

Standard model with gravity couplings

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In this paper we examine the coupling of matter fields to gravity within the framework of the standard model of particle physics. The coupling is described in terms of Weyl fermions of a definite chirality, and employs only (anti-)self-dual or left-handed spin connection fields. We review the general framework for introducing the coupling using these fields, and show that conditions ensuring the cancellation of perturbative chiral gauge anomalies are not disturbed. We also explore a global anomaly associated with the theory, and argue that its removal requires that the number of fundamental fermions in the theory must be multiples of 16. In addition, we investigate the behavior of the theory under discrete transformations P , C , and T , and discuss possible violations of these discrete symmetries, including CPT , in the presence of instantons and the Adler-Bell-Jackiw anomaly.

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I. INTRODUCTION

Ashtekar [1] has introduced a set of variables to describe gravity, which makes essential use of the chiral decomposition of the connection one-forms of the local Lorentz group. What has been shown is that, at least for Einstein manifolds without matter, the full set of field equations of general relativity can be recovered by use of only one of the two chiral projections of these connection forms. We may either use the self-dual or the anti-self-dual connections and their respective conjugate variables. The constraints and reality conditions of the theory then define what the other set has to be.

Chiral projections are used routinely in particle physics; indeed, the fermion fields that define the standard model are all chiral Weyl spinor fields. Within the Ashtekar context, left-handed spinor fields are coupled to one of these connection forms, say A^- , while right-handed ones are coupled to the other. Since only one of these is all one needs to define general relativity, the actual Lagrangian must be expressed entirely in terms of either left-handed or right-handed Weyl spinor fields.

That the standard model can be so described is of course already known. For example, in $SO(10)$ grand unification schemes [2], one employs a single 16-dimensional left-handed Weyl field to describe one generation of fermions. What is less clear is how the coupling of Ashtekar fields affects the resultant physics.

In what follows, we describe some facets of these consequences. Since Ashtekar gravity makes use of only one of the chirally projected Lorentz connections, there arises the question of whether anomalies which are normally present in such theories are under control. We show that the usual con-

ditions for anomaly cancellations for the standard model gauge groups remain true in the presence of Ashtekar gravity, and that the new fields do not introduce any further perturbative anomalies. However, they do introduce global anomalies. Cancellation of these obstructions for grand unified theories (GUT's) in the most general context results in the rather strong constraint that the total number of fundamental fermions in the theory must be a multiple of 16. Grand unification schemes based upon groups such as $SU(5)$ are therefore inconsistent when coupled to gravity. As a consequence, there is every likelihood that neutrinos must be massive when consistency with gravity is taken into account. In particular, the $SO(10)$ GUT with 16 fundamental Weyl fermions per generation is singled out as the preeminent and simplest choice free from the global anomaly, if one allows for generalized spin structures and arbitrary topologies.

We also discuss how the usual discrete symmetries are implemented in the presence of Ashtekar gravity. We shall show that it is possible to posit discrete transformation laws for the fermion and Ashtekar fields which are consistent both with our general notions of what parity, time-reversal, and charge conjugation transformations are, and with the fundamental canonical commutation relations of all of the fields. Discrete symmetries for bispinors can be implemented in the classical limit modulo certain reality conditions. However, what we shall show is that, inevitably, the underlying quantum theory is not invariant under parity due to the occurrence axial anomaly in quantum field theory. The question of CPT invariance will also be briefly discussed.

II. THE SAMUEL-JACOBSON-SMOLIN ACTION AND THE ASHTEKAR VARIABLES

We first consider spacetimes of Lorentzian signature $(-, +, +, +)$ and start with the gravitational action proposed by Samuel, and Jacobson and Smolin [3]

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$$\begin{aligned}\mathcal{S}_{\text{SJS}}^{\mp} &= \int_M L_G^{\mp} \\ &= \frac{1}{8\pi G} \int_M \Sigma^{\mp a} \wedge F_a^{\mp} \pm i \frac{\lambda}{3(16\pi G)} \int_M \Sigma^{\mp a} \wedge \Sigma_a^{\mp}.\end{aligned}\quad (1)$$

The (anti-)self-dual two-forms Σ^{\mp} which obey $*\Sigma^{\mp a} = \mp i \Sigma^{\mp a}$ are defined as

$$\Sigma^{\mp a} \equiv \left(-e^0 \wedge e^a \pm \frac{i}{2} \epsilon^a{}_{bc} e^b \wedge e^c \right). \quad (2)$$

F^{\mp} are the curvature two-forms of the $\text{SO}(3, \mathbb{C})$ Ashtekar connections, i.e.,

$$F_a^{\mp} = dA_a^{\mp} + \frac{1}{2} \epsilon^a{}_{bc} A_b^{\mp} \wedge A_c^{\mp}, \quad (3)$$

and $e_A, A=0, \dots, 3$ denote the vierbein one-forms in four dimensions, $\epsilon_{abc} \equiv \epsilon_{0abc}$, while λ is the cosmological constant. Latin indices label flat Lorentz indices while spacetime indices will be denoted by Greek indices. Lower case Latin indices run from 1 to 3 while upper case Latin indices range from 0 to 3.

The Ashtekar variables [1] are sometimes referred to as (anti-)self-dual variables because the equations of motion of the Samuel-Jacobson-Smolin action with respect to A^{\mp} ,

$$D^{\mp} \Sigma^{\mp a} = 0, \quad (4)$$

imply that A^{-} and A^{+} are the anti-self-dual and self-dual part of the spin connection ω , respectively: i.e.,

$$A_a^{\mp} = \pm i \omega_{0a} - \frac{1}{2} \epsilon^a{}_{bc} \omega_{bc}. \quad (5)$$

It is easy to see that the action reproduces the Ashtekar variables and constraints. For convenience, we work in the spatial gauge in which the components of the vierbein and its inverse can be written in the form

$$e_{A\mu} = \begin{bmatrix} N & 0 \\ N^j e_{aj} & e_{ai} \end{bmatrix}, \quad E^{\mu A} = \begin{bmatrix} N^{-1} & 0 \\ -(N^i/N) & \sigma^i_a \end{bmatrix}. \quad (6)$$

The form assumed in (6) is compatible with the Arnowitt-Deser-Misner (ADM) [4] decomposition of the metric

$$\begin{aligned}ds^2 &= e_{A\mu} e^A{}_{\nu} dx^{\mu} dx^{\nu} \\ &= -N^2 (dx^0)^2 + g_{ij} (dx^i + N^i dx^0) (dx^j + N^j dx^0)\end{aligned}\quad (7)$$

with the spatial metric $g_{ij} = e^a{}_i e_{aj}$. Thus we see that the choice (6) in no way compromises the values of the lapse and shift functions, N and N^i , which have geometrical interpretations in hypersurface deformations. With this decomposition, it is straightforward to rewrite

$$\begin{aligned}\mathcal{S}_{\text{SJS}}^{\mp} &= \frac{1}{16\pi G} \int d^4x \{ \pm 2i \tilde{\sigma}^{ia} \dot{A}_{ia}^{\mp} \pm 2i A_{0a}^{\mp} D_i \tilde{\sigma}^{ia} \pm 2i N^j \tilde{\sigma}^{ia} F_{ija}^{\mp} \} \\ &\quad - \frac{1}{16\pi G} \int d^4x \left\{ N \left(\epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} F_{ij}^{\mp c} + \frac{\lambda}{3} \epsilon_{abc} \xi_{ijk} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} \tilde{\sigma}^{kc} \right) \right\} + \text{boundary terms},\end{aligned}\quad (8)$$

with $\tilde{\sigma}$ and N defined as

$$\begin{aligned}\tilde{\sigma}^{ia} &\equiv \frac{1}{2} \tilde{\epsilon}^{ijk} \epsilon^{abc} e_{jb} e_{kc}, \\ N &\equiv \det(e_{ai})^{-1} N.\end{aligned}\quad (9)$$

The tildes above and below the variables indicate that they are tensor densities of weight 1 and -1 , respectively. Therefore, $\pm(2i\tilde{\sigma}^{ia}/16\pi G)$ are readily identified as the conjugate variables to A_{ia}^{\mp} , and we have the commutation relations

$$[\tilde{\sigma}^{ia}(\vec{x}, t), A_{jb}^{\mp}(\vec{y}, t)] = \pm(8\pi G) \delta_j^a \delta_b^i \delta^3(\vec{x} - \vec{y}). \quad (10)$$

The variables A_{0a}^{\mp} , N^i , and N are clearly Lagrange multipliers for the Ashtekar constraints, which can be identified as Gauss' law generating $\text{SO}(3)$ gauge invariance,

$$G^a \equiv 2i D_i \tilde{\sigma}^{ia} \approx 0, \quad (11)$$

and the supermomentum and super-Hamiltonian constraints

$$H_i \equiv 2i \tilde{\sigma}^{ja} F_{ija} \approx 0, \quad (12)$$

$$H \equiv \epsilon_{abc} \tilde{\sigma}^{ia} \tilde{\sigma}^{jb} \left(F_{ij}^c + \frac{\lambda}{3} \xi_{ijk} \tilde{\sigma}^{kc} \right) \approx 0,$$

respectively. Ashtekar showed that these constraints and their algebra, despite their remarkable simplicity, are equivalent in content to the constraints and constraint algebra of general relativity [1]. Note that both the self-dual and anti-self-dual versions, $\mathcal{S}_{\text{SJS}}^{\pm}$, describe *pure* gravity equally well.

III. COUPLING TO MATTER FIELDS

The coupling of matter fields to gravity described by the (anti-)self-dual Ashtekar variables have been considered by others before [5–7], and will be examined closely in this work. It should be emphasized that besides self-interactions in pure gravity, only fermions couple directly to the Ashtekar connections. They are hence direct sources for the Ashtekar connection. Conventional scalars and Yang-Mills fields have direct couplings only to the metric (hence, to $\tilde{\sigma}$) rather than

to the Ashtekar-Sen connection. Thus when one substitutes the conventional gravitational action with the Samuel-Jacobson-Smolin action, complications can come from the fermionic sector, primarily because one has to make a choice between the actions described by A^+ and A^- and once the choice is made, it is permissible to couple only right or left-handed fermions (but not both) to the theory.¹

We first look at the conventional Dirac action, with couplings to ordinary spin connections. We then describe couplings of Ashtekar fields to fermions, and explore the physical implications in the following sections.

Consider the conventional Dirac action for an electron or a single quark of a particular color in the presence of only the conventional gravitational field. The action can be written as

$$\mathcal{S}_D = -\frac{i}{2} \int_M (*1) \bar{\Psi} \gamma^A E_A \lrcorner D_\omega \Psi + \text{H. c.}, \quad (13)$$

where the covariant derivative with respect to the spin connection, ω , is defined by

$$D_\omega \Psi = dx^\mu (\partial_\mu + \frac{1}{2} \omega_{\mu BC} \mathcal{S}^{BC}) \Psi \quad (14)$$

with the generator $\mathcal{S}^{AB} = \frac{1}{4} [\gamma^A, \gamma^B]$. We adopt the convention

$$\{\gamma^A, \gamma^B\} = 2 \eta^{AB} \quad (15)$$

with $\eta^{AB} = \text{diag}(-1, +1, +1, +1)$.

We use the chiral representation henceforth for convenience and clarity. In the chiral representation

$$\gamma^A = \begin{pmatrix} 0 & i\tau^A \\ i\bar{\tau}^A & 0 \end{pmatrix}, \quad (16)$$

where $\tau^a = -\bar{\tau}^a$ ($a=1,2,3$) are Pauli matrices, and $\tau^0 = \bar{\tau}^0 = -I_2$. We also have

$$\begin{aligned} \gamma^5 &= i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \\ &= \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \end{aligned} \quad (17)$$

and the Dirac bispinor is expressed in terms of two-component left and right-handed Weyl spinors $\phi_{L,R}$ as

$$\Psi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}. \quad (18)$$

The contractions in (13) are defined by

$$E_A \lrcorner D \Psi \equiv E^\mu_A \mathcal{S}_\mu \Psi. \quad (19)$$

Note that the vierbein vector fields and one-forms, $E_A = E^\mu_A \partial_\mu$ and $e^A = e^A_\mu dx^\mu$, satisfy $E_A \lrcorner e^B = \delta_A^B$.

In the chiral representation, the covariant derivative can be written as

$$D \Psi \equiv dx^\mu \left\{ \partial_\mu I_4 - i \begin{pmatrix} A^+_{\mu a} \frac{\tau^a}{2} & 0 \\ 0 & A^-_{\mu a} \frac{\tau^a}{2} \end{pmatrix} \right\} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}, \quad (20)$$

and in conventional coupling of fermions to spin connections, A^\mp are, precisely,

$$A^\mp_a = \pm i \omega_{0a} - \frac{1}{2} \epsilon_a{}^{bc} \omega_{bc}. \quad (21)$$

Written in terms of the Weyl spinors, the Dirac action is

$$\begin{aligned} \mathcal{S}_D &= \underbrace{\int_M (*1) \left(-\frac{i}{2} \right) \phi_L^\dagger \tau^A E_A \lrcorner D^- \phi_L}_{(1)} + \underbrace{\int_M (*1) \left(\frac{i}{2} \right) E_A \lrcorner (D^- \phi_L)^\dagger \tau^A \phi_L}_{(2)} \\ &+ \underbrace{\int_M (*1) \left(-\frac{i}{2} \right) \phi_R^\dagger E_A \lrcorner D^+ \phi_R}_{(3)} + \underbrace{\int_M (*1) \left(\frac{i}{2} \right) E_A \lrcorner (D^+ \phi_R)^\dagger \tau^A \phi_R}_{(4)}, \end{aligned} \quad (22)$$

where

¹We shall choose the $-$ action. There is an ambiguity in the conventions of the Dirac matrices, γ^A , which allows one to couple either A^+ or A^- to left-handed Weyl spinors. We have adopted a choice which couples anti-self-dual spin connections and A^- to left-handed spinors. To the extent that there are no right-handed neutrinos in nature, we should describe nature with only left-handed Weyl spinors. So once the initial choice of coupling A^- to the neutrino is made, the theory cannot be described by A^+ and *right-handed* Weyl spinors.

$$D^- \phi_L \equiv \left(d - i A_a^- \frac{\tau^a}{2} \right) \phi_L, \quad D^+ \phi_R \equiv \left(d - i A_a^+ \frac{\tau^a}{2} \right) \phi_R,$$

$$(D^- \phi_L)^\dagger \equiv d \phi_L^\dagger + i \phi_L^\dagger A_a^+ \frac{\tau^a}{2}, \quad (23)$$

$$(D^+ \phi_R)^\dagger \equiv d \phi_R^\dagger + i \phi_R^\dagger A_a^- \frac{\tau^a}{2}.$$

Notice that in (22), terms (1) and (4) contain A^- but not A^+ , while (2) and (3) involve A^+ but not A^- . Furthermore, it is well known that right-handed spinors can be written in terms of left-handed ones through the relation

$$\phi_R = -i \tau^2 \chi_L^* \quad (24)$$

and vice versa, so the term (4) which involves A^- can be written in terms of totally *left-handed* (anti-self-dual) spin connections and Weyl spinors as

$$\int_M (*1) \left(-\frac{i}{2} \right) \chi_L^\dagger \tau^A E_{A|} D^- \chi_L. \quad (25)$$

Similar remarks apply to the term (2) with regard to right-handed spinors and the field A^+ .

In order to couple spinors to Ashtekar connections of only one chirality, we can now define modified ‘‘chiral’’ Dirac actions

$$\begin{aligned} \mathcal{S}_D^- &\equiv 2\{(1)+(4)\} \\ &= \int_M (*1) (-i) (\phi_L^\dagger \tau^A E_{A|} D^- \phi_L + \chi_L^\dagger \tau^A E_{A|} D^- \chi_L), \end{aligned} \quad (26)$$

$$\begin{aligned} \mathcal{S}_D^+ &\equiv 2\{(2)+(3)\} \\ &= \int_M (*1) (-i) [\phi_R^\dagger \bar{\tau}^A E_{A|} D^+ \phi_R + E_{A|} (D^- \phi_L)^\dagger \tau^A \phi_L]. \end{aligned} \quad (27)$$

Next, we make the identification that the quantities A^\mp , as anticipated by the notation, are precisely the connections introduced by Ashtekar in his simplification of the constraints of general relativity, and the very same variables in the Samuel-Jacobson-Smolin action of Eq. (1). Instead of the conventional sum of the Einstein-Hilbert-Palatini action and the full Dirac action, \mathcal{S}_D , the total action is now taken to be $(\mathcal{S}_D^- + \mathcal{S}_{\text{SJS}})$. Remarkably, it is possible to show [for instance, by using Eq. (37)] that this total action reproduces the correct classical equations of motion for general relativity with spinors [5,8].

IV. DISCRETE TRANSFORMATIONS AND SPINORS

The usual discrete transformations for the second quantized spinor fields are

$$P: \phi_L(x) \leftrightarrow -i \tau^2 \chi_L^*(P^{-1}(x)), \quad \text{i.e., } [\phi_L(x) \leftrightarrow \phi_R(P^{-1}(x))],$$

$$C: \phi_L(x) \leftrightarrow \chi_L(x), \quad \text{i.e., } (\Psi^c = C \bar{\Psi}^T),$$

$$T: \phi_L(x) \mapsto -i \tau^2 \phi_L(T^{-1}(x)). \quad (28)$$

A set of discrete transformations for the gravity variables consistent with the ones described above for fermion fields can then be defined as follows:

$$P: (\bar{\sigma}^{ia}(\vec{x}, t), A_{ia}^\mp(\vec{x}, t)) \mapsto (\bar{\sigma}^{ia}(P^{-1}(\vec{x}, t)), A_{ia}^\pm(P^{-1}(\vec{x}, t))),$$

$$C: (\bar{\sigma}^{ia}(\vec{x}, t), A_{ia}^\mp(\vec{x}, t)) \mapsto (\bar{\sigma}^{ia}(\vec{x}, t), A_{ia}^\mp(\vec{x}, t)), \quad (29)$$

$$T: (\bar{\sigma}^{ia}(\vec{x}, t), A_{ia}^\mp(\vec{x}, t)) \mapsto (\bar{\sigma}^{ia}(T^{-1}(\vec{x}, t)), A_{ia}^\mp(T^{-1}(\vec{x}, t))).$$

These transformations are consistent with the commutation relations (10). In dealing with curved spacetimes, it is more convenient to rewrite these transformations in terms of their action on the one-forms (e^A, A_a^\mp) . These give

$$P: (e^0, e^a; A_a^\mp) \mapsto (e^0, -e^a; A_a^\pm),$$

$$C: (e^A; A_a^\mp) \mapsto (e^A; A_a^\mp), \quad (30)$$

$$T: (e^0, e^a; A_a^\mp) \mapsto (-e^0, e^a; A_a^\mp).$$

P and T transformations for the Ashtekar variables for pure gravity have been discussed previously [9]. However, that discussion did not cover C and CPT , nor take into account the effect of coupling to fermions. Note also that P and T are orientation-reversing transformations. While P and C are to be implemented by unitary transformations, T is to be implemented anti-unitarily, so that under T , c numbers are complex conjugated.

We emphasize that the Ashtekar variables are however not necessarily orientation-reversal invariant. In fact there are four-manifolds with no orientation reversing diffeomorphisms. The signature invariant, τ , of a four-manifold is odd under orientation reversal. Therefore a manifold with nonvanishing τ cannot possess an orientation reversing diffeomorphism. One can show that for the Ashtekar variables, $\int_M F_a^- \wedge F^{-a} - \int_M F_a^+ \wedge F^{+a} \propto \tau$ [10]. Thus for manifolds with nonvanishing τ 's, A^+ can neither be diffeomorphic nor $\text{SO}(3, C)$ gauge equivalent to A^- .

To discuss the effect of the discrete transformations in a concise manner, we first define $A_{AB} = -A_{BA}$ such that

$$A_{0a} \equiv \frac{1}{2i} (A_a^- - A_a^+), \quad (31)$$

$$A_{bc} \equiv -\frac{1}{2} \epsilon^a{}_{bc} (A_a^- + A_a^+),$$

and keep in mind the effect of the discrete transformations displayed in (30).

The curvature

$$F_{AB} \equiv dA_{AB} + A_A^C \wedge A_{CB} \quad (32)$$

then has components

$$F_{0a} = \frac{1}{2i}(F_a^- - F_a^+), \quad F_{bc} = -\frac{1}{2}\epsilon^a{}_{bc}(F_a^- + F_a^+), \quad (33)$$

and the ‘‘torsion’’ $T_A(A^\mp)$, which depends on A^\mp , is defined as

$$T^A \equiv de^A + A^A{}_B \wedge e^B. \quad (34)$$

With all this, it can be shown that the ‘‘chiral’’ gravitational and Dirac actions are related to the conventional ones by

$$\begin{aligned} \mathcal{S}_{\text{SJS}}^\mp &= \frac{1}{(16\pi G)} \int_M e^A \wedge e^B \wedge *F_{AB} \\ &\mp \frac{i}{(16\pi G)} \int_M [-d(e^A \wedge T_A) + T^A \wedge T_A] \\ &- \frac{2\lambda}{16\pi G} \int_M e^0 \wedge e^1 \wedge e^2 \wedge e^3 \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathcal{S}_D^\mp &= \mathcal{S}_D + \int_M \left[\pm \frac{i}{4} \psi^A \epsilon_{ABCD} e^C \wedge e^D \wedge T^B \right. \\ &\left. \mp \frac{i}{2(3!)} d(\psi^A \epsilon_{ABCD} e^B \wedge e^C \wedge e^D) \right], \end{aligned} \quad (36)$$

where $\psi^A \equiv \phi_L^\dagger \tau^A \phi_L + \chi_L^\dagger \tau^A \chi_L$.

The combined fermionic and gravitational total action is, therefore,

$$\begin{aligned} \mathcal{S}_{\text{total}}^\mp &= \mathcal{S}_{\mathcal{G}}^\mp + \mathcal{S}_{\text{SJS}}^\mp \\ &= \mathcal{S}_{\mathcal{G}} + \frac{1}{(16\pi G)} \int_M e^A \wedge e^B \wedge *F_{AB} \\ &\mp \frac{i}{2} \int_M d \left\{ -\frac{1}{(8\pi G)} e^A \wedge T_A \right. \\ &\left. + \frac{1}{3!} (\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D) \right\} \\ &\mp \frac{i}{(16\pi G)} \int_M \Theta_A \wedge \Theta^A - \frac{2\lambda}{16\pi G} \int_M e^0 \wedge e^1 \wedge e^2 \wedge e^3, \end{aligned} \quad (37)$$

where

$$\Theta_A \equiv T_A + (2\pi G) \epsilon_{ABCD} \psi^B e^C \wedge e^D. \quad (38)$$

Observe that the sum of the conventional Einstein-Hilbert-Palatini and Dirac actions is given by the first two terms in the second line of (37).

We then find that under P (and also CP and CPT), the change is

$$\begin{aligned} \Delta \mathcal{S}_D^\mp &= P \mathcal{S}_D^\mp P^{-1} - \mathcal{S}_D^\mp \\ &= i \int_M \frac{1}{3!} d(\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D) \\ &- \frac{1}{2} \epsilon_{ABCD} \psi^A e^C \wedge e^D \wedge T^B \end{aligned} \quad (39)$$

and the change of the total action is then

$$\begin{aligned} \Delta \mathcal{S}_{\text{total}}^\mp &= \Delta \mathcal{S}_{\text{SJS}}^\mp + \Delta \mathcal{S}_D^\mp \\ &= i \int_M d \left\{ -\frac{1}{(8\pi G)} e^A \wedge T_A \right. \\ &\left. + \frac{1}{3!} (\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D) \right\} \\ &+ \frac{i}{(8\pi G)} \int_M \Theta_A \wedge \Theta^A. \end{aligned} \quad (40)$$

It should be noted that for spacetimes with Lorentzian signature, the Ashtekar variables are not real but rather may be required to satisfy reality conditions which are supposed to be enforced by a suitable inner product for quantum gravity [1]. Moreover, the actions are not explicitly real and their hermiticity could be tied to the inner product for the yet unavailable quantum theory of gravity.²

However, if we are interested in examining second quantized matter in a gravitational background (for that matter, in flat spacetime), we may enforce the reality conditions on A^- by hand. We assume for our present purposes, that even though we do not at the moment have an inner product for quantum gravity that will enforce these reality conditions, we can pass over to the second order formulation whereby we eliminate the Ashtekar connections in terms of the vierbein and spinors through the equations of motion for A^- and enforce the reality conditions by inspection.

So, varying the total first order action, $\mathcal{S}_D^\mp + \mathcal{S}_{\text{SJS}}^\mp$, with respect to A^- yields

$$\begin{aligned} D\Sigma^{-a} &= (4\pi G) \left[\frac{1}{3} \epsilon_{bcd} \psi^a e^b \wedge e^c \wedge e^d + \frac{1}{2} \epsilon^a{}_{bc} \psi^0 e^b \wedge e^c \wedge e_0 \right. \\ &\left. + i \psi^b e^a \wedge e_b \wedge e^0 \right] \end{aligned} \quad (41)$$

which implies that the Ashtekar connection is

$$A_a^- = i\omega_{0a} - \frac{1}{2} \epsilon_a{}^{bc} \omega_{bc} - (2\pi G) \epsilon_{abc} \left[\frac{1}{2} \epsilon^{bc}{}_{AB} \psi^A e^B - i \psi^b e^c \right]. \quad (42)$$

Upon imposing the usual Hermiticity requirements on the spinors (which implies that ψ^A is Hermitian), $(A^-)^\dagger = A^+$ and the requirement that the spin connection ω is real, one finds that $\Theta = 0$, which for the case of pure gravity, reduces to the torsionless condition. Thus passing over to the second order formulation where the Ashtekar connection has been eliminated and reality conditions have been imposed, we find that the change in the action under P (and also CP and CPT) is

²Recall that pure imaginary local Lorentz invariant pieces of the action are CPT odd.

$$\begin{aligned}
\Delta \mathcal{F}_{\text{total}} &= -\frac{i}{12} \int_M d\{\epsilon_{ABCD} \psi^A e^B \wedge e^C \wedge e^D\} \\
&= -\frac{i}{2} \int_M \partial_\mu [\det(e) j_5^\mu] d^4x \\
&= -\frac{i}{2} \int_M (\nabla_\mu j_5^\mu) \det(e) d^4x \\
&= -\frac{i}{2} \Delta Q_5,
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
j_5^\mu &= \bar{\Psi} E_A^\mu \gamma^A \gamma^5 \Psi \\
&= -E_A^\mu (\phi_L^\dagger \tau^A \phi_L - \phi_R^\dagger \bar{\tau}^A \phi_R) \\
&= -E_A^\mu (\phi_L^\dagger \tau^A \phi_L + \chi_L^\dagger \tau^A \chi_L)
\end{aligned} \tag{44}$$

is precisely the chiral current, and ΔQ_5 is the change in the chiral (axial) charge. Therefore, it would appear that if the chiral current is not conserved, the action fails to be *Hermitian* and invariant under P , CP , and CPT . In this respect, the Adler-Bell-Jackiw (ABJ) anomaly or the axial anomaly [11] induced by global instanton effects may lead to violations of CPT through chirality changing transitions. Chirality-changing transitions due to QCD instantons have been investigated by others before [12], including 't Hooft [13] in his resolution of the $U(1)_A$ problem. We shall postpone the discussion on the physical implications and bounds on the apparent non-Hermiticity of the action and the implied violations of discrete symmetries to a separate report. A related discussion on the effects of discrete transformations on actions which utilize self-dual variables in the description of four-dimensional gravity can be found in Ref. [14].

Although the expressions and discussions above are for a single quark or electron, it is easy to incorporate more matter fields into the theory simply by replacing ψ^A with

$$\psi^A \equiv \sum \Phi_L^\dagger \tau^A \Phi_L, \tag{45}$$

where the sum is over all Weyl fermions, including right-handed fermions which are to be written as left-handed ones. Conventional standard model Yukawa mass couplings and Higgs fields can be introduced without modifications.

V. THE STANDARD MODEL WITH ASHTEKAR FIELDS

We next examine how the quark and lepton fields of the standard model are to be coupled to the Ashtekar fields. Since we are allowing couplings to A^- rather than the whole spin connection, only left-handed fermions can be coupled to the theory. By writing all right-handed fields in terms of left-handed ones, it is possible to couple all the standard model quark and lepton fields to Ashtekar gravity in a consistent manner. We shall show in the next section that this coupling will not disturb the cancellations of all perturbative anomalies, given the multiplets of the standard model. The question of how to deal with the global anomaly that may be

present in the theory will be examined in the following section.

We shall label collectively by $W_i T^i$ the gauge connection one-forms of the standard model. The T^i 's denote generators of the standard model gauge group, and there should not be any confusion between this index i and spatial indices. The covariant derivative in $\mathcal{F}_{\mathcal{G}}$ in the action is then replaced by the total covariant derivative

$$D^- \phi_{L_I} = \left[\left(d - i A_a^- \frac{\tau^a}{2} \right) \delta_{IJ} - i W_i (T^i)_{IJ} \right] \phi_{L_J}, \tag{46}$$

where the index I associated with ϕ_{L_I} denotes internal ‘‘flavor’’ and/or ‘‘color.’’ In this modified model, the right-handed Weyl fields of the conventional standard model are to be written as left-handed Weyl fermions via the relation

$$\phi_{R_I} = -i \tau^2 \chi_{L_I}^*. \tag{47}$$

So an electron or a single quark of a particular color is represented as a pair of left-handed Weyl fermions (ϕ_L and χ_L), and a left-handed neutrino is represented by a single left-handed Weyl fermion.

The total action is still to be $\mathcal{F}_{\text{SIS}} + \mathcal{F}_{\mathcal{G}}$ with the total covariant derivative of (46) but summed over all left-handed multiplets, including ‘‘right-handed’’ multiplets which are written as left-handed ones. Using Eq. (47), right-handed currents can be written in terms of left-handed currents through

$$\begin{aligned}
J_R^{\mu i} &= \phi_{R_I}^\dagger \bar{\tau}^\mu (T^i)_{IJ} \phi_{R_J} \\
&= \chi_{L_I}^\dagger \tau^\mu (T^i)_{IJ} \chi_{L_J},
\end{aligned} \tag{48}$$

with

$$T_L^i = -(T_R^i)^T. \tag{49}$$

Notice however that terms containing the ordinary gauge connections W are Hermitian, so the factor 2 multiplying terms denoted by (1) and (4) in (22) takes care of the contributions from terms with W in (2) and (3) which would have been present had we used the conventional Dirac action. Similarly, this holds also for the left-handed neutrino although it is now described by only the term (1) (summed over the species) instead of (1) and (2). The net result of all this is that the difference between the ‘‘modified standard model’’ with gravity described by $\mathcal{F}_{\mathcal{G}} + \mathcal{F}_{\text{SIS}}$ with couplings only to left-handed fermions and the conventional action is still as described by (37).

VI. CANCELLATION OF PERTURBATIVE GAUGE ANOMALIES

We shall first demonstrate that the standard model defined entirely in terms of left-handed fermions, and the anti-self-dual Ashtekar fields A^- , is free of perturbative chiral gauge anomalies.

Recall that the anomalies can be determined via Fujikawa's Euclidean path integral method. We can expand the left-handed multiplet, $\Psi_{L_I} = (\phi_{L_I})^0$, and $\bar{\Psi}_{L_I}$, in terms of the

complete orthonormal set $\{X_n^L, X_n^R\}$ (see, for instance, Ref. [15]) with

$$\begin{aligned} X_n^L(x) &= \left(\frac{1-\gamma^5}{\sqrt{2}} \right) X_n(x), \quad \lambda_n > 0, \\ &= \left(\frac{1-\gamma^5}{2} \right) X_0(x), \quad \lambda_n = 0, \\ X_n^R(x) &= \left(\frac{1+\gamma^5}{\sqrt{2}} \right) X_n(x), \quad \lambda_n > 0, \\ &= \left(\frac{1+\gamma^5}{2} \right) X_0(x), \quad \lambda_n = 0. \end{aligned} \quad (50)$$

There can be more than one chiral zero modes with their left-right asymmetry governed by the Atiyah-Patodi-Singer index theorem. In our present instance, the curved space Dirac operator contains the full set of A^\pm connections besides W , and remains Hermitian provided $T_A = 0$; and X_n 's are the eigenvectors with eigenvalues λ_n of the full Euclidean Dirac operator. The chiral projections in (50) and (51) serve the purposes of defining a chiral fermion determinant for the effective action as well as selecting only the A^- Ashtekar fields in the action $\int (*1) \bar{\Psi}_L i \gamma^\mu D_\mu \Psi_L$. This is achieved by using the Euclidean expansions [16]

$$\begin{aligned} \Psi_L(x) &= \sum_{n: \lambda_n \geq 0} a_n X_n^L(x), \\ \bar{\Psi}_L(x) &= \sum_{n: \lambda_n \geq 0} \bar{b}_n X_n^{R\dagger}(x). \end{aligned} \quad (51)$$

The diffeomorphism invariant Euclidean measure for each left-handed multiplet,

$$\prod_x D[e^{1/2}(x) \bar{\Psi}_L(x)] D[e^{1/2}(x) \Psi_L(x)], \quad (52)$$

where $e \equiv \det(e)$, is (up to a phase) equivalent to (see Ref. [15])

$$d\mu = \prod_n da_n \prod_n d\bar{b}_n. \quad (53)$$

Under a gauge transformation with generator T^i ,

$$\Psi_L(x) \rightarrow e^{-i\alpha(x)T^i} \Psi_L(x) \quad (54)$$

the measure transforms as

$$d\mu \rightarrow d\mu \exp\left\{-i \int \alpha(x) \mathcal{A}^i(x) dx\right\}. \quad (55)$$

The anomaly can be computed as in Ref. [15]. This gives

$$\begin{aligned} \mathcal{A}^i(x) &= \sum_{\text{all } n} X_n^\dagger(x) \gamma^5 T^i X_n(x) \det(e) \\ &= -\frac{1}{16\pi^2} \text{Tr}\left\{T^i \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta} G_{\mu\nu}\right\}, \end{aligned} \quad (56)$$

and is proportional to $\text{Tr}(T^i\{T^j, T^k\})$. Here, $G_{\mu\nu}^i$ is the field strength associated with W_{μ}^i . So, the condition for cancellation of perturbative gauge anomalies is that $\text{Tr}(T^i\{T^j, T^k\})$ when summed over all fields coupled to $W_i T^i$ vanishes. But as we have seen, if ‘‘right-handed’’ spinors in the standard model coupled to $W_i T_R^i$ are written as left-handed spinors coupled to $W_i T_L^i$ such that the representations $T_L^i = -(T_R^i)^T$, then

$$\text{Tr}(T_L^i\{T_L^j, T_L^k\}) = -\text{Tr}(T_R^i\{T_R^j, T_R^k\}), \quad (57)$$

and, the condition for the perturbative chiral gauge anomalies of left and ‘‘right’’-handed fermions to cancel is

$$\text{Tr}(T_L^i\{T_L^j, T_L^k\}) + \text{Tr}(T_R^i\{T_R^j, T_R^k\}) = 0, \quad (58)$$

which is precisely equivalent to the well-known condition [17]

$$\text{Tr}(T_L^i\{T_L^j, T_L^k\}) = \text{Tr}(T_R^i\{T_R^j, T_R^k\}). \quad (59)$$

The Ashtekar fields do not give rise to perturbative anomalies because the generators of Ashtekar gauge group belong to $\mathfrak{su}(2)$, and $\text{SU}(2)$ is a ‘‘safe group’’ where $\text{Tr}(T^i\{T^j, T^k\}) = 0$ for any representation. The introduction of the Ashtekar fields therefore does not disturb the usual perturbative anomaly cancellation conditions. But the theory can still be afflicted with global anomalies, which is the issue we shall address next.

VII. GLOBAL ANOMALY AND FERMION CONTENT

Witten [18] showed that in four dimensions, theories with an odd number of Weyl fermion doublets coupled to gauge fields of the $\text{SU}(2)$ group are inconsistent. Without gravity, the standard model has four $\text{SU}(2)_{\text{weak}}$ doublets in each generation, and so it is not troubled by such a global anomaly. But what about the Ashtekar gauge fields? Strictly speaking for manifolds of Lorentzian rather than Euclidean signature, the group is complexified $\text{SU}(2)$ and isomorphic to $\text{SL}(2, \mathbb{C})$. But both these groups have the homotopy group $\Pi_4(G) = \mathbb{Z}_2$, and the nontrivial transformations of the complexified $\text{SU}(2)$ group in $\Pi_4(G)$ are associated with the rotation group. As Witten [18] has argued, the presence of nontrivial elements of this homotopy group can produce global $\text{SU}(2)$ anomalies. So it would appear that in four dimensions there could be further constraints on the particle content in order to ensure that the theory be free of the global anomaly associated with the Ashtekar gauge group. However, unlike the pure gauge case, gravitational instantons are strongly correlated with the topology of spacetime, and arguments based on $\Pi_4(G)$ cannot be carried over straightforwardly. In what follows, we present a unified treatment of both gravitational and pure gauge instanton contributions to the global anomaly for Weyl fermions.

We review the essential points of the global anomaly

[18,19]. To begin with, consider a suitable Wick rotation of the background spacetime into a Riemannian manifold. Not all manifolds with Lorentzian signature have straightforward continuations to Euclidean signature and vice versa. A more general setting for this section is the scenario of matter coupled to gravity in the context of path integral Euclidean quantum gravity [20]. The Dirac operator is then Hermitian with respect to the Euclidean inner product $\langle X|Y\rangle = \int d^4x \det(e) X^\dagger(x) Y(x)$ [15]. The fermions can then be expanded in terms of the complete set of the eigenfunctions of the Dirac operator. The expansions will be the same [16] as in Eq. (51) with the eigenfunctions normalized so that

$$\int_M d^4x \det(e) X_m^\dagger(x) X_n(x) = \delta_{mn}. \quad (60)$$

Consider next a chiral transformation by π which maps each two-component left-handed Weyl fermion $\Psi_L(x) \mapsto \exp(i\pi\gamma_5)\Psi_L(x) = \exp(-i\pi)\Psi_L(x) = -\Psi_L(x)$. Obviously such a map is a symmetry of the action. However, the measure is not necessarily invariant under such a chiral transformation because of the ABJ anomaly. Instead, for each left-handed fermion, the measure transforms as

$$d\mu \mapsto d\mu \exp\left(-i\pi \int_M d^4x \det(e) \sum_n X_n^\dagger(x) \gamma_5 X_n(x)\right). \quad (61)$$

The expression $\int_M d^4x \sum_n \det(e) X_n^\dagger(x) \gamma_5 X_n(x)$ is formally equal to $(n_+ - n_-)$, where n_\pm are the number of normalizable positive and negative chirality zero modes of the Dirac operator. Upon regularization, the expression works out to be

$$\begin{aligned} \sum_n \det(e) X_n^\dagger(x) \gamma_5 X_n(x) &\equiv \lim_{M \rightarrow \infty} \sum_n \det(e) X_n^\dagger(x) \gamma_5 e^{-(\lambda_n / \mathcal{M})^2} X_n(x) \\ &= \lim_{M \rightarrow \infty} \lim_{x' \rightarrow x} \text{Tr} \gamma_5 \det(e) e^{-(i\gamma^\mu D_{\mu'} / \mathcal{M})^2} \sum_n X_n(x) X^\dagger(x')_n \\ &= -\frac{1}{384\pi^2} R_{\mu\nu\sigma\tau}^* R^{\mu\nu\sigma\tau}. \end{aligned} \quad (62)$$

As a result, the index $(n_+ - n_-) = \sum_n \int_M d^4x \det(e) X_n^\dagger(x) \gamma_5 X_n(x) = -\tau/8$, where $\tau = (1/48\pi^2) \int_M R_{\mu\nu\sigma\tau}^* R^{\mu\nu\sigma\tau}$, is the signature invariant of the closed four-manifold M . (In this report, we restrict our attention to compact orientable manifolds without boundary. For manifolds with boundaries, boundary corrections to the Atiyah-Patodi-Singer index theorem must be taken into account.) Consequently, the change in the measure for *each* left-handed Weyl fermion under a chiral rotation of π is precisely $\exp(i\pi\tau/8)$. If there are altogether N left-handed Weyl fermions in the theory, the total measure changes by $\exp(iN\pi\tau/8)$. But, as emphasized in Ref. [19], the transformation of -1 on the fermions can also be considered to be an ordinary $SU(2)$ rotation of 2π . Since $SU(2)$ is a safe group with no perturbative chiral anomalies (there are no Lorentz anomalies in four dimensions), the measure must be invariant under all $SU(2)$ transformations. Thus there is an inconsistency unless this phase factor, $\exp(iN\pi\tau/8)$, is always unity.

It is known that for consistency of parallel transport of spinors for *topological four-manifolds*, τ must be a multiple of 8 for spin structures to exist [21]. This result also follows from $n_+ - n_- = -\tau/8$, since the index must always be an integer. However, a theorem due to Rohlin [22] states the signature invariant of a *smooth* simply connected closed spin four-manifold is divisible by 16. If one restricts to four-manifolds with signature invariants which are multiples of 16 in quantum gravity, or in semiclassical quantum field theory with left-handed spinors, one allows for these four-

manifold backgrounds only, then under a chiral rotation of π the measure is invariant regardless of N . Otherwise, if we allow for all manifolds with signature invariants which are multiples of 8, consistency with the global anomaly cancellation requires that N must be even. In either instance, the net index summed over all fermion fields is even.

It is not presently clear what the integration range for A^- should be. But, it is quite likely that it should extend over a wider class of manifolds than those which admit ordinary spin structures [23]. The considerations outlined above suggest that in order to do this, we would have to allow for couplings among the various fermion fields, for example via Yukawa couplings to spin-0 fields, and particularly via couplings to gauge fields, which is what occurs in grand unification theories (GUT's). The reason can be stated as follows. When τ is not a multiple of 8, it is not possible to lift the $SO(4)$ bundle to its double cover $Spin(4)$ bundle since the second Stiefel-Whitney class, w_2 , is nontrivial. However, given a general grand unification (simple) simply connected gauge group G with a $Z_2 = \{e, c\}$ in its center, it is possible to construct *generalized spin structures* with gauge group $Spin_G(4) = \{Spin(4) \times G\}/Z_2$ where the Z_2 equivalence relation is defined by $(x, g) \equiv (-x, cg)$ for all $(x, g) \in Spin(4) \times G$ [23]. Note that $Spin_G(4)$ is the double cover of $SO(4) \times (G/Z_2)$. Such spinors then transform according to double cover $Spin_G(4)$ group, and the parallel transport of fermions does not give rise to any inconsistency. However, the additional quantum field theoretic global anomaly cancellation condition must still be satisfied if the theory is to be consistent.

Now, in a scenario where all of the fermions are coupled to one another via gauge fields, the index theorem should be applied to the trace current involving the sum over all the fermion fields. The resultant restriction on N is the condition that the index for the *total* Dirac operator with $\text{Spin}_G(4)$ connections, $N_+ - N_-$, is even. Otherwise, Green's functions which contain $(N_+ - N_-)$ more $\bar{\Psi}_L$ than Ψ_L variables in the odd index sector will be inconsistent. The relevant index of the total Dirac operator in the presence of an additional grand unification gauge field other than the spin connection is

$$N_+ - N_- = -N \frac{\tau}{8} - \frac{1}{8\pi^2} \int_M \text{Tr}(F \wedge F), \quad (63)$$

where $F = F^i \mathcal{F}_i$ is the curvature of the internal grand unification gauge connection. The global anomaly consistency condition given above then implies that for arbitrary τ 's there can only be $16k$ fermions in total in the theory, where k is an integer; and the gauge group should be selected such that the instanton number, $(1/8\pi^2) \int_M \text{Tr}(F \wedge F)$, is always even for nonsingular configurations.

If one counts the number of left-handed Weyl fermions that are coupled to the Ashtekar connection for the standard model, one finds that the number is 15 per generation, giving a total of 45 for 3 generations. This comes about because each bispinor is coupled twice to the Ashtekar connection while each Weyl spinor is coupled once (e.g., for the first generation, the number is 2 for each electron and each up or down quark of a particular color, and 1 for each left-handed neutrino.) Thus even if one restricts to $\tau = 8k$, the global anomaly with respect to the Ashtekar gauge group seems to imply that there should be additional particle(s). For example, there could be a partner for each neutrino, making N to be 16 per generation, or a partner for just the τ neutrino, or even four generations. As a result, even with $\tau = 8k$, grand unification schemes based upon groups such as $\text{SU}(5)$ and odd number of Weyl fermions would be inconsistent when coupled to gravity.

The net conclusion is therefore that inclusion of manifolds with arbitrary τ 's in the integration range is possible, provided we allow for generalized spin structures, with a total of $N = 16k$ fermions, in order that the global anomaly is absent. In the event that the structures are defined by simple GUT groups, the simplest choice would be $\text{SO}(10)$. It is easy to check that the 16 left-handed Weyl fermions in the $\text{SO}(10) = \text{Spin}(10)/\mathbb{Z}_2$ GUT indeed belong to the 16-dimensional chiral representation of $\text{Spin}(10)$ and satisfy the generalized spin structure equivalence relation for $\{\text{Spin}(4) \times \text{Spin}(10)\}/\mathbb{Z}_2$.

It is worth emphasizing that this generalized spin structure implies an additional "isospin-spin" relation in that fermions must belong to $\text{Spin}(10)$ representations, while bosons must belong to $\text{SO}(10)$ representations of the GUT. This has implications for spontaneous symmetry breaking via fundamental bosonic Higgs fields, which cannot belong to the spinorial representations of $\text{SO}(10)$ if one allows for arbitrary τ 's. More generally, the presence of the extra particle(s) implied by global anomaly considerations can generate masses

for neutrinos, thereby giving rise to neutrino oscillations, and may also play a significant role in the cosmological issue of "dark matter."

A more mathematical treatment of the perturbative and global anomaly cancellation conditions we have discussed can be given. It is based on the fact that the chiral fermion determinant is actually a *section* of the determinant line bundle [24–27]. In order for the functional integral over the remaining fields to make sense, there must not be any obstruction to the existence of a global section. It is therefore necessary for the first Chern class of the determinant line bundle to vanish [24,25]. This gives the result for perturbative anomaly cancellation. However, a vanishing first Chern class does not necessarily imply a trivial connection for the determinant line bundle. There can be a further obstruction to a consistent definition of the chiral fermion determinant due to nontrivial parallel transport around a closed loop. Indeed, it can be shown that under suitable assumptions, the holonomy of the Quillen-Bismut-Freed connection of the determinant line bundle is given by $(-1)^{\text{index}[\text{Dirac}(M,A)]} \exp(-2i\pi\xi)$. Here $\xi = 1/2(\eta + h)$, with η and h being the spectral asymmetry and number of zero modes, respectively, of the Dirac operator on the five-dimensional manifold $M \times S^1$ endowed with a suitably scaled metric on S^1 [26,27]. In order to evaluate ξ , one can view $M \times S^1$ as the boundary of a six-dimensional manifold, M_6 , with a product metric on the boundary, and apply the Atiyah-Patodi-Singer index theorem to obtain

$$\text{index}(\text{Dirac})_{M_6} = - \int_{M_6} \hat{A} \wedge \text{ch}(F) - \xi(M \times S^1). \quad (64)$$

It is instructive to work out the terms and their implications explicitly. For the case relevant to us, M is four-dimensional and we pick out the six-forms in the integrand in the first term on the right-hand side. These are of the form $\text{Tr}(F \wedge F \wedge F)$ and $R_{AB} \wedge R^{AB} \wedge \text{Tr}(F)$. The former is the Abelian anomaly in $D+2$ dimensions (here $D=4$) which gives rise to the perturbative non-Abelian gauge anomaly in D dimensions [24,25,28]. It vanishes when the perturbative anomaly-free condition $\text{Tr}(\mathcal{F}_i \{ \mathcal{F}_j, \mathcal{F}_k \}) = 0$ is satisfied. The latter, which is the gravitational-gauge cross term, is required to vanish [$\text{Tr}(F) = 0$] for the absence of mixed Lorentz-gauge anomalies when there are couplings to Abelian fields [29]. These terms occur in the holonomy theorem of Bismut and Freed because the holonomy measures the nontriviality of the connection of the determinant line bundle, and perturbative anomalies which correspond to the nontriviality of the first Chern class must necessarily contribute. In this spirit, discussions of global anomalies make sense only if perturbative anomalies cancel. With the perturbative anomaly cancellation conditions, we therefore conclude that ξ is an integer since $\text{index}(\text{Dirac})_{M_6}$ is an integer. Thus the holonomy theorem tells us that the remaining obstruction, i.e., the global anomaly is due to $(-1)^{\text{index}[\text{Dirac}(M,A)]}$, and the consistency condition is again the requirement that the index of the total chiral Dirac operator in four dimensions with generalized spin structure is even.

We have presented our arguments of the global anomaly in terms of the Ashtekar variables for definiteness. We would like to end by emphasizing that our results would also obtain in the setting of Weyl fermions coupled to conventional gravity and ordinary spin connections instead of the Ashtekar connections [30]. This is essentially because the anomaly can be understood as coming from the inconsistent change in the fermionic measure or the chiral fermion determinant, and for both schemes, the change is $\exp[-i\pi(N_+ - N_-)] = (-1)^{\text{index (Dirac)}}$.

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