+\text{perturb. part.} \tag{40}$$

We follow [24] and choose the external momentum $q_0=m_Q-E$, $q_1=0$ very close to threshold. For positive values of E the correlation function can be written in the following way:

$$P(E)=\int_0^\infty e^{-Et}dt\langle0|\overline{q(t)}Pe^{ig\int_0^tA_0dt}q(0)|0\rangle+\text{perturb. part.} \tag{41}$$

Let us note that we could consider the difference of two correlation functions [like (23)]. In that case the perturbative contribution to (41) would vanish.

For large enough $E \gg m_0$ one can expand Eq. (41) in $1/E$ [21]

$$P(E)=\frac{1}{E}\left[\langle\bar{q}q\rangle-\frac{1}{E^2}\langle\bar{q}P_0^2q\rangle+\frac{1}{E^4}\langle\bar{q}P_0^4q\rangle-\cdots\right]+\text{perturb. part}, \tag{42}$$

where $P_0=iD_0$ is the time component of the momentum operator.

Our goal now is to substitute the asymptotic expression (39) for the matrix elements f_n and energies E_n into the dispersion relations [analogous to (18)]. These will determine the higher order corrections to the correlation function as well as the large n behavior of the vacuum condensates $\langle\bar{q}D^{2n}q\rangle$.

The appropriate dispersion relation states that

$$P(E)=\frac{N}{2\pi}m_0\sqrt{\pi}\sum_n\frac{f_n^2}{E+E_n}\sim\frac{N}{\pi}\sum\frac{1}{\sqrt{n}(\sqrt{n}+\epsilon)}, \tag{43}$$

where ϵ is external energy measured in $m_0\sqrt{\pi}$ units. In this formula we have taken into account the following key observation which we have used earlier in the deriviation of Eq. (21): the corrections $\sim 1/n$ to the asymptotic behavior of the residues f_n and energies E_n (39) might change the preasymptotic factor for the large order behavior. However, these corrections cannot change the main result—the factorial growth of the coefficients found below. This is the reason why we cannot calculate the corresponding coefficients *exactly*, but only the leading factorial factor. For the same reason we do not consider the special difference of two correlation functions [like (23)], where perturbative contribution is exactly canceled. In this case if we knew f_n, E_n exactly, we would calculate the nonperturbative condensates *exactly*. Unfortunately, this is not the case. Thus we ignore all complications related to the separation of pure nonperturbative contribution from the perturbative one.

Let us note that the sum in Eq. (43) is divergent at large n. This divergence is related to the necessity of a subtraction in the dispersion integral: $P(E)\to P(E)-P(0)$. Besides that, at large energy $E\gg m_0$ one can estimate the behavior $P(E\to\infty)\sim\ln E$, which corresponds to the pure perturbative one loop diagram. These same features were present in our analogous previous formula (21).

With these remarks in mind, our problem is reduced to the calculation of the coefficients c_k at large k in the following expansion:

$$P(\epsilon)-\text{subtractions}\sim\sum_n\frac{1}{\sqrt{n}+\epsilon}-\text{subtractions}\sim\ln\epsilon+\sum_k c_k\frac{1}{\epsilon^k}. \tag{44}$$

We note that the only difference with the previous formula (21) is the dependence of the sum on $\sqrt{n}$ rather than n itself.

The way to evaluate the coefficients c_k is as follows. We are going to use the standard idea (see, e.g., [30]) to present the sum in terms of the integral:

$$\sum_{n=1}^{n=\infty}F(n)=\int_0^\infty\frac{f(x)}{e^x-1}dx,\quad F(x)=\int_0^\infty f(x)e^{-xt}dt. \tag{45}$$

However, in our case we have $\sqrt{n}$ rather than n itself. The corresponding generalization is known as well:

$$\sum_{n=1}^{n=\infty}F[g(n)]=\int_0^\infty\int_0^\infty\frac{h(x,y)f(y)}{e^x-1}dx\,dy,$$
$$e^{-yg(p)}=\int_0^\infty h(x,y)e^{-xp}dx. \tag{46}$$

For our particular case this formula gives the following representation for the sum (44):

$$P(\epsilon)\sim\sum_n\frac{1}{\sqrt{n+c}+\epsilon}=\frac{1}{2\sqrt{\pi}}\int_0^\infty dx\int_0^\infty dy\frac{ye^{-(y^2/4x)-cx-\epsilon y}}{x^{3/2}(e^x-1)}, \tag{47}$$

where we have introduced a constant c for the future convenience as an auxiliary parameter. Such a parameter is actually present in original formulas (39); however, we believe that the main factorial dependence does not depend on it. Thus, after differentiating with respect to c, we put $c=0$ at the very end of calculation. One more remark regarding formula (47). Only even n should be taken into account in this formula. However, by redefinition of parameters c and energy ϵ, the problem can be reduced to the same integral. The result is an extra power dependence, which is beyond our scope of interest. Additionally, a subtraction which should be made in this formula to get a convergent result, has no influence on c_k at large k (44).

The integration over x in the formula (47) can be executed using the following expansion:

$$\frac{x}{e^x-1}=\sum_{k=0}^\infty B_k\frac{x^k}{k!}, \tag{48}$$

where B_k are Bernoulli numbers. Bearing in mind that the appropriate integral from (47) is known exactly,

$$\int_0^\infty\frac{e^{-(y^2/4x)-cx}}{\sqrt{x}}dx=\sqrt{\frac{\pi}{c}}e^{-y\sqrt{c}}, \tag{49}$$

and replacing x^k from the formula (48) by $(-1)^k(d/dc)^k$, we arrive at the following expression for the sum (47):

$$P(\epsilon)\sim\frac{1}{2}\int_0^\infty dy\;e^{-y\epsilon}y\sum_{k=2}^\infty\frac{B_k}{k!}(-1)^k\left(\frac{d}{dc}\right)^k\left(\sqrt{\frac{1}{c}}e^{-y\sqrt{c}}\right). \tag{50}$$

In this formula we ignore the few first terms (proportional to B_0, B_1) because they (a) do not contribute to large order coefficients c_k, $k\gg 1$; and (b) they are divergent and, thus, require some subtractions discussed previously.

The key observation is as follows: The nonperturbative part (which is our main interest) in the expansion (42) is determined by odd powers of $1/(\epsilon)^{2m+1}$. Such terms can be easily extracted from the formula (50) by expanding the exponent

$$\left(\sqrt{\frac{1}{c}}e^{-y\sqrt{c}}\right)=\sum_l \frac{(-1)^l y^l c^{(l-1)/2}}{l!} \quad (51)$$

and executing the integration over y:

$$\int_0^\infty dy\; e^{-\epsilon y} y \frac{y^l}{l!}=\frac{l+1}{\epsilon^{l+2}}. \quad (52)$$

It is now clear that only odd $l=2m-1$ terms in the formula (51) contribute to the coefficients related to the nonperturbative part (42). For small parameter c the coefficients c_k can be easily calculated[4] by noting that for $l=2m-1$ the appropriate terms from (51) take the form $c^{l-1/2} l! \sim c^{m-1}/(2m-1)!$. The nonzero result in this case [after differentiating $(d/dc)^k$ and taking the limit $c=0$] comes exclusively from the term $k=m-1$ in Eq. (50). Finally we arrive at the following asymptotic expression for the odd coefficients c_k from the series (44):

$$c_{2k-1}\frac{1}{\epsilon^{2k-1}}\sim\frac{(-1)^k B_k}{k\epsilon^{2k-1}}, \quad k\gg 1. \quad (53)$$

Comparison with the original series (42) suggests that as dimension of the operator grows, their vacuum expectation values grow factorially,

$$\langle \bar{q} P_0^{2n} q\rangle \sim \langle \bar{q}q\rangle (\pi m_0)^{2n} n! \quad (54)$$

From this formula one could naively think that half of the VEV's in (53) vanish because the odd Bernoulli numbers are zero. However, we think that this additional ''selection rule'' is accidental in its nature and, thus, it should not be considered seriously. We believe that this accidental vanishing of the coefficients c_k in the OPE is related to our approximation (39) for the matrix elements f_n and energies E_n. Thus we expect that half of the VEV's which formally vanish in the leading approximation are actually not zero, but suppressed by a factor $\sim 1/n$, where n is dimension of an operator.

Having demonstrated the main result of this section, a few comments are in order. First of all, the OPE for $P(\epsilon)$ is an asymptotic series as it must be in agreement with general arguments [21]. Besides that, we expect the same behavior for analogous vacuum condensates in four-dimensional QCD [27]. Additionally, the scale for the vacuum condensates is m_0. We shall see in the next section that different calculations of the same condensates give somewhat different results. We explain this puzzle later, but note for now that $n!$ behavior in (54) plays a crucial role in the explanation.

Finally, as we shall see, the $n!$ behavior has very general, essentially kinematical origin in the large N_c limit. This property is related to the so-called master field.

V. HIGH-DIMENSIONAL CONDENSATES FROM THE THEORY. PUZZLE

One can show [28] that the vacuum condensates which enter into the formula (42) at $N=\infty$ in the chiral limit $m_q\to 0$ in two dimensions can be reduced to a form which contains the field strength tensor $ig\epsilon_{\mu\nu}G_{\mu\nu}$ only. Indeed, the covariant derivatives $\langle \overline{qD}^n D_\mu D_\mu q\rangle$ placed at the very right and at the very left (near the quark fields) can be transformed into the operator $ig\epsilon_{\mu\nu}G_{\mu\nu}$ using the equation of motion $D_\mu\gamma_\mu q=0$. To do the same thing with operators D_μ which is placed somewhere in the middle of the expression, we need to act, for example, on the right until the quark field is reached. By doing so, step by step, we create many additional terms which are either commutator like $[D_\mu, D_\nu]=-igG_{\mu\nu}$ which is the field strength operator or commutators like $\sim[D_\lambda, \epsilon_{\mu\nu}G_{\mu\nu}]$. Fortunately, in two dimensions these terms are related to creation of the quark-antiquark fields and we discard them in the chiral limit because they do not give $1/m_q$ enhancement; see formula (56).[5] We discuss the exact correspondence in the Appendix, but for now we emphasize the existence of factor $n!$ in the corresponding formula:

$$\langle \overline{q(}x_\mu D_\mu)^{2n} q\rangle \sim (x^2)^n n! \langle \overline{q(}gE)^n q\rangle,$$
$$n\gg 1, \quad E\sim \epsilon_{\lambda\sigma}G_{\lambda\sigma}. \quad (55)$$

Thus we end up with the vacuum condensates $\langle \overline{q(}gE)^n q\rangle$ which are expressed exclusively in terms of the field strength tensor. Such vacuum condensates can be calculated *exactly* in the chiral limit [28]. The reason for this incredible simplification is the observation that in two-dimensional QCD a gluon is not a physical degree of freedom, but rather is a constrained auxiliary field which can and should be expressed in terms of the quark fields. At the same time, in the large N limit, the expectation value of a product of any invariant operators reduces to their factorized values [31]. Exactly this feature of the large N limit (based on analysis of the so-called master field) makes it possible to calculate the vacuum condensates exactly. Our final expression takes the form [28]

$$\frac{1}{2^n}\langle \overline{q(}ig\epsilon_{\lambda\sigma}G_{\lambda\sigma}\gamma_5)^n q\rangle=\left(-\frac{g^2\langle\bar{q}q\rangle}{2m_q}\right)^n\langle\bar{q}q\rangle. \quad (56)$$

We interpret this expression as follows: Each insertion of an additional factor proportional to field strength tensor gE gives one and the same numerical factor (56). This situation can be interpreted as having a *classical* master field [31]

[4]As we mentioned before the factorial behavior does not depend on the particular magnitude c. However, for $c=0$ the calculations are much easier to present. Nonzero values of the parameter c might change the preasymptotic behavior, which is beyond this method.

[5]Of course this is not the case in four dimensions where $[D_\mu^2, G_{\lambda\sigma}]$ is an independent operator which cannot be reduced to quark fields.

which we insert in place of gE in the vacuum condensates. Because of its classical nature, it gives one and the same numerical factor.

Secondly, it is important to note that the vacuum condensate of an arbitrary local operator can be reduced through the equation of motion and constraints to the fundamental quark condensate (9).

Finally, we must comment on the effective energy scale which enters into the expression (56): This is not m_0^2 as one could naively think, but rather $m_{eff}^2 = m_0^3/m_q \gg m_0^2$. The obvious technical reason for that is related to the fact that in the light cone gauge (where the theory has been quantized) $A_- = 1/\sqrt{2}(A_0 - A_1) = 0$ we have few constraints: the usual constraint in the gauge sector (Gauss law)[6]

$$\partial_- E^{ab} \sim g\left(q_+^{\dagger b} q_+^a - \frac{1}{N}\delta^{ab} q_+^{\dagger c} q_+^c \right), \tag{57}$$

with right moving fermions q_+ considered as dynamical degrees of freedom. The left-moving fermions q_- are nondynamical degrees of freedom in this gauge. The latter can be eliminated by the following constraint:

$$\frac{1}{\partial_-} q_+ \sim \frac{1}{m_q} q_- \tag{58}$$

(for more details, see, e.g., [15]). This relation explicitly explains the origin of the factor $1/m_q$ which is present in the formula (56) (see [28] for details).

Before formulating the puzzle, we would like to pause here in order to explain the definition of the high-dimensional vacuum condensates. As is known, they are perturbatively strongly ultraviolet (UV) divergent objects.

As usual, all vacuum condensates should be understood in a sense that the perturbative part is subtracted. The subtraction is organized by introducing of the so-called normalization parameter μ. In general, vacuum condensates do depend on this parameter μ. The gluon condensates of dimensions four, six, eight, etc. in four-dimensional QCD are perfect examples where perturbative parts are divergent, but nonperturbative parts (the remnants, which are left after subtraction) are perfectly defined. One could naively think that the instantons might spoil this picture, because they give ultraviolet divergent contribution to high order condensates. However, we do not think this is the case and one can argue that the definition we have formulated remains untouched even with small size instanton effects taken into account.

The argument is as follows: We should treat the small size instantons and the small size perturbative contributions on the same basis. So, we should subtract both these divergences at the same time in order to get the so-called ''nonperturbative'' condensates which define the large distance physics. Of course small-distances physics does not disappear when we do such a subtraction. According to Wilson OPE the corresponding small distance contribution (perturbative part and small instanton contributions) should be taken into account separately.

In two dimensions this problem of course is much simpler. However, the perturbative contributions to the condensates are divergent as well. Nevertheless, the high-dimensional condensates perfectly exist. Indeed, we have a formula (56), with mixed condensates expressed in terms of chiral condensate $\langle\overline{q}q\rangle$, where the latter are defined as the remnant after subtraction of the perturbative contribution (for the technical details, see [16]). Thus our formula (56) is understood as a nonperturbative one, when we treat $\langle\overline{q}q\rangle$ as the nonperturbative chiral condensate. Let us note that the gluon condensate in this model is finite and can be calculated exactly [12] (see also the recent paper [16] on this subject). In two-dimensional QCD it can be expressed in terms of the chiral condensate.

With this remark in mind, we are now ready to formulate the *puzzle*. The scale which enters into the OPE (42) presumably is determined by the coefficients c_k from (44). The leading contribution to these coefficients can be expressed in terms of the vacuum condensates (56)

$$c_{2k+1} \sim \left(-\frac{g^2\langle\overline{q}q\rangle}{2m_q}\right)^k \langle\overline{q}q\rangle \sim (m_0^3/m_q)^k k!, \quad k \gg 1.$$

Thus the characteristic scale of the problem is m_0^3/m_q. We see an extra factor $1/m_q$ in this scale. It has very clear origin (58) and might be very big in the chiral limit. At the same time, the characteristic scale which can be found from the spectrum (21), (42) is much smaller and proportional to m_0^2. This is the puzzle.

The explanation of this apparent paradox is as follows. If our series (21), (42) were convergent ones, we would be in trouble, but fortunately, our series are asymptotic ones. Thus, in order to make sense of these series we have to define them and here we will use the standard Borell summing prescription [23].[7] Once this prescription has been accepted, we can write down an expression which reproduces the asymptotic coefficients for large number k and at the same time is well defined everywhere:

$$\sum_k c_k \gamma^k \sim \int \frac{d\gamma'}{\gamma'(\gamma+\gamma')} e^{-1/\gamma'},$$

$$c_k \sim (-1)^k k! \sim (-1)^k \int \frac{d\gamma'}{(\gamma')^{k+2}} e^{-1/\gamma'}. \tag{59}$$

We did not specify the parameter γ in this equation on purpose. Suppose we have a large scale in the problem determined by the vacuum condensates (56). In this case dimensionless parameter γ has a large factor $1/m_q$:

$$\gamma \sim \frac{(g^2 N_c)^3}{m_q E^2}. \tag{60}$$

[6]This formula explicitly demonstrates why a quark condensate appears in (56) each time we insert an extra gluon field E^{ab}.

[7]We already mentioned in the Introduction that the Borel summability or its loss is not the crucial issue [10]. However, for the sake of definiteness (and for simplification), we assume in general discussion which follows that the series is Borel summable.

The prescription (59) in this case states that the sum of leading terms $\sim(1/m_q)^k$ gives a zero contribution (in the chiral limit)

$$\sum_k c_k\gamma^k \sim \int \frac{d\gamma'}{\gamma'(\gamma+\gamma')} e^{-1/\gamma'} \to 0 \quad (\text{at } m_q\to 0),$$

$$\gamma\sim\frac{1}{m_q} \qquad (61)$$

to the physical correlation function. We would like to note that the effect (61) does not crucially depend on the factorization properties for the condensates (56) as neither on our assumption of exact factorial dependence of the coefficients $c_k=k!(-)^k$ (59). Both of these effects presumably lead (apart to $k!$) to some mild k dependence which can be easily implemented into the formula (61) by introducing some smooth function $f(\gamma)$ whose moments

$$c_k=\frac{(-)^k}{k!}\int f(\gamma)\gamma^{-k-2}\exp\left(-\frac{1}{\gamma}\right)d\gamma\sim 1$$

exactly reproduce a k dependence of the coefficients as well as of the condensates. If this function is mild enough, it will not destroy the relation (61), but might change some numerical coefficients.

If we could calculate the vacuum condensates (56) exactly (and not only the leading terms at $m_q\to 0$), we would find that the terms of order one give contribution of order one to the correlation function. Thus subleading terms play a much more important role in the final formula than the leading ones $\sim 1/m_q$. The origin of this mystery of course is the factorial growth coefficients in the series. This observation actually resolves the puzzle announced in the beginning of the section.

We would like to make a few more remarks regarding this subject in the Conclusion which follows.

VI. CONCLUSION

We conclude this manuscript with the following lessons.

(a) The OPE is an asymptotic series.

(b) We were lucky in a sense that in two-dimensional QCD the scales for high dimensional condensates $\sim m_q^{-1}$ and for the spectrum ~ 1 were parametrically different. This was the reason why we noticed the difference very easily. The lesson we can learn from this example is that numerically leading terms in the asymptotic expansion may give somewhat negligible contribution into the final expression. We believe that this is not a specific result for two-dimensional physics, but rather is a general property of a field theory associated with an asymptotic series.

(c) Of course, we do not expect that in the real four-dimensional QCD the vacuum condensates might exhibit some parameters similar to $\sim m_q^{-1}$. However, it might happen that some subseries of condensates possess a large numerical factor, let us call it $L\gg 1$. In this case we would expect that the corresponding contribution into the final formula will be suppressed, L^{-1}. At the same time, summing of a subseries of terms which are order of one, would lead to the result which is order of one. This observation raises the following question.

(d) Suppose we have a condensate $\langle O_k\rangle$ of dimension k which has both parts: the enhanced part, proportional to L^k and the ''regular'' one of order 1^k. As we have learned, the subseries which have big factors like L^k do not contribute much to the final expression. At the same time they are the main contributions into the condensate with the given dimension k.

The moral is when we use truncated expansions or approximate approaches (like QCD sum rules), we inevitably study not the actual condensates, but rather, some *effective condensates*.[8] One simple consequence of this is that the lattice calculation of vacuum condensate might be different from QCD sum rules analysis.

(e) One may wonder what is the role of the scale $1/m_q$ in this model? The answer is very peculiar. Indeed this scale can not be seen in spectrum however, from exact identities like (38) one can see that the edge region of order m_q plays a very important role in maintaining self-consistency of the theory. The calculation of the condensate [12], where the region $x\sim m_q$ gives whole answer is another example of the same kind. Exactly this infrared region determines the scale for the condensates (56), but not the scale for integral characteristics like the spectrum itself. See also the remark after (39) on the same subject.

(f) Since OPE is an asymptotic series, it is a good idea to keep only a few first terms in the expansion (like people do in the standard QCD sum rules approach, [33]) and to stop at some point.[9] Any hopes to improve the standard QCD sum rules (like the idea advocated in [34]) by summing up of the certain subset of the power corrections, and ignoring all the rest, might lead to the results which are much worse than the ones which follow from a preimproved version of the approach. At least in two-dimensional QCD such a procedure, as we learned, gives a parametrically incorrect result.

We would like to make two more remarks which were not our main subjects, but look like interesting byproducts worthy of mention.

(g) We argued that the $n!$ behavior found in two-dimensional QCD is related to *instantons*. Even more, the action of the instanton in the theory with adjoint matter is a factor of two less than in the theory with fundamental quarks. The explicit realization of such a solution is still lacking.

(h) We analyzed the $Q\overline{q}$ system in two-dimensional QCD. We found that the edge behavior of the system is very peculiar and cannot be found from the truncated theory, where limit $m_Q=\infty$ is already taken. We believe that the same behavior is an inevitable feature of the real four-dimensional QCD.

[8] I thank Michael Peskin for the discussion of this subject. Actually, he raised the question of the effective nature for the condensates before this work was presented at the SLAC theory seminar.

[9] In asymptotic series with coupling constant λ one should stop with the number of terms of order λ^{-1}. In particular, in case of QCD sum rules, where $\lambda\sim 1/3$–$1/5$ to be determined by the scale where power corections are 20–30 %, one can estimate the maximum number of terms in the expansion is about 3–5.

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APPENDIX

The main goal of this Appendix is a demonstration of the factor $n!$ in formula (A1) in the chiral limit $m_q=0$,

$$\langle \overline{q(}x_\mu D_\mu)^{2n}q\rangle \sim (x^2)^n n! \langle \overline{q(}gE)^n q\rangle, \quad n\geqslant 1, \quad E\sim \epsilon_{\lambda\sigma}G_{\lambda\sigma}. \tag{A1}$$

We sketch the idea only, ignoring for simplicity all normalization factors.

We use the standard representation for γ_μ matrices satisfying

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}, \quad \gamma_0=\sigma_x, \quad \gamma_1=-i\sigma_y, \quad \gamma_5=\sigma_3, \tag{A2}$$

$$\gamma_\pm=\gamma_0\pm\gamma_1, \quad D_\pm=D_0\pm D_1, \quad \gamma_-^2=0,$$

$$\gamma_+^2=0, \quad \gamma_\mp\gamma_\mp\sim 1\mp\gamma_5.$$

Dirac equations take the following form in the chiral limit:

$$(\gamma_+D_- + \gamma_-D_+)q=0. \tag{A3}$$

By multiplying these equations by $\gamma_\pm$, we arrive at the following relations:

$$(1\pm\gamma_5)D_\pm q=0, \quad \overline{q}\overline{D}_\mp(1\pm\gamma_5)=0. \tag{A4}$$

Let us present our original condensate in the following form:

$$\langle \overline{q(}x_\mu D_\mu)^{2n}q\rangle \sim \left\langle \overline{q}\left(\frac{1+\gamma_5}{2}+\frac{1-\gamma_5}{2}\right) \times (x_+D_- + x_-D_+)^{2n} q\right\rangle. \tag{A5}$$

From Lorentz invariance it is clear that a nonzero result $\sim(x_-x_+)^n$ comes from terms with equal numbers of D_- *and* D_+ with all possible permutations. Our problem is how to count them. First, we pick up the projector $(1+\gamma_5)$ from the expression (A5). According to (A4) we should move D_+ to the right and D_- to the left (to reach the quark field) using the following commutation relations:

$$[D_+,D_-]\sim\epsilon_{\mu\nu}G_{\mu\nu}\sim E. \tag{A6}$$

Here the field strength E can be considered as constant (not operator), because acting the operators $D_\pm$ on E leads to the pair creation. We neglect everywhere such terms in the limit $N\to\infty, m_q\to 0$. The contribution with projection $(1-\gamma_5)$ gives exactly the same result after relabeling $D_-\Leftrightarrow D_+$ and repeating the procedure described above.

Now one can see that these calculations are very similar (in the algebraic sense) to the oscillator problem with ladder operators satisfying the standard relations

$$[a,a^+]=1, \quad a|0\rangle=0, \quad \langle 0|a^+=0. \tag{A7}$$

Our problem is the calculation of mean value of the operator $x^{2n}\sim(a+a^+)^{2n}$ for the ground state:

$$\langle 0|x^{2n}|0\rangle \sim \langle 0|(a+a^+)^{2n}|0\rangle \sim \Gamma\left(n+\frac{1}{2}\right)\sim n!, \quad n\geqslant 1. \tag{A8}$$

This concludes our explanation of the factor $n!$ (55) which we heavily used in our previous discussions.

Let us repeat again that all factors from (55), (56) have clear meaning and can be explained without detailed calculations. Three steps are involved. (1) The transition from the operator $\langle \overline{q(}x_\mu D_\mu)^{2n}q\rangle$ to the operator $\langle \overline{q(}gE)^n q\rangle$ with a factor $n!$ was explained in this Appendix. (2) The idea of a master field predicts that each insertion of E into the expression for the operator $\langle \overline{q(}gE)^n q\rangle$ gives one and the same constant $M_{\rm eff}^n$. (3) The constraints (57), (58) explicitly demonstrate that $M_{\rm eff}^2\sim m_q^{-1}$.

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