

From quantum field theory to hydrodynamics: Transport coefficients and effective kinetic theory

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(Received 11 December 1995)

The evaluation of hydrodynamic transport coefficients in relativistic field theory, and the emergence of an effective kinetic theory description, is examined. Even in a weakly coupled scalar field theory, interesting subtleties arise at high temperatures where thermal renormalization effects are important. In this domain, a kinetic theory description in terms of the fundamental particles ceases to be valid, but one may derive an effective kinetic theory describing excitations with temperature dependent properties. While the shear viscosity depends on the elastic scattering of typical excitations whose kinetic energies are comparable to the temperature, the bulk viscosity is sensitive to particle nonconserving processes at small energies. As a result, the shear and the bulk viscosities have very different dependence on the interaction strength and temperature, with the bulk viscosity providing an especially sensitive test of the validity of an effective kinetic theory description. [S0556-2821(96)07110-X]

PACS number(s): 11.10.Wx, 05.60.+w, 47.75.+f, 51.20.+d

I. INTRODUCTION

In a weakly coupled quantum field theory, one would expect to be able to compute most physical observables starting from first principles. However, at sufficiently high temperatures, in even the simplest scalar field theory, the correct evaluation of transport coefficients characterizing long wavelength hydrodynamic behavior is quite subtle. Only recently has a thorough diagrammatic analysis of the bulk and shear viscosity appeared [1], which is valid at temperatures where thermal renormalization effects are important. The purpose of this paper is to discuss the physical interpretation of the results of [1], and to describe the formulation of an effective kinetic theory which properly incorporates thermal renormalization effects and which generates the correct weak coupling behavior of both the bulk and shear viscosities.

Existing literature in this area is somewhat sparse, particularly on aspects which are unique to relativistic quantum field theories. Consequently, we have tried to make the presentation reasonably self-contained, and briefly review necessary background material. For simplicity, nearly all discussion will be limited to the case of a real scalar theory with cubic and quartic self-interactions,

$$-\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4, \quad (1.1)$$

with $\lambda \ll 1$, m_0^2 positive, and $g^2 = O(\lambda m_0^2)$. Since the theory is weakly coupled, the physical (zero temperature) mass of the resulting scalar particles equals m_0 (after renormalization) up to radiative corrections, $m_{\text{phys}} = m_0[1 + O(\lambda)]$.

At nonzero temperature, the equilibrium state of this theory may be regarded as a fluid (or gas) of interacting spinless bosons. For fixed values of the coupling constants, the pressure, energy density, and other thermodynamic ob-

servables depend only on the temperature.¹ It will be helpful to distinguish various ranges of temperature.

(i) $0 < T \ll m_{\text{phys}}$. The system is nonrelativistic and dilute. The equilibrium particle density is exponentially small, $n \sim (m_{\text{phys}}T)^{3/2} e^{-m_{\text{phys}}/T}$.

(ii) $m_{\text{phys}} \lesssim T \ll m_{\text{phys}}/\sqrt{\lambda}$. The system is relativistic, but thermal corrections to the effective particle mass (or scattering amplitudes) are negligible.

(iii) $T \approx m_{\text{phys}}/\sqrt{\lambda}$. The thermal correction to the particle mass, of order $\sqrt{\lambda}T$, is comparable to the zero temperature mass. The system may no longer be regarded as a weakly interacting collection of the underlying fundamental particles.

(iv) $m_{\text{phys}}/\sqrt{\lambda} \ll T \ll m_{\text{phys}}/\lambda$. The zero temperature mass is negligible compared to the thermal mass shift, but the zero temperature mass still dominates the trace anomaly of $\beta(\lambda)\phi^4/4!$ in $\langle T_{\mu}^{\mu} \rangle$.

(v) $T \gg m_{\text{phys}}/\lambda$. The zero temperature mass is negligible even in $\langle T_{\mu}^{\mu} \rangle$.

The most interesting domains, from a theoretical perspective, are the high temperature ranges (iii)–(v) where thermal renormalization effects are important. Table I summarizes the qualitative behavior of various quantities at these temperatures. The scaling behavior of the effective particle mass and thermal width will be essential ingredients in the following discussion. The thermal width is the inverse of the mean free time between scattering (up to statistical factors) and equals the displacement of the single particle pole away from the real frequency axis. The size of the thermal width follows directly from the imaginary part of the on-shell self energy divided by the particle energy. For weak coupling, the thermal width is small compared to the effective mass because the imaginary part of the on-shell self energy first arises from two-loop graphs, whereas the real part has one-loop contri-

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¹Since the scalar field is real, particles are their own antiparticles and there is no conserved number operator or charge to which one could couple a chemical potential.

TABLE I. Scaling behavior of various quantities in high temperature scalar field theory. The estimates hold in the domain $T \gtrsim m_{\text{phys}}/\sqrt{\lambda}$ where the one-loop thermal contribution dominates the (real part of the) single particle self-energy $\Sigma(p)$. If the scalar field has only quartic interactions, then the last result for the bulk viscosity acquires an additional factor of $\lambda^{-1/2}$. See Sec. V for more detailed expressions.

Particle density	$n = O(T^3)$
Energy density	$\varepsilon = O(T^4)$
Effective particle mass	$m_{\text{th}} = O(\lambda^{1/2}T)$
On-shell self energy	$\Sigma(p) = O(\lambda T^2) + iO(\lambda^2 T^2)$
Thermal width [$p = O(m_{\text{th}})$]	$\Gamma_p = O(\lambda^{3/2}T)$
Thermal width [$p = O(T)$]	$\Gamma_p = O(\lambda^2 T)$
Mean free time [$p = O(T)$]	$\tau_{\text{f}} = O(\lambda^{-2}T^{-1})$
Elastic cross section [$p = O(T)$]	$\sigma = O(\lambda^2 T^{-2})$
Speed of sound	$v_s = 1/\sqrt{3} + O(\lambda)$
Shear viscosity	$\eta = O(\lambda^{-2}T^3)$
Bulk viscosity [$T \gtrsim O(m_{\text{phys}}/\lambda)$]	$\zeta = O(\lambda T^3 \ln^2 \lambda)$
Bulk viscosity [$T = O(m_{\text{phys}}/\sqrt{\lambda})$]	$\zeta = O(m_{\text{phys}}^4 T^{-1} \lambda^{-5/2} \ln^2 \lambda)$

butions. The results displayed for the shear and bulk viscosities will be discussed in detail in Sec. V.

II. TRANSPORT COEFFICIENTS AND BASIC KINETIC THEORY

In a fluid with no conserved particle number, the stress-energy tensor $T_{\mu\nu}$ is the only locally conserved current, and fluctuations in the energy and momentum densities are the only hydrodynamic modes.² Two transport coefficients, the shear and bulk viscosities, (denoted η and ζ , respectively) characterize the resulting hydrodynamic response.³ If the system is slightly perturbed from equilibrium, then the non-equilibrium expectation of $T_{\mu\nu}$ will satisfy the constitutive relation (in a local fluid rest frame),

$$\begin{aligned} \langle T_{ij} \rangle = & \delta_{ij} \langle \mathcal{P} \rangle - \frac{\eta}{\langle \varepsilon + \mathcal{P} \rangle} (\nabla_i \langle T^0_j \rangle + \nabla_j \langle T^0_i \rangle - \frac{2}{3} \delta_{ij} \nabla^l \langle T^0_l \rangle) \\ & - \frac{\zeta}{\langle \varepsilon + \mathcal{P} \rangle} \delta_{ij} \nabla^l \langle T^0_l \rangle, \end{aligned} \quad (2.1)$$

together with the exact conservation law, $\partial_\mu \langle T^{\mu\nu} \rangle = 0$. Here T_{ij} is the spatial part of the stress-energy tensor, $\varepsilon \equiv T_{00}$ is the energy density, and $\langle \mathcal{P} \rangle$ is the local equilibrium pressure. The constitutive relation (2.1) is valid for small fluctuations in the limit in which the scale of the variation in $\langle T_{\mu\nu} \rangle$ is arbitrarily large compared to microscopic length scales (such as the mean free path of excitations).

The shear viscosity η characterizes the diffusive relaxation of transverse momentum density fluctuations;

²These are fluctuations whose relaxation time diverges as the wavelength of the fluctuation increases. Such fluctuations determine the behavior of the system at arbitrarily long times and large distances.

³Because there is no conserved particle number, thermal conductivity is not an independent transport coefficient.

$\eta/\langle \varepsilon + \mathcal{P} \rangle$ is the diffusion constant for such shear fluctuations. The bulk viscosity ζ characterizes the departure from equilibrium during a uniform expansion. If the divergence of the fluid flow is constant, then the pressure differs from the local equilibrium value by the bulk viscosity times the expansion rate, or $\zeta \nabla^i \langle T_i^0 \rangle / \langle \varepsilon + \mathcal{P} \rangle$. Both shear and bulk viscosity contribute to the attenuation of sound waves; the decay rate of sound waves with wavenumber \mathbf{k} is $\mathbf{k}^2 (\frac{4}{3} \eta + \zeta) / \langle \varepsilon + \mathcal{P} \rangle$ [2]. The bulk viscosity vanishes identically in a scale invariant theory [3]. This follows from the vanishing trace of the stress-energy tensor, $T_\mu^\mu = 0$, in any scale invariant theory, and reflects the fact that a uniform dilation of an equilibrium distribution function remains in equilibrium (at a modified temperature) if the dispersion relation is scale invariant.

Transport coefficients are proportional to the mean free path of the scattering processes responsible for relaxation of the associated hydrodynamic modes. The shear viscosity is proportional to the two body elastic scattering mean free path. In (scale noninvariant) relativistic theories, the bulk viscosity is proportional to the mean free path for particle number changing processes. This may be understood by noting that after a uniform expansion, a change in the total number of particles is required in order to reequilibrate at a different temperature.⁴ Note that decreasing the interaction strength will increase the mean free paths and thus normally increase transport coefficients. Consequently, the weak coupling expansion of viscosities typically begins with negative powers of the coupling.

Most textbook discussions of the evaluation of transport coefficients (such as [2]) begin by assuming the validity of a kinetic theory description of the interacting fluid. One argues that the system may be characterized by a distribution function $f(x, p)$ giving the phase space probability density of the fundamental particles comprising the fluid. Although written

⁴Under uniform expansion, a nonrelativistic gas of molecules relaxes by converting internal energy (vibrational or rotational) into kinetic energy. In contrast, a relativistic gas of structureless particles relaxes by converting rest-mass energy into kinetic energy.

as if it depends on an arbitrary four momentum, the distribution function is only defined for on-shell particles, for which $p^0 = E_p \equiv \sqrt{\mathbf{p}^2 + m_{\text{phys}}^2}$. The time dependence of the distribution function is governed by a Boltzmann equation,

$$\frac{p^\mu}{E_p} \frac{\partial}{\partial x^\mu} f(x, p) = \frac{1}{2} \int_{123} d\Gamma_{12 \leftrightarrow 3p} (f_1 f_2 (1 + f_3) (1 + f_p) - (1 + f_1) (1 + f_2) f_3 f_p), \quad (2.2)$$

where $d\Gamma_{12 \leftrightarrow 3p}$ is the differential transition rate for particles of momenta k_1 and k_2 to scatter into momenta k_3 and p ,

$$d\Gamma_{12 \leftrightarrow 3p} \equiv \frac{1}{2E_p} |\mathcal{S}(p, k_3; k_2, k_1)|^2 \prod_{i=1}^3 \frac{d^3 \mathbf{k}_i}{(2\pi)^3 (2E_{k_i})} \times (2\pi)^4 \delta(k_1 + k_2 - k_3 - p), \quad (2.3)$$

and $f_i \equiv f(x, k_i)$, $f_p \equiv f(x, p)$. The collision term [or the right-hand side of (2.2)] vanishes when the distribution function is an equilibrium Bose distribution with an (inverse) temperature β and flow velocity u^μ , or

$$f_{\beta}^{\text{eq}}(x, p) = n(|u^\mu p_\mu|), \quad (2.4)$$

with $n(E)$ the usual Bose distribution function at inverse temperature β ,

$$n(E) \equiv (e^{\beta E} - 1)^{-1}. \quad (2.5)$$

To extract transport coefficients, it is sufficient to consider perturbations away from equilibrium which are arbitrarily small and slowly varying. Writing the distribution function as a local equilibrium piece plus a nonequilibrium correction,

$$f(x, p) = f_{\beta(x)}^{\text{eq}}(x, p) \{1 - \chi(x, p) [1 + f_{\beta(x)}^{\text{eq}}(x, p)]\}, \quad (2.6)$$

one may linearize the Boltzmann equation and expand in powers of ∇u or $\nabla \beta$. After using the conservation relation, $\partial_\mu T^{\mu\nu} = 0$, to express time derivatives in terms of spatial gradients,⁵ one finds that (in the fluid rest frame at a particular point x) [2],

$$\chi(x, p) = \beta(x) A(x, p) \nabla \cdot \mathbf{u}(x) + \beta(x) B(x, p) [\hat{\mathbf{p}} \cdot \nabla(\mathbf{u}(x) \cdot \hat{\mathbf{p}}) - \frac{1}{3} \nabla \cdot \mathbf{u}(x)], \quad (2.7)$$

where the coefficient B multiplying the shear in the flow satisfies the linear inhomogeneous integral equation

$$p_i p_j - \frac{1}{3} \mathbf{p}^2 \delta_{ij} = \frac{E_p}{2} \int_{123} d\Gamma_{12 \leftrightarrow 3p} \times (1 + n_1) (1 + n_2) n_3 (1 + n_p)^{-1} \times [B_{ij}(p) + B_{ij}(k_3) - B_{ij}(k_2) - B_{ij}(k_1)], \quad (2.8)$$

with $B_{ij}(p) \equiv B(p) (\hat{p}_i \hat{p}_j - \frac{1}{3} \delta_{ij})$, and all quantities evaluated at the point x . The coefficient A multiplying the divergence of the flow satisfies an analogous integral equation,

$$\frac{1}{3} \mathbf{p}^2 - v_s^2 (\mathbf{p}^2 + m_{\text{phys}}^2) = \frac{E_p}{2} \int_{123} d\Gamma_{12 \leftrightarrow 3p} \times (1 + n_1) (1 + n_2) n_3 (1 + n_p)^{-1} \times [A(p) + A(k_3) - A(k_2) - A(k_1)], \quad (2.9)$$

with $v_s \equiv (\partial \mathcal{S} / \partial \varepsilon)^{1/2}$ the (local equilibrium) speed of sound, together with the constraint

$$0 = \int \frac{d^3 p}{(2\pi)^3} E_p n(E_p) [1 + n(E_p)] A(p). \quad (2.10)$$

Finally, inserting the distribution function into the kinetic theory stress-energy tensor,

$$T^{\mu\nu}(x) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} p^\mu p^\nu f(x, p), \quad (2.11)$$

and comparing with the constitutive relation (2.1) yields

$$\eta = \frac{\beta}{15} \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} \mathbf{p}^2 n(E_p) [1 + n(E_p)] B(\mathbf{p}) \quad (2.12a)$$

and

$$\zeta = \beta \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} [\frac{1}{3} \mathbf{p}^2 - v_s^2 (\mathbf{p}^2 + m_{\text{phys}}^2)] \times n(E_p) [1 + n(E_p)] A(\mathbf{p}). \quad (2.12b)$$

Hence, in this kinetic theory treatment, a quantitative evaluation of viscosities requires solving the linear integral equations (2.8) and (2.9) and then computing the final momentum integrals in (2.12). These results (2.9–2.12) for the viscosities can only be trusted within the domain of validity of the underlying Boltzmann equation (2.2). Basic assumptions underlying kinetic theory which must hold include the following.

(a) The collision time is negligible compared to the mean free time between collisions of the fundamental particles.

(b) Between collisions, particles may be regarded as propagating classically with definite momentum and energy.

⁵And imposing the Landau-Lifshitz condition $T^{\mu\nu} u_\nu = T_{\text{eq}}^{\mu\nu} u_\nu$ to make the decomposition (2.6) unique.

(c) The on-shell energy and momentum of particles in between collisions satisfy the zero temperature free particle dispersion relation, $E_p = \sqrt{\mathbf{p}^2 + m_{\text{phys}}^2}$.

When the system is nonrelativistic, $T \ll m_{\text{phys}}$, the mean free time is exponentially large (compared to the Compton time $\hbar/m_{\text{phys}}c^2$) and the above assumptions are well satisfied. In the relativistic domain, $T \gtrsim m_{\text{phys}}$, the situation is more involved. In this regime, the density of particles scales as T^3 , a typical two-body elastic scattering cross section is $\sigma \sim \lambda^2/T^2$, and so the mean free time is $O(1/\lambda^2 T)$ [or $O(1/\lambda^{3/2} T)$ for soft particles due to Bose-enhanced stimulated emission]. In contrast, the typical collision time (determined by the variation of the phase shift with energy) is $O(\lambda^2/T)$; hence condition (a) is satisfied as long as the theory is weakly coupled. Quantum uncertainties in the energy or momentum of a particle propagating between collisions are negligible provided the kinetic energy times the mean free time is large compared to \hbar . For particles with typical $O(T)$ energies, this condition again merely requires $\lambda \ll 1$. But since the mean free time becomes arbitrarily small as the temperature increases, “soft” particles with momentum of order of their rest mass cannot be viewed as propagating classically when $T \gtrsim m_{\text{phys}}/\lambda^2$. Moreover, standard kinetic theory fails long before this temperature is reached due to condition (c). The collision term in the Boltzmann equation summarizes the effects of scatterings in which particles change their momenta in a near-random manner which may be regarded as destroying phase coherence. It does *not* describe the coherent change in phase caused by exactly forward scattering. The amplitude for a soft particle to propagate through the surrounding medium will be modified due to phase shifts arising from forward scattering interactions, and this will change the dispersion relation from the zero temperature form.⁶ For a hot scalar theory, this is precisely the origin of the well-known thermal correction to the effective particle mass,

$$m_{\text{th}}(T)^2 = m_{\text{phys}}^2 + \frac{\lambda T^2}{24} \left[1 + O\left(\frac{m_{\text{phys}}}{T}\right) \right]. \quad (2.13)$$

For simplicity, contributions arising from the cubic coupling have not been displayed. Here (and henceforth) λ and g^2 are renormalized couplings evaluated at the scale T .⁷ The thermal mass correction is negligible when $T \lesssim O(m_{\text{phys}})$, but when $T \gtrsim O(m_{\text{phys}}/\sqrt{\lambda})$ the mass correction is significant and the standard Boltzmann equation fails to describe correctly the propagation of particles with soft $O(m_{\text{phys}})$ momenta. It is important to note that forward scattering effects will also change the effective cross sections of soft particles propagating through the medium.

When $T \gg m_{\text{phys}}$, one might expect the inapplicability of the Boltzmann equation for soft particles to be irrelevant, since particles with $O(m_{\text{phys}})$ momenta comprise a small $O[(m_{\text{phys}}/T)^2]$ fraction of the total. This is true for some

⁶This, of course, is exactly how the index of refraction for light is generated.

⁷A renormalization point of order T is needed to avoid large logarithms in higher order corrections.

physical observables (including thermodynamic quantities such as the pressure or energy density, and also the shear viscosity). However, as will be seen explicitly below, the bulk viscosity is predominately sensitive to soft $O(m_{\text{phys}})$ momenta.⁸

There is an additional problem with the kinetic theory treatment for the bulk viscosity. The integral equation (2.9) has no solution. The kernel of the equation has a zero mode (which is not orthogonal to the source term). The zero mode is a consequence of the conservation of particle number. The original $g\phi^3 + \lambda\phi^4$ theory does not have a conserved particle number, but the Boltzmann equation (2.2) with only two-particle scattering terms included obviously does conserve the number of particles. As stated earlier, the bulk viscosity, which characterizes the relaxation of the system after a uniform expansion, is directly sensitive to particle number changing processes since a (scale noninvariant) system undergoing uniform expansion cannot reequilibrate without changing the number of particles.⁹ Consequently, higher order particle number changing terms must be included in the Boltzmann equation (2.2) even though they are suppressed by additional powers of λ . This will be described more explicitly below.

In summary, standard kinetic theory (with number changing processes included) is adequate for calculating transport coefficients in a weakly coupled theory in the temperature regimes where $T \ll m_{\text{phys}}/\sqrt{\lambda}$, but not in the high temperature regimes with $T \gtrsim m_{\text{phys}}/\sqrt{\lambda}$. In order to derive transport coefficients in this domain, one should start directly from the underlying field theory.

III. DIAGRAMMATIC EVALUATION OF TRANSPORT COEFFICIENTS

The shear and bulk viscosities may be extracted from the zero momentum, small frequency limit of the spectral density of the equilibrium stress tensor–stress tensor correlation function. One finds that [1]

$$\eta = \frac{1}{20} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^4x e^{i\omega t} \langle [\pi_{lm}(t, \mathbf{x}), \pi_{lm}(0)] \rangle_{\text{eq}}, \quad (3.1a)$$

and

⁸This difference between the shear and bulk viscosity is easy to see from Eqs. (2.8)–(2.12). The factors of \mathbf{p}^2 in the shear viscosity integrand (2.12a) and the inhomogeneous term in (2.8) combine to suppress the contribution of soft momenta by four powers of $|\mathbf{p}|$ relative to the case of the bulk viscosity.

⁹If the number of particles is conserved (as in a nonrelativistic field theory), equilibrium states will depend on a chemical potential as well as the temperature, and the zero mode in (2.9) will be removed by the additional subsidiary condition on χ needed to make the local chemical potential uniquely defined. In such a theory, a uniform expansion will, of course, produce a change in the chemical potential.

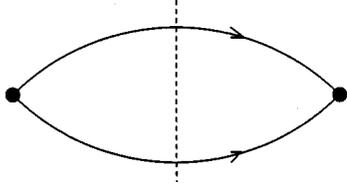


FIG. 1. The cut one-loop diagram contribution to the viscosity.

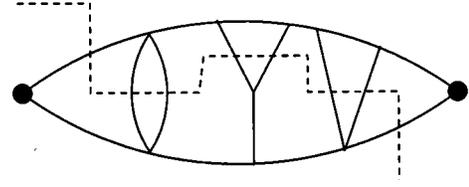
$$\zeta = \frac{1}{2} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \int d^4x e^{i\omega t} \langle [\bar{\mathcal{P}}(t, \mathbf{x}), \bar{\mathcal{P}}(0)] \rangle_{\text{eq}}. \quad (3.1b)$$

Here, $\pi_{lm} \equiv \nabla_l \phi \nabla_m \phi - \frac{1}{3} \delta_{lm} (\nabla \phi)^2$ is the traceless part of the stress tensor and $\bar{\mathcal{P}} \equiv \mathcal{P} - v_s^2 \varepsilon$ is the pressure minus the energy density times the square of the speed of sound.¹⁰ These Kubo relations provide the natural starting point for a field theory evaluation. The spectral density equals the discontinuity in the (Fourier transformed) stress-stress correlation function and has a perturbative expansion generated by the sum of cut diagrams with two insertions of $T_{\mu\nu}$ [4]. Naively, one would expect the leading weak coupling contribution to arise solely from the single one-loop diagram shown in Fig. 1.

This is correct for generic values of the external four-momentum, but is completely incorrect in the limit of vanishing external momentum and frequency. Finite temperature propagators have poles (in frequency) with both $+i\epsilon$ and $-i\epsilon$ prescriptions. When the external four-momentum vanishes, the product of propagators corresponding to the graph in Fig. 1 contains terms in which the contour of the frequency integration is pinched between coalescing poles, thereby producing an on-shell divergence. As always, such divergences have a simple physical origin [5]. When a small momentum is introduced by an insertion of $T_{\mu\nu}$, an on-shell (bare) particle in the thermal medium can absorb the external momentum and become slightly off-shell. The amplitude is proportional to the length of time the particle can remain off-shell. As the external momentum vanishes, the virtual particle moves on-shell and the integral over the propagation time diverges.

However, at nonzero temperature, no excitation can actually propagate indefinitely through the thermal medium without suffering collisions off other excitations. In a scalar field theory, a single particle excitation of momentum \mathbf{k} acquires a finite lifetime τ_k , or nonzero thermal width $\Gamma_k \equiv 1/\tau_k$, due to the $O(\lambda^2)$ imaginary part of the on-shell two-loop self-energy. To examine the limit of vanishing external momentum, one must resum the single particle self-energy insertions which will shift the poles in the single particle propagator from $\pm E_k^0 \pm i\epsilon$ to $\pm E_k^{\text{th}} \pm i\Gamma_k$ [where $E_k^{\text{th}} \equiv \sqrt{\mathbf{k}^2 + m(T)^2}$]. This serves to regulate the apparent on-

¹⁰The Kubo relation (3.1b) is equally correct if the pressure \mathcal{P} is used in place of $\bar{\mathcal{P}}$, since commutators of the energy density vanish at zero momentum (due to energy conservation). However, as shown in [1] and explained below, the particular choice of $\bar{\mathcal{P}}$ given is appropriate for deriving an “effective” kinetic theory description.

FIG. 2. A typical cut ladder diagram for the shear viscosity in $g\phi^3 + \lambda\phi^4$ theory containing $O(\lambda^2)$, $O(g^2\lambda)$, and $O(g^4)$ “rungs.”

shell singularity, and makes the one-loop diagram in Fig. 1 yield a finite result proportional to the single particle lifetime. However, since the lifetime is $O(1/\lambda^2)$ [for particles with $O(T)$ momenta] this means that higher loop diagrams can be just as important as the one-loop contribution if they are sufficiently infrared sensitive.

For the shear viscosity, a careful analysis shows that one must sum all cut “ladderlike” diagrams of the type illustrated in Fig. 2. See [1] for details. This is similar to the situation in nonrelativistic systems [6], except that instead of having ladders built from an instantaneous two body interaction, one must deal with ladder graphs containing far more complicated “rungs.” Nevertheless, one may formally sum all cut ladderlike graphs by introducing an effective vertex $\mathcal{D}_\pi(k, q-k)$ satisfying a linear equation of the form

$$\begin{aligned} \mathcal{D}_\pi(k, q-k) &= \mathcal{T}_\pi(k, q-k) + \int \frac{d^4p}{(2\pi)^4} \mathcal{M}(k-p) \\ &\quad \times \mathcal{F}(p, q-p) \mathcal{D}_\pi(p, q-p). \end{aligned} \quad (3.2)$$

The effective vertex actually has a four components (in order to represent the four different choices for which legs are above and below the cut), while $\mathcal{M}(k-p)$ and $\mathcal{F}(p, q-p)$ are 4×4 matrices representing the rungs and siderails of the ladder, respectively. These matrices have entries consisting of various products of cut and uncut propagators. The inhomogeneous term $\mathcal{T}_\pi(k, q-k)$ represents the vertex factors corresponding to an insertion of the traceless stress tensor. The explicit form of each of these quantities may be found in [1]. Closing the two legs of the effective vertex with a second insertion of the traceless stress tensor produces the sum of all ladderlike graphs contributing to the shear viscosity, so that

$$\begin{aligned} \eta &= \frac{\beta}{10} \lim_{q^0 \rightarrow 0} \lim_{\mathbf{q} \rightarrow 0} \int \frac{d^4k}{(2\pi)^4} \mathcal{T}_\pi(k, q-k) \cdot \mathcal{F}(k, q-k) \\ &\quad \times \mathcal{D}_\pi(k, q-k) [1 + O(\sqrt{\lambda})]. \end{aligned} \quad (3.3)$$

In the limit of vanishing external momentum q , one may perform the frequency integration and extract the leading order behavior from the nearly pinching-pole contributions [1]. Moreover, by using the finite temperature optical theorem [1,7] the 4×4 kernel $\mathcal{M}\mathcal{F}$ may be shown to equal a rank one matrix (up to corrections subleading in λ), thereby allowing one to reduce the equation to a single component, three dimensional integral equation. The result is identical to

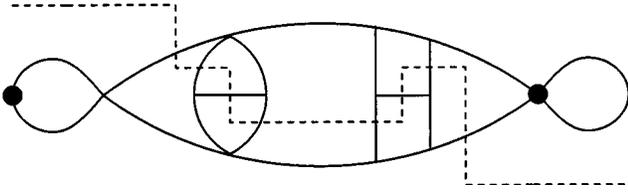


FIG. 3. A typical graph containing $O(g^2\lambda^2)$ and $O(g^6)$ two-to-three particle “rungs,” plus “thermal renormalization” of the stress tensor vertices. Graphs such as this contribute to the leading order weak coupling behavior of the bulk viscosity.

Eq. (2.8) for the spin-two part of the Boltzmann distribution function, and (2.12a) for the kinetic theory shear viscosity provided one (a) identifies the shear response $B(k)$ with the effective vertex divided by the imaginary part of the single particle self-energy, (b) uses the thermal mass $m_{\text{th}}(T)$ instead of the zero-temperature mass in the dispersion relation defining on-shell momenta, and (c) uses an effective temperature-dependent “scattering amplitude” equal to the usual tree-level amplitude but evaluated with finite temperature retarded propagators,¹¹

$$\begin{aligned} \mathcal{T}(p_1, p_2; p_3, p_4) \equiv & \lambda - \bar{g}^2 [G_R(p_1 + p_2) + G_R(p_1 - p_4) \\ & + G_R(p_1 - p_3)], \end{aligned} \quad (3.4)$$

where $G_R(p_1 + p_2) = [(p_1 + p_2)^2 + m_{\text{th}}^2]^{-1}$ and $\bar{g} = g + \lambda \langle \phi \rangle$ is the “shifted” cubic coupling constant that results when one shifts the field by its thermal expectation value $\langle \phi \rangle$ in order to remove tadpole diagrams.¹²

The calculation of bulk viscosity requires considerably more care than the shear viscosity. In addition to ladder diagrams of the type shown in Fig. 2, one must also sum diagrams containing iterations of higher order number-changing scattering processes, and include thermal “vertex renormalization” subgraphs [1]. Examples are shown in Fig. 3.

Nevertheless, one may again sum all the relevant diagrams by introducing an effective vertex $\mathcal{D}_{\bar{\mathcal{P}}}(k, q-k)$ satisfying a linear equation of the same form as in (3.2). The appropriate kernel now contains the previous $O(\lambda^2)$ subdiagrams plus $O(g^2\lambda^2)$ number changing subdiagrams.¹³ The inhomogeneous term receives $O(\lambda)$ corrections involving

the one-loop contributions to the thermal mass and the speed of sound. These vertex corrections cannot be neglected at high temperatures because the speed of sound (squared) approaches $1/3$, producing a cancellation in the leading $O(\mathbf{p}^2)$ part of the inhomogeneous term (2.9). Consequently, an insertion of $\bar{\mathcal{P}}$ (or pressure minus v_s^2 times the energy density) is $O(m_{\text{phys}}^2)$, even when the loop momentum is of $O(T)$. Hence vertex corrections which are $O(\lambda T^2)$ can be comparable to the zeroth order term.

Once again, one may perform the frequency integrations and extract the leading behavior from the nearly pinching-pole contributions,¹⁴ show that the resulting kernel is dominated by a rank one matrix, and reduce the equation to a single component, three dimensional integral equation. This has the same form as Eq. (2.9) [with $A(p)$ identified with the effective vertex divided by the imaginary part of the self-energy] except for the following.

(a) In addition to the two particle elastic scattering term, the right-hand side now contains a particle number changing term proportional to the square of the tree level two-to-three particle “scattering amplitude,”¹⁵

$$\begin{aligned} \mathcal{T}_{\Delta N} = & i\lambda \bar{g} \sum_{\{i,j\}} G_T(p_i + p_j) \\ & - i\bar{g}^3 \sum_{\{i,j\}, \{l,m\}} G_R(p_i + p_j) G_R(p_l + p_m), \end{aligned} \quad (3.5)$$

where again \bar{g} is the shifted cubic coupling constant.

(b) The thermal mass is used in the dispersion relation for on-shell momenta, and in the retarded propagators appearing in the “effective” thermal scattering amplitudes (3.4) and (3.5).

(c) The physical mass (squared) appearing in the source term is replaced by

$$\tilde{m}^2 \equiv m_{\text{th}}^2 - T^2 \frac{\partial m_{\text{th}}^2}{\partial T^2}. \quad (3.6)$$

The “subtracted” mass \tilde{m}^2 is a measure of the departure

¹¹Since the intermediate propagator in (3.4) cannot go on-shell, the $i\epsilon$ prescription in the retarded propagator is actually irrelevant. Note, however, that using the real time Feynman propagator in the scattering amplitude is incorrect as this differs (off-shell) by a $[1 + n(E_p)]$ Bose distribution factor.

¹²Scattering amplitudes, strictly speaking, do not exist at nonzero temperature, since all excitations have finite lifetimes. However, in this weakly coupled theory, the effective scattering amplitude (3.4) provides a meaningful characterization of scattering processes which occur on time scales short compared to the single particle lifetime.

¹³Or, for a pure $\lambda\phi^4$ theory, $O(\lambda^4)$ two-to-four particle subdiagrams.

¹⁴Subleading nonpinching pole terms in the kernel can be neglected only if the inhomogeneous term is orthogonal to the zero modes of the reduced pinching-pole kernel (as well as orthogonal to the zero modes of the full kernel). Imposing this condition forces the energy density coefficient in the source $\bar{\mathcal{P}} = \mathcal{P} - v_s^2 \epsilon$ to equal the speed of sound (including one-loop corrections) [1].

¹⁵Or, for a pure $\lambda\phi^4$ theory, the two-to-four particle amplitude $\mathcal{T}_{\Delta N} = -i\lambda^2 \sum_{\{i,j,k\}} G_R(p_i + p_j + p_k)$. Here all 6 momenta involved in the two-to-four scattering are regarded as incoming, and the sum runs over 10 distinct partitions of the six momenta into two groups of three momenta. Similarly, the sums in Eq. (3.5) run over partitions of the five momenta into sets of 2 and 3 momenta, or 2, 2 and 1 momenta, respectively.

from scale invariance. The subtraction cancels the leading temperature dependence in m_{th}^2 , so that \bar{m}^2 differs negligibly from m_{phys}^2 when $T \lesssim m_{\text{phys}}/\sqrt{\lambda}$, and approaches $m_{\text{phys}}^2 - \frac{1}{2}g^2/\lambda$ for $m_{\text{phys}}/\sqrt{\lambda} \ll T \ll m_{\text{phys}}/\lambda$. At asymptoti-

cally large temperatures, $T \gg m_{\text{phys}}/\lambda$, the running of the quartic coupling in (2.13) dominates and $\bar{m}^2 = \beta(\lambda)T^2/48$, up to $O(\sqrt{\lambda})$ corrections.

The resulting equation for the spin-0 response is

$$\begin{aligned} \frac{1}{3}\mathbf{p}^2 - v_s^2(\mathbf{p}^2 + \bar{m}^2) &= \frac{E_p}{2} \int_{123} d\Gamma_{12 \leftrightarrow 3p} (1+n_1)(1+n_2)n_3(1+n_p)^{-1} [A(p) + A(k_3) - A(k_2) - A(k_1)] \\ &+ \frac{E_p}{4} \int_{1234} d\Gamma_{12 \leftrightarrow 34p} (1+n_1)(1+n_2)n_3n_4(1+n_p)^{-1} [A(p) + A(k_4) + A(k_3) - A(k_2) - A(k_1)] \\ &+ \frac{E_p}{6} \int_{1234} d\Gamma_{123 \leftrightarrow 4p} (1+n_1)(1+n_2)(1+n_3)n_4(1+n_p)^{-1} [A(p) + A(k_4) - A(k_3) - A(k_2) - A(k_1)], \end{aligned} \quad (3.7)$$

with

$$d\Gamma_{12 \leftrightarrow 34p} \equiv \frac{1}{2E_p} \left| \mathcal{T}_{\Delta N}(p, k_4, k_3; k_2, k_1) \right|^2 \prod_{i=1}^4 \frac{d^3 \mathbf{k}_i}{(2\pi)^3 (2E_{k_i})} (2\pi)^4 \delta^4(P_{\text{in}}^{\text{tot}} - P_{\text{out}}^{\text{tot}}), \quad (3.8)$$

etc. In the pure quartic theory, the $2 \leftrightarrow 3$ particle terms are replaced by the corresponding $2 \leftrightarrow 4$ particle contributions. Closing the effective vertex with an insertion of $\bar{\mathcal{P}}$ yields the bulk viscosity,

$$\begin{aligned} \zeta &= \beta \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} \left[\frac{1}{3}\mathbf{p}^2 - v_s^2(\mathbf{p}^2 + \bar{m}^2) \right] \\ &\times n(E_p) [1 + n(E_p)] A(\mathbf{p}), \end{aligned} \quad (3.9)$$

which differs from (2.12b) by the replacement of m_{phys}^2 by \bar{m}^2 .¹⁶

IV. EFFECTIVE KINETIC THEORY

Before discussing the solutions of these linearized equations for the hydrodynamic response, we wish to show how one may construct an effective kinetic theory for quasiparti-

cle excitations which reproduces, at arbitrary temperature in a weakly coupled theory, the correct hydrodynamic response. As usual, the quasi-particle distribution function $f(x, p)$ will depend on an on-shell four-momentum p , but now the quasi-particle energy $p^0 \equiv E_p$ will be a function of both the spatial momentum \mathbf{p} and an effective mass $m(q)$, which in turn depends on a spacetime-dependent auxiliary field $q(x)$:

$$E_p(x) \equiv [\mathbf{p}^2 + m(q(x))^2]^{1/2}. \quad (4.1)$$

The auxiliary field q characterizes the effect of the forward scattering of a quasiparticle off other excitations in the medium, and depends self-consistently on the distribution function,

$$q(x) \equiv \int \frac{d^3 p}{(2\pi)^3} \frac{f(x, p)}{E_p(x)}. \quad (4.2)$$

This is just a nonequilibrium generalization of the usual thermal contribution to the scalar field propagator at coincident points, $\langle \phi(x)^2 \rangle$. The quasiparticle Boltzmann equation can be written as

$$0 = \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} (\mathbf{p}^2 + \bar{m}^2) n(E_p) [1 + n(E_p)] A(\mathbf{p}). \quad (3.11)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial E_p}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial E_p}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{p}} \right) f(x, p) = \Delta \Gamma(x, p). \quad (4.3)$$

¹⁶The solution of (3.7) for $A(p)$ is only unique up to the addition of a zero mode contribution proportional to E_p . This has no effect on the bulk viscosity (3.9) because the speed of sound satisfies the identity

$$0 = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left[\frac{1}{3}\mathbf{p}^2 - v_s^2(\mathbf{p}^2 + \bar{m}^2) \right] n(E_p) [1 + n(E_p)]. \quad (3.10)$$

Nevertheless, the ambiguity in $A(p)$ may be eliminated by imposing the Landau-Lifshitz condition for the effective theory described below. This reduces to the constraint

The dispersion relation (4.1) implies that $\partial E_p / \partial \mathbf{p} = \mathbf{p} / E_p$ and $\partial E_p / \partial \mathbf{x} = (m / E_p) \nabla m$. Hence the spatial gradient of the effective mass acts like an external force which changes the momentum of propagating excitations. The

collision term on the right-hand side is the usual Boltzmann collision term with both $2 \leftrightarrow 2$ and $2 \leftrightarrow 3$ (or $2 \leftrightarrow 4$ for a pure quartic theory) particle processes included,

$$\begin{aligned} \Delta \Gamma(x, p) = & \frac{1}{2} \int_{123} d\Gamma_{12 \leftrightarrow 3p} [f_1 f_2 (1 + f_3) (1 + f_p) - (1 + f_1) (1 + f_2) f_3 f_p] \\ & + \frac{1}{4} \int_{1234} d\Gamma_{12 \leftrightarrow 34p} [f_1 f_2 (1 + f_3) (1 + f_4) (1 + f_p) - (1 + f_1) (1 + f_2) f_3 f_4 f_p] \\ & + \frac{1}{6} \int_{1234} d\Gamma_{123 \leftrightarrow 4p} [f_1 f_2 f_3 (1 + f_4) (1 + f_p) - (1 + f_1) (1 + f_2) (1 + f_3) f_4 f_p]. \end{aligned} \quad (4.4)$$

The transition rates (for a given spacetime location x) are given by the usual definitions (2.3) and (3.8), with effective scattering amplitudes (3.4) and (3.5) computed using retarded free propagators containing the effective mass $m(q(x))$.

This effective Boltzmann equation is to be combined with a modified definition of the kinetic theory stress-energy tensor,

$$T^{\mu\nu}(x) = \left(\int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} p^\mu p^\nu f(x, p) \right) - g^{\mu\nu} U(q(x)). \quad (4.5)$$

A short exercise shows that the modified stress-energy tensor (4.5) is conserved provided the interaction energy $U(q)$ satisfies $\partial U / \partial q = -\frac{1}{2} q (\partial m^2 / \partial q)$, or

$$U(q) = \frac{1}{2} \int_0^q dq' [m^2(q') - m^2(q)]. \quad (4.6)$$

This is also the necessary consistency condition for ensuring that the variation of the total energy density with respect to the quasiparticle density yields the correct quasiparticle energy, $E_p(x) = \delta T^{00}(x) / \delta f(x, p)$ [8], and in equilibrium, that the pressure satisfies the correct thermodynamic identity $T(d\mathcal{P} / dT) = \varepsilon + \mathcal{P}$.

The final ingredient needed to complete the definition of the effective kinetic theory is the dependence of the effective mass on the auxiliary field q . This is completely determined by the dependence of the equilibrium thermal mass m_{th} on the one-loop ‘‘bubble’’ $\langle \phi(x)^2 \rangle$. In the pure quartic scalar theory, the thermal mass has the simple form,

$$m^2(q) = m_0^2 + \frac{1}{2} \lambda q \quad (4.7)$$

(up to corrections suppressed by powers of λ), while if cubic interactions are present one must first self-consistently expand the field about its thermal expectation value $c \equiv \langle \phi \rangle$, leading to

$$m^2(q) = m_0^2 + gc + \frac{1}{2} \lambda (c^2 + q), \quad (4.8)$$

with $0 = m_0^2 c + \frac{1}{2} g (c^2 + q) + \frac{1}{6} \lambda c (c^2 + 3q)$. As always, the coupling constants appearing in (4.7) and (4.8) should be evaluated at a scale appropriate to the physics under consideration; the running of the quartic coupling affects even leading order results when $q \gtrsim m_{\text{phys}}^2 / \lambda^2$. In equilibrium, $q \sim T^2 / 12$ when $T \gg m_{\text{phys}}$. Hence, the appropriate generalization is to regard the coupling as an implicit function of q satisfying (when $q \gg m_{\text{phys}}^2$)

$$q \frac{\partial \lambda}{\partial q} \equiv \frac{1}{2} \beta(\lambda) = \frac{1}{2} b_0 \lambda^2 + O(\lambda^3). \quad (4.9)$$

The resulting effective mass in, for example, the massless pure quartic theory is

$$m^2(q) = \frac{q}{b_0 \ln(\Lambda^2 / q)}, \quad (4.10)$$

where $\Lambda \equiv \mu e^{1/b_0 \lambda(\mu^2)}$ is the renormalization group invariant scale of massless ϕ^4 theory.

This effective kinetic theory provides a consistent description of the nonequilibrium dynamics of a weakly coupled scalar field theory, including the propagation of slowly moving excitations, even when the effective mass of the excitations differs substantially from the zero-temperature mass, or varies significantly in space or time. Expanding about a local equilibrium distribution, as in (2.6), and evaluating the effective stress energy tensor (4.5) (carefully keeping track of the implicit dependence on the distribution function hiding in every factor of energy), leads to the fairly simple result

$$\begin{aligned} T^{\mu\nu}(x) = & T_{\text{eq}}^{\mu\nu}(x) - \int \frac{d^3 p}{(2\pi)^3 E_p} n(E_p) [1 + n(E_p)] \chi(x, p) \\ & \times \left(p^\mu p^\nu - u^\mu u^\nu T^2 \frac{\partial m^2}{\partial T^2} \right), \end{aligned} \quad (4.11)$$

where $T_{\text{eq}}^{\mu\nu} \equiv u^\mu u^\nu (\varepsilon + \mathcal{P}) + g^{\mu\nu} \mathcal{P}$ is the local-equilibrium contribution. Expressing $\chi(x, p)$ in terms of the shear and bulk amplitudes [cf. Eq. (2.7)], and linearizing the effective Boltzmann equation in the hydrodynamic limit, yields ex-

actly the same equations obtained in the previous section for the amplitudes $A(x,p)$ and $B(x,p)$. When inserted into the stress tensor (4.11) one precisely obtains the previous results (2.12a) and (3.9) for the shear and bulk viscosities.

V. RESULTS FOR VISCOSITIES

Computing the bulk viscosity requires solving the integral equation (3.7). Unlike the case of the shear viscosity, solving this equation is trivial because the kernel has a single small eigenvalue which is only displaced from zero due the inclusion of number changing processes. Hence, the solution is dominated by the projection onto the near-zero mode, leading to

$$A(p) = \frac{F}{\Gamma_{\Delta N}} (1 - \alpha E_p), \quad (5.1)$$

where

$$F \equiv \int \frac{d^3 \mathbf{p}}{(2\pi)^3 E_p} [1 + n(E_p)] n(E_p) I_{\overline{\mathcal{A}}}(p), \quad (5.2)$$

with $I_{\overline{\mathcal{A}}}(p) \equiv \frac{1}{3} \mathbf{p}^2 - v_s^2 (\mathbf{p}^2 + \tilde{m}^2)$ the same source term as in Eq. (3.7), and $\Gamma_{\Delta N}$ the total $3 \rightarrow 2$ particle (or $4 \rightarrow 2$ for pure $\lambda \phi^4$) thermal reaction rate per unit volume,

$$\begin{aligned} \Gamma_{\Delta N} &= \frac{1}{12} \int \prod_{i=1}^5 \frac{d^3 \mathbf{k}_i}{(2\pi)^3 2E_{k_i}} |\mathcal{T}_{\Delta N}(\{k_i\})|^2 \\ &\times (2\pi)^4 \delta^4(k_1 + k_2 + k_3 - k_4 - k_5) \\ &\times [1 + n(E_1)][1 + n(E_2)] n(E_3) n(E_4) n(E_5). \end{aligned} \quad (5.3)$$

The constant α in (5.1) is undetermined by (3.7), but may be adjusted to satisfy (3.11). The bulk viscosity obtained by inserting (5.1) into (3.9) is simply

$$\zeta = \beta \frac{F^2}{\Gamma_{\Delta N}}. \quad (5.4)$$

The final evaluation of the shear viscosity requires a numerical solution of the integral equation (2.8) and the final integral (2.12a), while the bulk viscosity requires performing the rather involved phase space integral (5.3) for the particle number changing reaction rate. Details of this evaluation may be found in [1].

Despite the need to resum self-energy insertions in order to cut off singularities in the original diagrams, the introduction of thermal corrections in the dispersion relation and scattering amplitude is actually irrelevant for the leading behavior of the shear viscosity because the integrals (2.8) and (2.12a) are dominated by momenta of order T . This, however, is not the case for the bulk viscosity. At high temperature, the number changing reaction rate scales as $O(g^2 \lambda^2 T^5 / m_{\text{th}}^3)$ for the $g\phi^3 + \lambda\phi^4$ theory, and $O(\lambda^4 T^6 / m_{\text{th}}^2)$ for the pure $\lambda\phi^4$ theory due to its infrared sensitivity to the region where all momenta are $O(m_{\text{th}})$. The factor F appearing in the numerator is a measure of the violation of scale invariance of the theory, and behaves as

$O(\tilde{m}^2 T^2 \ln(T/m_{\text{th}}))$ when $m_{\text{phys}} \ll T$.¹⁷ Hence, the shear and bulk viscosities have very different behaviors throughout the high temperature region.¹⁸ In pure $\lambda\phi^4$ theory,

$$\eta = a \frac{T^3}{\lambda^2} \times [1 + O(\sqrt{\lambda}) + O(m_{\text{phys}}/T)], \quad (5.5)$$

while

$$\begin{aligned} \zeta &= b \frac{\tilde{m}^4 m_{\text{th}}^2}{\lambda^4 T^3} \ln^2 \left(\frac{\kappa m_{\text{th}}}{T} \right) \\ &\times [1 + O(\sqrt{\lambda}) + O(m_{\text{phys}}/T) + O(\lambda T / m_{\text{phys}})], \end{aligned} \quad (5.6)$$

when $m_{\text{phys}} \ll T \ll m_{\text{phys}}/\lambda$, and

$$\zeta = c \lambda \ln^2(\gamma \lambda) T^3 \times [1 + O(\sqrt{\lambda}) + O(m_{\text{phys}}/\lambda T)], \quad (5.7)$$

when $T \gg m_{\text{phys}}/\lambda$. The forms (5.5) and (5.7) remain valid if cubic interactions are present, but the bulk viscosity in the intermediate regime $m_{\text{phys}} \ll T \ll m_{\text{phys}}/\lambda$ acquires dependence on the relative strength of cubic and quartic couplings,

$$\begin{aligned} \zeta &= d \left(\frac{g^2}{\lambda m_{\text{th}}^2} \right) \frac{\tilde{m}^4 m_{\text{th}}^3}{g^2 \lambda^2 T^2} \ln^2 \left(\frac{\kappa m_{\text{th}}}{T} \right) \\ &\times [1 + O(\sqrt{\lambda}) + O(m_{\text{phys}}/T) + O(\lambda T / m_{\text{phys}})], \end{aligned} \quad (5.8)$$

with $d(x)$ a nontrivial dimensionless function.

A numerical evaluation of Eqs. (2.8), (2.12a), (5.3), and (5.4) for the pure quartic theory yields the values [1]:¹⁹

$$a = 3.04 \times 10^3, \quad (5.9a)$$

$$b = 5.5 \times 10^4, \quad (5.9b)$$

$$c = b / [6(32\pi)^4] = 8.9 \times 10^{-5}, \quad (5.9c)$$

$$\kappa = e^{15\zeta(3)/2\pi^2} / 2 = 1.2465, \quad (5.9d)$$

¹⁷One finds $v_s^2 = \frac{1}{3} - \frac{5}{12} \tilde{m}^2 / \pi^2 T^2$ and $F = -(\tilde{m}^2 T^2 / 6\pi^2) [\ln(2T/m_{\text{th}}) - \frac{15}{2} \zeta(3) / \pi^2]$, when evaluating (3.10) and (5.2) for $T \gg m_{\text{phys}}$, up to corrections suppressed by $\sqrt{\lambda}$ or m_{phys}/T .

¹⁸They are also very different at low temperature. When $T \ll m_{\text{phys}}$, the shear viscosity behaves like $\eta \sim m_{\text{phys}}^3 (T/m_{\text{phys}})^{1/2} / \lambda^2$, but the bulk viscosity diverges exponentially as $\zeta \sim e^{2m_{\text{phys}}/T} m_{\text{phys}}^6 / \lambda^4 T^3$ for pure $\lambda\phi^4$ theory, or $e^{m_{\text{phys}}/T} (m_{\text{phys}}/T)^{1/2} m_{\text{phys}}^6 / \lambda^2 g^2 T$ for $g\phi^3 + \lambda\phi^4$ theory. This is the bulk viscosity characterizing asymptotically long wavelength hydrodynamic fluctuations, appropriate for distances large compared to the mean free path for particle number changing interactions (which displays the same exponential divergence). Ordinary nonrelativistic hydrodynamics (with a conserved particle number) is valid at distances small compared to this number changing mean free path but large compared to the elastic mean free path. It is, of course, this region and not the strict asymptotic domain which has practical utility.

¹⁹The result (5.9c) was not included in [1]. In addition, the evaluation of $\Gamma_{\Delta N}$ in Ref. [1] contained a numerical error which affected the plot of ζ shown in that paper. Recomputed values have been used in our Fig. 5 and Eq. (5.9b).

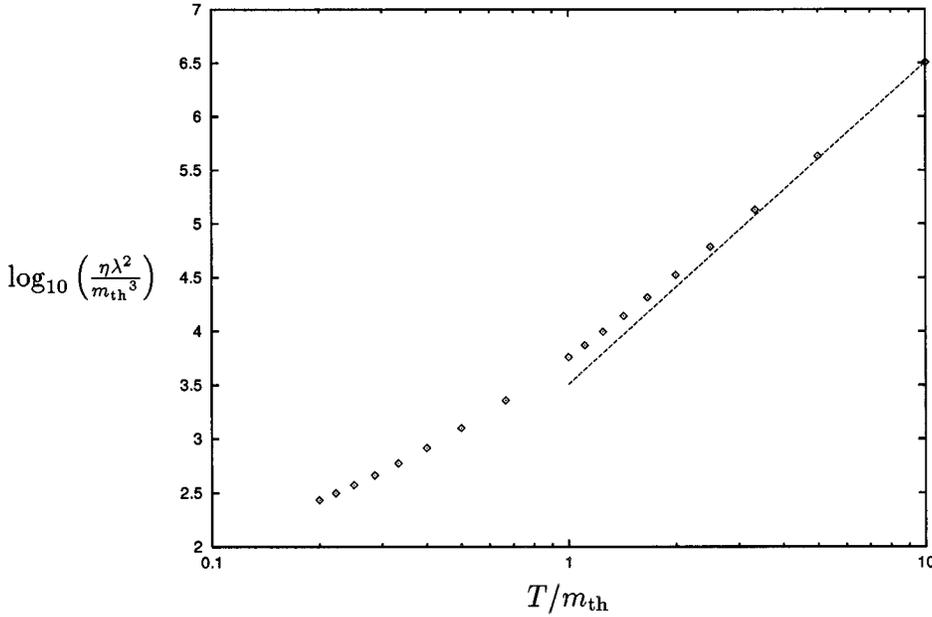


FIG. 4. Numerical results for the shear viscosity. The straight line shows the $O(T^3/\lambda^2)$ asymptotic behavior of η .

$$\gamma = e^{15\zeta(3)/\pi^2}/96 = 0.064736. \quad (5.9e)$$

Results in the relativistic crossover region $T \sim m_{th}$ are plotted in Figs. 4 and 5. If one ignores the need to sum all ladder diagrams and only includes the one-loop diagram of Fig. 1 (after resumming self-energy corrections) then one underestimates the shear viscosity by roughly a factor of four. The analogous error for the bulk viscosity leads to an $O(m_{phys}^4/\lambda^2 T)$ result which scales completely incorrectly with λ .

VI. CONCLUSIONS

The analysis of this simple scalar field theory illustrates a number of points which are applicable to any relativistic field theory.

(a) The diagrammatic evaluation of transport coefficients

is a remarkably inefficient approach. An infinite set of rather complicated diagrams must be summed, merely to obtain the leading weak coupling behavior.

(b) The bulk viscosity depends on particle number changing processes and is sensitive to soft momenta, whereas the shear viscosity is determined by two body elastic scattering cross sections at typical momenta. The ratio of the bulk to the shear viscosities varies from very small [$O(\lambda^3)$] to exponentially large depending on the temperature. Hence crude estimates such as $\zeta \sim \eta(v_s^2 - \frac{1}{3})^2$ which have appeared in the literature [9,10] cannot generally be trusted.

(c) At high temperature, the existence of an effective kinetic theory adequate for computing transport coefficients depends crucially on the theory being weakly coupled, so that mean free paths are large compared to the wavelengths of relevant excitations. In, for example, high temperature QCD, it is unclear if the bulk viscosity can be correctly com-

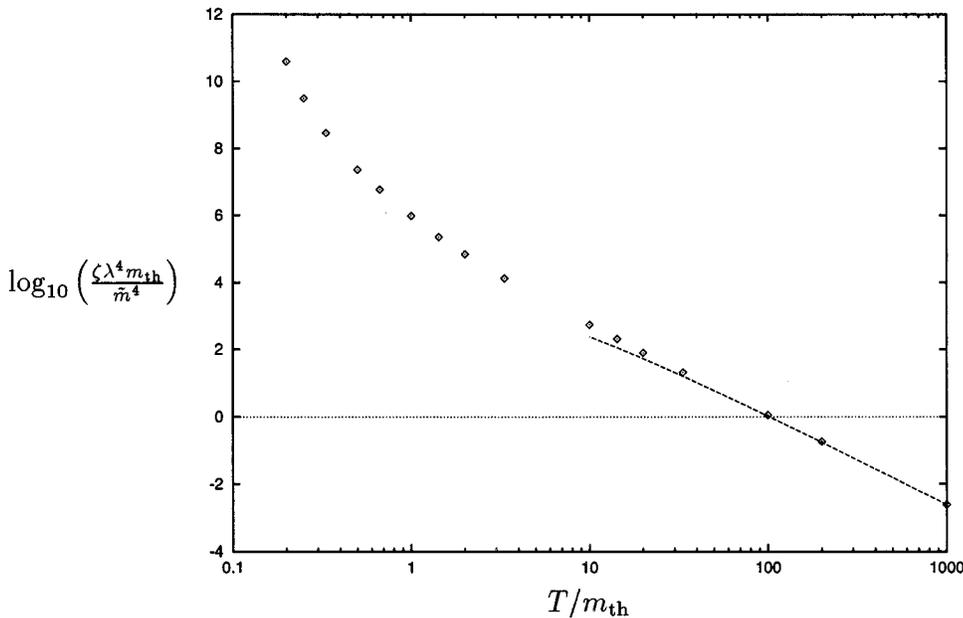


FIG. 5. Numerical results for the bulk viscosity. The solid line shows the $(m_{th}/T)^3 \ln^2(\kappa m_{th}/T)$ behavior of Eq. (5.6).

puted with any kinetic theory since the effective coupling of excitations with soft $O(g^2T)$ momenta is not small.

It is tempting to view the derivation of kinetic theory from the underlying field theory, and the derivation of the hydrodynamic constitutive equation (1.2) from the effective kinetic theory, as two different stages of a “real time renormalization group.” At each stage, one is eliminating irrelevant degrees of freedom from the description of dynamics at successively lower frequency or momentum scales. We have little doubt that this notion of a real time renormalization group is essentially correct. However, we are unaware of any useful framework for defining a real time renormalization group which will systematically transform the basic dynamical formulation from a quantum field theory to kinetic theory, or ultimately to classical hydrodynamics. In contrast to the situation for equilibrium Euclidean space observables [11], how to repackage the cumbersome diagrammatic analysis of [1] in simple renormalization group terms is poorly understood. The diagrammatic treatment does not cleanly separate different frequency scales, as shown, for example, by the necessity of resumming both the real and imaginary parts of the on-shell single particle self energy in order to

regulate individual cut diagrams, even though only the real part of the self energy appears explicitly in the resulting kinetic theory. The imaginary self energy, or single particle lifetime, should be viewed as an output of the effective kinetic theory, not an input parameter. A true real time renormalization group approach should allow one to derive completely the effective kinetic theory before treating any of the physics for which the kinetic theory description is adequate. Furthermore, a useful renormalization group framework should allow one to calculate corrections systematically, at least in weakly coupled theories. Although an effective kinetic theory did emerge in the analysis of the leading weak coupling behavior, it is unclear whether subleading corrections can be incorporated within a kinetic theory framework, since quantum coherence effects are only suppressed by a power of λ . We hope that future investigations will shed light on some of these issues.

ACKNOWLEDGMENTS

Helpful conversations with Peter Arnold and Lowell Brown are gratefully acknowledged.

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- [1] S. Jeon, Phys. Rev. D **52**, 3591 (1995).
 - [2] See, for example, S.R. de Groot, W.A. van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980), and references therein.
 - [3] See, for example, S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
 - [4] S. Jeon, Phys. Rev. D **47**, 4568 (1993).
 - [5] S. Coleman and R.E. Norton, Nuovo Cimento **38**, 438 (1965).
 - [6] See, for example, G.D. Mahan, *Many-Particle Physics* (Plenum, New York, 1981).
 - [7] R.L. Kobes and G.W. Semenoff, Nucl. Phys. **B272**, 329 (1986).
 - [8] For example, see E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, New York, 1980).
 - [9] A. Hosoya, M. Sakagami, and M. Takao, Ann. Phys. (N.Y.) **154**, 229 (1984), and references therein.
 - [10] R. Horsley and W. Schoenmaker, Nucl. Phys. **B280**, 716 (1987).
 - [11] See, for example, E. Braaten and A. Nieto, Phys. Rev. D **51**, 6990 (1995).