Vacuum energy in a spherically symmetric background field

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The vacuum energy of a scalar field in a spherically symmetric background field is considered. It is expressed through the Jost function of the corresponding scattering problem. The renormalization is discussed in detail and performed using the uniform asymptotic expansion of the Jost function. The method is demonstrated in a simple explicit example. [S0556-2821(96)01610-4]

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I. INTRODUCTION

The evaluation of quantum corrections to classical solutions plays an important role in several areas of modern theoretical physics. The classical solutions involved may be monopoles [1,2], sphalerons [3], and electroweak Skyrmions [4–12]. In general, the classical fields are inhomogeneous configurations. Thus, as a rule, the effective potential approximation to the effective action, where quantum fluctuations are integrated out about a constant classical field, is not expected to be adequate. The derivative expansion [13] improves on this by accounting for spatially varying background fields; being a perturbative approximation it has, however, its own limitations. Having in mind that even the classical solutions are often known only numerically, it is clearly desirable to have a numerical procedure to determine the quantum corrections. Some effort in this direction has already been undertaken [14–17].

The aim of the present article is to develop further a regular analytic approach which reduces the evaluation of quantum corrections to the corresponding quantum-mechanical scattering problem. This approach has been developed for theories in $1+1$ dimensions [18,19]; however, the difficult problem of the summation over the angular momentum necessary in $3+1$ dimensions with spherical symmetry has not been addressed there. It is thus the aim of the present article to develop further the regular analytic approach and apply it to theories in the physically most interesting $(3+1)$-dimensional spacetime. Connected with the angular momentum sum, some aspects of the renormalization are also more complicated here and are discussed in detail.

II. THE MODEL AND ITS RENORMALIZATION

Let us first describe our concrete model and its renormalization introducing also various notations used in the following. We will consider the Lagrangian

$$L=\frac{1}{2}\Phi(\Box-M^2-\lambda\Phi^2)\Phi+\frac{1}{2}\varphi(\Box-m^2-\lambda'\Phi^2)\varphi. \quad (1)$$

Here, the field Φ is a classical background field. By means of

$$V(x)=\lambda'\Phi^2, \quad (2)$$

it defines the potential in (1) for the field $\varphi(x)$, which should be quantized in the background of $V(x)$. The embedding into an external system is necessary in order to guarantee the renormalizability of the ground-state energy.

The complete energy

$$E[\Phi]=E_{\text{class}}[\Phi]+E_{\varphi}[\Phi] \quad (3)$$

of the system consists of the classical part and the one-loop contributions resulting from the ground-state energy of the quantum field φ in the background of the field Φ. The classical part reads

$$E_{\text{class}}[\Phi]=\frac{1}{2}V_g+\frac{1}{2}M^2V_1+\lambda V_2, \quad (4)$$

with the definitions $V_g=\int d^3x(\nabla\Phi)^2$, $V_1=\int d^3x\Phi^2$, and $V_2=\int d^3x\Phi^4$. Here, M^2 and λ are the bare mass and coupling constant, respectively, which need renormalizations as we will explain in the following. For the ground-state energy one defines [20]

$$E_{\varphi}[\Phi]=\frac{1}{2}\sum_{(n)}(\lambda_{(n)}^2+m^2)^{1/2-s}\mu^{2s}, \quad (5)$$

where μ is an arbitrary mass parameter and s is a regularization parameter which has to be put to zero after renormalization. Furthermore, $\lambda_{(n)}$ are the eigenvalues of the corresponding wave equation

$$[-\Delta+V(x)]\phi_{(n)}(x)=\lambda_{(n)}^2\phi_{(n)}(x). \quad (6)$$

For the moment we assume the space to be a large ball of radius R as an intermediate step to have discrete eigenvalues and thus a discrete multi index (n).

It is convenient to express the ground-state energy (5) in terms of the ζ function

$$\zeta_V(s)=\sum_{(n)}(\lambda_{(n)}^2+m^2)^{-s} \quad (7)$$

of the wave operator with potential $V(x)$ as defined in (6) by

$$E_{\varphi}[\Phi]=\frac{1}{2}\zeta_V(s-1/2)\mu^{2s}. \quad (8)$$

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In general, $\zeta_V(-1/2)$ will be a divergent quantity and a renormalization procedure for the definition of $E_\varphi[\Phi]$ is needed. It is easily discussed in terms of the asymptotic $t\to 0$ heat-kernel expansion associated with the wave equation (6):

$$K(t)=\sum_{(n)} \exp(-\lambda_{(n)}^2 t) \underset{t\to 0}{\sim} \left(\frac{1}{4\pi t}\right)^{3/2} e^{-tm^2}\sum_{j=0}^{\infty} A_j t^j. \quad (9)$$

The heat-kernel expansion in Eq. (9) will also contain boundary terms which, in addition to the terms written down explicitly, will lead to half integer powers in t. However, ultimately we are interested in the $R\to\infty$ limit and we will subtract the Minkowski space contribution in order to normalize $E_\varphi[\Phi=0]=0$. For the case that $V(r)\sim r^{-2-\epsilon}$ for $r\to\infty$, $\epsilon>0$, these terms will then disappear.

For the renormalization, only the first three terms are relevant which read explicitly $A_0=\int d^3x$, $A_1=-\int d^3x V(x)$, and $A_2=(1/2)\int d^3x V^2(x)$. Their contributions to the ultraviolet divergencies in the ground-state energy read

$$\begin{aligned} E_\varphi^{\rm div}[\Phi]=&-\frac{m^4}{64\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-\frac{1}{2}\right)A_0\\ &+\frac{m^2}{32\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-1\right)A_1\\ &-\frac{1}{32\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-2\right)A_2. \end{aligned} \quad (10)$$

The first term is a constant independent of the background field Φ and it can be dropped (especially it will be absent after subtraction of the Minkowski space contribution). In a more general context of a gravitational background field, this term would yield a renormalization of the cosmological constant. The second term can be absorbed in a renormalization of the mass M of the background field

$$M^2\to M^2+\frac{\lambda' m^2}{16\pi^2}\left(-\frac{1}{s}+1+\ln\frac{m^2}{4\mu^2}\right) \quad (11)$$

and the third one in the coupling constant λ by

$$\lambda\to\lambda+\frac{\lambda'^2}{64\pi^2}\left(-\frac{1}{s}+2+\ln\frac{m^2}{4\mu^2}\right). \quad (12)$$

The kinetic term V_g in $E[\Phi]$ suffers no renormalization. By defining

$$E_\varphi^{\rm ren}=E_\varphi[\Phi]-E_\varphi^{\rm div}, \quad (13)$$

one obtains the finite ground-state energy, which is normalized in a way that the functional dependence on Φ^2 present in the classical energy is now absent in the quantum corrections $E_\varphi^{\rm ren}$.

Let us note that this is just the well-known general renormalization scheme written down here explicitly in the notation needed in our case.

III. SCATTERING THEORY AND JOST FUNCTIONS

Let us now restrict to a spherically symmetric background field $\Phi(r)$. Then the multi index $(n)\to n,l,m$ consists of the main quantum number n, the angular momentum number l, and the magnetic quantum number m. In polar coordinates, the ansatz for a solution of the wave equation (6) reads

$$\phi_{(n)}(x)=\frac{1}{r}\phi_{n,l}(r)Y_{lm}(\theta,\varphi), \quad (14)$$

where the radial wave equation takes the form

$$\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)+\lambda_{n,l}^2\right]\phi_{n,l}(r)=0. \quad (15)$$

Now, we use the standard scattering theory within $r\in[0,\infty)$ and have the momentum p instead of the discrete $\lambda_{n,l}$. Let $\phi_{p,l}(r)$ be the so-called regular solution which is defined as to have the same behavior at $r\to 0$ as the solution without potential

$$\phi_{p,l}(r)\underset{r\to 0}{\sim} j_l(pr) \quad (16)$$

with the spherical Bessel function j_l [21]. This regular solution defines the Jost function f_l through its asymptotics as $r\to\infty$:

$$\phi_{l,p}(r)\underset{r\to\infty}{\sim}\frac{i}{2}[f_l(p)\hat{h}_l^-(pr)-f_l^*(p)\hat{h}_l^+(pr)], \quad (17)$$

where $\hat{h}_l^-(pr)$ and $\hat{h}_l^+(pr)$ are the Riccati-Hankel functions [21].

Now, we use the Jost function to transform the frequency sum in Eq. (7) in a contour integral. Let us assume for a moment that the support of the potential is contained in the cavity of radius R. Then, the above Eq. (17) gets exact at $r=R$ and may be interpreted as an implicit equation for the eigenvalues $p=\lambda_{n,l}$. Choosing Dirichlet boundary conditions at $r=R$, $\phi_{p,l}(R)=0$, it reads, explicitly,

$$f_l(p)\hat{h}_l^-(pR)-f_l^*(p)\hat{h}_l^+(pR)=0. \quad (18)$$

As already mentioned, ultimately we are interested in the limit $R\to\infty$ and in that limit the results will not depend on the boundary condition chosen, once we assume that $V(r)\sim r^{-2-\epsilon}$ for $r\to\infty$.

Let us now consider the ground-state energy associated with the eigenvalues determined by (18). It is convenient to represent the frequency sum in (18) by a contour integral, the basic idea being explained in detail, for example, in [22,23]. Using Eq. (18), one immediately finds

$$E_{\varphi}[\Phi]=\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_{\gamma}\frac{dp}{2\pi i}(p^2+m^2)^{1/2-s}\frac{\partial}{\partial p}\ln[f_l(p)\hat{h}_l^-(pR)-f_l^*(p)\hat{h}_l^+(pR)]+\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\sum_n(m^2-\kappa_{n,l}^2)^{1/2-s}, \tag{19}$$

with $-\kappa_{n,l}^2$ as the energy eigenvalues of the bound states with given orbital momentum l. The contour γ is chosen counterclockwise enclosing all real solutions of Eq. (18) on the positive real axis. The division of the discrete eigenvalues within the large ball into positive (inside γ) and negative ones ($-\kappa_{n,l}^2$), is determined by the conditions that $V(r)\to 0$ for $r\to\infty$. In that way, in the limit of the infinite space, the negative eigenvalues become the usual bound states and the $\lambda_{n,l}>0$ turn into the scattering states.

For the calculation of (19), as the next step, one deforms the contour γ to the imaginary axis. A contour coming from $i\infty+\epsilon$, crossing the imaginary axis at some positive value smaller than the smallest κ_n and going to $i\infty-\epsilon$ results first. Shifting the contour over the bound state values κ_n which are the zeros of the Jost function on the imaginary axis, the bound state contributions in Eq. (19) are canceled and in the limit $R\to\infty$, subtracting the Minkowski space contributions, one finds

$$E_{\varphi}[\Phi]=-\frac{\cos\pi s}{\pi}\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_m^{\infty}dk[k^2-m^2]^{1/2-s}\times\frac{\partial}{\partial k}\ln f_l(ik). \tag{20}$$

This is the representation of the ground-state energy [and by means of (21) as well of the ζ function] in terms of the Jost function, which is the starting point of our following analysis. It has the nice property that the dependence on the bound states is not present explicitly; it is however contained in the Jost function by its properties on the positive imaginary axis. To the authors' knowledge, this representation of the ground-state energy, respectively, for the ζ function is not known in the literature. Its analogue proved to be useful for explicit numerical calculations already in the case of a potential depending only on one coordinate [19]. It is connected with the more conventional representations by means of the analytic properties of the Jost function. Expressing them by the dispersion relation

$$f_l(ik)=\prod_n\left(1-\frac{\kappa_{n,l}^2}{k^2}\right)\exp\left(-\frac{2}{\pi}\int_0^{\infty}\frac{dq\,q}{q^2+k^2}\delta_l(q)\right),$$

where $\delta_l(q)$ is the scattering phase, we obtain, from (20),

$$E_{\varphi}[\Phi]=\mu^{2s}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\Bigg\{-\sum_n(m^{1-2s}-\sqrt{m^2-\kappa_{n,l}^2}^{\,1-2s})-\frac{1-2s}{\pi}\int_0^{\infty}dq\frac{q}{\sqrt{q^2+m^2}^{\,1-2s}}\delta_l(q)\Bigg\}, \tag{21}$$

which gives the expression of the ground-state energy through the scattering phase. From here, one can pass to the representation through the mode density by integrating by part. Note, that this representation can be obtained also directly from (19) in the limit $R\to\infty$ by deforming the contour γ.

Let us add a discussion on the sign of the ground-state energy. In (21) the first contribution results from the bound states and is completely negative. The second contribution which contains the scattering phase $\delta_l(q)$ is positive (negative) for an attractive (repulsive) potential, i.e., for $V(r)<0$ $[V(r)>0]$ for all[1] r. So, the regularized (still not renormalized) ground-state energy $E_{\varphi}[\Phi]$ (21) is positive for a potential which is repulsive for all r (there is no bound state in this case) and it is negative for a potential which is attractive for all r. Now, if we perform the renormalization in accordance with (11) and (12), we obtain

$$E^{\mathrm{ren}}=E_{\varphi}[\Phi]+\frac{m^2}{8\pi}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-1\right)\int_0^{\infty}dr\,r^2V(r)+\frac{1}{16\pi}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-2\right)\int_0^{\infty}dr\,r^2V(r)^2. \tag{22}$$

The contribution containing A_0 had already been subtracted in $E_{\varphi}[\Phi]$ (20). This expression is finite for $s\to 0$, i.e., when removing the regularization. But because of the subtracted terms, there is no longer any definite result on the sign. Note, that this is in contrast with the case of a one-dimensional potential where it had been possible to express the subtracted terms through the scattering phase [19].

IV. UNIFORM ASYMPTOTICS OF THE JOST FUNCTION

Let us continue with a detailed analysis of the ground-state energy, Eq. (20). As we have easily seen using heat-kernel techniques, the nonrenormalized vacuum energy $E_{\varphi}[\Phi]$ contains divergencies in $s=0$ [see Eq. (10)], which are removed by the renormalization prescription given in Eqs. (11)–(13) [see Eq. (22)]. The poles present are by no means obvious in the representation (20) of $E_{\varphi}[\Phi]$. However, in order to actually perform the renormalization, Eq. (13), it is necessary to represent the ground-state energy equation (20) in a form which makes the subtraction of the divergencies explicit. This will be our first task.

As is known from general ζ function theory, as well as one sees from simply counting the large momentum behavior, the representation equation (20) of $E_{\varphi}[\Phi]$ will be convergent for Re$s>2$. However, for the calculation of the ground-state energy, we need the value of Eq. (20) in $s=0$,

[1]This is a well-known fact from potential scattering, see, e.g., [21].

thus an analytical continuation to the left has to be constructed. The basic idea is the same as the one presented in [22,23]: adding and subtracting the leading uniform asymptotics of the integrand in Eq. (20). Let

$$E_\varphi[\Phi]=E_f+E_{\rm asym}, \tag{23}$$

where

$$E_f=-\frac{\cos(\pi s)}{\pi}\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_m^\infty dk[k^2-m^2]^{1/2-s}\times\frac{\partial}{\partial k}[\ln f_l(ik)-\ln f_l^{\rm asym}(ik)], \tag{24}$$

and

$$E_{\rm asym}=-\frac{\cos(\pi s)}{\pi}\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_m^\infty dk[k^2-m^2]^{1/2-s}\times\frac{\partial}{\partial k}\ln f_l^{\rm asym}(ik). \tag{25}$$

The idea is that as many asymptotic terms are subtracted as to allow to put $s=0$ in the integrand of E_f. This term will then (in general) be evaluated numerically. In $E_{\rm asym}$ the analytic continuation to $s=0$ can be done explicitly showing that the pole contributions cancel when subtracting $E_\varphi^{\rm div}[\Phi]$, Eq. (23). Note, that the contribution resulting from A_0 has already been dropped in Eq. (20).

The first task thus is to obtain the asymptotics of the Jost functions. This may be done by using the integral equation (Lippmann-Schwinger equation) known from scattering theory [21]. For the Jost function, one has ($\nu\equiv l+1/2$)

$$f_l(ik)=1+\int_0^\infty dr\,rV(r)\phi_{l,ik}(r)K_\nu(kr), \tag{26}$$

with the regular solution given by the integral equation

$$\phi_{l,ik}(r)=I_\nu(kr)+\int_0^r dr'r'[I_\nu(kr)K_\nu(kr')-I_\nu(kr')K_\nu(kr)]V(r')\phi_{l,ik}(r'). \tag{27}$$

General ζ function theory tells us that the divergence at $s=0$ contains at most terms of order V^2. Thus, one might expand $\ln f_l(ik)$ in powers of V and take into account only the asymptotics of terms up to $O(V^2)$. The expansion in powers of V is easily obtained. Using Eqs. (26) and (27), one finds

$$\ln f_l(ik)=\int_0^\infty dr\,rV(r)K_\nu(kr)I_\nu(kr)-\int_0^\infty dr\,rV(r)K_\nu^2(kr)\int_0^r dr'r'V(r')I_\nu^2(kr')+O(V^3). \tag{28}$$

Now, the uniform asymptotics for $l\to\infty$ of $\ln f_l(ik)$ is essentially reduced to the well-known uniform asymptotics of the modified Bessel functions K_ν and I_ν [(9.7.7) and (9.7.8) in Ref. [24]]. With the notation $t=1/\sqrt{1+(kr/\nu)^2}$ and $\eta(k)=\sqrt{1+(kr/\nu)^2}+\ln\{(kr/\nu)/[1+\sqrt{1+(kr/\nu)^2}]\}$, one finds, for $\nu\to\infty$, $k\to\infty$ with k/ν fixed,

$$I_\nu(kr)K_\nu(kr)\sim\frac{t}{2\nu}+\frac{t^3}{16\nu^3}(1-6t^2+5t^4)+O(1/\nu^4),$$

$$I_\nu(kr')K_\nu(kr)\sim\frac{1}{2\nu}\frac{e^{-\nu[\eta(k)-\eta(kr'/r)]}}{[1+(kr/\nu)^2]^{1/4}[1+(kr'/\nu)^2]^{1/4}}\times[1+O(1/\nu)].$$

Using these terms in the right-hand side (RHS) of Eq. (28), we define

$$\ln f_l^{\rm asym}(ik)=\frac{1}{2\nu}\int_0^\infty dr\frac{rV(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{1/2}}+\frac{1}{16\nu^3}\int_0^\infty dr\frac{rV(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{3/2}}\left[1-\frac{6}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]}+\frac{5}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^2}\right]-\frac{1}{8\nu^3}\int_0^\infty dr\frac{r^3V^2(r)}{\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{3/2}}, \tag{29}$$

thereby the r' integration in the term quadratic in V has been performed by the saddle-point method using the monotony of $\eta(k)$. Now, by means of (29), the limit $s\to0$ can be performed in Eq. (24), and we obtain

$$E_f=-\frac{1}{\pi}\sum_{l=0}^{\infty}(l+1/2)\int_m^\infty dk\sqrt{k^2-m^2}\frac{\partial}{\partial k}[\ln f_l(ik)-\ln f_l^{\rm asym}(ik)], \tag{30}$$

a form which is suited for a numerical evaluation.

For $E_{\rm asym}$ at $s=0$, one might explicitly find the analytical continuation. First of all, the k integrations may be done, using

$$\int_m^\infty dk[k^2-m^2]^{1/2-s}\frac{\partial}{\partial k}\left[1+\left(\frac{kr}{\nu}\right)^2\right]^{-n/2}$$

$$=-\frac{\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(\frac{3}{2}-s\right)}{\Gamma(n/2)}\frac{\left(\frac{\nu}{mr}\right)^n m^{1-2s}}{\left(1+\left(\frac{\nu}{mr}\right)^2\right)^{s+(n-1)/2}} \tag{31}$$

to yield Eq. (A1) in the Appendix A. From this, the renormalization, Eq. (26), can be carried out and we arrive at

$$E_{\rm asym}-E_\varphi^{\rm div}=\frac{1}{48\pi}\int_0^\infty dr V(r)\left[6m^2r^2-12\zeta_R'(-1)-3\ln 2 -6r^2\ln(16m^2r^2)\left(m^2+\frac{1}{2}V(r)\right) -\gamma[1+12m^2r^2+6r^2V(r)]\right]+E_{\rm asym}^{\rm sum}.$$

The complete energy $E[\Phi]$, Eq. (16), consisting of $E_{\rm class}[\Phi]$, Eq. (17), of the background field Φ and of the renormalized loop contribution (26),

$$E_\varphi^{\rm ren}=E_\varphi[\Phi]-E_\varphi^{\rm div}=E_f+E_{\rm asym}-E_\varphi^{\rm div}, \tag{32}$$

reads

$$E[\Phi]=\frac{1}{2}V_g+\frac{1}{2}M_{\rm ren}^2V_1+\lambda_{\rm ren}V_2+\frac{1}{48\pi}\int_0^\infty dr V(r)\{6m^2r^2 -12\zeta_R'(-1)-3\ \ln 2-6r^2\ln(16m^2r^2)[m^2+\tfrac{1}{2}V(r)] -\gamma[1+12m^2r^2+6r^2V(r)]\}+E_f+E_{\rm asym}^{\rm sum}. \tag{33}$$

The remaining task for the analysis of the ground-state energy in the presence of a spherically symmetric potential is the numerical analysis of the above quantity. To achieve that, a (as a rule numerical) knowledge of the Jost function $f_l(ik)$ is necessary. Apart from this, only integrals of the potentials and convergent sums have to be dealt with which present no problem.

V. EXAMPLE: SQUARE-WELL POTENTIAL

If the potential has a compact support, there is a formalism how to obtain the Jost function (at least in principle). Starting point is the observation that one may write the regular solution in the form

$$\phi_{l,p}(r)=u_{l,p}(r)\Theta(R-r)+\frac{i}{2}[f_l(p)\hat{h}_l^-(pr) -f_l^*(p)\hat{h}_l^+(pr)]\Theta(r-R), \tag{34}$$

where $V(r)=0$ for $r\geqslant R$ is assumed. The matching conditions then read

$$u_{l,p}(R)=\frac{i}{2}[f_l(p)\hat{h}_l^-(pR)-f_l^*(p)\hat{h}_l^+(pR)],$$

$$u_{l,p}'(R)=\frac{i}{2}p[f_l(p)\hat{h}_l^{-\prime}(pR)-f_l^*(p)\hat{h}_l^{+\prime}(pR)].$$

Combining the two equations and using the fact that the Wronskian determinant of $\hat{h}_l^\pm$ is $2i$, one arrives at

$$f_l(p)=-\frac{1}{p}[pu_{l,p}(R)\hat{h}_l^{+\prime}(pR)-u_{l,p}'(R)\hat{h}_l^+(pR)], \tag{35}$$

which gives the expression of the Jost function for a potential with a compact support through the wave function.

For the square-well potential, $V(r)=V_0\Theta(R-r)$, it is easily seen that

$$u_{l,p}(r)=\left(\frac{p}{\tilde{q}}\right)^{l+1}\hat{j}_l(\tilde{q}r)$$

with $\hat{j}_l$ the Riccati-Bessel function and $\tilde{q}=\sqrt{p^2-V_0}$.

So we obtain the well-known formula

$$f_l^{\rm SW}(ik)=R\left(\frac{k}{q}\right)^\nu[qI_\nu'(qR)K_\nu(kR)-kI_\nu(qR)K_\nu'(kR)], \tag{36}$$

with $q=\sqrt{k^2+V_0}$. This has to be used for the numerical evaluation of Eq. (33). Because of the simple form of the potential $V(r)$, the r integrals may be done explicitly and numerically, and easy tractable expressions result. For E_f, we obtain

$$E_f=-\frac{1}{\pi}\sum_{l=0}^\infty (l+1/2)\int_m^\infty dk[k^2-m^2]^{1/2}\frac{\partial}{\partial k}\left\{\ln f_l^{\rm SW}(ik)-\frac{V_0\nu}{2k^2}\left(\left[1+\left(\frac{kR}{\nu}\right)^2\right]^{1/2}-1\right)+\frac{V_0}{16\nu k^2}\left(\left[1+\left(\frac{kR}{\nu}\right)^2\right]^{-1/2} -2\left[1+\left(\frac{kR}{\nu}\right)^2\right]^{-3/2}+\left[1+\left(\frac{kR}{\nu}\right)^2\right]^{-5/2}\right)+\frac{V_0^2}{8\nu k^2}\left[\frac{R^2+2(\nu/k)^2}{[1+(kR/\nu)^2]^{1/2}}-2\left(\frac{\nu}{k}\right)^2\right]\right\}, \tag{37}$$

which is, as one might easily check, a finite expression. For the renormalized contributions of the asymptotic terms, we obtain

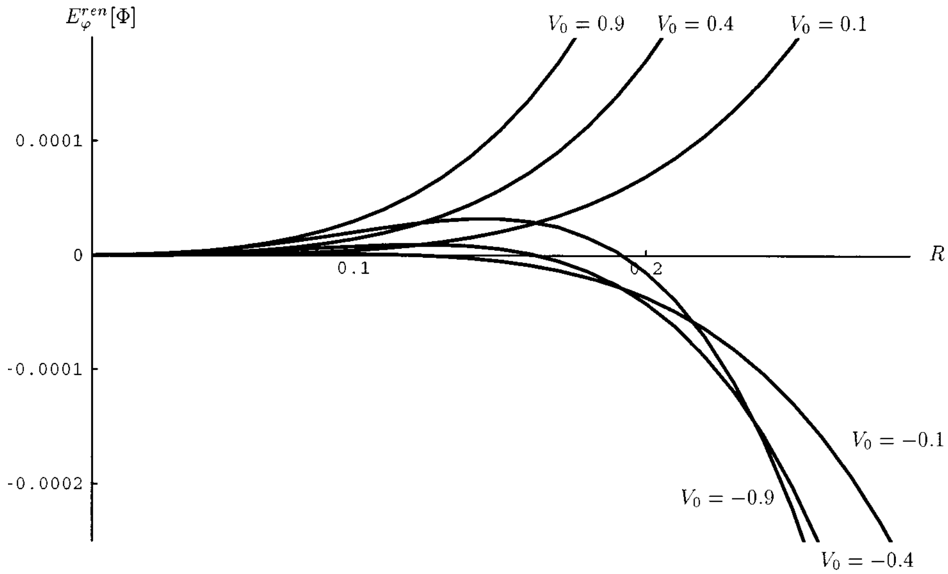

FIG. 1. The complete energy as a function of the radius R for different values of the height of the potential wall V_0 for $m=1$.

$$E_{\rm asym}-E_\varphi^{\rm div}=\frac{V_0R}{48\pi}\bigg[-12\zeta_R'(-1)+\frac{10}{3}m^2R^2-3\ \ln 2-2m^2R^2\ln(16m^2R^2)+\frac{2}{3}V_0R^2-V_0R^2\ln(16m^2R^2)$$
$$-\gamma(1+4m^2R^2+2V_0R^2)\bigg]+E_{\rm asym,SW}^{\rm sum}, \tag{38}$$

where $E_{\rm asym,SW}^{\rm sum}$ is given in Appendix B. It is presented in a form of quickly convergent series which allow an easy numerical analysis.

The results for $E_\varphi^{\rm ren}[\Phi]$ are presented in Fig. 1. For small values of the radius R of the support of the potential, it is seen that even for negative values of the potential, $V_0<0$, a positive vacuum energy results. The reason is that, for small values of R, the bound state energies are located closely to zero, giving only small negative contributions. For increasing R, their number and their values increase, leading necessarily to negative vacuum energies.

The behavior described is the one we expected from the one-dimensional results presented in [19]. However, the absolute orders of magnitude are lower by 2 than in the corresponding one-dimensional considerations.

VI. CONCLUSIONS

In this article we reduced the task of calculating the vacuum energy of a scalar field in the presence of a spherically symmetric background field to the corresponding quantum-mechanical scattering problem. We were able to present the renormalized vacuum energy solely in terms of the quantum-mechanical scattering dates summarized by the Jost function. For the example of the square-well potential, we showed that a direct numerical analysis of the vacuum energy is possible.

Several extensions of our approach are necessary. From the physical point of view, the consideration of higher-spin fields is necessary and envisaged. In addition, in order to apply our formalism to classical background fields such as, for example, sphalerons and electroweak Skyrmions, a numerical analysis of Eq. (30) is necessary for cases when the Jost function is known only numerically. However, the scattering theory developed during the last few decades provides many techniques and results so that here also progress seems possible.

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APPENDIX A: CONTRIBUTIONS OF THE ASYMPTOTIC TERMS TO THE ζ FUNCTION

Here, we calculate the analytic continuation of $E_{\rm asym}$ (25) using the asymptotics of the Jost functions $\ln f_l^{\rm asym}(ik)$ (29). After carrying out the k integration by means of (31), we obtain

$$E_{\rm asym}=-\frac{\Gamma(s)\mu^{2s}}{2\sqrt{\pi}m^{2s}\Gamma(s-1/2)}\int_0^\infty dr V(r)\sum_{l=0}^\infty \nu\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s}-\frac{m^{1-2s}\mu^{2s}}{2\sqrt{\pi}\Gamma(s-1/2)}\int_0^\infty dr r V(r)\sum_{l=0}^\infty \left\{\frac{\Gamma(s+1)}{4(mr)^3}\right.$$
$$\times\nu\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s-1}-\frac{\Gamma(s+2)}{(mr)^5}\nu^3\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s-2}+\left.\frac{\Gamma(s+3)}{3(mr)^7}\nu^5\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s-3}\right\}$$
$$+\frac{\Gamma(s+1)\mu^{2s}}{4\sqrt{\pi}m^{2s+2}\Gamma(s-1/2)}\int_0^\infty dr V^2(r)\sum_{l=0}^\infty \nu\left[1+\left(\frac{\nu}{mr}\right)^2\right]^{-s-1}. \tag{A1}$$

The analytical continuations of the above expressions to the value $s=0$ are easily obtained and listed below:

$$E_{\rm asym}=\frac{1}{96\pi}\int_0^\infty dr V(r)(1-12m^2r^2)\left(\frac{1}{s}+\ln r^2\mu^2\right)+\frac{1}{48\pi}\int_0^\infty dr V(r)[-1-12\zeta_R'(-1)+12m^2r^2(1-\gamma-3\ \ln 2)]$$
$$-\frac{1}{96\pi}\int_0^\infty dr V(r)\left(\frac{1}{s}+\ln r^2\mu^2\right)+\frac{1}{48\pi}\int_0^\infty dr V(r)[1-\gamma-3\ \ln 2]-\frac{1}{16\pi}\int_0^\infty dr r^2V^2(r)\left(\frac{1}{s}+\ln r^2\mu^2\right)$$
$$+\frac{1}{8\pi}\int_0^\infty dr r^2V^2(r)(1-\gamma-3\ \ln 2)+E_{\rm asym}^{\rm sum}+O(s), \tag{A2}$$

where the sums in

$$E_{\rm asym}^{\rm sum}=\frac{1}{4\pi}\int_0^\infty dr V(r)\sum_{l=0}^\infty \nu\left[\left(\frac{mr}{\nu}\right)^2-\ln\left(1+\left(\frac{mr}{\nu}\right)^2\right)\right]+\frac{1}{4\pi}\int_0^\infty dr V(r)\sum_{l=0}^\infty \frac{1}{\nu}\left\{\frac{1}{4}\left[1+\left(\frac{mr}{\nu}\right)^2\right]^{-1}-\left[1+\left(\frac{mr}{\nu}\right)^2\right]^{-2}\right.$$
$$\left.+\frac{2}{3}\left[1+\left(\frac{mr}{\nu}\right)^2\right]^{-3}+\frac{1}{12}\right\}-\frac{1}{8\pi}\int_0^\infty dr r^2V^2(r)\sum_{l=0}^\infty \frac{1}{\nu}\left(\left[1+\left(\frac{mr}{\nu}\right)^2\right]^{-1}-1\right) \tag{A3}$$

converge.

These expressions have been the basis to give the results for the special example listed in Appendix B.

APPENDIX B: ASYMPTOTIC CONTRIBUTIONS FOR THE SQUARE-WELL POTENTIAL

In this appendix, we give the result for $E_{\rm as,SW}^{\rm sum}$ which reads

$$E_{\rm asym,SW}^{\rm sum}=\frac{V_0R}{4\pi}\sum_{l=0}^\infty \nu\left\{\frac{1}{3}\left(\frac{mR}{\nu}\right)^2-\ln\left[1+\left(\frac{mR}{\nu}\right)^2\right]+2-\frac{2\nu}{Rm}\arctan\left(\frac{Rm}{\nu}\right)\right\}+\frac{V_0R}{4\pi}\sum_{l=0}^\infty \frac{1}{\nu}\left\{-\frac{1}{4}\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-1}\right.$$
$$\left.+\frac{1}{6}\left[1+\left(\frac{mR}{\nu}\right)^2\right]^{-2}+\frac{1}{12}\right\}-\frac{V_0^2R^3}{8\pi}\sum_{l=0}^\infty \frac{1}{\nu}\left\{\left(\frac{\nu}{mR}\right)^2-\left(\frac{\nu}{mR}\right)^3\arctan\left(\frac{mR}{\nu}\right)-\frac{1}{3}\right\}.$$

These sums have been used for the numerical analysis of the ground-state energy in the presence of a square-well potential.

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