Solitonic strings and BPS saturated dyonic black holes

Mirjam Cvetič[1,*] and Arkady A. Tseytlin[2,†]
[1]*School of Natural Science, Institute for Advanced Study, Princeton, New Jersey 08540*
[2]*Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*
(Received 7 December 1995)

We consider a six-dimensional solitonic string solution described by a conformal chiral null model with nontrivial $N=4$ superconformal transverse part. It can be interpreted as a five-dimensional dyonic solitonic string wound around a compact fifth dimension. The conformal model is regular with the short-distance (''throat'') region equivalent to a Wess-Zumino-Witten (WZW) theory. At distances larger than the compactification scale the solitonic string reduces to a dyonic static spherically symmetric black hole of toroidally compactified heterotic string. The new four-dimensional solution is parametrized by five charges, saturates the Bogomol'nyi bound, and has a nontrivial dilaton-axion field and moduli fields of a two-torus. When acted on by combined T- and S-duality transformations it serves as a generating solution for *all* the static spherically symmetric Bogomol'nyi-Prasad-Sommerfield saturated configurations of the low-energy heterotic string theory compactified on a six-torus. Solutions with regular horizons have the global space-time structure of extreme Reissner-Nordström black holes with the non-zero thermodynamic entropy which depends only on conserved (quantized) charge vectors. The *independence* of the thermodynamic entropy on moduli and axion-dilaton couplings strongly suggests that it should have a microscopic interpretation as counting degeneracy of underlying string configurations. This interpretation is supported by arguments based on the corresponding six-dimensional conformal field theory. The expression for the level of the WZW theory describing the throat region implies a renormalization of the string tension by a product of magnetic charges, thus relating the entropy and the number of oscillations of the solitonic string in compact directions. [S0556-2821(96)07010-5]

PACS number(s): 04.50.+h, 04.20.Jb, 11.25.Sq

I. INTRODUCTION

String theory is bound to have important implications for the physics of four-dimensional black holes. It is likely that certain fundamental properties of ''realistic'' black holes can be understood by studying a special class of supersymmetric Bogomol'nyi-Prasad-Sommerfield (BPS) saturated backgrounds which for large enough supersymmetry do not receive quantum corrections. Examples of such backgrounds are provided by pure electrically or pure magnetically charged solutions [1] of lowest-order effective field theory (for a review see [2] and references therein).

To embed an effective field theory solution into *string theory* one is to find the corresponding world-sheet conformal σ model whose couplings reduce to the given background fields at scales larger than the compactification and string scales (see, e.g., [3] and references therein). Thus four-dimensional effective field theory backgrounds generically appear to be only large-distance approximations to higher-dimensional string solutions. In particular, all supersymmetric (BPS-saturated) electric black hole solutions of toroidally compactified heterotic (or type II superstring) theory [4] correspond to conformal chiral null σ models [5–9].

The latter can be interpreted as describing higher-dimensional fundamental string backgrounds, i.e., external long-range fields produced by stable classical string sources of elementary closed strings, which are in general charged, oscillate in one, e.g., left-moving, sector and are wound around a compact spatial dimension [10,11,8,9]. The leading-order solution is singular at the core[1] but α' corrections most likely provide an effective smearing of the δ-function source at the quantum string scale $\sqrt{\alpha'}$ [12].[2] What appears to a distant four-dimensional observer as an extreme electric black hole has actually an internal structure of a higher-dimensional fundamental string. This interpretation suggests a natural way of understanding the thermodynamic black-hole entropy in terms of degeneracy of string configurations [14], which give rise to black holes with the same values of asymptotic charges [8,9]. For example, a higher-dimensional string ''oscillating'' in a compact internal direction reduces to a family of black holes with the same asymptotic charges but different nonvanishing *massive* Kaluza-Klein fields which are invisible at scales larger than the compactification scale.[3]

In order to make this qualitative picture quantitative, i.e.,

*On sabbatic leave from the University of Pennsylvania. Electronic address: cvetic@sns.ias.edu

†On leave from Lebedev Institute, Moscow. Electronic address: tseytlin@ic.ac.uk

[1]In fact, it remains singular to all orders in α' if one chooses to ignore the source altogether what, from the world-sheet σ model point of view, formally seems to be a legitimate alternative [6,3].

[2]In addition, a first-principle conformal field theory interpretation of solutions with string sources should probably involve a ''thin handle''-type resummation of string loop expansion [13].

[3]In order to try to reproduce the black hole entropy as a statistical entropy $[\ln d(N)]$ it is important that the oscillating object should be stringlike, i.e., having an exponentially growing number $d(N)$ of states at a given oscillator level N; it is not enough to consider just a Kaluza-Klein theory.

to compute the black hole entropy directly from string theory, one has to address the question of string loop and α' corrections. A nice property of the extreme electric black hole (fundamental string) solution is that the dilaton, i.e., the effective string coupling, goes to zero as one approaches the origin, so that string loop corrections may be ignored. This is not true, however, for the α' corrections since the curvature of the leading-order solution blows up near the origin. The expectation [12] that α' corrections should smear the singularity of the fundamental string solution at scales of order $\sqrt{\alpha'}$ is in this context closely related to the suggestion [14] that the thermodynamic black-hole entropy, which vanishes when evaluated at the singular horizon ($r=0$) of the leading-order effective field theory solution, should instead be computed at the ''stretched'' horizon at $r=\sqrt{\alpha'}$ [15]. The resulting expression then matches the statistical string entropy [14,16]. Though very plausible, it may be hard to implement this idea in a first-principle calculation of the entropy.

Magnetically charged supersymmetric extreme black holes have very different properties. The leading-order solution [1] has a nonsingular string metric with the origin at $r=0$ being transformed into a ''throat'' region. Now α' corrections can be ignored provided the magnetic charge P is large, i.e., $P\gg\sqrt{\alpha'}$. Indeed, these configurations have the string-theory representation [17,18] in terms of a higher-dimensional string soliton, with magnetic charge having Kaluza-Klein origin (for a review, see [19,20] and references therein). They are described by a regular, source-free superconformal field theory [21] which reduces in the throat region to a Wess-Zumino-Witten (WZW) model supplemented with a linear dilaton. For that type of magnetic solitons the dilaton blows up near the origin, and thus string loop corrections cannot be ignored. This prevents one from computing the black-hole entropy by counting different solitonic configurations with the same four-dimensional large-distance behavior.

Given the fact that the presence of an electric charge seems to regularize the short-distance behaviour of the dilaton while the presence of a magnetic charge leads to a regular string metric, one may speculate that to obtain solutions, where both string loop and α' corrections are under control, both electric and magnetic charges should be nonvanishing. Remarkably, this is indeed what happens in the case of four-dimensional supersymmetric *dyonic black hole solutions* [22,23] of leading-order effective field equations corresponding to toroidally compactified heterotic string (see also [24] for a review and references). At the string-theory level they correspond to the world-sheet conformal theory [25] which is a hybrid of ''electric'' chiral null model and ''magnetic'' $N=4$ superconformal model, thus combining the best features of the pure electric and pure magnetic models. This conformal theory describes a higher ($D\geqslant 6$) dimensional string soliton with all the background fields regular everywhere (for $r\geqslant 0$). The magnetic charge plays the role of a short-distance regulator, providing an effective shift $r\to r+P$, analogous to the shift $r\to r+\sqrt{\alpha'}$ expected to happen in the exact fundamental string solution.

As in the purely magnetic case, the short-distance region is a throat described by a regular WZW-type conformal field theory, but now with a *constant* dilaton. In fact, the dilaton varies smoothly between constant values at large and small distances and its $r=0$ value is given by the ratio of the magnetic and electric charges. The approximate constancy of the dilaton ensures that the resulting four-dimensional dyonic black holes are black holes indeed; these solutions have the global space-time structure of extreme Reissner-Nordström black holes.

As a result, it may be possible to choose the charges so that both the world-sheet α' and the string loop corrections *remain small everywhere*, thus suggesting that in the case of dyonic charges one should be able to reproduce the expression for the black hole entropy by a semiclassical computation. Indeed, the thermodynamic entropy determined by the area of the horizon is now proportional to the product of the electric and magnetic charges and thus is no longer vanishing. By analogy with the corresponding counting of degenerate (fundamental string) states for purely electric black holes [14,8,9] one may expect that the entropy should now have an interpretation in terms of counting of degenerate solitonic string states [26].

To implement this suggestion it is important to understand which solitonic string states correspond to a given set of asymptotic dyonic black hole charges. One should be able to do this by starting directly with the underlying conformal field theory of the dyonic soliton. As we shall argue below, for large magnetic charges the level of the WZW-type conformal field theory which describes the horizon (throat) region is large and thus the counting of states should be the same as in flat space up to a renormalization of the string tension by magnetic charges, as anticipated in [26]. As a result, one indeed reproduces the thermodynamic entropy by semiclassical, string-theory considerations.

The dyonic model studied in [25] was a six-dimensional supersymmetric chiral null model with curved transverse part which is a hybrid of the five-dimensional fundamental-string type model [giving rise upon dimensional reduction along the compact ''string'' direction to extreme black holes with Kaluza-Klein (Q_1) and two-form (Q_2) electric charges] and an $N=4$ superconformal model [which generalizes both the Kaluza-Klein monopole (P_1) and the H-monopole (P_2) models]. It has a remarkable covariance property with respect to T duality in the two compactified dimensions and with respect to S duality, interchanging the electric and magnetic couplings. To get a better understanding of general features of this class of solitonic conformal models, in particular, of their possible marginal perturbations, we shall generalize the model of [25] to include one extra coupling function, specifying the electric charge (q) of the solitonic string (Sec. II). The throat region of the resulting background is described by (an orbifold of) the six-dimensional $\mathrm{SL}(2,R)\times\mathrm{SU}(2)$ WZW model with the level proportional to the product of the two magnetic charges P_1P_2. The relation to the WZW model also implies quantization conditions on charges (Sec. II B).

In the simplest spherically-symmetric case the corresponding four-dimensional dyonic solution (Sec. III) is parameterized by the two magnetic $P_1^{(1)}=P_1, P_1^{(2)}=P_2$ and the two electric $Q_2^{(1)}=Q_1, Q_2^{(2)}=Q_2$ charges and by one new parameter q specifying the electric charges $Q_1^{(1)}=-Q_1^{(2)}=q$ [the upper and lower indices 1,2 indicate

the Kaluza-Klein and two-form U(1) gauge fields and the first and the second compactified toroidal coordinates, respectively].

Like its $q=0$ limit, corresponding to the four-parameter solution of [22,23], the five-parameter dyonic solution saturates the Bogomol'nyi bound. It has nontrivial dilaton, axion and moduli fields of the compactified two-torus and serves, when acted on by the T- and S-duality transformations, as a generating solution for *all* the static spherically-symmetric BPS-saturated solutions of the effective heterotic string compactified on a six-torus. These solutions are parameterized by *unconstrained* 28 electric and 28 magnetic charges (Sec. IV). Solutions with regular event horizons have the Reissner-Nordström global space-time with zero temperature and nonzero thermodynamic entropy. We derive the general T- and S-duality invariant expression for the entropy, which depends only on conserved (quantized) electric and magnetic charges and is *independent* of the asymptotic values of the dilaton-axion and moduli fields. This result supports the expectation [26] that the entropy is counting the number of string degrees of freedom which should not change under adiabatic variations of couplings of the theory.

The statistical interpretation of the entropy is discussed in Sec. V by considering the string-theory (conformal model) interpretation of the five-parameter solution. We present an argument relating the thermodynamic entropy to the statistical entropy which counts the degeneracy of the dyonic solitonic string configurations ''oscillating'' in a compact direction. Our approach generalizes the suggestion of [26] and explains the renormalization of string tension by magnetic charges by direct consideration of the underlying conformal model in the horizon (throat) region.

II. SIX-DIMENSIONAL SOLITONIC STRING CONFORMAL MODEL

The string soliton we are going to discuss is described by a supersymmetric chiral null model with curved transverse space. The chiral null models [6,3] are a class of two-dimensional (2D) σ models which generalize both plane wave type and fundamental string type models and are defined by the following string Lagrangian:[4]

$$L=F(x)\partial u[\overline{\partial v}+K(u,x)\overline{\partial u}+2\mathscr{A}_i(u,x)\overline{\partial x}^i] +(G_{ij}+B_{ij})(x)\partial x^i\overline{\partial x}^j+\mathscr{R}\Phi(x). \tag{1}$$

Here u,v are ''light-cone'' coordinates; x^i are ''transverse space'' coordinates. $x^i=(x^s,y^n)$ where x^s $(s=1,2,3)$ are three noncompact spatial coordinates and y^n are toroidally compactified (Kaluza-Klein) coordinates which may also include the chiral scalar coordinates of the internal 16-torus of the heterotic string. $\mathscr{R}\equiv\frac{1}{4}\alpha'\sqrt{g^{(2)}}R^{(2)}$ is proportional to the world-sheet curvature.

One can also consider generalizations of this model by including u-dependence in functions Φ, F, G_{ij},B_{ij}. Examples of such conformal σ models (1) with u-dependent Φ [6,3], F [8], and G_{ij},B_{ij} [27] were considered in the literature.

There exists a renormalization scheme in which (1) is conformal to all orders in α' provided (i) the ''transverse'' σ model $(G_{ij}+B_{ij})\partial x^i\overline{\partial x}^j$ is conformal when supplemented with a dilaton coupling $\phi(x)$ and (ii) the functions $F^{-1},K,\mathscr{A}_i,\Phi$ satisfy the following conditions (as in [25] we shall assume that the transverse theory has $N=4$ extended world-sheet supersymmetry so that the conformal invariance conditions preserve their one-loop form):

$$-\tfrac{1}{2}\nabla^2F^{-1}+\partial^i\phi\partial_iF^{-1}=0,$$

$$-\tfrac{1}{2}\nabla^2K+\partial^i\phi\partial_iK+\partial_u\nabla_i\mathscr{A}^i=0, \tag{2}$$

$$-\tfrac{1}{2}\hat{\nabla}_i\mathscr{F}^{ij}+\partial_i\phi\mathscr{F}^{ij}=0,\ \text{i.e.,}$$

$$\nabla_i(e^{-2\phi}\mathscr{F}^{ij})-\tfrac{1}{2}e^{-2\phi}\mathscr{F}_{ik}H^{ikj}=0, \tag{3}$$

with

$$\hat{\nabla}\equiv\nabla(\hat{\Gamma}),\quad \hat{\Gamma}^i_{jk}=\Gamma^i_{jk}+\tfrac{1}{2}H^i_{jk},\quad \mathscr{F}_{ij}\equiv\partial_i\mathscr{A}_j-\partial_j\mathscr{A}_i,$$

$$\Phi=\phi+\tfrac{1}{2}\ln F, \tag{4}$$

where ∇_i is covariant derivative defined with respect to the transverse metric G_{ij} and $H_{ijk}=3\partial_{[i}B_{jk}$. The Maxwell-type equation for $\mathscr{F}_{ij}$ is the conformal invariance condition in the (ux^i) direction. The linear equations for K and $\mathscr{A}_i$ can be viewed as marginality conditions on the corresponding ''perturbations'' of the conformal model specified by F,G_{ij},B_{ij},ϕ.

Given a u-independent solution of the above equations one can construct its u-dependent generalization, e.g., by replacing the constant parameters in K and $\mathscr{A}_i$ by functions of u, i.e., $K(x;c)\to K(x;c(u))$, and $\mathscr{A}_i(x;q)\to h_i(u)+\mathscr{A}_i(x;q(u))$. If $\partial_u\nabla_i\mathscr{A}^i\neq0$ the function h_i will be related to K by the second equation in (2). In the case of the fundamental string solution with $u=x_9-t,v=x_9+t$, and $G_{ij}+B_{ij}=\delta_{ij}$ this corresponds to adding [28,6,8,9] traveling waves of momentum along the string as well as arbitrary left-moving oscillations both in compact y^n (charge) directions and noncompact spatial directions x^s. Such solutions are in correspondence with BPS-saturated states of the heterotic string spectrum at the vacuum level in the right-moving sector (right-moving oscillator number $N_R=\frac{1}{2}$) and an arbitrary level in the left-moving sector (arbitrary left-moving oscillator number N_L) [8,9].

Let us now specialize to the particular case of six-dimensional σ model (1) of the type

[4]We shall use the following notation. The string world-sheet action is normalized so that $I=(\pi\alpha')^{-1}\int d^2\sigma\partial x\bar{\partial}x=(4\pi\alpha')^{-1}\int d^2\sigma\partial_a x\partial^a x$. The string effective action is $S_{10}=c\int d^{10}x\sqrt{-G}e^{-2\Phi}(R+\cdots)$ and after reduction to four dimensions $S_4=(16\pi G_N)^{-1}\int d^4x\sqrt{-G'}e^{-2\Phi'}(R'+\cdots)$, where the $D=4$ dilaton $\Phi'(x)$ will be assumed to have trivial asymptotic value, $e^{\Phi'(x\to\infty)}=1$ (so that both string-frame and Einstein-frame $D=4$ metrics approach $\eta_{\mu\nu}$ at infinity), with $e^{\Phi'_\infty}$ being absorbed in the Newton's constant $G_N=\frac{1}{8}e^{2\Phi'_\infty}\alpha'$. In Secs. III and IV we shall set $\alpha'=2$ and the compactification radii $R_n=\sqrt{\alpha'}=\sqrt{2}$. In Sec. III we shall also assume that $e^{\Phi'_\infty}=1$, i.e., that $G_N=1/4$.

$$L=F(x)\partial u(\partial\overline{v}+K(x)\partial\overline{u}+2A(x)[\partial\overline{x}^4+a_s(x)\partial\overline{x}^s])+\tfrac{1}{2}\mathscr{R}\ln F(x)+L_\perp, \tag{5}$$

$$L_\perp=f(x)k(x)[\partial x^4+a_s(x)\partial x^s][\partial\overline{x}^4+a_s(x)\partial\overline{x}^s]+f(x)k^{-1}(x)\partial x^s\partial\overline{x}^s+b_s(x)(\partial x^4\partial\overline{x}^s-\partial\overline{x}^4\partial x^s)+\mathscr{R}\phi(x), \tag{6}$$

where $x^s=(x^1,x^2,x^3)$ are three noncompact spatial dimensions, while x^4 and u will be compact coordinates. We shall assume that all the fields depend only on x^s and f,k,a_s,b_s,ϕ are subject to[5]

$$\partial_s\partial^s f=0,\quad \partial_s\partial^s k^{-1}=0, \tag{7}$$

$$\partial_p b_q-\partial_q b_p=-\epsilon_{pqs}\partial^s f,\quad \partial_p a_q-\partial_q a_p=-\epsilon_{pqs}\partial^s k^{-1},\qquad \phi=\tfrac{1}{2}\ln f, \tag{8}$$

where $p,q,s=1,2,3$.

In (5) we have made a special choice of the field $\mathscr{A}_i$

$$\mathscr{A}_s=Aa_s,\mathscr{A}_4\equiv A, \tag{9}$$

which makes the model covariant under the duality transformation in the x^4 direction. The 2D duality transformation $x^4\to\tilde{x}^4$ can be performed by gauging the shifts in x^4, adding $B\partial\tilde{x}^4-B\partial\tilde{x}^4$, gauge-fixing $x^4=0$, and integrating out B and $\overline{B}$. One finds that indeed this duality transformation maps the model (5) into itself with[6]

$$f\to k^{-1},k\to f^{-1},a_s\to b_s,b_s\to a_s,A\to(fk)^{-1}A, \tag{10}$$

where we have assumed that $a_{[p}b_{q]}=0$. The choice of $\mathscr{A}_i$ (9) also leads to the absence of a Taub-NUT term in the metric of the corresponding four-dimensional spherically-symmetric background.

Let us note that case of six target space dimensions is special in that here the duality transformation applied to a rank-two antisymmetric tensor gives again a rank-two tensor and thus can be represented as a formal map of one string σ model into another. In contrast to T duality, however, this transformation cannot be realised directly at the world-sheet level.[7] As was pointed out in [25], the model (5),(6) with $A=0$ has a remarkable covariance property under the six-dimensional S-duality ($G\to e^{-2\Phi}G$, $dB\to e^{-2\Phi}*dB$, $\Phi\to-\Phi$): when formally applied to the background fields of the σ model this transformation simply interchanges the functions F and f. When $A\neq0$ the above six-dimensional model is still covariant under the T-duality in x_4 and u directions. However, it is no longer covariant under the S-duality. The reason is that for $A\neq0$ the components H_{u4s} and H_{upq} of the torsion become nonvanishing but under the duality they are transformed into H'_{v4s} and H'_{vpq}. That means that the S-duality induces the torsion terms $\sim\partial v\,\partial x^4$ and $\sim\partial v\,\partial x^s$ in the σ-model action. Though the resulting background will again represent a leading-order solution of the string effective equations, now it is not clear whether it will remain an exact solution to all orders in α'.

The conditions (2) on F,K and on $\mathscr{A}_i$ (i.e., on A) can be put into the form

$$\partial_s\partial^s F^{-1}=0,\quad \partial_s\partial^s K=0, \tag{11}$$

$$\partial_s(k^{-1}f^{-1}\partial^s A)+k^{-1}\partial_s k\partial^s(k^{-1}f^{-1})A=0,\ \text{i.e.,}\qquad \partial_s[k^{-3}f^{-1}\partial^s(kA)]=0. \tag{12}$$

In deriving (12) from (3) we have used that a_s,b_s satisfy (8) and that f and k^{-1} are harmonic. The model is thus parametrized by four harmonic functions F^{-1},K,f,k^{-1} and the function A satisfying (12). The functions a_s and b_s are then determined by f,k according to (8).

The model discussed above is a generalization of the one introduced in [25] where the fifth function A was turned off. Let us emphasize that unlike F,K and f,k terms, which coexist in (5) without influencing the equations of each other, the introduction of the new coupling A leads to the equation (12) which *depends* on the couplings f,k of the transverse part of the σ model (5).

A. Solution of conformal invariance conditions

While F^{-1},K,f,k^{-1} are *independent* harmonic functions, A is specified by (12) which has the following solution:

$$A=q_1k^{-1}+q_2f^2k,q_{1,2}=\text{const.} \tag{13}$$

This is the general solution in the case of one-center spherically-symmetric harmonic functions f,k^{-1}. For more general f,k^{-1}, e.g., multicenter harmonic functions, the solution of the linear equation (12) [which is equivalent to the scalar Laplace equation for kA in curved three-dimensional space with conformally-flat metric $ds_3^2=\rho^2(x)dx^sdx^s,\rho=k^{-3}f^{-1}$] will look more complicated.

If we further assume the asymptotic flatness conditions on the functions, i.e., that $k\to1,f\to1,A\to0$ for $r^2\equiv x^sx_s\to\infty$, then (13) becomes

$$A=q_0k(k^{-2}-f^2),\quad q_0=q_1=-q_2. \tag{14}$$

[5]More generally, one may consider a similar six-dimensional model with the functions depending on all four transverse coordinates. Special cases will be the six-dimensional fundamental string ($F^{-1}=1+Q/x^2$, $K=f=k=1$) [11], its S-dual solitonic string solution [20,29,30] ($F=K=1,A=0,f=1+P/x^2,k=1$) which also corresponds to a six-dimensional reduction of the five-brane solution [21,31] of the ten-dimensional theory and the dyonic six-dimensional string [32] ($F^{-1}=1+Q/x^2,f=1+P/x^2,k=K=1$).

[6]Note that under the duality in u direction $F\to K^{-1}$, $K\to F^{-1},\Phi(F)\to\Phi(K^{-1})$ while other functions remain unchanged.

[7]This transformation maps one solution (with vanishing vector fields) of the toroidally compactified to six dimensions heterotic (or type II) string into another. In the context of string-string duality between heterotic string on a four-torus and type II string on $K3$ surface [33,34] this transformation should be supplemented by identification of the vector fields in the Neveu-Schwarz–Neveu-Schwarz sector of the heterotic string with the vector fields in the Ramond-Ramond sector of the type II string.

Note that under the T-duality transformation (10) the (q_1,q_2) solution for A is mapped into the (q_2,q_1) solution, i.e., q_0 in (14) changes sign.

In the special case of one-center harmonic functions we get the following explicit form of the solution:[8]

$$F^{-1}=1+\frac{Q_2}{r},\ \ K=1+\frac{Q_1}{r},\ \ f=1+\frac{P_2}{r},\ \ k^{-1}=1+\frac{P_1}{r}, \tag{15}$$

$$A=\frac{q[r+\frac{1}{2}(P_1+P_2)]}{r(r+P_1)}, \tag{16}$$

$$a_s dx^s=P_1(1-\cos\theta)d\varphi,\quad b_s dx^s=P_2(1-\cos\theta)d\varphi, \tag{17}$$

$$e^{2\Phi}=Fe^{2\phi}=\frac{r+P_2}{r+Q_2}, \tag{18}$$

where the parameter q is related to q_0 of (14) as $q\equiv 2q_0(P_1-P_2)$. Note that the expression for A in terms of q is valid also for $P_1=P_2$, i.e., for $fk=1$, when it becomes just the harmonic function $A=q/r$. The one-center solution is thus specified by the five parameters P_1,P_2, Q_1,Q_2, and q.

Let us note that for positive P_2 and Q_2 the string dilaton (18) is regular and is constant both at large $r\to\infty$ and small $r\to 0$ distances. Thus one can, in principle, make the effective string coupling small everywhere by choosing $e^{\Phi_\infty}<1$ and $P_2\sim Q_2$.

B. Throat region

An important property of the six-dimensional σ model (5),(6), (15)–(18) is that in contrast to the six-dimensional chiral null model with flat transverse part ($f=k=1$) which is singular at $r=0$ (and describes a fundamental string type configuration), in the case of nontrivial transverse part with non-vanishing parameters $P_1>0$ and $P_2>0$ the singularity at the core $r=0$ disappears. It gets replaced by a ''throat'' or ''semi-wormhole'' region [21].

In the throat region $r\to 0$ the Lagrangian (5) [with the functions given by (15)–(18) and P_1,P_2,Q_1,Q_2 all positive] takes the form

$$\begin{aligned}L_{r\to 0}=&P_1P_2\partial z\overline{\partial z}+e^{-z}\partial u\overline{\partial v}+Q_1Q_2^{-1}\partial u\overline{\partial u}\\&+2Q_2^{-1}q\,\partial u[\overline{\partial y}_1+P_1(1-\cos\theta)\overline{\partial\varphi}]\\&+P_1^{-1}P_2[\partial y_1+P_1(1-\cos\theta)\partial\varphi][\overline{\partial y}_1\\&+P_1(1-\cos\theta)\overline{\partial\varphi}]+P_1P_2(\partial\theta\overline{\partial\theta}+\sin^2\theta\partial\varphi\overline{\partial\varphi})\\&+P_2(1-\cos\theta)(\partial y_1\overline{\partial\varphi}-\overline{\partial y}_1\partial\varphi),\\&z\equiv-\ln\frac{r}{Q_2}\to\infty.\end{aligned} \tag{19}$$

In the case of $q=0$ discussed in [25] we get a regular conformal model which, up to a factorization over a discrete subgroup, is the WZW theory based on a direct product of the SL(2,R) and SU(2) groups.[9] An important feature is that, in contrast to, e.g., the five-brane model [21], here the dilaton (18) is *constant* in the throat region, i.e., the string coupling is not blowing up and thus the solution can be trusted everywhere.[10]

For a nonzero q it looks as if the Lagrangian (19) describes a globally nontrivial ''mixture'' of the SL(2,R) and SU(2) theories. However, the central charge retains its free-theory value, i.e., the dilaton Φ is still constant at $r=0$, and it is easy to see that (19) can, in fact, be put in the same form as in the $q=0$ case [25] by redefining the coordinates. Changing the notation for coordinates to $u=y_2,v=2t,x_4=y_1$ where y_1 and y_2 will be assumed to be circular coordinates with periods $2\pi R_1$ and $2\pi R_2$ we get [up to a total-derivative term $\propto q(\partial y_2\partial y_1-\partial y_1\partial y_2)$]

$$\begin{aligned}L_{r\to 0}=&(p\,\partial z\overline{\partial z}+p'\partial\tilde{y}_2\overline{\partial\tilde{y}}_2+2^{e^{-z}}\partial\tilde{y}_2\overline{\partial\tilde{t}})\\&+p(\partial\tilde{y}_1\overline{\partial\tilde{y}}_1+\partial\varphi\overline{\partial\varphi}+\partial\theta\overline{\partial\theta}-2\cos\theta\partial\tilde{y}_1\overline{\partial\varphi}).\end{aligned} \tag{20}$$

Here

$$\tilde{y}_1=P_1^{-1}y_1+qP_2^{-1}\tilde{y}_2+\varphi,\quad \tilde{y}_2=Q_2^{-1}y_2,\quad \tilde{t}=Q_2t,$$

$$p=P_1P_2,\quad p'=Q_1Q_2-q^2P_1P_2^{-1}. \tag{21}$$

The role of q is thus to mix the two compact coordinates y_1 and y_2, i.e., in the throat region q plays a role of a modulus, which turns on the off-diagonal components of the metric of the two-torus corresponding to y_1,y_2.

The throat region model (20) is thus equivalent to a direct product of the SL(2,R) and SU(2) WZW theories [corresponding to the terms in each of the parentheses in (20)] divided by discrete subgroups. The levels of the SL(2,R) and SU(2) models are both equal to

$$\kappa=\frac{4}{\alpha'}p=\frac{4}{\alpha'}P_1P_2. \tag{22}$$

Since the level of SU(2) must be integer, we get the quantisation condition $P_1P_2=\frac{1}{4}\alpha'\kappa$.

When $q=0$ one can follow [36,17] and construct an orbifold of SU(2) by identifying the coordinate $\tilde{y}_1$, which in

[8]The relation to the notation used in [25] is $Q_1=\mathbf{Q}_2^{(1)},Q_2=\mathbf{Q}_2^{(2)},P_1=\mathbf{P}_1^{(1)},P_2=\mathbf{P}_1^{(2)}$.

[9]In the special case of $P_1=P_2$, $Q_1=Q_2$, $q=0$ similar throat region model was considered previously in [35].

[10]One manifestation of the regularity of the dilaton in this model is that the central charge of this conformal field theory, which has a free-theory value in the supersymmetric σ model case, can be easily computed either in $r=\infty$ or in $r=0$ regions [25].

the standard SU(2) WZW model must be 4π periodic, in the following way: $\tilde{y}_1 \equiv \tilde{y}_1 + 4\pi/m$, where m is an integer. This is possible provided[11]

$$2P_1 = mR_1 . \tag{23}$$

Then the modular invariance of the orbifold $\mathrm{SU}(2)_\kappa/Z_m$ demands [36] that $\kappa = nm$ where n is an integer. Since here the level κ of SU(2) is itself proportional to the product of the two magnetic charges, we also get the following quantization condition for P_2:

$$2P_2 = \frac{n\alpha'}{R_1} . \tag{24}$$

For $q \neq 0$ we demand that the coordinate $\tilde{y}_1$ should still have the same period $4\pi/m$ and get an additional condition $2P_2Q_2 = lmqR_2$, where l is an integer, i.e.,

$$\frac{Q_2}{q} = \frac{lm}{n} = \frac{lP_1}{P_2} . \tag{25}$$

T-duality in y_2 direction, which implies $Q_1 \to Q_2$, $R_2 \to \alpha'/R_2$, yields

$$\frac{Q_1}{q} = \frac{l'm}{n} = \frac{l'P_1}{P_2} . \tag{26}$$

Thus the consideration of the throat region leads to the relations which *mix* the quantization conditions on q, Q_n, and P_n.

III. FOUR-DIMENSIONAL DYONIC BLACK HOLES

The six-dimensional σ model of the previous section can be interpreted as describing a dyonic five-dimensional (u,v,x_s) solitonic string solution. The string has both electric (q) charge (with the gauge field one-form $A\,du$) and magnetic (P_1) charge, both resulting from the couplings to the compact Kaluza-Klein $x_4 \equiv y_1$ direction. There is also a momentum (Q_1) along the string and a special perturbation (P_2) in curved noncompact spatial directions x_s. The new coupling A preserves the same amount of the world-sheet supersymmetry of the σ-model string action, and thus the same amount of the space-time supersymmetry of the target space background, as in the case of $A=0$ [25].

Like the fundamental string solution [10,11] $(f=k=1)$ can be viewed as a field of stable elementary winding string mode in flat background, this solution may be interpreted as representing a particular state of a dyonic string soliton. This interpretation is consistent with the linear form of the equations for F^{-1}, K, A (11),(12) which can be viewed as conditions of marginality for (exact) perturbations of the six-dimensional model defined as a direct product of the trivial chiral null (''electric'') part, specified by u,v, and nontrivial transverse (''magnetic'') part, specified by x_s, x_4.

At the same time, this model has also a four-dimensional dyonic black hole interpretation. We shall now derive the explicit expressions for the corresponding canonical four-dimensional fields.

A. Dimensional reduction

Following [6], we set $u = y_2, v = 2t, x_4 = y_1$ where y_1, y_2 are two compact toroidal coordinates.[12] One can rewrite the σ model (5), i.e., $L = (G_{MN} + B_{MN})\partial x^M \overline{\partial} x^N + \mathscr{R}\Phi$, in the form[13]

$$\begin{aligned} L = {} & (G'_{\mu\nu} + B'_{\mu\nu})(x)\,\partial x^\mu \overline{\partial} x^\nu + (G_{mn} + B_{mn})(x) \\ & \times [\partial y^m + A^{(1)m}_\mu(x)\,\partial x^\mu][\overline{\partial} y^n + A^{(1)n}_\nu(x)\,\overline{\partial} x^\nu] \\ & + A^{(2)}_{n\mu}(x)(\partial y^n \overline{\partial} x^\mu - \overline{\partial} y^n \partial x^\mu) + \mathscr{R}\Phi'(x), \end{aligned} \tag{27}$$

where $x^\mu = (t, x^s)$, $s = 1,2,3$, $n,m = 1,2$. The four-dimensional string-frame space-time metric $G'_{\mu\nu}$, the two form-field $B'_{\mu\nu}$ and the dilaton Φ', which includes the shift resulting from ''integrating out'' y^n, as well as the canonical vector potentials $A^{(1)n}_\mu$ and $A^{(2)}_{n\mu}$ of the respective Kaluza-Klein and two-form U(1) gauge fields are related to the fields of the six-dimensional σ model (5) in the following way:

$$\begin{aligned} G'_{\mu\nu} &= G_{\mu\nu} - G_{mn}A^{(1)m}_\mu A^{(1)n}_\nu , \\ B'_{\mu\nu} &= B_{\mu\nu} - B_{mn}A^{(1)m}_\mu A^{(1)n}_\nu , \end{aligned} \tag{28}$$

$$A^{(1)n}_\mu = G^{nm}G_{m\mu}, \quad A^{(2)}_{n\mu} = B_{n\mu} - B_{nm}A^{(1)m}_\mu, \quad \Phi' = \Phi - \tfrac{1}{4}\Delta,$$

$$\Delta \equiv \det G_{mn} . \tag{29}$$

The four-dimensional Einstein-frame metric is

$$g_{\mu\nu} = e^{-2\Phi'} G'_{\mu\nu}, \tag{30}$$

and the gauge-invariant torsion can be written as [37]

$$\begin{aligned} H'_{\mu\nu\lambda} = {} & H_{\mu\nu\lambda} \\ & - (A^{(1)n}_\mu H_{n\nu\lambda} - A^{(1)m}_\mu A^{(1)n}_\nu H_{mn\lambda} + \text{cycl. perms.}), \end{aligned} \tag{31}$$

where H_{MNK} is the field strength of the antisymmetric tensor B_{MN} in (5). $H'_{\mu\nu\lambda}$ is related to the four-dimensional axion Ψ by

---

[11] The quantization of P_1 can also be understood as a consequence of the requirement of regularity of the metric $(\sim [dy_1 + P_1(1-\cos\theta)d\varphi]^2 + \cdots)$ of the full six-dimensional model (5): to avoid the Taub-NUT singularity one should be able to identify y_1 with period $4\pi P_1$, which is possible if $2P_1/R_1 = m$. By T-duality, the same constraint should also apply to P_2, i.e., $2P_2 = n\tilde{R}_1 = n\alpha'/R_1$.

[12] In the case of the fundamental string interpretation the direction of the string winding is the ''boosted'' compact coordinate $y'_2 = y_2 + t$, i.e., $u = y'_2 - t, v = y'_2 + t$. Then, e.g., for the one-center solution one gets $K = Q_1/r$ instead of $K = 1 + Q_1/r$ used here.

[13] In the purely ''electric'' case of $f=k=1$, i.e., the case of the chiral null model with flat transverse part, similar dimensional reduction was discussed in [7]. There, an additional term $2F\mathscr{A}_s \partial u \overline{\partial} x^s$ was also included, leading to nonstatic, e.g., Taub-NUT or rotating, four-dimensional space-time metric.

$$H'^{\mu\nu\lambda} \equiv \frac{e^{4\Phi'}}{\sqrt{-g}} \epsilon^{\mu\nu\lambda\rho} \partial_\rho \Psi, \tag{32}$$

where the indices are raised using $g_{\mu\nu}$ and $g = \det g_{\mu\nu}$.

B. Four-dimensional background

We shall now express the four-dimensional fields in terms of harmonic functions F^{-1}, K, f, k^{-1}, the function A and the functions a_s, b_s.

The Einstein-frame metric is found to be of the following form:

$$ds_E^2 = g_{\mu\nu} dx^\mu dx^\nu = -\lambda(r) dt^2 + \lambda^{-1}(r)(dr^2 + r^2 d\Omega^2). \tag{33}$$

The structure of the space-time metric is that of an *extreme* static spherically symmetric configuration. This indicates that this background corresponds to a BPS-saturated state.

The metric function λ, the dilaton Φ', and the moduli G_{mn}, B_{mn} of the two-torus are given by

$$\lambda = Fk\Delta^{-1/2}, \quad e^{2\Phi'} = Ff\Delta^{-1/2}, \quad \Delta = FKfk - A^2F^2, \tag{34}$$

$$G_{11} = fk, \quad G_{22} = FK, \quad G_{12} = -B_{12} = AF. \tag{35}$$

The description in terms of the four-dimensional fields is valid in the spatial region, where F, K, f, k and the volume of the two-torus Δ are all positive. The constraint $\Delta > 0$ implies a constraint on the function A: $F^{-1}Kfk > A^2$.

The four four-dimensional U(1) Kaluza-Klein and two-form gauge fields have the following components:

$$A_\mu^{(1)1} = (-AF^2\Delta^{-1}, a_s), \quad A_\mu^{(1)2} = (Ffk\Delta^{-1}, 0), \tag{36}$$

$$A_{1\mu}^{(2)} = (AF^2 fk\Delta^{-1}, b_s), \quad A_{2\mu}^{(2)} = (F^2 Kfk\Delta^{-1}, 0).$$

The axion Ψ is determined by

$$\partial_s \Psi = Af^{-2} k \partial_s (fk^{-1}). \tag{37}$$

C. One-center four-dimensional solution

In the case of spherically-symmetric one-center harmonic functions F^{-1}, K, f, k^{-1} the explicit solution (15)–(18) yields a class of static spherically-symmetric four-dimensional backgrounds specified by the five parameters P_1, P_2, Q_1, Q_2, and q.[14] These parameters determine the magnetic $P_m^{(1,2)}$ and electric $Q_m^{(1,2)}$ charges of the corresponding Kaluza-Klein $A_\nu^{(1)m}$ and two-form $A_{m\nu}^{(2)}$ gauge fields, i.e., $A_{mt}^{(i)} \to -Q_m^{(i)}/r$ as $r^2 \equiv x_s x^s \to \infty$, and $A_{m\phi}^{(i)} = (1 - \cos\theta) P_m^{(i)}$. As follows from (15)–(18) and the expressions for the gauge fields (36), the physical charges are related to these five parameters in the following way:[15]

$$(Q_1^{(1)}, P_1^{(1)}) = (q, P_1), \quad (Q_2^{(1)}, P_2^{(1)}) = (Q_1, 0),$$

$$(Q_1^{(2)}, P_1^{(2)}) = (-q, P_2), \quad (Q_2^{(2)}, P_2^{(2)}) = (Q_2, 0). \tag{38}$$

Note that the magnetic charges arise from the transverse part of the σ model (5) and all the electric charges arise from the chiral null part of (5). When there is no A-coupling term ($q = 0$) the electric and magnetic charges are orthogonal, i.e., they are associated with gauge fields originating from two different compactified directions. The A coupling induces new electric charges, but only in a left-moving direction: it leads to nonzero left-moving electric charge $Q_{L\,1} \equiv \frac{1}{2}(Q_1^{(1)} - Q_1^{(2)}) = q$ along the magnetic charge direction. The right-moving charges, i.e., $P_{Rn} \equiv \frac{1}{2}(P_n^{(1)} + P_n^{(2)})$ and $Q_{Rn} \equiv \frac{1}{2}(Q_n^{(1)} + Q_n^{(2)})$, still remain orthogonal.

The explicit form of the space-time metric function λ, the dilaton Φ', the axion Ψ and the moduli G_{mn}, B_{mn} of the internal two-torus is

$$\lambda = \frac{r^2}{\{(r+Q_1)(r+Q_2)(r+P_1)(r+P_2) - q^2[r + \frac{1}{2}(P_1+P_2)]^2\}^{1/2}}, \tag{39}$$

$$e^{2\Phi'} = \frac{(r+P_1)(r+P_2)}{\{(r+Q_1)(r+Q_2)(r+P_1)(r+P_2) - q^2[r + \frac{1}{2}(P_1+P_2)]^2\}^{1/2}}, \tag{40}$$

[14]The five-parameter extreme (BPS-saturated) solution as well as nonextreme solutions of the effective heterotic string action compactified on a six-torus were obtained independently in [38] by performing a subset of O(8,24) symmetry transformations of the three-dimensional effective action on the Schwarzschild black hole background. They serve as generating solutions for all the static spherically-symmetric configurations of the heterotic string theory compactified on a six-torus. The BPS-saturated solution obtained in [38] is related to the one described in this section through a subset of $SO(2) \times SO(2) \subset O(2,2)$ (T-duality) and $SO(2) \subset SL(2,R)$ (S-duality) transformations.

[15]In this section we assume that $\alpha' = 2$, $e^{\Phi'_\infty} = 1$, the Newton's constant $G_N = \frac{1}{8}\alpha' e^{\Phi'_\infty} = \frac{1}{4}$ and the compactification radii $R_1 = R_2 = \sqrt{\alpha'} = \sqrt{2}$.

$$\Psi=\frac{q}{2P_1}\left[\frac{P_2-P_1}{r+P_1}+\frac{2(P_1+P_2)}{r}-3\frac{P_2}{P_1}\ln\left(1+\frac{P_1}{r}\right)-\frac{P_1}{P_2}\ln\left(1+\frac{P_2}{r}\right)\right], \tag{41}$$

$$G_{11}=\frac{r+P_2}{r+P_1},\quad G_{22}=\frac{r+Q_1}{r+Q_2},\quad G_{12}=-B_{12}=\frac{q[r+\frac{1}{2}(P_1+P_2)]}{(r+Q_2)(r+P_1)}. \tag{42}$$

Note that this one-center solution is written with the following choice of the asymptotic ($r\to\infty$) values for the fields: $\Phi'_\infty=\Psi_\infty=G_{12\,\infty}=B_{12\,\infty}=0$ and $G_{11\,\infty}=G_{22\,\infty}=1$. Solutions with other aymptotic values of the axion-dilaton and moduli fields are related to this one by a particular SL(2,R) (S-duality) and O(2,2) (T-duality of two-torus) transformations, respectively.

Regular solutions, i.e., solutions with event horizons, are determined by choosing the four parameters $P_{1,2},Q_{1,2}$ to be *positive*:

$$P_1>0,\quad P_2>0,\quad Q_1>0,\quad Q_2>0, \tag{43}$$

and q satisfying the following constraint:

$$Q_1Q_2-q^2>0,\quad (Q_1Q_2-q^2)P_1P_2-\tfrac{1}{4}q^2(P_1-P_2)^2>0. \tag{44}$$

Regular solutions,[16] i.e., those satisfying all the inequalities (43),(44), have an event horizon at $r=0$ and a timelike singularity at negative $r=r_{\rm sing}$ ($-\min\{P_1,P_2,Q_1,Q_2\}<r_{\rm sing}<0$ for $q\neq0$ and $r_{\rm sing}=-\min\{P_1,P_2,Q_1,Q_2\}$ for $q=0$), i.e., the global space-time is that of extreme Reissner-Nordström black holes. In the case when any of the charge combinations in (43),(44) is zero, the singularity is null and located at $r=0$, i.e., the horizon and the singularity coincide. In the case of only one nonzero parameter in (43) the singularity at $r=0$ is naked. When at least one of the charge combinations in (43),(44) becomes negative the solutions are singular with a naked singularity at $r>0$.

However, we would like to emphasize that for small r the effective four-dimensional description breaks down and the solution becomes effectively six-dimensional. Therefore the nature of singularities should be readdressed from the point of view of the six-dimensional string theory, using string-frame metric (cf. [39]). For $r\geqslant0$, the regular solutions satisfying (43),(44) are always *nonsingular*. This is a reflection of the regularity of the underlying six-dimensional conformal σ model discussed in Sec. II.

In the following we shall concentrate on regular solutions. The asymptotic value of the metric coefficient λ (39) is of the form

$$\lambda=1-\frac{M_{\rm ADM}}{2r}+O(r^{-2}), \tag{45}$$

where the ADM mass

$$M_{\rm ADM}=Q_1+Q_2+P_1+P_2 \tag{46}$$

does not depend on q. It saturates the Bogomol'nyi bound [22,40] and corresponds to the BPS-saturated state that preserves $\frac{1}{4}$ of the original $N=4$ target space supersymmetry.

Scalar fields (40)–(42) have the following asymptotic behavior

$$e^{2\Phi'}=1+\frac{(P_1+P_2-Q_1-Q_2)}{2r}+O(r^{-2}),$$

$$\Psi=\frac{q(P_1+P_2)}{r^2}+O(r^{-3}), \tag{47}$$

$$G_{11}=1+\frac{(P_2-P_1)}{r}+O(r^{-2}),$$

$$G_{22}=1+\frac{(Q_1-Q_2)}{r}+O(r^{-2}), \tag{48}$$

$$G_{12}=-B_{12}=\frac{q}{r}+O(r^{-2}). \tag{49}$$

Note that the dilaton and all the two-torus moduli have nonzero scalar charges, while the axion charge is zero.

The area of the event horizon, i.e., $\mathbf{A}\equiv4\pi(\lambda^{-1}r^2)_{r=0}$, is

$$\mathbf{A}=4\pi\left[(Q_1Q_2-q^2)P_1P_2-\frac{1}{4}q^2(P_1-P_2)^2\right]^{1/2}. \tag{50}$$

For $q\neq0$ the area is decreased compared to its value for $q=0$.

At the horizon at $r=0$ the axion Ψ (41) blows up while the dilaton Φ' (40) and the moduli (42) are constant. Note that unlike for pure electric or pure magnetic configurations, where Φ' grows either at small or at large distances, here $e^{\Phi'}$ can be chosen to be small in the whole region $r\geqslant0$.

The T-self-dual case with $Q_1=Q_2=Q,P_1=P_2=P$ deserves a special discussion. In this case the moduli G_{11} and G_{22} (42) remain constant. For $q=0$ and $Q=P$ it corresponds to the extreme Reissner-Nordström-type dyonic black hole with all scalar fields being constant [22,25]. In this case the six-dimensional σ model (5),(27) takes a particularly simple form discussed in [25]. For $Q=P$ the six-dimensional dilaton (18) is constant, but if $q\neq0$ the moduli $G_{12}=-B_{12}$, and thus also the four-dimensional dilaton Φ', are no longer constant. The area of the horizon in this case is given by $\mathbf{A}=4\pi P\sqrt{Q^2-q^2}$.

[16]The space-time properties of regular solutions with $q=0$ were studied in Ref. [22].

IV. ALL BPS-SATURATED STATIC BLACK HOLE SOLUTIONS OF HETEROTIC STRING COMPACTIFIED ON A SIX-TORUS

The five-parameter solution obtained in Sec. III turns out to be a generating solution for *all* static spherically-symmetric BPS-saturated configurations of the four-dimensional heterotic string compactified on a six-torus. These solutions can be obtained by applying a subset of T-duality [O(6,22)] and S-duality [SL(2,R)] transformations to the generating solution. It should be noted, however, that while the generating solution is described by an exact conformal σ model (5), (6), these more general BPS-saturated backgrounds are guaranteed only to be solutions of the leading-order effective string equations. Indeed, in contrast to the T-duality transformations, the S-duality transformations do not, in general, map one conformal σ model into another.

A. Effective four-dimensional action

The effective four-dimensional heterotic string compactified on a six-torus has $N=4$ supersymmetry. The bosonic part of the leading term in the effective action has the following form (for a review see [41] and references therein):

$$S=\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\Big[R(g)-2\partial_\mu\Phi'\partial^\mu\Phi' -\frac{1}{12}\,\mathrm{e}^{-4\Phi'}H'_{\mu\nu\rho}H'^{\mu\nu\rho}-\frac{1}{4}\,\mathrm{e}^{-2\Phi'}\mathscr{F}^i_{\mu\nu}(LML)_{ij}\mathscr{F}^{j\mu\nu} +\frac{1}{8}\,\mathrm{Tr}(\partial_\mu ML\partial^\mu ML)\Big]. \tag{51}$$

The action (51) depends on massless four-dimensional bosonic fields, which are determined in terms of the following dimensionally reduced ten-dimensional fields: Zehnbein $\hat{E}^A_M$, dilaton Φ, two-form field B_{MN}, and $\mathrm{U}(1)^{16}$ gauge fields A^I_M ($M,N=0,\ldots,9$; $I=1,\ldots,16$). The ansatz for the Zehnbein is of the form

$$\hat{E}^A_M=\begin{pmatrix} e^{\Phi'}e^\alpha_\mu & A^{(1)m}_\mu e^a_m \\ 0 & e^a_m \end{pmatrix},$$

where $A^{(1)m}_\mu$ ($m=1,\ldots,6;\mu=0,\ldots,3$) are Kaluza-Klein U(1) gauge fields, $\Phi'=\Phi-\frac{1}{2}\ln\det e^a_m$ is the four-dimensional dilaton field, and $g_{\mu\nu}=e^\alpha_\mu e^\alpha_\nu$ is the Einstein frame metric. Other components of 28 U(1) gauge fields $\mathscr{A}^i_\mu\equiv(A^{(1)m}_\mu,A^{(2)}_{\mu m},A^{(3)I}_\mu)$ are defined as $A^{(2)}_{\mu m}\equiv B_{\mu m}+B_{mn}A^{(1)n}_\mu+\frac{1}{2}a^I_mA^{(3)I}_\mu$, $A^{(3)I}_\mu\equiv A^I_\mu-a^I_mA^{(1)m}_\mu$ with the field strengths $\mathscr{F}^i_{\mu\nu}=\partial_\mu\mathscr{A}^i_\nu-\partial_\nu\mathscr{A}^i_\mu$. The two-form field with the field strength $H'_{\mu\nu\rho}=3(\partial_{[\mu}B_{\nu\rho]}+\frac{1}{2}\mathscr{A}_{[\mu}L\mathscr{F}_{\nu\rho]})$ is equivalent to a pseudoscalar (the axion) Ψ through the duality transformation $H'^{\mu\nu\rho}=e^{4\Phi'}/\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\partial_\sigma\Psi$. A symmetric O(6,22) matrix M of scalar (moduli) fields can be expressed in terms of the following O(6,22) matrix V [37]:

$$M=V^TV,\quad V=\begin{pmatrix} E^{-1} & E^{-1}C & E^{-1}a^T \\ 0 & E & 0 \\ 0 & a & I_{16}\end{pmatrix}, \tag{52}$$

where $E\equiv[e^a_m]$ (the Sechsbein of the internal metric G_{mn}), $C\equiv[\frac{1}{2}A^I_mA^I_n+B_{mn}]$, and $a\equiv[A^I_m]$. V plays a role of a Vielbein in the O(6,22) target space.

The four-dimensional effective action is invariant under the O(6,22) transformations (T-duality) [37,41]:

$$M\to\Omega M\Omega^T,\quad \mathscr{A}^i_\mu\to\Omega_{ij}\mathscr{A}^j_\mu,\quad g_{\mu\nu}\to g_{\mu\nu},\quad S\to S, \tag{53}$$

where $S\equiv\Psi+ie^{-2\Phi'}$ and $\Omega\in\mathrm{O}(6,22)$ is an O(6,22) invariant matrix,

$$\Omega^TL\Omega=L,\quad L=\begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16}\end{pmatrix}. \tag{54}$$

In addition, the corresponding equations of motion and Bianchi identities are invariant under the SL(2,R) transformations (S-duality) [41]:

$$S\to\frac{aS+b}{cS+d},\quad M\to M, g_{\mu\nu}\to g_{\mu\nu},$$

$$\mathscr{F}^i_{\mu\nu}\to(c\Psi+d)\mathscr{F}^i_{\mu\nu}+ce^{-2\Phi'}(ML)_{ij}\tilde{\mathscr{F}}^j_{\mu\nu}, \tag{55}$$

where $\tilde{\mathscr{F}}^{i\mu\nu}=1/2\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\mathscr{F}^i_{\rho\sigma}$ and $a,b,c,d\in R$ satisfy $ad-bc=1$. At the quantum level, the parameters of both T- and S-duality transformations become integer valued.

B. General class of dyonic solutions generated by duality transformations

Static spherically-symmetric solutions corresponding to the effective action (51) are described by the ansatz (33) for the four-dimensional space-time metric, the dilaton-axion field S and the moduli fields M, which depend only on the radial coordinate r, and by 28 electric and 28 magnetic U(1) gauge fields. The Maxwell's equations and the Bianchi identities determine the components of the U(1) field strength to be

$$\sqrt{2}\mathscr{F}^i_{tr}=4G_N\frac{e^{2\Phi'(r)}\lambda(r)}{r^2}M_{ij}(\alpha_j+\Psi\beta_j),$$

$$\sqrt{2}\mathscr{F}^i_{\theta\phi}=L_{ij}\beta_j\sin\theta, \tag{56}$$

which are expressed in terms of the conserved charge vectors $\vec{\alpha}$ and $\vec{\beta}$. The electric and magnetic charges

$$\vec{P}^T\equiv(P^{(1)}_m;P^{(2)}_m;P^{(3)}_I),\quad \vec{Q}^T\equiv(Q^{(1)}_m;Q^{(2)}_m;Q^{(3)}_I), \tag{57}$$

are related to the charge vectors $\vec{\alpha}$ and $\vec{\beta}$ in the following way [42]:

$$\sqrt{2}Q_i=e^{2\Phi'_\infty}M_{ij\infty}(\alpha_j+\Psi_\infty\beta_j),\quad \sqrt{2}P_i=L_{ij}\beta_j, \tag{58}$$

where the subscript ∞ refers to the asymptotic ($r\to\infty$) values of M and Ψ and we have assumed that $\alpha'=2, G_N=\frac{1}{8}\alpha' e^{2\Phi'_\infty}=\frac{1}{4}e^{2\Phi'_\infty}$.

The equations of motion are invariant under both T- and S-duality transformations. Therefore, one can generate new supersymmetric solutions by applying O(6,22) and SL(2,R) transformations to some known solution. This is the technique[17] [14,22,38] which was used previously to obtain a general class of BPS-saturated backgrounds. In particular, starting with the four-parameter BPS-saturated solution one finds [22] a general class of BPS-saturated black hole solutions with 28 electric and 28 magnetic charges subject to one constraint. Here we follow the same procedure to obtain the most general BPS-saturated solution starting with the five-parameter solution of Sec. III. We shall consider regular solutions with event horizons. In particular, we shall determine the expression for the ADM mass formula and for the area of the event horizon for the most general BPS-saturated configuration in this class.

Without loss of generality one can bring [41] arbitrary asymptotic values of the moduli and axion-dilaton fields to the form $M_\infty=I$ and $S_\infty=i$ by performing the following O(6,22) and SL(2,R) transformations:

$$M_\infty\to\hat\Omega M_\infty\hat\Omega^T=I,\quad S_\infty\to(aS_\infty+b)/d=i. \tag{59}$$

Here $\hat\Omega\in$ O(6,22), $ad=1$, and in quantized theory the charge lattice vectors will belong to the new transformed lattice. Then the subsets of O(6,22) and SL(2,R) transformations that preserve the above new asymptotic values of M_∞ and S_∞ are O(6)$\times$O(22) and SO(2) transformations, respectively. Note that configurations obtained in that manner have the same four-dimensional space-time structure and thus the same singularity and thermal properties as the generating solution. To find solutions with arbitrary asymptotic values of M and S one has to undo the above transformations.

The four-dimensional black hole background corresponding to the solitonic string solution described in Secs. II and III is parameterized by the two magnetic $P_1^{(1,2)}$ and the four electric $Q_2^{(1,2)}$ and $Q_1^{(1)}=-Q_1^{(2)}\equiv q$ charges, i.e.,

$$\vec Q^T=(q,Q_1,0,\ldots,0;-q,Q_2,0,\ldots,0;0,\ldots,0),$$

$$\vec P^T=(P_1,0,\ldots,0;P_2,0,\ldots,0;0,\ldots,0).$$

In Sec. III the asymptotic values of the moduli and the dilaton-axion fields were already chosen to be of the form $M_\infty=I$, $S_\infty=i$. This background can now be used as a generating solution for the most general set of solutions in this class.

As a first step, one applies a subset of O(6)$\times$O(22) $\subset$O(6,22) transformations which correspond to SO(6)/SO(4) transformations with 9 parameters and SO(22)/SO(20) transformations with 41 parameters, which, along with 5 original charges, give a configuration with 56 (28 electric $\vec Q$ and 28 magnetic $\vec P$) charges subject to one constraint. After one has undone the transformation (59), so that M_∞ and S_∞ become arbitrary, this constraint can be cast into the following O(6,22) (T-duality) invariant form:

$$\vec P^T\mathscr{M}_+\vec Q=0,\quad \mathscr{M}_\pm\equiv LM_\infty L\pm L. \tag{60}$$

The ADM mass (46) of the generating solution can be written in the following O(6,22) invariant form:

$$M^2_{\rm ADM}=e^{-4\Phi'_\infty}[(\vec Q^T\mathscr{M}_+\vec Q)^{1/2}+(\vec P^T\mathscr{M}_+\vec P)^{1/2}]^2. \tag{61}$$

The area of the event horizon **A** (50) for the regular generating solution can also be cast in the O(6,22) invariant form:

$$\mathbf{A}=2\pi\left[(\vec P^TL\vec P)(\vec Q^TL\vec Q)-\frac{1}{4}(\vec P^T\mathscr{M}_-\vec Q)^2\right]^{1/2}. \tag{62}$$

Using the charge constraint (60) one can replace the term $(\vec P^T\mathscr{M}_-\vec Q)$ in (62) by $2[\vec P^T(LM_\infty L)\vec Q]$ and represent (62) in the following SL(2,R) (S-duality) invariant form:

$$\mathbf{A}=2\pi[(\vec P^TL\vec P)(\vec Q^TL\vec Q)-(\vec P^T(LM_\infty L)\vec Q)^2]^{1/2}. \tag{63}$$

The subsequent SO(2)$\subset$SL(2,R) transformation provides one more parameter, which removes the charge constraint (60). The most general configuration in this class has then 56 parameters specified by *unconstrained* 28 electric $\vec Q$ and 28 magnetic $\vec P$ charges. This configuration thus corresponds to the most general spherically-symmetric static BPS-saturated black hole solution consistent with the no-hair theorem.

The SO(2)$\subset$SL(2,R) transformation allows one to write the ADM mass formula in the following O(6,22) and SL(2,R) invariant form [22,40]:

$$M^2_{\rm ADM}=e^{-4\Phi'_\infty}\{\vec Q^T\mathscr{M}_+\vec Q+\vec P^T\mathscr{M}_+\vec P +2[(\vec P^T\mathscr{M}_+\vec P)(\vec Q^T\mathscr{M}_+\vec Q)-(\vec P^T\mathscr{M}_+\vec Q)^2]^{1/2}\}. \tag{64}$$

Note that when the magnetic and electric charges are parallel in the SO(6,22) sense, i.e., $\vec P\propto\vec Q$, the ADM mass (64) corresponds to the mass of the BPS-saturated black holes which preserve $\frac{1}{2}$ of $N=4$ supersymmetry (see, e.g., [41]). In the case when the magnetic and electric charges are not parallel, the mass is larger and the configurations preserve $\frac{1}{4}$ of $N=4$ supersymmetry.

The area of the event horizon (63) is already invariant under the SL(2,R) transformations and thus remains of the same form. The general expression for the area reduces to the special form when the charge configurations are constrained. Regular configurations with $\vec P\propto\vec Q$, i.e., BPS-saturated states which preserve $\frac{1}{2}$ of $N=4$ supersymmetry, have zero area of the event horizon.

Another example is provided by the most general solutions with *zero* axion [22]. Those are backgrounds obtained from the four-parameter generating solution with $q=0$ by applying a subset of O(6,22) transformations. They are specified by 28 electric and 28 magnetic charges subject to *two* constraints: $\vec P^T\mathscr{M}_+\vec Q=0$ and $\vec P^T\mathscr{M}_-\vec Q=0$ [22]. Thus,

[17]Analogous techniques, employing the symmetries of the effective four-dimensional as well as three-dimensional action of the $(4+n)$-dimensional Abelian Kaluza-Klein theory were used [43] to obtain all static spherically-symmetric black holes in that theory.

in this case $\vec{P}^T(LM_\infty L)\vec{Q}=0$, and only the first term in the expression (63) is present, as pointed out in [26]. In general, the area of the event horizon is *decreased* by an additional positive definite term which measures the orthogonality of the magnetic and electric charge vectors.

C. ADM mass and area of horizon in terms of conserved charges

As the last step, we can express the ADM mass formula (64) and the area of the event horizon (63) in terms of the conserved electric $\vec{\alpha}$ and magnetic $\vec{\beta}$ charge vectors, thus allowing to study their dependence on the asymptotic values of the the axion-dilaton S_∞ and moduli M_∞ fields. Since $\vec{P}$ and $\vec{Q}$ are related to the conserved charge vectors $\vec{\alpha}$ and $\vec{\beta}$ through (58), the ADM mass formula (64) can be written as

$$M^2_{\rm ADM}=\tfrac{1}{2}\vec{\tilde{\alpha}}^T\mu_+\vec{\tilde{\alpha}}+\tfrac{1}{2}e^{-4\Phi'_\infty}\vec{\beta}^T\mu_+\vec{\beta} + e^{-2\Phi'_\infty}[(\vec{\beta}^T\mu_+\vec{\beta})(\vec{\alpha}^T\mu_+\vec{\alpha})-(\vec{\beta}^T\mu_+\vec{\alpha})^2]^{1/2}, \tag{65}$$

where $\vec{\tilde{\alpha}}\equiv\vec{\alpha}+\Psi_\infty\vec{\beta}$ and $\mu_\pm\equiv M_\infty\pm L$. Similarly, the area of the event horizon (63) and thus the black hole entropy can be represented as[18]

$$\mathbf{A}=\pi e^{2\Phi'_\infty}[(\vec{\beta}^TL\vec{\beta})(\vec{\alpha}^TL\vec{\alpha})-(\vec{\beta}^TL\vec{\alpha})^2]^{1/2},\quad \mathbf{S}=\frac{\mathbf{A}}{4G_N}. \tag{66}$$

Both the ADM mass (65) and the entropy (66) can be cast in the *manifestly T- and S*-duality invariant form as[19]

$$M^2_{\rm ADM}=(8G_N)^{-1}\{\mathscr{M}_{\infty ab}(\vec{v}^{aT}\mu_+\vec{v}^b) + [2\mathscr{S}_{ac}\mathscr{S}_{bd}(\vec{v}^{aT}\mu_+\vec{v}^b)(\vec{v}^{cT}\mu_+\vec{v}^d)]^{1/2}\}, \tag{67}$$

$$\mathbf{S}=\frac{\mathbf{A}}{4G_N}=\pi[\tfrac{1}{2}\mathscr{S}_{ac}\mathscr{S}_{bd}(\vec{v}^{aT}L\vec{v}^b)(\vec{v}^{cT}L\vec{v}^d)]^{1/2}. \tag{68}$$

Here $\mathscr{M}_\infty$ is the asympotic value of the axion-dilaton matrix $\mathscr{M}$, and $\mathscr{S}$ is the SL(2,R) invariant matrix. $\mathscr{M}$ and $\mathscr{S}$ are of the following form:

$$\mathscr{M}=e^{2\Phi'}\begin{pmatrix}1 & \Psi\\ \Psi & \Psi^2+e^{-4\Phi'}\end{pmatrix},\quad \mathscr{S}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{69}$$

and the (quantized) charge vector $\vec{v}^T=(\vec{v}^{1\,T},\vec{v}^{2\,T})\equiv(\vec{\alpha}^T,\vec{\beta}^T)$. $\mathscr{M}$ and $\vec{v}$ transform under the SL(2,R) transformation $\omega=\left(\begin{smallmatrix}d & c\\ b & a\end{smallmatrix}\right)$ $(ad-bc=1)$ as [41]

$$\mathscr{M}\to\omega\mathscr{M}\omega^T,\quad \vec{v}\to\mathscr{S}\omega\mathscr{S}^T\vec{v}. \tag{70}$$

An important observation is that while for fixed values of the charge vectors $\vec{\alpha}$ and $\vec{\beta}$ the ADM mass (67) changes under the variations of the moduli and string coupling, *the entropy remains the same as one moves in the moduli and coupling space*. The fact the entropy is an invariant quantity is consistent with the expectation that the internal structure of a BPS-saturated black hole should not change under variations of moduli and couplings.

This invariance indicates [26] that the classical entropy may have a statistical interpretation in terms of a number of degenerate black-hole configurations: being an integer such a number would not change under adiabatic variations of moduli and axion-dilaton couplings. Indeed, the combination of charges $(\vec{\beta}^TL\vec{\beta})(\vec{\alpha}^TL\vec{\alpha})-(\vec{\beta}^TL\vec{\alpha})^2$ which appears in (68) is expected to be an (even) integer. Following [41], one may attempt to justify this by using the analogy with the level matching condition for the elementary BPS-saturated string states of toroidally compactified heterotic string and the Dirac-Schwinger-Zwanziger-Witten (DSZW) [42] quantization condition. In the case of the generating solution described by the conformal model discussed in Sec. II, the quantization of charges is implied by the consideration of the conformal theory corresponding to the throat region (cf. Sec. II B).

Note that the purely electric BPS-saturated black holes preserve $\frac{1}{2}$ of $N=4$ supersymmetry and have the same quantum numbers [8,9] as the elementary BPS-saturated string states with no excitations in the right sector ($N_R=\frac{1}{2}$). In the electric case the quantized charge vector $\vec{\alpha}$ is constrained to lie on an even self-dual lattice with the norm [14]

$$\vec{\alpha}^TL\vec{\alpha}=2N_L-2=-2,0,2,\ldots. \tag{71}$$

The DSZW charge quantization condition then implies an analogous constraint for $\vec{\beta}^TL\vec{\beta}$.

The necessary conditions for a BPS-saturated configuration to be regular with a nonzero area of the event horizon are

$$\vec{\alpha}^TL\vec{\alpha}>0,\quad \vec{\beta}^TL\vec{\beta}>0,\quad (\vec{\beta}^TL\vec{\beta})(\vec{\alpha}^TL\vec{\alpha})-(\vec{\beta}^TL\vec{\alpha})^2>0. \tag{72}$$

The latter constraint becomes an equality for the BPS-saturated configurations preserving $\frac{1}{2}$ of $N=4$ supersymmetry (i.e., for quantized charges, when $\vec{\alpha}\propto\vec{\beta}$ with magnetic and electric charge vector components being coprime integers [41]).

V. STRING ORIGIN OF DYONIC BLACK HOLE ENTROPY

One of the motivations behind the above discussion of the five-parameter dyonic static spherically-symmetric BPS-saturated black hole as a four-dimensional ''image'' of an exact solitonic string solution in six dimensions is to try to use information about the underlying conformal theory in order to give a statistical interpretation to the black hole entropy. The aim is to extend and amplify the recent interesting proposal [26] along these lines which generalize an earlier suggestion [14] (see also [15]). The discussion in [26] was based on a subset of BPS-saturated dyonic black holes of

[18]The four-dimensional dilaton value at the event horizon is also a moduli- and coupling-independent quantity: $e^{2\Phi'(0)}=\vec{\beta}^TL\vec{\beta}\,[(\vec{\beta}^TL\vec{\beta})(\vec{\alpha}^TL\vec{\alpha})-(\vec{\beta}^TL\vec{\alpha})^2]^{-\frac{1}{2}}$.

[19]We would like to thank C. Hull for discussions on that point.

heterotic string on a six-torus, namely the most general configurations with zero axion. Those can be obtained by T-duality transformations on the four-parameter, i.e., $q=0$, generating solution considered in [22,25]. The expressions for the area and the entropy of these solutions correspond to a special case of (68) with $\vec{\beta}^T L \vec{\alpha}=0$. We expect that the inclusion of the new parameter q should reveal certain more general aspects of the relation between the entropy and the degeneracy of black hole states.

Just as in the case of purely electric BPS-saturated black holes whose entropy may be qualitatively understood as being a consequence of degeneracy of states originating from oscillations of the fundamental string [8,9], one would like to explain the entropy of the dyonic BPS-saturated black holes in terms of degeneracy of states originating from small-scale oscillations of the underlying six-dimensional string soliton.

It was already emphasized in Sec. I that an important advantage of the dyonic black holes, studied in Secs. III and IV, over the electric ones, considered in [14,16,8,9], is that the magnetic charges provide a short-distance ''regularization'' of the metric and that the dilaton is approximately constant.[20] As a result, one may hope to understand the statistical origin of the black hole entropy starting directly from string theory and using only semiclassical considerations.

A. Statistical entropy and magnetic renormalization of α'

Let us consider the case when all the charges are large and the magnetic charges P_1, P_2 are approximately equal. Since the value of the dilaton at the $r=0$ horizon is $e^{2\Phi_0}=e^{2\Phi_\infty}P_2 Q_2^{-1}$ [cf. (18)] to get a small value for the string coupling in the horizon region one needs to assume that Q_2 is also large. Then the expression for the thermodynamic black-hole entropy, proportional to the area of the horizon (50), is of the form[21]

$$\mathbf{S}=\frac{\mathbf{A}}{4G_N}\approx\frac{\pi}{G_N}[P_1P_2(Q_1Q_2-q^2)]^{1/2}. \tag{73}$$

One would like to relate the combination of charges in (73) to the number of ''microscopic'' string configurations giving rise to the same black hole solution at large distances. In the case of the fundamental string states of toroidally compactified heterotic string the combination of charges $Q_1Q_2-q^2$ would be related to the number of the left-moving string oscillation modes N_L, i.e., $(Q_1Q_2-q^2)\approx\frac{1}{4}\alpha' N_L$ (we assume that both charges and N_L are large and set $G_N=\frac{1}{8}\alpha'$, $e^{\Phi'_\infty}=1$). The key observation is that in the present case the horizon (throat) region $r\to 0$ is actually described by the SL(2,R)$\times$SU(2) WZW-type model (20) with the level, i.e., the coefficient in front of the action, $\kappa=4/\alpha' P_1P_2$ (22). For large P_1P_2, the level κ is large and the spectrum of string excitations in this region should be approximately the same as in the flat space, but with the *renormalized* string tension (in the ''transverse'' part of the action)

$$\frac{1}{\alpha'}\to\frac{1}{\alpha'_*}=\frac{P_1P_2}{\alpha' R_1^2}=\frac{P_1P_2}{\alpha'^2}, \tag{74}$$

where we have set $R_1=\sqrt{\alpha'}$. Then $Q_1Q_2-q^2\approx\frac{1}{4}\alpha'_* N_L$, or, equivalently,

$$P_1P_2(Q_1Q_2-q^2)\approx\tfrac{1}{4}\alpha'^2 N_L. \tag{75}$$

At the same time, the value of the Newton's constant is determined by the asymptotic $r\to\infty$ region and thus remains unchanged, i.e., $G_N=\frac{1}{8}\alpha'$. As a result, the thermodynamic entropy (73) takes the form of the statistical entropy[22]

$$\mathbf{S}=\ln d(N_L)_{N_L\gg 1}\approx 4\pi\sqrt{N_L}. \tag{76}$$

This argument generalizes the one in [26] to the case of an extra electric parameter q and also explains the ''magnetic'' renormalization of the string tension [26] by direct consideration of the underlying conformal model in the horizon (throat) region.

It was also suggested in [26] that there may exist an interpolating formula for the entropy which would be valid for arbitrary values of charges and would reproduce the stretched horizon entropy [14,16] in the limit of vanishing magnetic charges. The idea was to use the S-duality for the

[20]The presence of the magnetic charges provides a ''regularization'' making it unnecessary to resort to ''stretched horizon'' considerations used in the case of purely electric extreme black holes [14,16]. There is a certain analogy between the present dyonic model and the conjectured modification (by world-sheet α' corrections) of the purely electric model [14]. Consideration of the stretched horizon region corresponds effectively to a shift $r\to r+\sqrt{\alpha'}$ in (part of) the metric. Analogous regularization of singularities of the fundamental string and extreme electric black hole solutions was suggested in [12]. Turning on magnetic charges $P_1=P_2=P$ can be represented as a replacement of one of the r^2 factors in the metric by $(r+P)^2$. Then the $r=0$ region becomes nonsingular provided all four charges (Q_n, P_n) are nonvanishing.

[21]The assumption $P_1\approx P_2$ implies that the second term under the square root in (50),(68) can be neglected.

[22]The number of BPS states in the free heterotic string spectrum with a given left-moving oscillator number $N_L\gg 1$ is $d(N_L)_{N_L\gg 1}\approx a N_L^{\nu}\exp(2\pi\sqrt{\frac{1}{6}c_{\rm eff}N_L})$, where $c_{\rm eff}=D_{\rm crit}-2=24$. In type II theory $c_{\rm eff}=\frac{3}{2}(D_{\rm crit}-2)=12$, however, there are both left- and right-moving BPS states in the free string spectrum. In contrast to the free string case, the number of BPS oscillation states counted in our case should be the same in the heterotic and type II theories. Namely, the relevant marginal perturbations are only ''left-moving,'' not ''right-moving,'' i.e., the functions in the sigma-model action can depend only on u, not on v to preserve the conformal invariance when both F and K functions are nontrivial. This is also related to the fact that the background is ''chiral'' and has the same amount of space-time supersymmetry in both theories, and therefore only ''left-moving'' perturbations will be supersymmetric. As a result, one should expect that the entropy should be the same in heterotic and type II cases, in agreement with the fact that the corresponding black hole solution and thus also its thermodynamic entropy is the same in two theories. This suggests that the effective value of the product $c_{\rm eff}N_L$ should be the same in the two theories. How this actually happens for the σ model describing the horizon region remains to be understood.

specific example of charge configurations, obtained from the generating solution with $q=0$, and to conjecture that in general the P_1P_2 factor should be replaced by $P_1P_2+\alpha'$, so that the renormalized string tension should be of the form $1/\alpha'_*=1/\alpha'(1+1/\alpha' P_1P_2)$. The ''quantum'' shift by α' can be viewed as a modification of the purely electric model (where the area of the horizon at $r=0$ is zero) corresponding to the prescription of evaluation of the entropy at the stretched horizon at $r=\sqrt{\alpha'}$. This proposal, however, does not seem to apply to the general class of solutions obtained from the five-parameter generating solution with $q\neq 0$. In this case one gets a more general expression for the area of the event horizon (50),(68) than the one assumed in [26].[23] The general quantum formula for the entropy, valid for large as well as small magnetic charges and thus interpolating between the classical general expression (68) and the one for purely electric configurations evaluated at the stretched horizon ($\mathbf{A}=2\pi\sqrt{2\vec{\alpha}^T L\vec{\alpha}}$) [14,16], should involve a nontrivial mixture of electric and magnetic charges.

B. Origin of degeneracy: More general ''oscillating'' solutions

The dyonic black hole is an approximate four-dimensional description of the six-dimensional string soliton represented by the conformal model (5). The origin of the ''internal degrees of freedom'' of the black hole or a degeneracy of configurations with fixed values of global charges, which explains the statistical nature of its entropy, should be related to the existence of different six-dimensional string configurations which have the same structure at scales larger than compactification scale. Such solutions should be represented by marginal deformations of the soliton theory which do not change the values of the asymptotic black hole charges.

As in the case of purely electric BPS-saturated black holes described by the five dimensional fundamental string solutions, one should look for more general conformal models which include (left-moving) oscillations, e.g., in a compact dimension [8,9]. Since these more general solutions explicitly depend on a compact internal coordinate, they can be represented as solutions of lower-dimensional theory with *massive* Kaluza-Klein fields having nontrivial background values. At scales larger than the compactification scale these backgrounds will have the same structure as the BPS-saturated black hole, but the degeneracy will be lifted once one starts measuring external fields with resolution comparable to the compactification scale [8].

Like the oscillating versions of the fundamental string solution correspond to the excited (but still supersymmetric BPS-saturated) states of the heterotic string with flat transverse space [10,9], similar generalizations of the model (5),(6) should represent the BPS-saturated excited states of the string soliton with nontrivial transverse (''magnetic'') space.

Remarkably, a class of supersymmetric generalisations of the soliton model (5),(6) can be obtained in the same way as in the fundamental string case, by allowing the functions K and $\mathscr{A}_i$ in (1) [e.g., K and A in (5)] to depend also on u.[24] In the case of the flat transverse space this corresponds to the generalised fundamental string solution with waves traveling along the string as well as fluctuations in the compact and noncompact flat spatial directions (see [8,9] and references therein).

Starting with the spherically-symmetric one-center model (15),(16) the simplest possibility is to add a u dependent, but linear in x^s term in K [note that the equation for K (11) does not depend on the functions of the transverse theory] and to replace the constants Q_1 and q in K and A by arbitrary functions of u [there is no extra constraint since the term $\partial_u\nabla_i\mathscr{A}^i$ in the equation for K in (2) vanishes in the spherically-symmetric case]

$$K(u,x)=1+f_s(u)x^s+\frac{Q_1(u)}{r},$$

$$A(u,x)=\frac{q(u)[r+\frac{1}{2}(P_1+P_2)]}{r(r+P_1)}. \tag{77}$$

Here $Q_1(u)=Q_1+\tilde{Q}_1(u), q(u)=q+\tilde{q}(u)$.[25]

For simplicity, let us ignore oscillations in the noncompact dimensions, i.e., $f_s=0$. In the fundamental string case ($P_1=P_2=0$) one expects the ''matching condition''[26]

$$Q_1(u)Q_2-q^2(u)=0, \tag{78}$$

which, after averaging in the compact coordinate u, can be put in the form of the (classical) level matching condition for the elementary string states[27]

---

[23]According to [26] the level matching condition should remain essentially the same as in flat space, i.e., $\alpha' N_L=4(Q_1Q_2-q^2+\cdots)$, while the magnetic charges should enter through the modification of the string tension mentioned above. This would imply that $\mathbf{A}\approx 4\pi\sqrt{N_L}$, $N_L=(1+4/\alpha' P_1P_2)\times[1+4/\alpha'(Q_1Q_2-q^2)]$. The general expression for the area (50),(68) does not seem to be consistent with such a factorization.

[24]There exists, in principle, a possibility of including a u dependence also in the functions G_{ij}, B_{ij}, ϕ in (1) [i.e., in the functions f,k in (6)] which define the transverse conformal theory. In this case one finds nontrivial second-order differential equations in u [27] which should be satisfied by $f(u,x), k(u,x)$. As a result, only a special dependence on u may be allowed in the ''magnetic'' part of the model.

[25]The asymptotic flatness of the background can be restored by a coordinate transformation [$x^s\to x^s-\tilde{f}^s(u), \partial_u^2\tilde{f}^s\sim f^s$, etc.] as in [8,9]. Note also that in general the u-dependent part of K can be traded for the $\mathscr{A}_i$-perturbation in (1) by making a redefinition of v [6].

[26]It can be imposed either by requiring that the $r=0$ singularity of the higher-dimensional background should be null [8] or by using string source considerations [9] and T-duality (see also the next footnote).

[27]This relation should already hold in the bosonic string case. In fact, Q_1 plays the role of the momentum along the string. Since Q_1 and Q_2 are interchanged by T-duality in $u=y_2$ direction, Q_2 should be an analogue of the winding number. Then their product should be proportional to the difference of the left-moving (N_L) and right-moving (N_R) oscillation numbers. In heterotic string case there are no classical oscillations in the right-moving sector, i.e., $N_R=0$.

$$\frac{4}{\alpha'}(Q_1Q_2-q^2)=N_L\,,\quad N_L\equiv\frac{4}{\alpha'}\,\langle\widetilde{q}^2(u)\rangle. \tag{79}$$

Analogous level matching condition should exist for excited states of the soliton theory with nonvanishing $P_{1,2}$.[28] Since now the horizon region is described by a well-defined conformal theory, the constraint should be just the level matching condition for the corresponding states of the generalized WZW-type theory (20). For example, replacing q in (19),(21) by a periodic function of $u=y_2$, $q\to q+\widetilde{q}(y_2)$ corresponds to adding to (19) the perturbation [cf. (20),(21)]

$$2\widetilde{q}(y_2)\partial\widetilde{y}_2[\overline{\partial y}_1+P_1(1-\cos\theta)\overline{\partial\varphi}]=2P_1\widetilde{q}(y_2)\partial\widetilde{y}_2\overline{J}_3\,. \tag{80}$$

It is marginal for any $\widetilde{q}(y_2)$ because of the presence of the $e^{-z}\partial y_2\partial t$ term in (20). By analogy with the fundamental string case we expect that for large level κ (22) the corresponding state in the soliton string spectrum should also satisfy

$$\kappa(Q_1Q_2-q^2)=\frac{4}{\alpha'}P_1P_2(Q_1Q_2-q^2)\sim<\widetilde{q}^2(u)>\sim N_L\,. \tag{81}$$

The main difference compared to (79) is again the renormalization of the string tension by P_1P_2, in agreement with the suggestion in [26]. The relation (81) provides an interpretation of (part of) the degeneracy N_L in (75) in terms of (classical) oscillations of the underlying soliton in the internal y_2 direction. Further study of marginal BPS-saturated perturbations of the soliton model is important for making the statistical interpretation of the black hole entropy (73) more quantitative.

[28]Although the higher-dimensional soliton background is nonsingular at $r=0$ if $Q_1Q_2P_1P_2\neq0$ it may still be possible to derive such a level matching condition from some geometrical considerations; cf. [26].

ACKNOWLEDGMENTS

We would like to thank K. Chan, C. Hull, F. Larsen, A. Peet, A. Sen, K. Sfetsos, F. Wilczek, D. Youm, and especially E. Witten for useful discussions. A.A.T. is grateful to the HEP group of Physics Department of Princeton University and Theory Division of CERN for hospitality while this work was in progress. The work of M.C. is supported by the Institute for Advanced Study funds and J. Seward Johnson foundation, U.S. Department of Energy Grant No. DOE-EY-76-02-3071, and the National Science Foundation Career Advancement Award PHY95-12732. The work of A.A.T. is supported by PPARC and ECC Grant No. SC1*-CT92-0789. We also acknowledge the support of the NATO collaborative research Grant No. CGR 940870.

[1] G.W. Gibbons, Nucl. Phys. **B207**, 337 (1982); G.W. Gibbons and K. Maeda, *ibid.* **B298**, 741 (1988); D. Garfinkle, G.T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888(E) (1992).
[2] G.T. Horowitz, in *Proceedings of the 1991 Trieste Spring School on String Theory and Quantum Gravity* (World Scientific, Singapore, 1993).
[3] A.A. Tseytlin, Class. Quant. Grav. **12**, 2365 (1995).
[4] A. Sen, Nucl. Phys. **B440**, 421 (1995).
[5] G.T. Horowitz and A.A. Tseytlin, Phys. Rev. Lett. **73** , 3351 (1994).
[6] G.T. Horowitz and A.A. Tseytlin, Phys. Rev. D **51**, 2896 (1995).
[7] K. Behrndt, Nucl. Phys. **B455**, 188 (1995).
[8] C.G. Callan, J.M. Maldacena, and A.W. Peet, Report No. PUPT-1565, hep-th/9510134 (unpublished).
[9] A. Dabholkar, J.P. Gauntlett, J.A. Harvey, and D. Waldram, Report No. CALT-68-2028, hep-th/9511053 (unpublished).
[10] A. Dabholkar and J.A. Harvey, Phys. Rev. Lett. **63**, 478 (1989).
[11] A. Dabholkar, G.W. Gibbons, J.A. Harvey, and F. Ruiz-Ruiz, Nucl. Phys. **B340**, 33 (1990).
[12] A.A. Tseytlin, Phys. Lett. B **363**, 223 (1995).
[13] A.A. Tseytlin, Phys. Lett. B **251**, 530 (1990).
[14] A. Sen, Mod. Phys. Lett. A **10**, 2081 (1995).
[15] L. Susskind, Report No. RU-93-44, hep-th/9309145 (unpublished); L. Susskind and J. Uglum, Phys. Rev. D **50**, 2700 (1994); J. Russo and L. Susskind, Nucl. Phys. **B437**, 611 (1995).
[16] A.W. Peet, Nucl. Phys. **B456**, 732 (1995).
[17] W. Nelson, Phys. Rev. D **49**, 5302 (1994).
[18] R. Kallosh and T. Ortin, Phys. Rev. D **50**, 7123 (1994).
[19] C.G. Callan, J.A. Harvey, and A. Strominger, in *Proceedings of the 1991 Trieste Spring School on String Theory and Quantum Gravity*, edited by J.A. Harvey *et al.* (World Scientific, Singapore, 1992).
[20] M.J. Duff, R.R. Khuri, and J.X. Lu, Phys. Rep. **259**, 213 (1995).
[21] C.G. Callan, J.A. Harvey, and A. Strominger, Nucl. Phys. **B359**, 611 (1991).
[22] M. Cvetič and D. Youm, Phys. Rev. D **53**, 584 (1996).
[23] M. Cvetič and D. Youm, Phys. Lett. B **359**, 87 (1995).
[24] M. Cvetič and D. Youm, *Proceedings of STRINGS 95: Future Perspectives in String Theory* (World Scientific, Singapore, in press).
[25] M. Cvetič and A.A. Tseytlin, Phys. Lett. B **366**, 95 (1996).
[26] F. Larsen and F. Wilczek, Report No. PUPT-1576, hep-th/9511064 (unpublished).
[27] A.A. Tseytlin, Nucl. Phys. **B390**, 153 (1993).
[28] D. Garfinkle, Phys. Rev. D **46**, 4286 (1992).
[29] A. Sen, Nucl. Phys. **B450**, 103 (1995).
[30] J.A. Harvey and A. Strominger, Nucl. Phys. **B449**, 535 (1995).
[31] M.J. Duff and J.X. Lu, Nucl. Phys. **B354**, 141 (1991).
[32] M.J. Duff, S. Ferrara, R.R. Khuri, and J. Rahmfeld, Phys. Lett. B **356**, 479 (1995).
[33] C.M. Hull and P.K. Townsend, Nucl. Phys. **B438**, 109 (1995).
[34] E. Witten, Nucl. Phys. **B443**, 85 (1995).

[35] D.A. Lowe and A. Strominger, Phys. Rev. Lett. **73**, 1468 (1994).
[36] S. Giddings, J. Polchinski, and A. Strominger, Phys. Rev. D **48**, 5784 (1993).
[37] J. Maharana and J.H. Schwarz, Nucl. Phys. **B390**, 3 (1993).
[38] M. Cvetič and D. Youm, Report No. IASSNS-HEP-95-107, hep-th/9512127 (unpublished).
[39] G.W. Gibbons, G.T. Horowitz, and P.K. Townsend, Class. Quantum Grav. **12**, 297 (1995).
[40] M.J. Duff, J.T. Liu, and J. Rahmfeld, Nucl. Phys. **B459**, 125 (1996).
[41] A. Sen, Int. J. Mod. Phys. A **9**, 3707 (1994).
[42] E. Witten, Phys. Lett. **86B**, 283 (1979).
[43] M. Cvetič and D. Youm, Phys. Rev. D **52**, 2144 (1995); Phys. Rev. Lett. **75**, 4165 (1995).