Dimensional regularization in configuration space

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Dimensional regularization is introduced in configuration space by Fourier transforming in ν dimensions the perturbative momentum space Green functions. For this transformation, the Bochner theorem is used; no extra parameters, such as those of Feynman or Bogoliubov and Shirkov, are needed for convolutions. The regularized causal functions in x space have ν-dependent moderated singularities at the origin. They can be multiplied together and Fourier transformed (Bochner) without divergence problems. The usual ultraviolet divergences appear as poles of the resultant analytic functions of ν. Several examples are discussed. [S0556-2821(96)03810-6]

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I. INTRODUCTION

When using Feynman diagram techniques in perturbative quantum field theory, the bare propagators are Feynman causal functions $\Delta(x)$ (with or without mass). For reasons of simplicity one usually works in momentum space where the well known convolution theorem takes the form

$$(2\pi)^{\nu}F\{\Delta_1(x)\Delta_2(x)\}=F\{\Delta_1(x)\}\times F\{\Delta_2(x)\}=P_1\times P_2,$$

$$P_i=\int d^{\nu}x\Delta_i(x)e^{ipx}, \tag{1}$$

where P_i is propagator in momentum space.

In (1) the singular behavior at the origin of $\Delta_1(x)\Delta_2(x)$ manifests itself as the famous ''ultraviolet divergence'' present at high momentum in the convolution integration.

To deal with (1) in momentum space, use is made of Feynman parameters [1] or the Bogoliubov-Shirkov method [2] to get an expression which is spherically symmetric. The integral is then regularized to obtain sensible results.

We will here use a different method. The propagator will be regularized in configuration space where no use will be required of extra parameters to get spherically symmetric functions.

The singular character of the convolution in (1) has a different aspect if we have the number of dimensions ν as a free regularizing parameter [3,4]. In this case, the Fourier antitransform of the propagator in momentum space is an analytic function of ν:

$$F^{-1}\{P_i\}=\Delta_i(x;\nu), \tag{2}$$

$$F^{-1}\{P_1\times P_2\}=(2\pi)^{\nu}\Delta_1(x;\nu)\Delta_2(x;\nu), \tag{3}$$

$$P_1\times P_2=(2pi)^{\nu}F\{\Delta_1(x;\nu)\Delta_2(x;\nu)\}. \tag{4}$$

The form of the propagators given by Eq. (2) in coordinate space implies that the singularity at the origin has been ''tempered'' or ''moderated.'' The usual ultraviolet divergences appear as poles of the analytic function of ν defined by (3) or (4). [Note that the right hand side of (3) is a product of distributions.]

The Feynman or Bogoliubov-Shirkov trick is a simple and elegant way to cast the convolution integration into a spherically symmetric expression, leaving the complications to a final integration over the extra auxiliary parameters.

We want to show that by the use of the regularized expressions in configuration space [right hand side of (2), (3), and (4)] we can obtain sensible (and correct) results. Furthermore, we have also the possibility of using a simple treatment for the case of more than just two propagators (see below).

II. FOURIER TRANSFORM OF SPHERICAL FUNCTIONS

First, we want to point out that the quantum causal propagators in configuration space are functions of $t^2-r^2+i\varepsilon$. Their Fourier transforms are functions of $E^2-\rho^2-i\varepsilon$ in momentum space. Then we can do all calculations in the Euclidean metric. When the final expression is obtained a time dilatation Kt must be done followed by an analytic continuation in the coefficient K to the value $K=i\pm\varepsilon$ (see Ref. [5]). In the Euclidean metric, the propagators are spherically symmetric functions. They depend only on the distance.

For the evaluation of the Fourier transform of spherically symmetric functions we will systematically use the well known Bochner theorem [6].

If a function $f(x_1x_2\cdots x_{\nu})$ depends only on the single variable

$$x=(x_1^2+x_2^2+\cdots+x_{\nu}^2)^{1/2}$$

then its Fourier transform

*Deceased.

$$g(y_1 y_2 \cdots y_\nu) = \int dx^\nu f(x) e^{iy\cdot x} \tag{5}$$

depends only on

$$y = (y_1^2 + y_2^2 + \cdots + y_\nu^2)^{1/2}$$

and can be written as a Bessel transform [7]

$$g(y) = \frac{(2\pi)^{-\nu}}{y^{\nu/2-1}} \int_0^\infty f(x) x^{\nu/2} J_{\nu/2-1}(xy)\, dx, \tag{6}$$

J_α being the Bessel function of the first kind and order α.

To take advantage of these properties of spherically symmetric functions we will proceed in the following way: starting in momentum space with a causal function of p we take its Fourier antitransform with the aid of (6),

$$f(x,\nu) = \frac{(2\pi)^{-\nu}}{x^{\nu/2-1}} \int_0^\infty g(p) p^{\nu/2} J_{\nu/2-1}(xp)\, dp. \tag{7}$$

In (7) the singularity at the origin depends analytically on ν. For example, using Ref. [8] we get the following results [note that, when using (6) for F, the factor $(2\pi)^\nu$ has to be suppressed for F^{-1}]: (a) a massless propagator $g(p) = p^{-2}$,

$$f(x,\nu) = 2^{\nu/2-2} \Gamma\left(\frac{\nu}{2} - 1\right) x^{2-\nu}, \tag{8}$$

or, more generally, $g(p) = p^{-2\alpha}$,

$$f(x,\nu) = 2^{\nu/2-2\alpha} \frac{\Gamma\left(\frac{\nu}{2} - \alpha\right)}{\Gamma(\alpha)} x^{2\alpha-\nu} \tag{9}$$

(Ref. [9], p. 365); (b) a massive propagator

$$g(p) = (p^2 + m^2)^{-1},$$

$$f(x,\nu) = m^{\nu/2-1} x^{1-\nu/2} K_{\nu/2-1}(mx) \tag{10}$$

(K_α = a Bessel function of the third kind) or, for arbitrary powers $g(p) = (p^2+m^2)^{-\lambda}$

$$f(x,\nu) = \frac{2^{1-\lambda}}{\Gamma(\lambda)} m^{\nu/2-\lambda} x^{\lambda-\nu/2} K_{\nu/2-\lambda}(mx) \tag{11}$$

(Ref. [9], p. 365).

Equations (9) and (11) are appropriate for analytically regularized propagators [5].

In all these cases, the singularity at the origin in configuration space is proportional to a ν-dependent power of x. This analytic dependence allows the definition of a product in a certain region of the ν plane and the subsequent extension, by analytic continuation to other regions, according to the method adopted in Ref. [9].

Let us now take two (causal) functions $g_1(p)$ and $g_2(p)$. We can multiply together the Bochner transforms Eq. (7):

$$\begin{aligned} f_3(x,\nu) &= f_1(x,\nu) f_2(x,\nu) \\ &= x^{2-\nu} \int_0^\infty dp_1 p_1^{\nu/2} g_1(p_1) J_{\nu/2}(p_1 x) \\ &\quad \times \int_0^\infty dp_2 p_2^{\nu/2} g_2(p_2) J_{\nu/2-1}(p_2 x). \end{aligned} \tag{12}$$

This procedure defines directly the distribution $f_3(x,\nu)$ in a convenient region of the ν plane. By analytic continuation we can define f_3 for other regions.

Since $f_3(x,\nu)$ depends only on x we can use Eq. (6) again to find the convolution of g_1 and g_2 in momentum space. In this way we obtain

$$\begin{aligned} g_1(p) * g_2(p) &= \frac{p^{1-\nu/2}}{(2\pi)^\nu} \int_0^\infty dx x^{2-\nu/2} J_{\nu/2-1}(px) \\ &\quad \times \left\{ \int_0^\infty dp_1 p_1^{\nu/2} g_1(p_1) J_{\nu/2-1}(p_1 x) \right. \\ &\quad \times \left. \int_0^\infty dp_2 p_2^{\nu/2} g_2(p_2) J_{\nu/2-1}(p_2 x) \right\}. \end{aligned} \tag{13}$$

But in Ref. [8], p. 696 (657-9), we find that

$$\int_0^\infty dx x^{1-\alpha} J_\alpha(ax) J_\alpha(bx) J_\alpha(cx) = \frac{2^{\alpha-1} \Delta^{2\alpha-1}}{(abc)^\alpha \Gamma\left(\alpha + \frac{1}{2}\right) \Gamma(\alpha) \Gamma\left(\frac{1}{2}\right)}, \tag{14}$$

where $\Delta = \Delta(abc)$ is the area of the triangle whose sides are a, b, and c. When a, b, and c cannot form a triangle the integral is zero. (Do not ask us about how the mathematicians got this result.) It is not difficult to see that

$$\Delta = \frac{1}{4}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)^{1/2}. \tag{15}$$

Formula (14) allows us to write

$$\begin{aligned} g_1(p) * g_2(p) &= \frac{2^{\nu/2-2} \pi^{-1/2} p^{2-\nu}}{(2\pi)^\nu \Gamma[(\nu-1)/2]} \int_0^\infty dp_1 p_1 \\ &\quad \times \int_0^\infty dp_2 p_2 g_1(p_1) g_2(p_2) \Delta^{\nu-3}. \end{aligned} \tag{16}$$

Equation (16) can be considered an extension of Bochner theorem to the convolution of two spherically symmetric functions.

If we choose to integrate first with respect to p_2 we can write

$$\Delta = \frac{1}{4}[p_2^2 - (p-p_1)^2]^{1/2} [(p+p_1)^2 - p_2^2]^{1/2} \tag{17}$$

so that

$$g_1*g_2=\frac{2^{\nu/2-2}\pi^{-1/2}p^{2-\nu}}{(2\pi)^\nu\Gamma[(\nu-1)/2]4^{\nu-3}}\int_0^\infty dp_1p_1g_1(p_1)$$
$$\times\int_{p-p_1}^{(p+p_1)}dp_2g_2(p_2)p_2[p_2^2-(p-p_1)^2]^{1/2}$$
$$\times[(p+p_1)^2-p_2^2]^{1/2},\qquad(18)$$

where we took into account that $\Delta=0$ when $p_2\leqslant(p-p_1)$, or $p_2\geqslant p+p_1$.

If we take, for example, $g_2=p_2^{2\lambda}$ we have to evaluate

$$I=\int_{a^2}^{b^2}dq^2q^{2\lambda}(q^2-a^2)^{(\nu-3)/2}(b^2-q^2)^{(\nu-3)/2}.$$

Or, changing variables to $x=q^2-a^2$,

$$I=\int_0^{b^2-a^2}dx x^{(\nu-3)/2}(x+a^2)^\lambda(b^2-a^2-x)^{(\nu-3)/2},$$

whose value can be found in Ref. [8], p. 287 (3.19-8). Note that for the evaluation of the convolution [Eqs. (16) or (18)] no extra parameter needs to be introduced. Now we have two equivalent ways for the evaluation of the convolution in momentum space. One of them operates in p space [Eqs. (16) or (18)]. The other consists in the use of the Bessel transform [Eq. (6)] of the product of the dimensional regularized [Eq. (7)] $f(x,\nu)$ functions in coordinate space. The latter is a better method when there are more than two functions to be multiplied together.

III. SOME APPLICATIONS

Here we shall illustrate how the method works in some particular cases.

A. Massless propagator

A massless propagator in p space, p^{-2}, has the form given by (8) in x space. If we have to evaluate the self-energy of a particle for which a simple loop is to be considered, then in coordinate space the square of (8) is involved. More generally, if the coupling is such that at the first vertex n massless quanta are produced which are annihilated at the second vertex, then in coordinate space we have $f^n(x,\nu)$ for the self-energy.

For the simplest case $(n=2)$ we get, from (8),

$$f^2(x,\nu)=2^{\nu-4}\Gamma^2\left(\frac{\nu}{2}-1\right)x^{4-2\nu}.\qquad(19)$$

In momentum space, the convolution of the two propagators can be found by a Bessel transformation of (19) [cf. Eq. (6)]:

$$\frac{1}{p^2}\times\frac{1}{p^2}=\frac{2^{\nu-4}(2\pi)^{-\nu}}{p^{\nu/2-1}}\Gamma^2\left(\frac{\nu}{2}-1\right)$$
$$\times\int_0^\infty dx x^{4-2\nu}x^{\nu/2}J_{\nu/2-1}(px).\qquad(20)$$

From Ref. [8], p. 684, we get

$$\int_0^\infty dx x^\mu J_\rho(ax)=2^\mu a^{-\mu-1}\frac{\Gamma[(1+\rho+\mu)/2]}{\Gamma[(1+\rho-\mu)/2]}.\qquad(21)$$

So, with appropriate substitutions we obtain

$$\frac{1}{p^2}\times\frac{1}{p^2}=2^{-\nu/2}(2\pi)^{-\nu}\Gamma^2\left(\frac{\nu}{2}-1\right)\frac{\Gamma(2-\nu/2)}{\Gamma(\nu-2)}p^{\nu-4}.\qquad(22)$$

It costs practically the same effort to evaluate the contribution in p space of n massless propagators joining two vertices,

$$f^n(x,\nu)=2^{\nu n/2-2n}\Gamma^n(\nu/2-1)x^{(2-\nu)n}\qquad(23)$$

whose Fourier transform is

$$g(p)=\frac{2^{\nu n/2-2n}\Gamma^n(\nu/2-1)}{(2\pi)^\nu p^{\nu/2-1}}\int_0^\infty dx x^{(2-\nu)n}x^{\nu/2}J_{\nu/2-1}(xp).\qquad(24)$$

And, utilizing (21),

$$g(p)=\frac{2^{(1-n)\nu/2}}{(2\pi)^\nu}\Gamma^n\left(\frac{\nu}{2}-1\right)\Gamma\left((1-n)\frac{\nu}{2}+n\right)$$
$$\times\Gamma^{-1}\left[\left(\frac{\nu}{2}-1\right)n\right]p^{(n-1)\nu-2n}.\qquad(25)$$

Note that this result can also be considered to be a consequence of the work of Ref. [10], where a systematic method for the computation of multiloop massless diagrams is developed, by using expansions in Gegenbauer polynomials (in configuration space [11]).

B. Massive propagator

A massive propagator $(p^2+m^2)^{-1}$ is given in coordinate space by Eq. (10) (see also Ref. [12]). The second-order self-energy produced by a massless quantum is obtained by multiplying together (8) and (10).

To evaluate the self-energy produced by the simultaneous emission and reabsorption of n massless particles, we have to take the nth power of (8) together with (10):

$$f(x,\nu)=2^{\nu/2-2}m^{\nu/2-1}x^{(2-\nu)n+1-\nu/2}K_{\nu/2-1}(mx).\qquad(26)$$

For the Bessel transform of (26) we need the formula (Ref. [8], p. 693, 6.576.3)

$$\int_0^\infty dx x^{-\lambda}K_\mu(ax)J_\rho(bx)$$
$$=\frac{b^\rho}{a^{\rho-\lambda+1}}\frac{\Gamma[(\rho-\lambda+\mu+1)/2]\Gamma[(\rho-\lambda-\mu+1)/2]}{2^{\lambda+1}\Gamma(1+\rho)}$$
$$\times F\left(\frac{\rho-\lambda+\mu+1}{2},\frac{\rho-\lambda-\mu+1}{2};1+\rho;-\frac{b^2}{a^2}\right).\qquad(27)$$

For the simple convolution $(n=1)$ we get

$$\frac{1}{p^2}\times\frac{1}{p^2+m^2}=\frac{m^{\nu-4}2^{-\nu/2}}{(2\pi)^\nu}\frac{\Gamma(\nu/2-1)\Gamma(2-\nu/2)}{\Gamma(\nu/2)}\times F\left(1,2-\frac{\nu}{2};\frac{\nu}{2};-\frac{p^2}{m^2}\right). \tag{28}$$

For the convolution with n massless particles, we get from (23), (26), and (27)

$$g(p)=m^{(\nu-2)n-2}2^{-\nu/2}\frac{\Gamma^n(\nu/2-1)}{\Gamma(\nu/2)}\Gamma\left(\frac{2-(\nu-2)n}{2}\right)\times\Gamma\left(\frac{\nu-(\nu-2)n}{2}\right)\times F\left(\frac{\nu}{2}(1-n)+n,\left(1-\frac{\nu}{2}\right)n+1;\frac{\nu}{2};-\frac{p^2}{m^2}\right). \tag{29}$$

In (27), (28), and (29) $F(a,b;c;z)$ is the Gauss hypergeometric function.

IV. DISCUSSION

In a way, the method adopted above is the most natural one. It is based on the generalized Fourier transform of causal distributions and the systematic use of Bochner's theorem.

Usually, one works in momentum space and the integrations needed to evaluate loops in Feynman diagrams are regularized to get sensible results. Here, all causal functions in p space are Fourier transformed in a ν-dimensional space, so that propagators (causal functions) in configuration space are ν dependent. This is a natural procedure; the form of the p propagator is kept fixed or ν independent, for example, $(p^2)^{-1}$ or $(p^2+m^2)^{-1}$. This means that the equations of motion are, respectively, the wave equation and the Klein-Gordon equation, whatever the dimensionality of space-time.

The Bochner theorem provides us with a simple tool for the ν-dimensional Fourier transform of spherically symmetric functions; namely, the Bessel transform. We can also handle, at practically no extra cost, the simultaneous presence, between two vertices, of several massless quanta. Furthermore, no use need be made of auxiliary parameters, such as those introduced by Feynman or Bogoliubov and Shirkov.

The principal property of the dimensionally regularized causal functions in x space is the fact that the singularity at the origin is reduced or moderated in a ν-dependent way. The choice of an appropriate region of the ν plane allows the definition of products (or powers) that can be transformed to p space without divergence problems; of course, the usual ultraviolet divergences appear as poles of the analytic functions of ν, for $\nu=4$, or, more generally, for integer ν (although some diagrams that are divergent with ν even are convergent in ν odd).

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