

Geometric interpretation and classification of global solutions in generalized dilaton gravity

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Two-dimensional gravity with torsion is proved to be equivalent to special types of generalized 2D dilaton gravity. For example, in one version, the dilaton field is shown to be expressible by the extra scalar curvature, constructed for an independent Lorentz connection corresponding to a nontrivial torsion. Elimination of that dilaton field yields an equivalent torsionless theory, nonpolynomial in curvature. These theories, although locally equivalent, exhibit quite different global properties of the general solution. We discuss the example of a (torsionless) dilaton theory equivalent to the $R^2 + T^2$ model. Each global solution of this model is shown to split into a set of global solutions of generalized dilaton gravity. In contrast to the theory with torsion, the equivalent dilaton one exhibits solutions which are asymptotically flat in special ranges of the parameters. In the simplest case of ordinary dilaton gravity we clarify the well-known problem of removing the Schwarzschild singularity by a field redefinition. [S0556-2821(96)06210-8]

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I. INTRODUCTION

The renewed interest in two-dimensional gravity models can be traced to the advent of string theories [1] with the especially promising aspect to study dynamical models of black holes. Generalized dilaton theories [2] or equivalent theories with higher powers in curvature with the dilaton eliminated have enriched our knowledge about even more complicated singularity structures. However, the common aspect of the latter generalizations, as well as of other, simpler models of 2D gravity [3], is the fact that in most of these cases a physical or geometrical interpretation of particular models does not seem to be evident. On the other hand, a corresponding interpretation obviously is present in theories quadratic in curvature R and in torsion T [4]: Such a theory resembles a (noncompact) gauge theory with second order field equations in the zweibein e^a_μ and the Lorentz connection $\omega^a_{\mu b}$, if the latter is considered as an independent variable. The globally complete solutions of those by now well-studied models with torsion [4,5] show a rich singularity structure [6].

A common feature of all such models, though, — in the matterless case — is the large number of unphysical degrees of freedom. Therefore, the question arises whether the (classical) equivalence between generalized dilaton theories and higher derivative curvature models [2] in the torsion-free case may not even extend to theories with nonvanishing torsion, thereby yielding a geometric interpretation of certain dilaton theories [7].

This seems to be especially attractive in view of the fact

that a singularity resembling most closely the black hole in $d=4$ has been the main motivation for the interest in dilaton theories. On the other hand, after coupling matter fields to the original dilaton black hole, so far the hopes have not materialized that, e.g., the problem of the depletion of a black hole by Hawking radiation can be understood in more than a semiclassical way, nor the even more fundamental question of information loss [7]. Thus, by extending the range of 2D models which are classically solvable and which contain the Schwarzschild singularity among their global solutions, possibly — after introducing couplings to matter — some new model may show advantages also in the quantum case.

Here, we only present the first step in such a program. We show how a theory quadratic in torsion and curvature may indeed be reformulated as an equivalent dilaton theory, first (Sec. II) by a suitable identification of the scalar curvature in $R^2 + T^2$ theory with the dilaton field of generalized dilaton theory. In this case the use of the field equation necessitates a careful check of the admissibility of the (nonlocal) transformation involved. In Sec. III we show this equivalence by a *local* method starting from a first order formalism for the $R^2 + T^2$ theory [5]. Here, the Lorentz connection is eliminated in favor of the torsion which turns into a nondynamical field variable. After the discussion of the general solution (Sec. IV) we present the classification in Sec. V. Essential steps of the corresponding mathematical analysis are given in Sec. VI. The relation of the global solutions for the dilaton theory equivalent to the $R^2 + T^2$ model [4] we present in Sec. VII. In the final Sec. VIII — beside a summary — also the role of field redefinitions and the creation of the singularity for the ordinary dilaton theory is discussed.

II. NONLOCAL EQUIVALENCE

The basis of our considerations is the Lagrangian of two-dimensional gravity with torsion, containing two coupling

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constants α, β and a cosmological constant Λ ($e = \det e_\mu^a$) [4]

$$\mathcal{L}_G = \frac{e}{8\beta} R^2 - \frac{e}{2\alpha} T_\alpha T^\alpha + e\Lambda. \quad (1)$$

The scalar curvature R is constructed from an independent zweibein e_α^a , $\alpha=0,1$, $a=0,1$, and Lorentz connection ω_α

$$\partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha = -\frac{1}{2} \epsilon_{\alpha\beta} R, \quad (2)$$

$$\epsilon^{\alpha\beta} = \epsilon^{ab} e_\alpha^a e_\beta^b = \frac{-\tilde{\epsilon}^{\alpha\beta}}{e}. \quad (3)$$

Here, $\epsilon_{ab} = -\epsilon_{ba}$, $\epsilon_{01} = 1$ is the totally antisymmetric tensor and $\tilde{\epsilon}^{\alpha\beta}$ a corresponding tensor density. Greek indices denote world tensors while Latin indices describe tensors transforming under local Lorentz rotation. The zweibein simply connects Greek indices with Latin ones and vice versa. The torsion

$$T_{\alpha\beta}^a = \partial_\alpha e_\beta^a - \partial_\beta e_\alpha^a + (\omega_\alpha e_\beta^b - \omega_\beta e_\alpha^b) \epsilon_b^a \quad (4)$$

is expressed in terms of zweibein and connection. Its trace reads

$$T_\alpha = e_b^\beta (\partial_\beta e_\alpha^b - \partial_\alpha e_\beta^b) + \omega_b \epsilon^b_\alpha. \quad (5)$$

Now, we may interpret the definition of the scalar curvature (2) as an equation for the Lorentz connection. In two dimensions the integrability conditions for these equations are trivially satisfied. So the scalar curvature and metric uniquely define a Lorentz connection up to a gradient of an arbitrary scalar field ψ

$$\omega_\alpha = \omega_{\perp\alpha} + \partial_\alpha \psi, \quad \partial_\alpha \omega_{\perp}^\alpha = 0, \quad (6)$$

where ω_{\perp}^α is the divergence-free solution of (2) and ψ can be readily identified with a local Lorentz rotation. This shows that — at least locally — an arbitrary Lorentz connection can always be parametrized by the scalar curvature and a Lorentz angle. Let us stress that this already happens at the kinematical level without using the equations of motion.

We now proceed to an alternative formulation of model (1) in terms of the metric

$$g_{\alpha\beta} = e_\alpha^a e_\beta^b \eta_{ab}, \quad \eta_{ab} = \text{diag}(+, -), \quad (7)$$

and scalar curvature R . To reformulate the model we start with the original equations of motion for zweibein and Lorentz connection from (1)

$$\frac{1}{2\beta} \partial_\alpha R + \frac{1}{\alpha} T_\alpha = 0, \quad (8)$$

$$\frac{1}{\alpha} \nabla_\alpha T_\beta + g_{\alpha\beta} \left(\frac{R^2}{8\beta} - \frac{T_\alpha T^\alpha}{2\alpha} - \Lambda \right) = 0, \quad (9)$$

where ∇_α denotes the covariant derivative corresponding to nontrivial torsion. The second term in (1) now is divided into two pieces, parametrized by a constant a

$$T_\alpha T^\alpha = (1-a) T_\alpha T^\alpha + a T_\alpha T^\alpha, \quad a = \text{const}. \quad (10)$$

In the first term of (10) the equation of motion (8) is used to express only one factor T_α as a gradient of the scalar curvature. We then integrate by parts and use the identity

$$2\tilde{\nabla}_\alpha T^\alpha = \tilde{R} - R,$$

where a tilde sign means that the corresponding quantities (in our case the metrical connection entering the covariant derivative and the scalar curvature) are computed in terms of the metric for *vanishing* torsion only. In the second term of (10) Eq. (8) is used twice to arrive at a kinetic term for the scalar curvature. In the resulting Lagrangian

$$\mathcal{L} = \frac{1}{8\beta} (1-a) \tilde{R} R - \frac{\alpha}{8\beta^2} a g^{\alpha\beta} \partial_\alpha R \partial_\beta R + \frac{1}{8\beta} a R^2 + \Lambda, \quad (11)$$

we may now consider metric and scalar curvature as new independent variables. Varying (11) with respect to the scalar curvature R and metric, one obtains new second order equations of motion

$$\frac{1}{2} (1-a) \tilde{R} + \frac{\alpha a}{\beta} \tilde{\nabla}^\alpha \partial_\alpha R + a R = 0, \quad (12)$$

$$(1-a) \tilde{\nabla}_\alpha \partial_\beta R - \frac{\alpha}{\beta} a \partial_\alpha R \partial_\beta R + \frac{1}{2} g_{\alpha\beta} \left(\frac{\alpha}{\beta} a \partial^\gamma R \partial_\gamma R + a R^2 + 8\beta \Lambda \right) = 0. \quad (13)$$

Although Lagrangian (11) was obtained from the Lagrangian of two-dimensional gravity with torsion, the resulting model is, in general, inequivalent to (1) for two reasons. First, the transformation of Lorentz connection to scalar curvature because of Eq. (2) is nonlocal and second, the equations of motions were used. Nevertheless, both models can be shown to be equivalent for $a = -1$ according to the following arguments.

It is proved easily that any zweibein and Lorentz connection satisfying the equations of motion of two-dimensional gravity with torsion [(8) and (9)] satisfy also Eqs. (12) and (13). Metric and scalar curvature are uniquely determined by zweibein and Lorentz connection. In fact, taking the trace of (9) and comparing it with Eqs. (12) and (13), one finds that Eq. (12) is satisfied if and only if $a = -1$. Then eliminating the torsion from (9) by means of Eq. (8), one indeed gets Eq. (13).

The inverse statement that (8) and (9) follow from (12) and (13) for $a = -1$, is more subtle. Equation (12) for $a = -1$ can be rewritten in the form

$$\tilde{\nabla}^\alpha \left(\frac{\partial_\alpha R}{2\beta} + \frac{1}{\alpha} T_\alpha \right) = 0.$$

Its general solution

$$\frac{\partial_\alpha R}{2\beta} + \frac{T_\alpha}{\alpha} = \epsilon_\alpha^\beta \partial_\beta \varphi$$

depends on an arbitrary function φ . Here, we note that both Eqs. (7) and (2) define zweibein and Lorentz connection up to an arbitrary local Lorentz rotation entering a general solution of Eqs. (7) and (2) as two independent arbitrary functions. So without loss of generality, we may use one of these functions, say ψ in (6) to set $\varphi=0$. Then the Lorentz rotation of zweibein and Lorentz connection must be performed simultaneously, and Eq. (8) indeed is the consequence of (12), and Eq. (9) immediately follows from (13).

In order to bring (11) into the familiar form of dilaton gravity, we parametrize the scalar curvature by a dilaton field ϕ and rescale the metric ($\sigma = \pm 1$)

$$R = \sigma 4\beta e^{-2\phi}, \quad g_{\alpha\beta} \rightarrow g_{\alpha\beta} e^{2\phi}. \quad (14)$$

This transformation is clearly canonical and thus leads to the equivalent generalized dilaton gravity described by the Lagrangian

$$\begin{aligned} \mathcal{L}_D^{(1)} = & e^{-2\phi} [\sigma \tilde{R} + (4\sigma + 8\alpha e^{-2\phi}) g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 2\beta \\ & + \Lambda e^{4\phi}] \sqrt{-g}. \end{aligned} \quad (15)$$

This coincides with the standard dilaton gravity Lagrangian [1] if $\sigma = +1$, $2\beta = -4\lambda^2$, $\Lambda = 0$, $\alpha = 0$. By leaving the sign σ in (15) open we are able to cover both regions $R > 0$ and $R < 0$ for a fixed sign of β .

III. LOCAL EQUIVALENCE

Instead of eliminating the (independent) Lorentz connection by the nonlocal transformation defined by Eq. (2), its components may also be solved *algebraically* in terms of the two independent components of the torsion represented by the Hodge dual of (4). In the definition

$$T^\pm = (\partial_\mu \pm \omega_\mu) e^\pm_\nu \tilde{\epsilon}^{\mu\nu} \quad (16)$$

we introduce light cone coordinates $T^\pm = 1/\sqrt{2}(T^0 \pm T^1)$ in the Lorentz indices. Note that here we just retain $\tilde{\epsilon}^{\mu\nu}$, as defined in (3). Equation (1) is equivalent to the first order action with Lagrangian

$$\mathcal{L}^{(1)} = X^+ T^- + X^- T^+ + X(\tilde{\epsilon}^{\mu\nu} \partial_\mu \omega_\nu) - e(\alpha X^+ X^- + V), \quad (17)$$

where

$$V = \frac{\beta}{2} X^2 - \Lambda. \quad (18)$$

The equations of motion for X and X^\pm are

$$e\beta X = \frac{R}{2}, \quad e\alpha X^\pm = T^\pm \quad (19)$$

and therefore X and X^\pm are proportional to curvature and torsion for $\beta \neq 0$ and $\alpha \neq 0$. It should be noticed that the subsequent steps hold for general $V = V(X)$, i.e., a theory quadratic in torsion but with arbitrary powers in curvature. Now, instead of ω_μ the T^\pm in (16) are introduced as new variables:

$$\tilde{\epsilon}^{\mu\nu} \partial_\mu \omega_\nu = \tilde{\epsilon}^{\mu\nu} \partial_\mu \tilde{\omega}_\nu + \tilde{\epsilon}^{\mu\nu} \partial_\mu \left(\frac{e^-_\nu T^+ + e^+_\nu T^-}{e} \right). \quad (20)$$

The first term on the right-hand side (RHS) of (20) is proportional to a torsionless curvature \tilde{R} ,

$$\tilde{\epsilon}^{\mu\nu} \partial_\mu \tilde{\omega}_\nu = -\frac{\tilde{R}e}{2}, \quad (21)$$

where $\tilde{\omega}$ is considered as a function of the zweibein and its derivatives corresponding to a torsionless case. Inserting (20) into (17), after shifting the derivatives in the second term of (20) onto X exhibits the nondynamical nature of T^\pm which may be ‘‘integrated out’’ by solving their (algebraic) equations of motion. At this point the identification $X = R/(2\beta e)$ immediately yields (11). With a definition of the dilaton field similar to (14)

$$\frac{X}{2} = e^{-2\phi}, \quad X > 0 \quad (22)$$

and after reexpressing the factors e^\pm_ν/e from the square brackets of (20) in terms of the inverse zweibeins combining them into $g^{\alpha\beta} = e^{+\alpha} e^{-\beta} + e^{+\beta} e^{-\alpha}$, the Lagrangian $\mathcal{L}^{(1)}$ is found to be equivalent to

$$\begin{aligned} \mathcal{L}^{(2)} = & \sqrt{-g} [-e^{-2\phi} \tilde{R} + 8\alpha \cdot e^{-4\phi} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \\ & - V(2e^{-2\phi})]. \end{aligned} \quad (23)$$

The case $X < 0$ will be discussed below [cf. (41)]. In addition, using the identity

$$\begin{aligned} g_{\mu\nu} = & e^{2\varphi} \hat{g}_{\mu\nu}, \\ \sqrt{-g} R = & \sqrt{-\hat{g}} \hat{R} - 2\partial_\alpha [\sqrt{-\hat{g}} \hat{g}^{\alpha\beta} \partial_\beta \varphi], \end{aligned} \quad (24)$$

which also represents a local transformation, allows one to write down the most general dilaton theory equivalent to (1):

$$\begin{aligned} \mathcal{L}^{(3)} = & \sqrt{-\hat{g}} [-e^{-2\phi} \hat{R} + 4\hat{g}^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \\ & + 2\alpha e^{-4\phi} \partial_\beta \phi] - e^{2\varphi} V(2e^{-2\phi}). \end{aligned} \quad (25)$$

The choice $\varphi = \phi$ immediately yields the case of $\sigma = -1$ of (15), while for $\varphi = -\phi$

$$\begin{aligned} \mathcal{L}^{(4)} = & -\sqrt{-\hat{g}} e^{-2\phi} [\hat{R} + 4(1 - 2\alpha e^{-2\phi})(\nabla \phi)^2 \\ & + V(2e^{-2\phi})], \end{aligned} \quad (26)$$

the deviation from ordinary dilaton theory ($\alpha = 0$, $V = 4\lambda^2$) is most obvious. Of course, the dilaton field may be eliminated altogether as well, if in (25) (for constant α)

$$\varphi = \varphi(\phi) = \alpha e^{-2\phi} = \frac{\alpha X}{2} \quad (27)$$

is chosen. In that case it seems more useful to retain the variable X instead of ϕ :

$$\mathcal{L}^{(5)} = -\sqrt{-\hat{g}} \left[\frac{X \hat{R}}{2} + e^{\alpha X} V(X) \right]. \quad (28)$$

Comparing (28) to a torsionless theory (17) with $\alpha=0$ but modified V , the difference now just resides in the additional exponential $e^{\alpha X}$ [13]. It should be mentioned that the ‘‘equivalence’’ shown in this and the preceding section is purely classical and even there only true locally. The global (classical) inequivalence will be addressed in the following section. We will show there in examples how one global solution of 2D gravity with torsion decouples into two or more global solutions of the ‘‘equivalent’’ generalized dilaton model.

Of course, at the quantum level questions like the influence of the Jacobian appearing as a consequence of the redefinition of fields will play a role. Such considerations remain outside the scope of our present work.

IV. GENERAL SOLUTION OF GENERALIZED DILATON GRAVITY

The study of global properties for 2D theories is based upon the extension of the solution which is known at first only in local patches, continued maximally to global ones. The analysis uses null directions which become the coordinates of Penrose diagrams which are sewed together appropriately. The continuation across horizons and the determination of singularities can be based upon extremals or geodesics. The physical interpretation of an extremal is the interaction of the space–time manifold with a pointlike test particle, ‘‘feeling’’ the metric $g_{\alpha\beta}$ through the Christoffel symbol [9]. After torsion has been eliminated, there is no ambiguity for our analysis which only has extremals at its disposal. That interaction with extremals, however, crucially depends on the choice of the ‘‘physical’’ metric to be used: the one computed from the e^a_μ of (1), or any $\hat{g}_{\alpha\beta}$ which is a result of different field transformations involving the dilaton field? Clearly, the torsionless dilaton theory (23) has the same metric as (1), e.g., the global analysis of [6] applies directly and the different types of solutions are exhausted by those studied in [6].

However, from the point of view of a ‘‘true’’ dilaton theory, one could argue that with a redefined metric as in (26), $\hat{g}_{\alpha\beta} = e^{2\phi} g_{\alpha\beta} = 2g_{\alpha\beta}/X$ has some physical justification as well. In fact, for Witten’s black hole $g_{\alpha\beta}$ is flat and the interesting (black hole) singularity structure just results from the factor $2/X$. Now, in the original $R^2 + T^2$ theory [6], there are solutions (G3) resembling, e.g., the black hole but not completely: Their singularity resides at lightlike lines and they are not asymptotically flat in the Schwarzschild sense. Thus, the factor $2/X$ may well yield improvements on that situation.

Here we shall analyze a generalized dilaton gravity

$$\mathcal{L} = \sqrt{-\hat{g}} e^{-2\phi} [\hat{R} + 4(1 - 2\alpha e^{-2\phi})(\nabla\phi)^2 + 2\beta e^{-4\phi} + 4\lambda^2] \tag{29}$$

which is obtained from (17) by taking

$$X = 2e^{-2\phi}, \quad g_{\mu\nu} = \hat{g}_{\mu\nu} e^{-2\phi}, \quad \Lambda = -4\lambda^2, \tag{30}$$

and omitting an overall minus sign. We need to consider only the cases for $\beta =$ positive, negative, or 0. The absolute value of a nonvanishing β may always be absorbed by rescaling X and ω to $X \rightarrow \sqrt{|\beta|}X$ and $\omega \rightarrow \omega/\sqrt{|\beta|}$. Let us start

with a positive value for β , e.g., $+2$. All global solutions with nonconstant curvature are most easily obtained by the known general solution [5] of the equations of motion of (17) for the zweibein e^a ($X^+ \neq 0$) in an arbitrary gauge

$$e^+ = X^+ e^{\alpha X} df, \tag{31}$$

$$e^- = \frac{dX}{X^+} + X^- e^{\alpha X} df, \tag{32}$$

where f , X , and X^+ are arbitrary functions except for the requirement that df and dX define a basis for one forms. The line element from (31) and (32) reads

$$(ds)^2 = 2e^+ e^- = 2e^{\alpha X} df \otimes (dX + X^+ X^- e^{\alpha X} df). \tag{33}$$

As shown in [4], the Lagrangian in all such models gives rise to an absolutely conserved quantity

$$C = X^+ X^- e^{\alpha X} + w(X), \tag{34}$$

$$w(X) = \int_{X_0}^X V(y) e^{\alpha y} dy, \tag{35}$$

where the lower limit $X_0 = \text{const}$ clearly has to be determined appropriately so that (inside a certain patch) the integral exists for a certain range of X . C can be used to eliminate $X^+ X^-$ in (33). Using (30) and defining coordinates $v = -4f$, $u = \phi$ yields the line element of the generalized dilaton Lagrangian (29)

$$(d\hat{s})^2 = e^{2u} (ds)^2 = g(u) [2dv du + l(u) dv^2] \tag{36}$$

with

$$l(u) = \frac{e^{2u}}{8} \left[C - g(u) \left(\frac{4e^{-4u}}{\alpha} - \frac{4e^{-2u}}{\alpha^2} + C_0 \right) \right], \tag{37}$$

$$g(u) = e^{2\alpha e^{-2u}}, \tag{38}$$

$$C_0 = \frac{2}{\alpha^3} + \frac{4\lambda^2}{\alpha}, \tag{39}$$

which automatically implies the convention for the constant of integration in (35) to be used in the following.

Note that transformation (22) is defined only for positive X while the original model contains arbitrary values of X . To cover the negative case one would have to replace (22) by

$$\frac{X}{2} = -e^{-2\phi}, \quad X < 0. \tag{40}$$

Since the metric (36) is invariant under

$$e^{-2u} \rightarrow -e^{-2u}, \quad \alpha \rightarrow -\alpha, \quad C \rightarrow -C, \tag{41}$$

one may consider the line element (36) for $\alpha > 0$ and $\alpha < 0$ to cover all patches of the original $R^2 + T^2$ theory where X is positive and negative.

In the metric of the form

$$\hat{g}_{\mu\nu} = g(u) \begin{pmatrix} 0 & 1 \\ 1 & l(u) \end{pmatrix}, \tag{42}$$

the Killing direction is $\partial/\partial v$ and the norm of the Killing vector

$$k^\mu = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

becomes $g(u)l(u)$. By a redefinition of the variable u it would be easy to reduce (42) to the standard light cone gauge with $g=1$, used, e.g., in the second Ref. [6]. However, because of $g(u)>0$ (except at $u\rightarrow\infty$), the horizons are determined by the zeros of $l(u)$ only. Therefore, we have found it technically easier to retain (42).

The conformal gauge $(d\hat{s})^2 = F(u)d\bar{u}'d\bar{v}'$ in (42) is obtained by "straightening" the null extremals

$$v = \text{const}, \quad (43)$$

$$\frac{dv}{du} = -\frac{2}{l} \text{ for all } u \text{ with } l(u) \neq 0, \quad (44)$$

$$u = u_0 = \text{const} \text{ for } l(u_0) = 0, \quad (45)$$

by means of a diffeomorphism

$$\bar{u}' = v + f(u), \quad \bar{v}' = v, \quad (46)$$

$$f(u) \equiv \int \frac{u 2dy}{l(y)}. \quad (47)$$

A subsequent one $\bar{v}' \rightarrow \tan \bar{v}'$ and another appropriately chosen one for \bar{u}' produce the Penrose diagram. It is valid for a certain patch where (44) is well defined. Clearly, the shape of those diagrams depends crucially on the (number and kind of) zeros and on the asymptotic behavior of $l(u)$.

V. CLASSIFICATION OF GLOBAL SOLUTIONS

The analysis of all possible cases as described by the ranges of parameters α, β, C , and λ^2 is straightforward but tedious. We shall first give the classification of all global solutions. Comments on the construction of Penrose diagrams will be presented in the next section. Apart from C_0 , defined in (38), also

$$C_1 = \frac{2}{\alpha^2} e^{2\alpha\sqrt{-\lambda^2}} \left(\frac{1}{\alpha} - 2\sqrt{-\lambda^2} \right) \quad (48)$$

plays a role for $\alpha \neq 0$ and $\lambda^2 < 0$, discriminating the possible cases with two zeros, with one double zero and without zero in l , i.e., the presence of two nondegenerate or one degenerate Killing horizon. The qualitatively distinct cases for $\alpha > 0$ and $\alpha < 0$ are listed in (49) and (50):

$\alpha > 0, \beta > 0$:

$$\begin{aligned} D1^+ &: C > C_0, \\ D2^+ &: C = C_0, \quad \lambda^2 < 0, \\ D3^+ &: C = C_0, \quad \lambda^2 \geq 0, \\ D4^+ &: C < C_0, \quad C > C_1, \quad \lambda^2 < 0, \\ D5^+ &: C < C_0, \quad C = C_1, \quad \lambda^2 < 0, \\ D6^+ &: C < C_0, \quad C < C_1, \end{aligned} \quad (49)$$

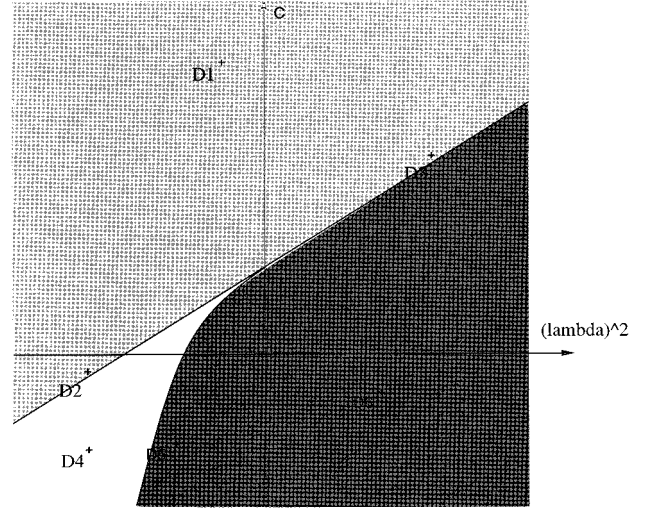


FIG. 1. Range of solutions for $\alpha > 0$.

$\alpha < 0, \beta > 0$:

$$\begin{aligned} D1^- &: C > C_0, \quad C \geq 0, \\ D2^- &: C > C_0, \quad C < 0, \\ D3^- &: C = C_0, \quad C \geq 0, \\ D3_r^- &: C = C_0, \quad C < 0, \quad \lambda^2 \geq 0, \\ D4^- &: C = C_0, \quad C < 0, \quad \lambda^2 < 0, \\ D1_r^- &: C < C_0, \quad C < C_1, \\ D5^- &: C < C_0, \quad C = C_1, \\ D6^- &: C < C_0, \quad C > C_1, \quad C < 0, \\ D2_r^- &: C < C_0, \quad C > C_1, \quad C \geq 0, \end{aligned} \quad (50)$$

with the index r indicating a rotation by 90° . The classification Dn^\pm denotes the Penrose diagrams for generalized dilaton gravity. Figures 1 and 2 show their range of parameters and in Figs. 3 and 4 the corresponding Penrose diagrams are depicted. These cases for nonzero α are locally equivalent to $R^2 + T^2$ gravity defined by the actions (1) and (17).

The limit $\alpha = 0$ — corresponding to ordinary dilaton gravity with an additional potential $\beta e^{4\phi}$ — is defined only for the action in first order form (17). It is equivalent to (higher derivative) R^2 gravity without torsion, whose global solutions are analyzed in the second Ref. of [6]. To obtain the classification of the global solutions for the locally equivalent dilaton theory we use (33) and (34) with $\alpha = 0$ and again $\beta = 2$ to get the line element

$$(d\hat{s})^2 = 2dudv + e^{2u} \left(\frac{C}{8} - \lambda^2 e^{-2u} - \frac{e^{-6u}}{3} \right) dv^2. \quad (51)$$

In terms of

$$C_1^0 = -\frac{16}{3} |\lambda^2|^{3/2}, \quad (52)$$

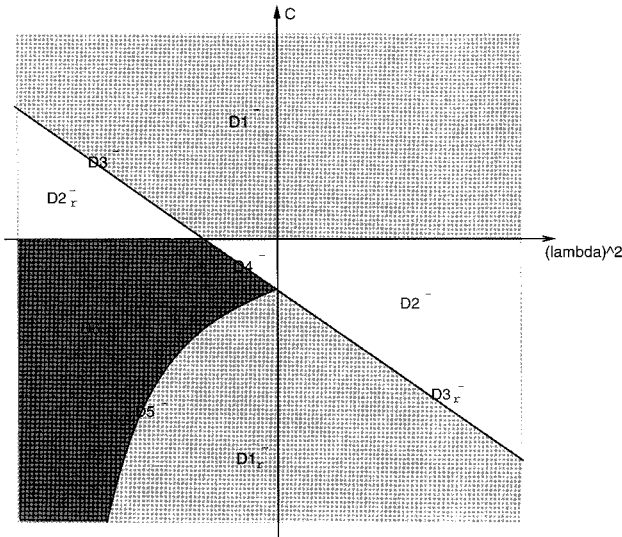


FIG. 2. Range of solutions for $\alpha < 0$.

qualitatively this yields exactly the same cases as for $\alpha > 0$ for the following conditions on the parameters:

$\alpha = 0, \beta > 0:$

- $D1^+ : C > 0,$
 - $D2^+ : C = 0, \lambda^2 < 0,$
 - $D3^+ : C = 0, \lambda^2 \geq 0,$
 - $D4^+ : C < 0, C > C_1^0,$
 - $D5^+ : C < 0, C = C_1^0,$
 - $D6^+ : C < 0, C < C_1^0.$
- (53)

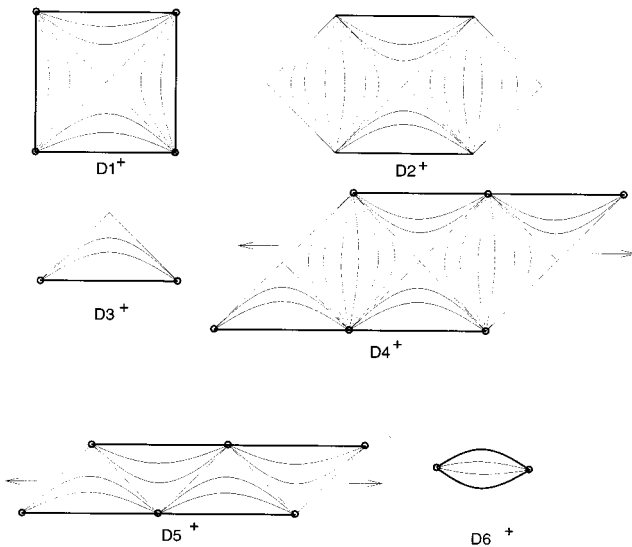


FIG. 3. Penrose diagrams for $\alpha > 0$. Thick lines indicate an incomplete singular boundary ($R \rightarrow \infty$). Thin lines show (complete) boundaries with finite or vanishing curvature; inside the diagrams they correspond to lines of constant curvature. Dots represent complete corners and Killing horizons are drawn as dashed lines.

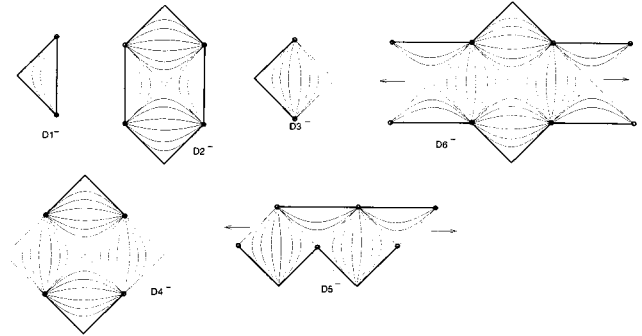


FIG. 4. Penrose diagrams for $\alpha < 0$. Conventions as above.

A sign change of β to $-\beta$ together with sign changes of C and λ^2 , followed by $dv \rightarrow -dv$, produces the same global solutions except that timelike and spacelike directions are reversed ($ds^2 \rightarrow -ds^2$). Since all possible signs for C and λ^2 were considered above, we obtain the solutions for negative β 's simply by rotating the diagrams by 90° and keeping the sign changes of C and λ^2 in mind.

Next, we consider the teleparallel limit of (17) corresponding to $\beta = 0$. In this limit the curvature is equal to zero and one has T^2 gravity [12]. We still obtain the line element of (36) with a simplified

$$l = \frac{e^{2u}}{8} \left(C - e^{2\alpha e^{-2u}} \frac{4\lambda^2}{\alpha} \right). \tag{54}$$

The Penrose diagrams for the corresponding global solutions have already been obtained in the above cases. Here, they are related to the following ranges of parameters ($D = C - 4\lambda^2/\alpha$):

$\alpha \neq 0, \beta = 0:$

- $D6_r^+ : D > 0, \lambda^2 < 0,$
 - $D1^- : D > 0, \lambda^2 = 0,$
 - $D1^+ : D > 0, \lambda^2 > 0,$
 - $D1_r^+ : D < 0, \lambda^2 < 0,$
 - $D1_r^- : D < 0, \lambda^2 = 0,$
 - $D6^+ : D < 0, \lambda^2 > 0,$
 - $D3^+ : D = 0, \lambda^2 > 0,$
 - $D3_r^+ : D = 0, \lambda^2 < 0,$
 - $D1^- : D > 0, C \geq 0,$
 - $D2^- : D > 0, C < 0,$
 - $D2_r^- : D < 0, C > 0,$
 - $D1_r^- : D < 0, C \leq 0,$
 - $D3^- : D = 0, \lambda^2 < 0,$
 - $D3_r^- : D = 0, \lambda^2 > 0.$
- (55)

The case $\lambda = 0, C = 0$ yields just the conformally flat case.

The last case $\alpha = \beta = 0$ of (30) describes ordinary dilaton gravity. Its general solution is

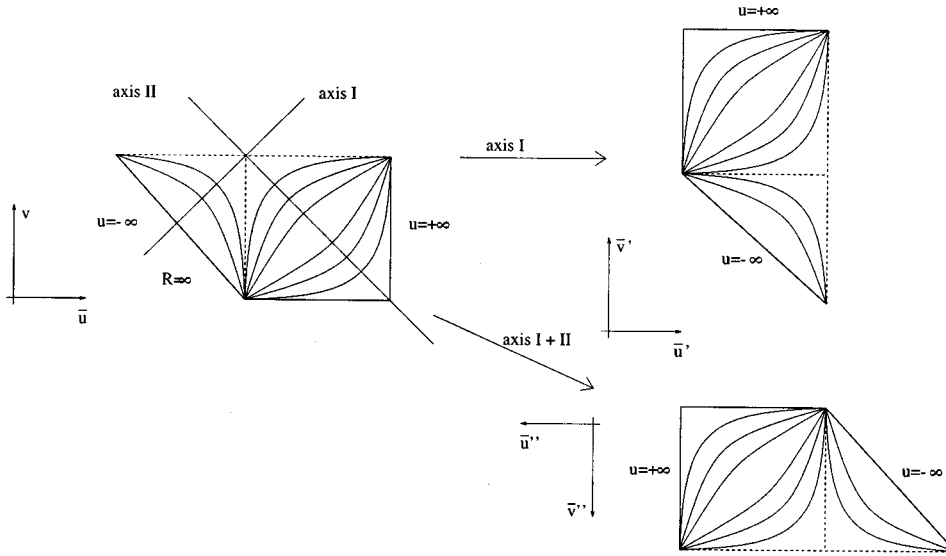


FIG. 5. Different types of solutions for $D2^+$.

$$(d\hat{s})^2 = 2dudv + e^{2u}(Ce^{2u} - \lambda^2)dv^2 \tag{56}$$

and contains three types of global solutions:
 $\alpha = \beta = 0$:

$$\begin{aligned} D2^+ : & C > 0, \quad \lambda^2 > 0, \\ D2_r^+ : & C < 0, \quad \lambda^2 < 0, \\ D3^+ : & C > 0, \quad \lambda^2 \leq 0, \\ D3_r^+ : & C < 0, \quad \lambda^2 \geq 0, \end{aligned} \tag{57}$$

and flat space-time for $C = 0$.

VI. CONSTRUCTION OF PENROSE DIAGRAMS

For each set of the parameters as summarized in (49), (50), (53), (55), and (57), the global solution is obtained by gluing (wherever necessary) solutions of the line element (36), which as a rule defines only a local solution. Another local solution in conformal coordinates is obtained by interchanging the role of the null directions. The transformation

$$\begin{aligned} \tilde{u}'' &= u, \\ \tilde{v}'' &= -f(u) - w, \end{aligned} \tag{58}$$

with $f(u)$ from (44) may be easily verified to do this job. The situation may be visualized best in a specific example, say, $D2^+$ [10] as shown in Fig. 5. Equations (58) essentially correspond to a reflection of the diagram with respect to an axis orthogonal to the lines $u = \text{const}$, i.e., transversal to the Killing directions $u = \text{const}$. In Fig. 5 this leads from (a) to (b), resp. (c). Further solutions are obtained by simple reflections in the (\tilde{u}, \tilde{v}) coordinates. Now, those patches may be glued together along certain parts of their boundaries by identifying either the triangle or the square. For that purpose the completeness of all extremals at these edges must be analyzed by checking whether there is at least one extremal reaching a certain boundary (or corner of a boundary) at a finite value of the affine parameter. By definition, a point not

to be reached by any extremal is complete. Using the conservation law for a metric of type (42) from its Killing direction

$$g \frac{du}{d\tau} + gl \frac{dv}{d\tau} = \sqrt{A} = \text{const}, \tag{59}$$

and identifying the affine parameter $d\tau$ with the $d\hat{s}$ in (36), these extremals are found to obey

$$\frac{dv}{du} = -\frac{1}{l} [1 \mp (1 - lg/A)^{-1/2}] \tag{60}$$

by simply solving a quadratic equation. In addition, from (59) and (60) the affine parameter is determined by

$$\hat{s}(u) = \frac{1}{\sqrt{|A|}} \int^u dy g \left(1 - \frac{lg}{A}\right)^{-1/2}. \tag{61}$$

For $A > 0$, resp. $A < 0$ the parameter s is a timelike, resp. spacelike quantity. Extremals along the null directions [(43) and (44)] are contained as the special case $A \rightarrow \infty$ in (60), with (61) replaced by similar relation without the square-root terms. In addition to (59), (60), and (61), the equation for extremals is satisfied by the horizons (45) and by degenerate extremals [see (63) below] for which $lg = A$ and therefore (61) does not hold. Clearly, the additional condition $l'(u_0) = 0$ identifies double zeros (degenerate Killing horizons) as, e.g., in $D5^-$. Incompleteness of the lines $u_0 = \text{const}$ for single zeros at one end is established easily. A line at a double zero as in $D5^-$ instead has incomplete end points. From considering the extremals for finite $A \neq 0$, in all cases the edges with $u = \pm\infty$ are found to be incomplete, *except* for $C = C_0$ (cf., e.g., our example $D2^+$ or $C = 0$ in $D2^0$) where for $u \rightarrow +\infty$, the boundary is complete. We re-

mark that the completeness of null and non-null extremals differs on some boundaries. For example, at $u \rightarrow -\infty$, $\alpha > 0$ null extremals are complete while non-null extremals are incomplete. Contrary to this situation, the completeness of all types of extremals is the same in $R^2 + T^2$ theory. It can be shown that this difference is because of the conformal rescaling of the metric (24).

Except for the cases $D2^+$, $D3^\pm$, $D3_r^-$, $D4^-$ (where $R \rightarrow 0$ for $u \rightarrow +\infty$), the scalar curvature diverges at $u \rightarrow \pm\infty$ which can be seen directly evaluating

$$R = \frac{1}{g^2} \left(\frac{g_{,u}^2}{g} l - g_{,uu} l - g_{,u} l_{,u} - g l_{,uu} \right) \quad (62)$$

and using (37). Thus, all singular boundaries are incomplete while the boundaries with zero curvature are always complete. This means that generalized dilaton gravity includes asymptotically flat solutions, something not encountered in the original $R^2 + T^2$ theory.

Special consideration require the corners of a boundary formed by lines with both $u = +\infty$ or both $u = -\infty$. For example, in Fig. 5 the lower right one can only be reached by extremals (59) with (61) when $A = 0$. Actually, for $D2^+$ that point is complete — just as the adjoining lines $u = +\infty$. In some diagrams such corners are formed by lines $u = +\infty$ and $u = -\infty$, both singular in curvature. These corners are essential singularities and can only be reached by degenerate extremals, which are parallel to the v axis and go through those points where

$$\left. \frac{\partial(lg)}{\partial u} \right|_{u=u_0} = 0 \quad (63)$$

is satisfied. The existence and number of such extremals depends on the values of the parameters. They are always complete and so are the corresponding corners. If (63) has no solution then the corner is not reached by any extremal and thus is complete by definition. In Figs. 3 and 4 these points are indicated by full dots. With these tools, patches as in Fig. 5 may now be glued together. For $D2^+$ this leads to the well-known shape of the ‘‘classical’’ black hole in the corresponding global solution drawn in Fig. 3. A full line in this diagram — as in the others — represents an (incomplete) singularity of the curvature at $u = \pm\infty$, a thin line at the external boundary denotes a complete asymptotically flat space-time which may occur at $u = +\infty$ only. Internal lines indicate $u = \text{const}$, with a broken line for a Killing horizon. Arrows indicate directions into which one should imagine periodic continuation.

VII. COMPARISON OF THE MODELS

It seems instructive to compare our present global structure to the one studied for other theories. The basic observation is that each global solution of the original $R^2 + T^2$ theory naturally splits into a set of global solutions in generalized dilaton gravity. As a generic example we show in Figs. 6(a), 6(b), and 6(c) the three different ways of splitting of the diagram $G3$ (in the notation of [6]) where the separation is determined solely by the value of the constants. The Penrose diagram $G3$ covers all values of X . Since the dilaton field is

introduced separately for positive and negative X , the Penrose diagram $G3$ is split by the lines $X = 0$, where the conformal transformation in (30) becomes singular. The resulting diagrams directly turn into the global solutions of generalized dilaton gravity. It should be noted that in the $R^2 + T^2$ theory all non-null boundaries can be straightened by a suitable choice of coordinates, while in generalized dilaton gravity some boundaries cannot be straightened without affecting other boundaries. For example, in the square diagram $D1^+$ or in the ‘‘eye’’-diagram $D6^+$ a redefinition of the coordinates cannot straighten all boundaries simultaneously, whereas this is possible in the diagram $D2^-$. This phenomenon has been observed before (cf. the second Ref. of [6]).

It is also important that the conformal transformation of the metric $\hat{g}_{\alpha\beta} = 2g_{\alpha\beta}/X$ changes the character of the boundaries. The boundary $X = 0 (u \rightarrow +\infty)$ becomes singular $\tilde{R}(\hat{g}) \rightarrow \infty$, and remains incomplete for $C \neq C_0$, while for $C = C_0$ it becomes complete and asymptotically flat $\tilde{R}(\hat{g}) \rightarrow 0$. The boundaries $X \rightarrow \pm\infty$ remain singular but always become incomplete.

In Fig. 6(d) the ‘‘black hole’’ solution $D2^+$ emerges. It illustrates how the diagram $G11$ that had to be continued in the plane splits into the diagrams $D2^+$ and $D4^-$ which cannot be continued. The decomposition in the case of ordinary dilaton gravity is shown in Fig. 7. Model (17) for $\alpha = \beta = 0$ has a unique global solution which is flat Minkowskian space-time represented by a rhombic Penrose diagram. Performing the conformal transformation yields a singular metric and curvature at the lines of vanishing X , thereby producing three global solutions of the dilaton model. $D2^+$ represents the famous black hole solution [1] whose singularity is reached by timelike extremals for a finite value of the affine parameter. The completeness of the other extremals is the same as in the above cases. The original $R^2 + T^2$ theory contains one solution resembling the ‘‘real’’ Schwarzschild black hole only in a very approximate sense. Its solution $G3$ exhibits an (incomplete) singularity, but into null directions, the ‘‘asymptotically flat’’ direction is replaced by a singularity of the curvature, albeit at an infinite distance (complete case). In the present dilaton theory, precisely the example $D2^+$, whose derivation from $G11$ was discussed more explicitly above, is Schwarzschild-like. Other solutions with similar properties, but more complicated singularity structure are $D3^-$ (naked singularities) and $D4^-$. On the other hand, the ‘‘eye’’ diagram $D6^+$ appears here, as well as the square diagrams $D1^+$ of R^2 gravity [6]. $D4^-$ represents an interesting variety of a manifold where the ordinary black hole is replaced by a ‘‘light’’-like singularity.

VIII. SUMMARY AND OUTLOOK

By showing explicitly the local equivalence of certain (torsionless) dilaton theories with a 2D theory quadratic in curvature and torsion, a certain class of such generalizations of the original Witten model has now been found to acquire a better geometric foundation. At the same time, however, we observe that local equivalence does not guarantee the equivalence of global solutions. Indeed, we have shown in two generic examples how one global solution of 2D gravity with torsion splits into a set of global solutions of general-

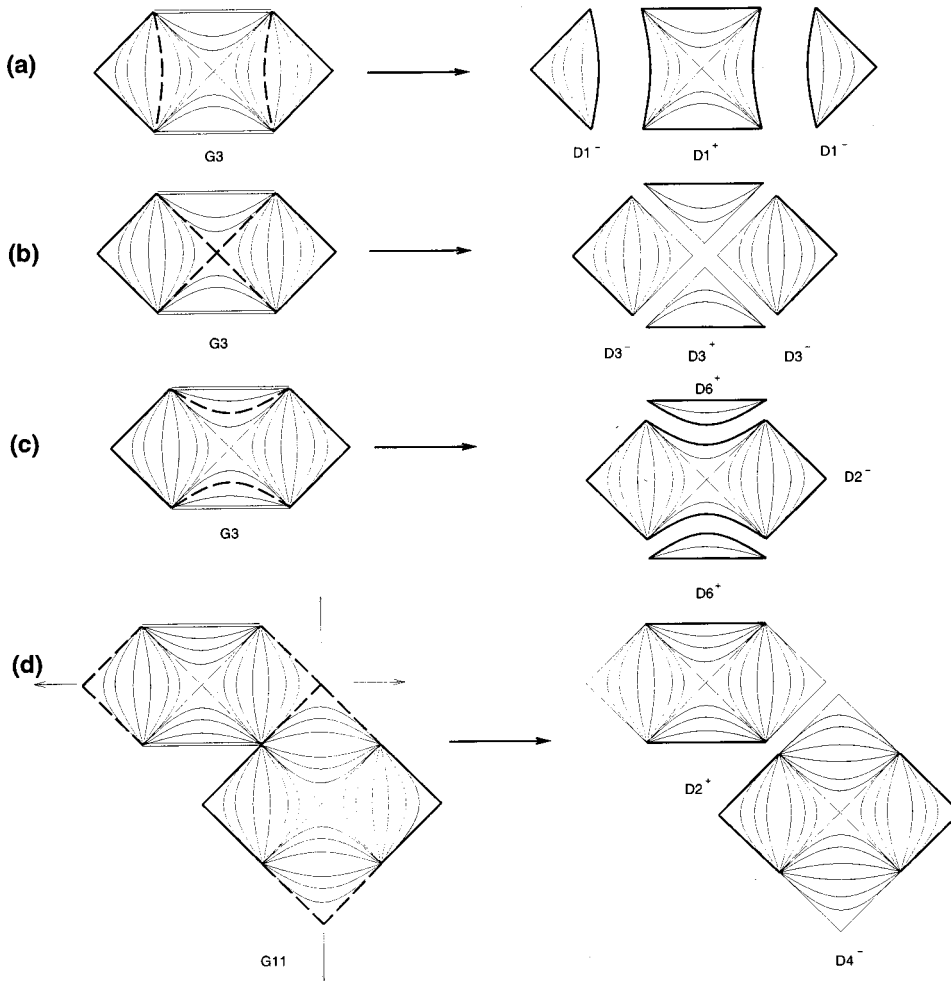


FIG. 6. Examples of the splitting of the $R^2 + T^2$ diagrams into the diagrams of generalized dilaton theory. The double lines indicate complete singular boundaries and thick dashed lines correspond to $X=0$. (a), (b), and (c) show different splittings of the same diagram G3 for $\lambda^2 > 0$ which depend on the parameters as follows: (a) $\alpha C_0 < \alpha C$, (b) $C = C_0$, (c) $0 < \alpha C < \alpha C_0$. (d) occurs for $\lambda^2 < 0$, $\alpha C = \alpha C_0 > 0$ and demonstrates the appearance of the black hole solution $D2^+$ in the generalized dilaton theory.

ized dilaton gravity. This separation occurs along lines $X=0$ where the conformal transformation of the metric $\hat{g}_{\alpha\beta} = 2g_{\alpha\beta}/X$ becomes singular [14].

We find that in the equivalent dilaton theories one of the main advantages of the original “minimal” dilaton theory [1], the presence of an asymptotically flat black hole solu-

tion, resembling the 4D case, is retained. Since the complete classical solution is known in all cases, the range of models thus has been extended considerably for which in a next step quantum effects can be studied, after interactions with matter have been added. Of course, we encounter here once more the problem, familiar from Brans-Dicke-Jordan-type theories

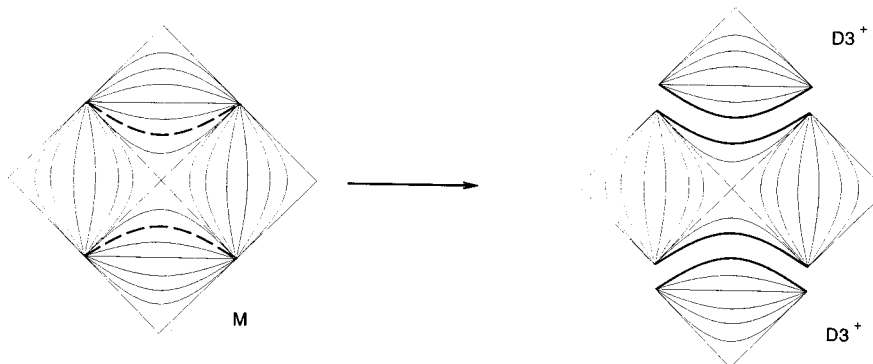


FIG. 7. Splitting of Minkowskian space-time M into the black hole of ordinary dilaton gravity. The thin lines in the left diagram are the lines of $X = \text{const}$ and especially the thick dashed line denotes $X=0$ along which the splitting occurs.

[11]: Our analysis implicitly assumes that our “test particle” obeying the equations for the extremals in Sec. VI for the “equivalent theory” is coupled to a metric of the Jordan version of the theory, i.e., to the redefined metric \hat{g} and not to the metric derived from the original R^2+T^2 theory. It must be admitted that, therefore, the argument of geometric interpretability is somewhat weakened. Nevertheless, we are convinced that the class of such models introduced here may serve as a field for promising further studies.

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