

Testing gravity to second post-Newtonian order: A field-theory approach

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A field-theory-based framework for discussing and interpreting experimental tests of relativistic gravity, notably at the second post-Newtonian (2PN) level, is introduced. Contrary to previous frameworks which attempted at parametrizing any conceivable phenomenological deviation from general relativity, we focus on the most general class of gravity models of the type suggested by unified theories: namely, models in which gravity is mediated by a tensor field together with one or several scalar fields. The 2PN approximation of these “tensor-multiscalar” theories is obtained thanks to a diagrammatic expansion which allows us to compute the Lagrangian describing the motion of N bodies. In contrast with previous studies which had to introduce many phenomenological parameters, we find that, within this tensor-multiscalar framework, the 2PN deviations from general relativity can be fully described by introducing only two new 2PN parameters ε and ζ beyond the usual (Eddington) 1PN parameters $\bar{\beta} \equiv \beta - 1$ and $\bar{\gamma} \equiv \gamma - 1$. It follows from the basic tenets of field theory, notably the absence of negative-energy excitations, that $\bar{\beta}$, ε , and ζ (as well as all the further parameters entering higher post-Newtonian orders) must tend to zero with $\bar{\gamma}$. It is also found that ε and ζ do not enter the 2PN equations of motion of light. Therefore, within our field-theory framework, second-order light-deflection or time-delay experiments cannot probe any 2PN deviation from general relativity. On the other hand, these experiments can give a clean access to $\bar{\gamma}$, which is of greatest significance as it measures the basic coupling strength of matter to the scalar fields. Because of the importance of self-gravity effects in neutron stars, binary-pulsar experiments are found to constitute a unique testing ground for the 2PN structure of relativistic gravity. A simplified analysis of current data on four binary-pulsar systems already leads to significant constraints on the two 2PN parameters $|\varepsilon| < 7 \times 10^{-2}$, $|\zeta| < 6 \times 10^{-3}$.

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I. INTRODUCTION

The past three decades have been the golden era of experimental gravity: from Pound and Rebka to Hulse and Taylor, many complementary aspects of general relativity have been successfully tested. In particular, solar-system experiments allowed one to map out fairly completely weak-field gravity at the first post-Newtonian (1PN) approximation, i.e., to put stringent numerical constraints on a large class of possible deviations from general relativity at order $1/c^2$. Let us recall the useful role played in this respect by the first-order parametrized post-Newtonian (PPN) formalism [1–8] which introduced, in its extended versions, about 10 independent phenomenological parameters to describe possible non-Einsteinian 1PN effects. Improved experiments are now planned to reach the second post-Newtonian (2PN) level, order $1/c^4$, such as microsecond level light deflection experiments. Let us also mention that a 2PN treatment of the periastron advance is already significant for the binary pulsar PSR 1913+16 [9]. It is therefore timely to undertake a systematic theoretical study of gravitational theories at this approximation.

The 2PN limit of general relativity has already been studied in depth [10–18,9,19], but we also need to know what can be the possible deviations from these results in alternative theories of gravity. An ambitious program developed by

Nordtvedt and Benacquista in [20–22] tries to extend directly the PPN formalism at order $1/c^4$; i.e., it aims at introducing a large number of parameters describing any possible relativistic theory at this order. Although it has only been partially implemented at the present time, this approach allowed one to derive some relations between these 2PN parameters by imposing the concept of “extended Lorentz invariance” (i.e., by requiring that the gravitational physics of subsystems, influenced by external masses, exhibit Lorentz invariance) [23,22]. In spite of its partial achievements, the ability of such a general phenomenological approach to delineate the physically most important structures at the 2PN level is unclear. For instance, it was claimed in [21] that 10 “parameters” are required to map Lorentz-invariant theories of gravity at the 2PN level; however, a careful reading of this article shows that several of these “parameters” are in fact *functions* of the distances between massive bodies, and could depend *a priori* on an infinite number of real parameters.

In the present paper, we shall follow an entirely different methodology by developing a “theory-dependent” approach initiated in [24]: Instead of considering any conceivable phenomenological deviation from general relativity, we focus on the simplest and best motivated class of non-Einsteinian theories, in which gravity is mediated by a tensor field ($g_{\mu\nu}^*$) together with one or several scalar fields (φ^a). These “tensor-multiscalar” theories arise naturally in theoretical

attempts at quantizing gravity or at unifying it with other interactions (Kaluza-Klein and superstrings theories). Moreover, they are the only consistent field theories, containing only fields of infinite range, able to satisfy the weak equivalence principle (universality of free fall of laboratory-size objects) [25].¹ Indeed, massless gravitational theories incorporating, besides the metric $g_{\mu\nu}^*$, vector fields, a second symmetric tensor field or an antisymmetric tensor field are known to present in general many flaws, such as discontinuities in the field degrees of freedom, negative-energy modes, causality violations, ill posedness of the Cauchy problem, etc., not to mention the lack of theoretical motivations for considering equivalence-principle-preserving couplings for such fields. By contrast, tensor-multiscalar theories are well motivated, consistent and simple enough to allow their observational predictions to be fully worked out [24]. Moreover, we believe that these field theories are the only ones satisfying the extended Lorentz invariance required in [20–23]. It would be an interesting program to prove it rigorously. In that case, our field-theoretical approach gives a technically much more efficient way of controlling their structure than that of Refs. [20–23], as exemplified by our results below.

A detailed study of the 1PN limit of tensor-multiscalar theories has been performed in [24], as well as the generalization of this approximation to the case of compact bodies (such as neutron stars), called the first post-Keplerian (1PK) limit. We recall some of our results in Sec. II below. Out of the 10 post-Newtonian parameters describing conceivable deviations from general relativity at the 1PN level, only two do not vanish in this class of theories: the parameters $\bar{\beta} \equiv \beta - 1$ and $\bar{\gamma} \equiv \gamma - 1$, introduced long ago (on different

grounds) by Eddington [1].² In [27,26], it was shown that the cosmological evolution generically drives these parameters towards values $\lesssim 10^{-7}$ at our present epoch. This class of theories gives therefore a natural explanation (requiring no fine tuning nor the *a priori* presence of small parameters) to the bounds $|\bar{\gamma}| < 2 \times 10^{-3}$ [28] and $|\bar{\beta}| < 6 \times 10^{-4}$ [29] found at the 1PN level in solar-system experiments, and furnish a motivation for increasing the precision of these measurements to the 10^{-7} level. Such an increase in accuracy down to a level comparable to 2PN effects ($Gm_{\odot}/R_{\odot}c^2 \sim 10^{-6}$) makes it necessary to determine how 2PN effects can influence such high-precision measurements, i.e., whether there are new and *a priori* unknown 2PN parameters which could complicate the interpretation of 1PN experiments. An example of this is given by higher-order light-deflection experiments which have been claimed to involve a new 2PN parameter [12,13,15]. We shall, however, prove below that this claim is incorrect in the framework of tensor-scalar theories.

The questions we shall address are thus: What are the new degrees of freedom describing the possible deviations of tensor-multiscalar theories from general relativity at the 2PN order, and can the corresponding effects be separated from those associated with $\bar{\beta}$ and $\bar{\gamma}$? On the other hand, do experimental bounds on $\bar{\beta}$ and $\bar{\gamma}$ give constraints on possible 2PN non-Einsteinian effects?

Before entering into a detailed study of the 2PN limit of tensor-scalar theories, let us quote one of our main results: Two, and only two, new parameters arise at the 2PN level. We have denoted them by ε and ζ , cf. Eqs. (3.30) below. The possible 2PN *deviations* from the general relativistic physical metric tensor are given by

$$\begin{aligned} \delta g_{00}(x) = & \frac{\varepsilon}{3c^6} U^3(\mathbf{x}) + \frac{\varepsilon}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')U^2(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{2\zeta}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \int d^3\mathbf{x}'' \frac{G\sigma(\mathbf{x}'')U(\mathbf{x}'')}{|\mathbf{x}'-\mathbf{x}''|} \\ & + \frac{2\zeta}{c^6} U(\mathbf{x}) \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')U(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + O\left(\frac{\bar{\beta}}{c^6}, \frac{\bar{\gamma}}{c^6}\right) + O\left(\frac{1}{c^8}\right), \end{aligned} \quad (1.1a)$$

$$\delta g_{0i}(x) = O\left(\frac{\bar{\beta}}{c^5}, \frac{\bar{\gamma}}{c^5}\right) + O\left(\frac{1}{c^7}\right), \quad (1.1b)$$

$$\delta g_{ij}(x) = O\left(\frac{\bar{\beta}}{c^4}, \frac{\bar{\gamma}}{c^4}\right) + O\left(\frac{1}{c^6}\right), \quad (1.1c)$$

where $\sigma(\mathbf{x})$ is the mass density and $U(\mathbf{x}) = \int d^3\mathbf{x}' G\sigma(\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$ the Newtonian potential. To increase the readability of Eqs. (1.1), we have suppressed the tilde which should decorate all the quantities appearing in it: $\tilde{g}_{00}(\tilde{x}), \tilde{U}, \tilde{G}, \tilde{\sigma}, \dots$; see below. The same parameters ε, ζ will be found to define the 2PN renormalizations of various

Newtonian or 1PN quantities under the influence of self-gravity or external gravitational fields.

In Sec. II, we recall the action and the equations of motion of tensor-multiscalar theories, as well as a few useful results concerning their 1PN limit. We also recall how the

¹Note, however, that the scalar couplings coming out naturally from unifying theories violate the equivalence principle [26].

²The intuitively preferred role played by $\bar{\beta}$ and $\bar{\gamma}$ in the PPN formalism is a further argument for working in the framework of tensor-scalar theories.

motion of self-gravitating bodies can be described in these theories. The main discussion of our paper, in Sec. III, is devoted to the Lagrangian describing the motion of N massive bodies at the 2PN level. Our main technical tool is a diagrammatic expansion, which allows us to compute straightforwardly all the 2PN effects. In Sec. IV we derive the 2PN metric corresponding to the Lagrangian of Sec. III, and we verify and complement our results by considering the metric generated by one static and spherically symmetric body, whose exact solution has been derived in [24]. In Sec. V, we discuss the impact of our findings on future relativistic experiments. We summarize our results and give our conclusions in Sec. VI. To relieve the tedium, technical details are relegated to various appendixes. Appendix A gives the explicit diagrammatic calculation of the 2PN Lagrangian. In Appendix B, we discuss the renormalizations of the Newtonian and 1PN coupling parameters due to 2PN effects. Finally, Appendix C derives the explicit 2PN formulas for the deflection of light and the perihelion shift of test masses.

II. TENSOR-MULTISCALAR THEORIES

In this section we define our notation for dealing with tensor-scalar theories, and recall the results of [24] that we need below to study their 2PN approximation.

A. Action and field equations

For simplicity, we consider in the present paper only theories respecting exactly the weak equivalence principle, i.e., theories in which matter is universally coupled to *one* second rank symmetric tensor, say $\tilde{g}_{\mu\nu}(x^\lambda)$. The action describing matter can then be written as a functional:

$$S_m[\psi_m, \tilde{g}_{\mu\nu}], \quad (2.1)$$

where ψ_m denotes globally all matter fields, including gauge bosons. Actually, from the perspective of modern unified theories, this class of models seems rather *ad hoc*. For instance, string theory does suggest the possibility that there exist long-range scalar fields contributing to the interaction between macroscopic bodies, but all such scalar fields have composition-dependent couplings. However, a recent study of a large class of superstring-inspired tensor-scalar models [26] has found that (because of deep physical facts) the composition-dependent effects represent only fractionally small ($\sim 10^{-5}$) corrections to standard post-Newtonian effects.

At a fundamental level, the matter action S_m should be chosen as the curved-spacetime version of the action of the standard model of electroweak and strong interactions, obtained by replacing the flat metric $f_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ by $\tilde{g}_{\mu\nu}$ and partial derivatives by \tilde{g} -covariant ones. At a phenomenological level, the action describing a system of N (non-self-gravitating) pointlike particles is

$$S_m = - \sum_{A=1}^N \int \tilde{m}_A c d\tilde{s}_A, \quad (2.2)$$

where $d\tilde{s}_A \equiv [-\tilde{g}_{\mu\nu}(x_A^\lambda) dx_A^\mu dx_A^\nu]^{1/2}$, and the \tilde{m}_A 's denote the (constant) inertial masses of the different particles. The uni-

versal coupling to $\tilde{g}_{\mu\nu}$ implies in particular that laboratory rods and clocks measure this metric, which will therefore be called the ‘‘physical metric’’ [the names Jordan, Fierz, or Pauli metric are also used in the literature].

The difference with general relativity lies in that the physical metric $\tilde{g}_{\mu\nu}$, instead of being a pure spin-2 field, is in tensor-scalar theories a mixing of spin-2 and spin-0 degrees of freedom. More precisely, it can be written as

$$\tilde{g}_{\mu\nu} = A^2(\varphi^a) g_{\mu\nu}^*, \quad (2.3)$$

where $A(\varphi^a)$ is a function of n scalar fields (we choose $A > 0$ to simplify some equations below). The dynamics of the pure spin-2 field $g_{\mu\nu}^*$, usually called the ‘‘Einstein metric,’’ is described by the Einstein-Hilbert action

$$S_{\text{spin } 2} = \frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} \sqrt{g_*} \frac{R^*}{4}, \quad (2.4)$$

where G_* is a constant (the bare gravitational constant), R^* is the scalar curvature of $g_{\mu\nu}^*$ (with the sign conventions of [30]), and $g_* \equiv |\det g_{\mu\nu}^*|$. On the other hand, the action describing the n scalar fields φ^a reads

$$S_{\text{spin } 0} = \frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} \sqrt{g_*} \left[-\frac{1}{2} g_*^{\mu\nu} \gamma_{ab}(\varphi^c) \partial_\mu \varphi^a \partial_\nu \varphi^b \right], \quad (2.5)$$

where $g_*^{\mu\nu}$ is the inverse of $g_{\mu\nu}^*$, the indices a, b, c, \dots vary from 1 to n , and $\gamma_{ab}(\varphi^c)$ is an n -dimensional (σ -model) metric in the internal scalar space spanned by the φ^a 's. [γ_{ab} must be positive-definite to get positive kinetic-energy terms.] A tensor-multiscalar theory contains in general $1 + n(n-1)/2$ arbitrary functions of n variables: the ‘‘coupling function’’ $A(\varphi^a)$ involved in Eq. (2.3), and the $n(n+1)/2$ components of γ_{ab} from which must be subtracted n arbitrary functions parametrizing arbitrary changes of scalar-field variables $\varphi'^a = f^a(\varphi^b)$. In the simplest case where there is only one scalar field ($n=1$), the only arbitrary function in the problem is $A(\varphi)$, the unique component of the σ -model metric being always reducible to the trivial form $\gamma_{11}(\varphi^1) = 1$. The reader should note that the consideration of multiple scalar fields, far from complicating uselessly our analysis, is in fact a technically powerful tool for delineating the structure of the possible deviations from general relativity. Once one is used to some notation, working with several fields is anyway not more difficult than working with only one.

We could also have added a potential term $V(\varphi^a)$ in the action (2.5), but we will restrict our attention to infinite-range fields in the present paper (see [24] for more details). Note that $g_{\mu\nu}^*$ and the n scalar fields φ^a are considered as forming the gravitational sector of the theory, by contrast with the matter sector described by the fields ψ_m of Eq. (2.1).

Although one should always keep in mind that the metric measured by normal physical standards is $\tilde{g}_{\mu\nu}$, it will be convenient in the following to formulate the theory in terms of the pure spin-2 and spin-0 fields $g_{\mu\nu}^*$ and φ^a . The Einstein-frame infinitesimal lengths ℓ , time intervals t , and masses m will therefore be related to the physical (measured) ones by

$$\ell = A^{-1}(\varphi)\tilde{\ell}, \quad t = A^{-1}(\varphi)\tilde{t}, \quad m = A(\varphi)\tilde{m}. \quad (2.6)$$

For instance, the action (2.2) describing N (non-self-gravitating) pointlike particles can be rewritten as

$$\begin{aligned} S_m &= - \sum_{A=1}^N \int \tilde{m}_A c A(\varphi^a(x_A)) \sqrt{-g_{\mu\nu}^*(x_A)} dx_A^\mu dx_A^\nu \\ &= - \sum_{A=1}^N \int m_A(\varphi^a(x_A)) c ds_A^*, \end{aligned} \quad (2.7)$$

where the Einstein-frame masses $m_A(\varphi^a) \equiv A(\varphi^a)\tilde{m}_A$ are no longer constant, as opposed to the \tilde{m}_A 's.

The field equations deriving from the total action $S_{\text{spin } 2} + S_{\text{spin } 0} + S_m$ read

$$R_{\mu\nu}^* = 2\gamma_{ab}(\varphi)\partial_\mu\varphi^a\partial_\nu\varphi^b + \frac{8\pi G^*}{c^4} \left(T_{\mu\nu}^* - \frac{1}{2}T^*g_{\mu\nu}^* \right), \quad (2.8a)$$

$$\square_{(g^*,\gamma)}\varphi^a = -\frac{4\pi G^*}{c^4}\alpha^a(\varphi)T^*, \quad (2.8b)$$

$$\delta S_m[\psi_m, \tilde{g}_{\mu\nu}]/\delta\psi_m = 0. \quad (2.8c)$$

Here $R_{\mu\nu}^*$ is the Ricci tensor of $g_{\mu\nu}^*$, $T_{\mu\nu}^* \equiv (2c/\sqrt{g_*})\delta S_m[\psi_m, A^2g_{\mu\nu}^*]/\delta g_{\mu\nu}^*$ is the Einstein-frame energy tensor (related to the conserved ‘‘physical’’ energy tensor $\tilde{T}^{\mu\nu}$ by $T_{\mu\nu}^* = A^6\tilde{T}^{\mu\nu}$; see [24]), and the d'Alembertian $\square_{(g^*,\gamma)}$ is covariant with respect to both space-time and σ -model indices, i.e., involves the Levi-Civita connections of both $g_{\mu\nu}^*$ and $\gamma_{ab}(\varphi)$, denoted as $\Gamma_{\mu\nu}^{*\lambda}$ and $\gamma_{bc}^a(\varphi)$, respectively:

$$\square_{(g^*,\gamma)}\varphi^a \equiv g_{\mu\nu}^{*\lambda}[\partial_\mu\partial_\nu\varphi^a - \Gamma_{\mu\nu}^{*\lambda}\partial_\lambda\varphi^a + \gamma_{bc}^a(\varphi)\partial_\mu\varphi^b\partial_\nu\varphi^c]. \quad (2.9)$$

In Eqs. (2.8) and everywhere else in this paper, the various indices are moved by their corresponding metric, for instance $T_{\mu\nu}^* = g_{\mu\alpha}^*g_{\nu\beta}^*T^{*\alpha\beta}$, $T^* = g_{\mu\nu}^*T^{*\mu\nu}$, $\tilde{T}_{\mu\nu} = \tilde{g}_{\mu\alpha}\tilde{T}^{\alpha\nu}$, etc., but also $\alpha^a(\varphi) = \gamma^{ab}(\varphi)\alpha_b(\varphi)$ where γ^{ab} is the inverse of γ_{ab} and where we have introduced the notation

$$\alpha_a(\varphi) \equiv \frac{\partial \ln A(\varphi)}{\partial \varphi^a}. \quad (2.10)$$

Note that, in view of the third equation (2.6), the definition of $\alpha_a(\varphi)$ can be rewritten as

$$\alpha_a(\varphi) \equiv \frac{\partial \ln m(\varphi)}{\partial \varphi^a}, \quad (2.11)$$

where $m(\varphi)$ is the mass of any non-self-gravitating particle in the Einstein conformal frame. This second way of defining α_a is quite general as it encompasses both self-gravity effects [24] (see below) and possible composition-dependent effects [26].

B. 1PN approximation

As is clear from Eq. (2.8b), the quantity $\alpha^a(\varphi)$ (which is a vector field in the internal ‘‘ σ -model’’ space spanned by

the φ^a 's) plays the crucial role of measuring the *coupling strength* of the scalar fields to matter. As we shall see below, all post-Newtonian deviations from general relativity (of any post-Newtonian order) can be expressed in terms of the asymptotic value of $\alpha^a(\varphi)$ at spatial infinity (i.e., far from all material sources) and of its successive scalar-field derivatives. Denoting by D_a the covariant derivative with respect to the internal metric γ_{ab} , we define

$$\beta_{ab} \equiv D_a\alpha_b = \frac{\partial\alpha_b}{\partial\varphi^a} - \gamma_{ab}^c\alpha_c, \quad (2.12a)$$

$$\beta'_{abc} \equiv D_a D_b \alpha_c. \quad (2.12b)$$

Denoting by α_a^0 the (cosmologically imposed) background values of the scalar fields, we then set $\alpha_a^0 \equiv \alpha_a(\varphi_0)$, $\beta_{ab}^0 \equiv \beta_{ab}(\varphi_0)$, $\beta'_{abc}{}^0 \equiv \beta'_{abc}(\varphi_0)$, etc., the index 0 always meaning that these σ -model tensors are calculated at φ_0 . As shown in [24], the effects of the scalar fields on any observable effect at the first post-Newtonian level depend only on two contractions of α_a^0 and β_{ab}^0 : namely,³

$$\alpha_0^2 \equiv \alpha_a^0\alpha_a^0 = \alpha_a^0\gamma^{ab}(\varphi_0)\alpha_b^0, \quad (2.13a)$$

$$(\alpha\beta\alpha)_0 \equiv \alpha_0^a\beta_{ab}^0\alpha_b^0. \quad (2.13b)$$

Indeed, the effective gravitational constant between two massive particles is given by

$$\tilde{G} \equiv G_* A_0^2 (1 + \alpha_0^2), \quad (2.14)$$

[with $A_0 \equiv A(\varphi_0)$] instead of the bare constant G_* involved in the action (2.4), and the Eddington parameters $\bar{\beta}$, $\bar{\gamma}$ read

$$\bar{\gamma} = -2\alpha_0^2/(1 + \alpha_0^2), \quad (2.15a)$$

$$\bar{\beta} = \frac{1}{2}(\alpha\beta\alpha)_0/(1 + \alpha_0^2)^2, \quad (2.15b)$$

instead of their general relativistic values $\bar{\beta} = \bar{\gamma} = 0$. Let us recall again that $\bar{\beta}$ and $\bar{\gamma}$ are usually denoted as $(\beta - 1)$ and $(\gamma - 1)$ in the literature. We will avoid the notation β , γ in the present paper to prevent a possible confusion with the σ -model tensors β_{ab} and γ_{ab} . Note, however, from Eqs. (2.13) and (2.15), that our notation has been chosen so that $\bar{\gamma} \propto \gamma_{ab}\alpha^a\alpha^b$ and $\bar{\beta} \propto \beta_{ab}\alpha^a\alpha^b$. As shown in [24], the results (2.14)–(2.15) can be simply interpreted (and remembered) in terms of the exchange of gravitons and scalar particles between material sources. For instance, Eq. (2.14) is the sum of the usual contribution of a graviton exchange, G_* , together with the contributions of scalar exchanges, $\sum_a G_* \alpha_a^0 \alpha_a^0$; see Fig. 1. The global factor A_0^2 is due to the change of units between the Einstein metric and the physical one $\tilde{g}_{\mu\nu} = A^2(\varphi)g_{\mu\nu}^*$ used to measure forces [see Eqs. (2.6)

³Note that in the simplest one-scalar case $\varphi = \varphi^1$, these contractions reduce to simple products, $\alpha_0^2 = \alpha_0 \times \alpha_0$, $(\alpha\beta\alpha)_0 = \alpha_0 \times \beta_0 \times \alpha_0$, where $\alpha_0 = \alpha(\varphi_0) = \partial \ln A(\varphi_0)/\partial \varphi_0$, and $\beta_0 = \beta(\varphi_0) = \partial \alpha(\varphi_0)/\partial \varphi_0$.

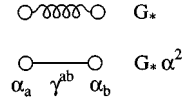


FIG. 1. Diagrammatic interpretation of the effective gravitational constant $\tilde{G} = G_* A_0^2 (1 + \alpha_0^2)$.

above]. Similarly, the parameter $(\alpha\beta\alpha)_0$ involved in (2.15b) corresponds to an exchange of scalar particles between three massive bodies, as shown in Fig. 2. The method that we will use in Sec. III below to study the second post-Newtonian approximation is a straightforward generalization of these diagrammatic observations.

C. Self-gravitating bodies

When studying the motion of massive bodies in the solar system, several dimensionless ratios happen to be small. If we denote by m , v , and R the typical mass, orbital velocity, and radius of a body, and by r the typical distance between two bodies, we find $Gm/rc^2 \sim v^2/c^2 \lesssim 2 \times 10^{-8}$ for the fastest planets, while $Gm/Rc^2 \approx 2 \times 10^{-6}$ for the Sun and $\approx 7 \times 10^{-10}$ for the Earth. This is the reason why the formal ‘‘post-Newtonian’’ expansion in powers of $1/c^2$ is so useful for analyzing the predictions of relativistic theories of gravity in the solar system. However, in situations involving compact bodies, like neutron stars (pulsars) in binary systems, it is necessary to distinguish the self-gravity parameter $s \sim Gm/Rc^2 \sim 0.2$ from the orbital parameters $Gm/rc^2 \sim v^2/c^2 \ll 1$. In that case, one can still describe the motion of the bodies by means of an expansion in powers of $Gm/rc^2 \sim v^2/c^2 \sim 1/c_{\text{orbital}}^2$, but one must not expand in powers of the compactnesses s . Such an expansion scheme has been called ‘‘post-Keplerian’’ in [24], since it is closely linked with the phenomenological approach to the analysis of binary pulsar data, introduced in [31,32] under the name of ‘‘parametrized post-Keplerian’’ formalism.

In the present paper, our main goal is to analyze tensor-multiscalar theories of gravity at the second post-Newtonian level, i.e., including all terms of formal order $1/c^4$ (be them of ‘‘orbital,’’ ‘‘self-gravity,’’ or mixed type). We found that the most efficient way of doing so is to derive first the second post-Keplerian (2PK) limit of these theories by means of a diagrammatic method. The 2PN approximation is then obtained by expanding our general 2PK results in powers of the compactnesses, up to the required order in s .

Let us recall how the motion of self-gravitating bodies is described in tensor-scalar theories. Following a suggestion of Eardley [33], one skeletonizes extended self-gravitating bodies as pointlike particles whose inertial masses \tilde{m}_A depend on

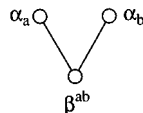


FIG. 2. Diagrammatic interpretation of the contraction $(\alpha\beta\alpha)_0$ involved in the Eddington parameter $\bar{\beta} = \frac{1}{2} (\alpha\beta\alpha)_0 / (1 + \alpha_0^2)^2$.

the scalar fields φ^a , as opposed to the constant \tilde{m}_A 's in Eq. (2.2) describing non-self-gravitating bodies. This scalar dependence of the inertial mass is due to the influence of the local scalar field background on the equilibrium configuration of the body. The action describing N self-gravitating bodies is thus written as

$$S_m = - \sum_{A=1}^N \int \tilde{m}_A(\varphi^a) c d\tilde{s}_A = - \sum_{A=1}^N \int m_A(\varphi^a) c ds_A^*, \quad (2.16)$$

where $m_A(\varphi) \equiv A(\varphi) \tilde{m}_A(\varphi)$. The validity of the skeletonized action (2.16) has been justified in Appendix A of [24] by a matching argument.

Note that the second expression (2.16), in terms of the Einstein line element ds_A^* , is formally identical to Eq. (2.7) describing non-self-gravitating bodies. However, the important difference is that $m_A(\varphi)$ can now be a nonuniversal (body-dependent) function of the scalar fields, instead of being merely proportional to $A(\varphi)$. It is therefore convenient to generalize to the case of self-gravitating bodies the σ -model tensors defined in Eqs. (2.10)–(2.12) above:

$$\alpha_a^A \equiv \frac{\partial \ln m_A(\varphi)}{\partial \varphi^a} = \alpha_a(\varphi) + \frac{\partial \ln \tilde{m}_A(\varphi)}{\partial \varphi^a}, \quad (2.17a)$$

$$\beta_{ab}^A \equiv D_a \alpha_b^A, \quad (2.17b)$$

$$\beta'_{abc}{}^A \equiv D_a D_b \alpha_c^A. \quad (2.17c)$$

(As above, we will raise and lower the σ -model indices a, b, \dots with γ^{ab} or γ_{ab} .) Here again α_a^A plays the fundamental role of measuring the coupling strength of the scalar fields to the self-gravitating body A . Indeed, equation (2.8b) now reads

$$\square_{(g^*, \gamma)} \varphi^a = - \frac{4\pi G_*}{c^4} \sum_A \alpha_a^A T_A^*, \quad (2.18)$$

where $T_A^* = -m_A c^2 (ds_A^*/dx^0) \delta^{(3)}(\mathbf{x} - \mathbf{x}_A) / \sqrt{g^*(x_A)}$ is localized on the position of the A th ‘‘particle.’’ Note that the body-dependent quantities (2.17) reduce to the definitions (2.10)–(2.12) for non-self-gravitating bodies, since the inertial masses \tilde{m}_A are constant in that case.

As shown in [24], the 1PK approximation of tensor-multiscalar theories can then be expressed very simply in terms of contractions of α_a^A and β_{ab}^A . More precisely, the Lagrangian describing the motion of N (spherical) self-gravitating bodies at the 1PK level reads (in Einstein-frame units)

$$L = \sum_A L_A^{(1)} + \frac{1}{2} \sum_{A \neq B} L_{AB}^{(2)} + \frac{1}{2} \sum_{B \neq A \neq C} L_{BC}^{(3)A} + O\left(\frac{1}{c^4}\right), \quad (2.19)$$

where the notation $B \neq A \neq C$ excludes $A=B$ and $A=C$ but not $B=C$, and where

$$L_A^{(1)} = -m_A^0 c^2 \sqrt{1 - \mathbf{v}_A^2/c^2} = -m_A^0 c^2 + \frac{1}{2} m_A^0 \mathbf{v}_A^2 + \frac{1}{8c^2} (\mathbf{v}_A^2)^2 + O\left(\frac{1}{c^4}\right), \quad (2.20a)$$

$$L_{AB}^{(2)} = \frac{G_{AB} m_A^0 m_B^0}{r_{AB}} \left[1 + \frac{3}{2c^2} (\mathbf{v}_A^2 + \mathbf{v}_B^2) - \frac{7}{2c^2} (\mathbf{v}_A \cdot \mathbf{v}_B) - \frac{1}{2c^2} (\mathbf{n}_{AB} \cdot \mathbf{v}_A)(\mathbf{n}_{AB} \cdot \mathbf{v}_B) + \frac{\bar{\gamma}_{AB}}{c^2} (\mathbf{v}_A - \mathbf{v}_B)^2 \right], \quad (2.20b)$$

$$L_{BC}^{(3)A} = -\frac{G_{AB} G_{AC} m_A^0 m_B^0 m_C^0}{r_{AB} r_{AC} c^2} (1 + 2\bar{\beta}_{BC}^A). \quad (2.20c)$$

Here we have set $m_A^0 \equiv m_A(\varphi_0)$, and we have denoted by $\mathbf{n}_{AB} \equiv \mathbf{r}_{AB}/r_{AB}$ (with $\mathbf{r}_{AB} \equiv \mathbf{x}_A - \mathbf{x}_B$) the unit vector directed from body B to body A . This Lagrangian involves three body-dependent parameters generalizing the effective gravitational constant (2.14) and the 1PN Eddington parameters (2.15): namely,

$$G_{AB} \equiv G_* [1 + (\alpha_A \alpha_B)_0], \quad (2.21a)$$

$$\bar{\gamma}_{AB} \equiv -2 \frac{(\alpha_A \alpha_B)_0}{1 + (\alpha_A \alpha_B)_0}, \quad (2.21b)$$

$$\bar{\beta}_{BC}^A \equiv \frac{1}{2} \frac{(\alpha_B \beta_A \alpha_C)_0}{[1 + (\alpha_A \alpha_B)_0][1 + (\alpha_A \alpha_C)_0]}, \quad (2.21c)$$

where⁴ $(\alpha_A \alpha_B) \equiv \alpha_A^a \gamma_{ab} \alpha_B^b$, $(\alpha_B \beta_A \alpha_C) \equiv \alpha_B^a \beta_{ab}^A \alpha_C^b$, and the index 0 means that these contractions are calculated at $\varphi^a = \varphi_0^a$. Of course, when using physical units related to the asymptotic metric $\tilde{g}_{\mu\nu}$, the effective gravitational constant reads $\tilde{G}_{AB} = G_* A_0^2 [1 + (\alpha_A \alpha_B)_0]$, and the parameters $\bar{\gamma}_{AB}$, $\bar{\beta}_{BC}^A$ do not change since they are dimensionless. Note that Figs. 1 and 2 give again a diagrammatic interpretation of G_{AB} and $(\alpha_B \beta_A \alpha_C)_0$, with the coupling coefficients α^a and β^{ab} of each body being replaced by their strong-field counterparts α_A^a and β_A^{ab} .

The pointlike description (2.16) neglects all finite size effects. The finite size corrections induced by tensor-mediated couplings to the mass and spin multipole moments of order $\ell \geq 1$ (i.e., the general relativistic multipole interactions) have been recently worked out in detail [34], and we assume that they are properly added. One must still worry about scalar-mediated finite-size effects whose existence has been established by Nordtvedt [35]. Such effects can be incorporated by a suitable formal generalization of Eq. (2.16): it suffices to consider m_A (or \tilde{m}_A) as a *functional* of φ (and $g_{\mu\nu}^*$) rather than as a function of the local value of φ along the A th world line. (In particle physics language, the functional $m_A[\varphi]$ is the ‘‘form factor’’ describing the nonlocal interaction of the extended body A with the field φ .) Then the quasilocal expansion of the functional $m_A[\varphi]$ in increasing number of derivatives of φ along the worldline corresponds to the usual expansion in powers of the small finite-size parameter $\epsilon \equiv R/r$. If one considers extended bodies

which would, in absence of interaction, be spherical, the first finite-size effects must, for symmetry reasons, contain two derivatives of φ , i.e., they must be of order $\epsilon^2 = (R/r)^2$. We leave to future work a study of finite-size effects by means of such an effective action approach. In the case of weakly self-gravitating bodies, direct calculations at the 1PN approximation have found that finite-size effects are proportional to $\eta = 4\bar{\beta} - \bar{\gamma}$ [35], and therefore introduce, in the equations of motion, essentially only small fractional corrections (of order $\epsilon^2 = (R/r)^2 \ll 1$) to the usual orbital post-Einsteinian effects proportional to $\bar{\gamma}/c^2$ and $\bar{\beta}/c^2$. Such fractionally small corrections can be safely neglected when looking for the numerically dominant deviations from general relativity. In the case of strongly self-gravitating bodies, finite-size effects are even more negligible than in the solar-system case because $\epsilon^2 \sim (10 \text{ km}/10^6 \text{ km})^2 \sim 10^{-10}$ is extremely small in all observed binary pulsars.

III. N-BODY LAGRANGIAN

Before entering the technical details of our derivation of the N -body Lagrangian, let us clarify the approximation methods we shall employ. As discussed in the previous section, we shall first study the second post-Keplerian approximation of tensor-multiscalar theories, before particularizing our results to the second post-Newtonian case. In other words, the compactnesses $s \sim Gm/Rc^2$ of the bodies will not be considered *a priori* as small parameters. In order to construct the Lagrangian describing the motion of N self-gravitating bodies, we will eliminate the field degrees of freedom, i.e., we will solve for the Einstein metric $g_{\mu\nu}^*$ and the scalar fields φ^a in terms of the material sources, using the field equations (2.8a), (2.8b). To perform this elimination, we shall consider that the interaction is propagated by a time-symmetric (half-retarded–half-advanced) Green’s function. This leaves out radiation damping effects. However, the latter are negligible when studying weakly self-gravitating bodies at 2PN order. Indeed, in general relativity, the leading dissipative effects occur at order $1/c^5$, because of the well-known quadrupolar radiation of gravitational waves. In tensor-scalar theories of gravity, the leading radiation emitted by systems of *compact* bodies is dipolar and occur at order s^2/c^3 . In the solar system case, $s \sim 1/c^2$ and the dipolar radiation is negligible compared to the $O(1/c^5)$ effects due to the spin-2 quadrupolar waves and spin-0 monopolar and quadrupolar waves [24]. In either case, we compute the conservative part of the gravitational interaction and assume that (tensor and scalar) radiation damping effects are added separately when they are not negligible small.

⁴Again, in the simplest case of one scalar field $\varphi \equiv \varphi^1$, these quantities reduce simply to $\alpha_A \times \alpha_B$ and $\alpha_B \times \beta_A \times \alpha_C$, where $\alpha_A = \partial \ln m_A / \partial \varphi$, $\beta_A = \partial \alpha_A / \partial \varphi$.

A. Diagrammatic expansion

We want to construct a ‘‘Fokker’’ Lagrangian [36,17] describing the motion of N massive bodies by eliminating the field degrees of freedom from the total Lagrangian of the theory. In order to prove formally that such a construction reproduces the correct dynamics of the bodies, let us introduce a global notation for the fields $\Phi \equiv (g_{\mu\nu}^* - f_{\mu\nu}, \varphi^a - \varphi_0^a)$, where $f_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the flat metric and φ_0^a are the background values of the scalar fields. Similarly, we denote globally by σ the matter variables, i.e., the N massive worldlines x_A involved in the matter action (2.16). The total action of the theory can therefore be written in the schematic form

$$S_{\text{tot}}[\sigma, \Phi] = S_{\Phi}[\Phi] + S_m[\sigma, \Phi], \quad (3.1)$$

from which we want to eliminate the fields Φ by expressing them in terms of the matter variables σ . This elimination cannot be done when working directly with the Einstein-Hilbert action (2.4) because of its invariance under diffeomorphisms (technically, the kinetic term of the gravitons is noninvertible). We need to reduce the field equations by means of a specific coordinate condition in order to solve for $g_{\mu\nu}^*$. In the language of particle physics, we need to fix the gauge in order to define the propagator of the gravitons. We will choose the g^* -harmonic gauge for our explicit calculations. We then replace $S_{\Phi}[\Phi]$ by its gauge-fixed version $S_{\Phi}^{\text{GF}}[\Phi] = S_{\Phi}[\Phi] + \text{gauge-fixing terms}$ (see Appendix A below), and the field equations deriving from the total action $S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]$ read

$$\frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \Phi} = \frac{\delta S_{\Phi}^{\text{GF}}[\Phi]}{\delta \Phi} + \frac{\delta S_m[\sigma, \Phi]}{\delta \Phi} = 0, \quad (3.2a)$$

$$\frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \sigma} = \frac{\delta S_m[\sigma, \Phi]}{\delta \sigma} = 0. \quad (3.2b)$$

Equation (3.2a) can now be solved perturbatively, and we denote by $\bar{\Phi}[\sigma]$ its solution. The Fokker action (which is a functional of the matter variables only) is then defined as

$$S_F[\sigma] \equiv S_{\text{tot}}^{\text{GF}}[\sigma, \bar{\Phi}[\sigma]] = S_{\Phi}^{\text{GF}}[\bar{\Phi}[\sigma]] + S_m[\sigma, \bar{\Phi}[\sigma]], \quad (3.3)$$

and its variation with respect to σ reads

$$\begin{aligned} \frac{\delta S_F[\sigma]}{\delta \sigma} &= \left(\frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \sigma} \right)_{\Phi = \bar{\Phi}[\sigma]} \\ &+ \left(\frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \Phi} \right)_{\Phi = \bar{\Phi}[\sigma]} \frac{\delta \bar{\Phi}[\sigma]}{\delta \sigma}. \end{aligned} \quad (3.4)$$

Since $\bar{\Phi}[\sigma]$ is precisely the solution which annuls the functional derivative $\delta S_{\text{tot}}^{\text{GF}}/\delta \Phi$, the second term on the right-hand side vanishes, and one finally gets

$$\frac{\delta S_F[\sigma]}{\delta \sigma} = \left(\frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \sigma} \right)_{\Phi = \bar{\Phi}[\sigma]} = \left(\frac{\delta S_m[\sigma, \Phi]}{\delta \sigma} \right)_{\Phi = \bar{\Phi}[\sigma]}. \quad (3.5)$$

Comparing (3.5) with (3.2b), we see that we have formally proved that the Fokker action (3.3) gives indeed the correct equations of motion, i.e., describes the actual dynamics of the matter variables σ in presence of the fields $\bar{\Phi}[\sigma]$. It should be noted that $S_F[\sigma]$ is *not* simply given by the matter action $S_m[\sigma, \bar{\Phi}[\sigma]]$ embedded in the self-consistent background field $\bar{\Phi}[\sigma]$, but that the field action $S_{\Phi}^{\text{GF}}[\bar{\Phi}[\sigma]]$ also contributes to the dynamics of the material bodies.

To start with, we do not need to assume that the bodies are moving slowly with respect to the velocity of light. In technical terms, we shall expand the Lagrangian in powers of G , i.e., Gm/rc^2 (‘‘nonlinearity expansion’’), while keeping unrestricted the magnitudes of v/c and s . We need to retain the terms up to order G^3 in the nonlinearity expansion of the Fokker action (3.3). The zeroth order term in S_F is of course the action describing free bodies $S_0[\sigma] \equiv S_m[\sigma, \Phi = 0]$, i.e., the matter action computed for $g_{\mu\nu}^* = f_{\mu\nu}$ and $\varphi^a = \varphi_0^a$. It can be written explicitly as $S_0 = \sum_A \int dt L_A^{(1)}$, where $L_A^{(1)}$ was given in Eq. (2.20a) and has the structure $mc^2(1 + v^2/c^2 + v^4/c^4 + v^6/c^6 + \dots)$. The next approximation, first power of G , has the structure $mc^2(Gm/rc^2) \times (1 + v^2/c^2 + v^4/c^4 + \dots)$ and describes the two-body interaction, starting with the Newtonian term $G_{AB}m_A m_B / r_{AB}$ [with the self-gravity-modified effective gravitational constant G_{AB} of Eq. (2.21a)]. The second power of G , $mc^2(Gm/rc^2)^2(1 + v^2/c^2 + \dots)$, describes three-body interactions starting with the self-gravity-modified 1PK term $\propto G_{AB}G_{AC}m_A m_B m_C / r_{AB}r_{AC}c^2$. Finally, the G^3 level corresponds to four-body interactions: $G^3 m_A m_B m_C m_D / r^3 c^4 +$ self-gravity and velocity modifications. Instead of counting powers of G , we can count the number of matter source terms involved: we must keep up to four powers of σ , i.e., of the masses. Since the matter action $S_m[\sigma, \Phi]$ is linear in σ [see Eq. (2.16)], the solution $\bar{\Phi}[\sigma]$ of the field equations (3.2a) starts at order σ . We need therefore to expand the total action $S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]$ up to orders $O(\Phi^4)$ and $O(\sigma\Phi^3)$ included, before replacing Φ by its solution $\bar{\Phi}[\sigma]$.

Let us first expand $S_{\text{tot}}^{\text{GF}}$ in powers of the fields Φ , and define

$$S_{\text{tot}}^{\text{GF}}[\sigma, \Phi] = S_0[\sigma] + S_1[\sigma, \Phi] + S_2[\sigma, \Phi] + S_3[\sigma, \Phi] + \dots, \quad (3.6)$$

where the term $S_i[\sigma, \Phi]$ involves the i th power of Φ (and zero or one power of the material sources σ). For instance, S_1 is the linear interaction term between the fields and matter, and has the formal structure $S_1 = \alpha \sigma \Phi$, where α is a coupling constant. On the other hand, the term quadratic in Φ has the form $S_2 = -\frac{1}{2} \Phi \mathcal{P}^{-1} \Phi + \frac{1}{2} \beta \sigma \Phi^2$, and involves both the kinetic operator $\propto \square$ (or ‘‘inverse propagator’’ \mathcal{P}^{-1}) of the fields and a vertex describing the interaction of matter with two fields (with a coupling constant β). It will be convenient to introduce a diagrammatic notation for this expansion. (Bertotti and Plebanski [37] were the first to introduce a similar diagrammatic notation for solving Einstein’s

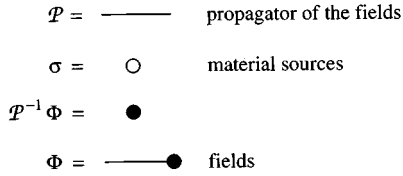


FIG. 3. Diagrammatic notations for the material sources, the fields, and their propagator.

equations perturbatively.) Let us denote the propagator \mathcal{P} by a straight line, the material source σ by a white blob, and the term $(\mathcal{P}^{-1}\Phi)$ by a black one; see Fig. 3. As this diagrammatic representation will be an important tool in the present paper, let us explain it in detail with a simpler example.

Let us consider the action (in Minkowski spacetime)

$$S[\varphi] = \int d^4x \left[-\frac{1}{8\pi} (\partial\varphi)^2 + \frac{g}{3} \varphi (\partial\varphi)^2 + \frac{\lambda}{4} \varphi^4 + \sigma(x) \varphi(x) \right], \quad (3.7)$$

where $(\partial\varphi)^2 \equiv f^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$ and where $\sigma(x)$ is a given (spacetime distributed) source for $\varphi(x)$. Integrating by parts, the kinetic terms for $\varphi(x)$ read $+(1/8\pi) \int d^4x \varphi(x) \square_f \varphi(x)$, where \square_f is the flat d'Alembertian. Identifying this with $-\frac{1}{2} \varphi \mathcal{P}^{-1} \varphi$ [which is a symbolic notation⁵ for $-\frac{1}{2} \int \int d^4x d^4y \varphi(x) \mathcal{P}_{xy}^{-1} \varphi(y)$, where \mathcal{P}_{xy}^{-1} is the kernel of an operator acting on functions of x^μ], we get $\mathcal{P}_{xy}^{-1} = -(4\pi)^{-1} \square_x \delta^{(4)}(x-y)$. The inverse of the operator \mathcal{P}_{xy}^{-1} is the propagator $\mathcal{P}_{xy} = \mathcal{G}(x-y)$, where $\mathcal{G}(x-y)$ is a (translation-invariant) Green function, solution

of $\square_x \mathcal{G}(x-y) = -4\pi \delta^{(4)}(x-y)$. The cubic vertex V_3 is defined as the distributional kernel entering the term $3S_3 \equiv \int dx g \varphi (\partial\varphi)^2$, i.e., $3S_3 = \int dx_1 dx_2 dx_3 V_3(x_1, x_2, x_3) \times \varphi(x_1) \varphi(x_2) \varphi(x_3)$. Requiring this kernel to be symmetric in its arguments leads to the explicit expression

$$V_3(x_1, x_2, x_3) = \frac{g}{3} \left[\frac{\partial}{\partial x_1^\mu} \delta(x_1 - x_2) \frac{\partial}{\partial x_1^\mu} \delta(x_1 - x_3) + \frac{\partial}{\partial x_2^\mu} \delta(x_2 - x_3) \frac{\partial}{\partial x_2^\mu} \delta(x_2 - x_1) + \frac{\partial}{\partial x_3^\mu} \delta(x_3 - x_1) \frac{\partial}{\partial x_3^\mu} \delta(x_3 - x_2) \right], \quad (3.8)$$

where the factor 1/3 comes from the average over the three different permutations needed to symmetrize $V_3(x_1, x_2, x_3)$. Similarly, the quartic vertex, defined by $4S_4 \equiv \int dx_1 dx_2 dx_3 dx_4 V_4(x_1, x_2, x_3, x_4) \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4)$, is

$$V_4(x_1, x_2, x_3, x_4) = \lambda \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(x_1 - x_4). \quad (3.9)$$

In the diagrammatic notation of Fig. 3, the blobs denote some spacetime functions [$\sigma(x)$ for the white blob and $-(4\pi)^{-1} \square \varphi(x)$ for the black one], and a line denotes a propagator $\mathcal{P}_{xy} = \mathcal{G}(x-y)$. Connecting a line to a blob or to a vertex (which is a cluster of several infinitesimally close points) means that one ‘‘contracts’’ (i.e., integrates) over the points at the extremities of the line. For instance the T-shaped diagram on the left of the third line of Fig. 7 below would represent, in the model (3.7),

$$\begin{aligned} & \int dx_1 dx_2 dx_3 dy_1 dy_2 dy_3 V_3(x_1, x_2, x_3) \mathcal{G}(x_1 - y_1) \mathcal{G}(x_2 - y_2) \mathcal{G}(x_3 - y_3) \sigma(y_1) \sigma(y_2) \sigma(y_3) \\ &= g \int dx dy_1 dy_2 dy_3 \mathcal{G}(x - y_1) \frac{\partial}{\partial x^\mu} \mathcal{G}(x - y_2) \frac{\partial}{\partial x^\mu} \mathcal{G}(x - y_3) \sigma(y_1) \sigma(y_2) \sigma(y_3). \end{aligned} \quad (3.10)$$

Similarly, the X diagram on the left of the last line of Fig. 7 would denote

$$\begin{aligned} & \int dx_1 dx_2 dx_3 dx_4 dy_1 dy_2 dy_3 dy_4 V_4(x_1, x_2, x_3, x_4) \mathcal{G}(x_1 - y_1) \mathcal{G}(x_2 - y_2) \mathcal{G}(x_3 - y_3) \mathcal{G}(x_4 - y_4) \sigma(y_1) \sigma(y_2) \sigma(y_3) \sigma(y_4) \\ &= \lambda \int dx dy_1 dy_2 dy_3 dy_4 \mathcal{G}(x - y_1) \mathcal{G}(x - y_2) \mathcal{G}(x - y_3) \mathcal{G}(x - y_4) \sigma(y_1) \sigma(y_2) \sigma(y_3) \sigma(y_4). \end{aligned} \quad (3.11)$$

Having explained the precise meaning of our symbolic notation, let us come back to the general action (3.6).

The different terms of the expansion (3.6) can be repre-

⁵Note that in the operator notation used here, any ‘‘contraction of spacetime indices’’ means an integration over the corresponding spacetime coordinates, e.g., $(\mathcal{P}\varphi)(x) \equiv \int d^4y \mathcal{P}_{xy} \varphi(y)$.

sented as in Fig. 4, which defines the different diagrams. Note that, as in our example, we have conventionally factorized a coefficient 1/i in defining the i-linear vertex V_i from the $O(\Phi^i)$ action: $S_i = V_i/i$. This coefficient is chosen in order to simplify the field equations (3.2a), whose diagrammatic expansion is displayed in Fig. 5. [The reader is invited to derive for himself the field equations of Fig. 5 from the action of Fig. 4, keeping in mind that the multilinear forms in Φ appearing in Fig. 4 are supposed to be symmetric in

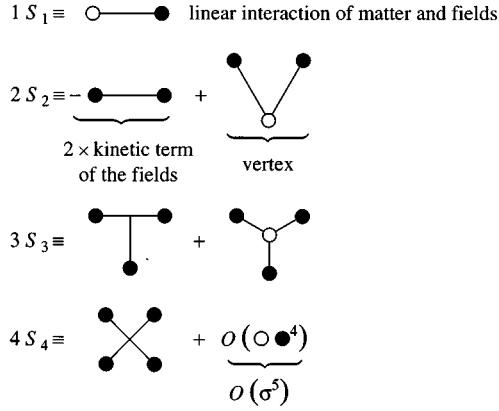


FIG. 4. Diagrammatic expression of the Φ^i -linear terms of the total action (3.6), for $i=1,2,3,4$.

$\Phi(x_1), \dots, \Phi(x_i)$.] Thanks to Euler's theorem on homogeneous functions, the field equations imply the useful result

$$\Phi \times \frac{\delta S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]}{\delta \Phi} = S_1 + 2S_2 + 3S_3 + \dots + iS_i + \dots = 0. \quad (3.12)$$

In diagrammatic terms, this corresponds to inserting a black blob at the free ends of the propagators in Fig. 5, i.e., to express the kinetic term of the fields, $\frac{1}{2} \Phi \mathcal{P}^{-1} \Phi \equiv \frac{1}{2} (\mathcal{P}^{-1} \Phi) \mathcal{P}(\mathcal{P}^{-1} \Phi)$, in terms of the other diagrams.

The Fokker action (3.3) can now be written straightforwardly by replacing in the total action (3.6) the fields Φ by their solution $\bar{\Phi}[\sigma]$, i.e., by replacing the black blobs in terms of the white ones through an iterative use of Fig. 5. The most delicate term to compute would be the contribution due to the kinetic term of the fields in S_2 , because one must expand up to order σ^3 the two fields Φ it involves. Fortunately, one can avoid estimating this term by using the Euler identity (3.12) to eliminate it from the Fokker action:

$$\begin{aligned} S_F[\sigma] &= [(S_0 + S_1 + S_2 + \dots) \\ &\quad - \frac{1}{2} (S_1 + 2S_2 + 3S_3 + \dots)]_{\Phi = \bar{\Phi}[\sigma]} \\ &= S_0 + [\frac{1}{2} S_1 - \frac{1}{2} S_3 - S_4]_{\Phi = \bar{\Phi}[\sigma]} + O(\sigma^5). \end{aligned} \quad (3.13)$$

The result of inserting Fig. 5 into Eq. (3.13) is displayed in Fig. 6. [The different diagrams have been drawn so that angles appear only at the vertices involving matter sources.] In the following, we will designate these diagrams by the letter they most naturally evoke, so that the final result for the Fokker action reads

$$\text{---} \bullet = \text{---} \circ + \text{---} \circ \text{---} + \text{---} \circ \text{---} + \text{---} \circ \text{---} + \text{---} \circ \text{---} + o(\sigma^4)$$

FIG. 5. Equation (3.2a) satisfied by the field $\bar{\Phi}[\sigma]$.

$$\begin{aligned} S_F[\sigma] &= S_0[\sigma] + \left\{ \frac{1}{2} \text{---} \circ \text{---} - \frac{1}{6} \text{---} \circ \text{---} - \frac{1}{6} \text{---} \circ \text{---} - \frac{1}{4} \text{---} \circ \text{---} \right\}_{\Phi = \bar{\Phi}[\sigma]} + o(\sigma^5) \\ &= S_0[\sigma] + \left(\frac{1}{2} \text{---} \circ \text{---} \right) + \left(\frac{1}{2} \text{---} \circ \text{---} + \frac{1}{3} \text{---} \circ \text{---} \right) \\ &\quad + \left(\frac{1}{3} \text{---} \circ \text{---} + \frac{1}{2} \text{---} \circ \text{---} + \frac{1}{2} \text{---} \circ \text{---} + \frac{1}{4} \text{---} \circ \text{---} \right) + o(\sigma^5) \end{aligned}$$

FIG. 6. Diagrammatic expansion of the Fokker action (3.13).

$$\begin{aligned} S_F[\sigma] &= S_0[\sigma] + \left(\frac{1}{2} I \right) + \left(\frac{1}{2} V + \frac{1}{3} T \right) + \left(\frac{1}{3} \epsilon + \frac{1}{2} Z + F + \frac{1}{2} H \right. \\ &\quad \left. + \frac{1}{4} X \right) + O(\sigma^5). \end{aligned} \quad (3.14)$$

The explicit form of this action is now only a question of straightforward algebra: one must expand $S_{\text{tot}}^{\text{GF}}[\sigma, \Phi]$ up to order Φ^4 included to get the expressions of the field propagator \mathcal{P} and of the vertices defined by Fig. 4, and merely replace them in Fig. 6, i.e., Eq. (3.14). In doing so, one must take into account the fact that Φ is a global notation for both the gravitons $h_{\mu\nu} \equiv g_{\mu\nu}^* - f_{\mu\nu}$ and the scalar fields $\varphi^a - \varphi_0^a$, and therefore that each propagator link in Fig. 6 is to be replaced by a sum of terms corresponding to the different fields. To simplify the expansion of the scalar-field action (2.5), it is convenient to choose Riemann normal coordinates at φ_0^a , so that the metric $\gamma_{ab}(\varphi)$ can be written as

$$\begin{aligned} \gamma_{ab}(\varphi) &= \gamma_{ab}(\varphi_0) + 0 \times (\varphi^a - \varphi_0^a) - \frac{1}{3} R_{abcd}(\varphi_0) (\varphi^c - \varphi_0^c) \\ &\quad \times (\varphi^d - \varphi_0^d) + O((\varphi - \varphi_0)^3), \end{aligned} \quad (3.15)$$

where R_{abcd} is the Riemann curvature of γ_{ab} . This choice cancels the term of order $\varphi \partial \varphi \partial \varphi$ in $S_{\text{spin}0}$, i.e., the ‘‘T’’

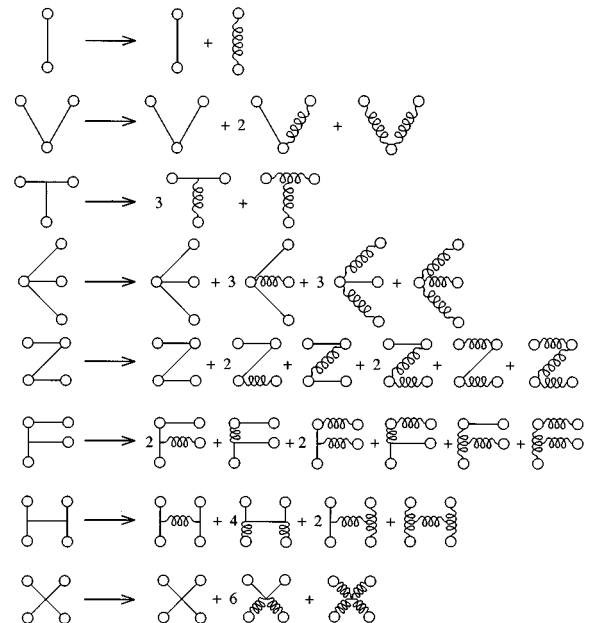


FIG. 7. Expression of the diagrams of Fig. 6 when the graviton and scalar propagators are represented respectively as curly and straight lines.

vertex connecting three scalar fields. The different diagrams of Eq. (3.14) can thus be decomposed into the elementary diagrams displayed in Fig. 7, where curly and straight lines represent the graviton and scalar propagators, respectively. The coefficients appearing in this figure are simple binomial coefficients coming from the various ways of choosing the lines (see Appendix A). The diagrams involving only gravitons give the $O(G^3)$ approximation of general relativity, which has been studied in the literature (see notably [14,17]). The diagrams involving at least one scalar propagator give therefore all the looked for deviations from general relativity predicted by tensor-scalar theories at this order. Their explicit calculation is performed in Appendix A below. Their

global structure is easy to grasp. Denoting by \mathcal{G} a Green function, solution of $\square_f \mathcal{G}(x) = -4\pi \delta^{(4)}(x)$ where \square_f is the flat d'Alembertian, each scalar propagator is $\mathcal{P}_\varphi^{ab} \propto \mathcal{G}(x_A - x_B) \gamma^{ab}$, while each graviton propagator is $\mathcal{P}_{\alpha\beta\gamma\delta}^h \propto \mathcal{G}(x_A - x_B) (f_{\alpha\gamma} f_{\beta\delta} + f_{\alpha\delta} f_{\beta\gamma} - f_{\alpha\beta} f_{\gamma\delta})$. Each matter vertex containing a single scalar coupling brings a factor α_a^A , those containing double scalar couplings bring a factor $\beta_{ab}^A + \alpha_a^A \alpha_b^A$, while triple scalar couplings bring a factor β'^{abc}_A together with extra β_{ab}^A and α_a^A terms [see Eqs. (A11)–(A14) in Appendix A below]. Each link implies a contraction over the internal indices.⁶ Finally, the global structure of the action is

$$\sum_A \sum_B \cdots \int d\tau_A \int d\tau_B \cdots \mathcal{G}(x_A - x_B) \mathcal{G} \cdots f[u_{A,B}^\mu, \dots; \alpha_{A,B}^a, \dots(\varphi_0), \beta_{A,B}^{ab}, \dots(\varphi_0), \beta'^{abc}_{A,B}, \dots(\varphi_0), R_{abcd}(\varphi_0)], \quad (3.16)$$

where τ_A is the Minkowski proper time along the world line $x_A^\mu(\tau_A)$, and $u_A^\mu \equiv dx_A^\mu / c d\tau_A$. We use in our calculations the symmetric Green function $\frac{1}{2}(\mathcal{G}_{\text{retarded}} + \mathcal{G}_{\text{advanced}})$, given by [34]

$$\mathcal{G}_{\text{sym}}(x_A - x_B) = \delta(f_{\mu\nu}(x_A^\mu - x_B^\mu)(x_A^\nu - x_B^\nu)) = [\delta(x_A^0 - x_B^0 - r_{AB}) + \delta(x_A^0 - x_B^0 + r_{AB})] / 2r_{AB}. \quad (3.17)$$

The power of our diagrammatic approach shows up in the fact that one can identify the theory-dependent parameters appearing in each term of the action without doing any calculation. For instance, the interaction between two bodies A and B is described by the two “ I ” diagrams of Fig. 7, and involves therefore (because of the first diagram) the contraction $(\alpha_A \alpha_B)_0$. This explains why the effective gravitational constant G_{AB} and the 1PK Eddington parameter $\bar{\gamma}_{AB}$ appearing in $L_{AB}^{(2)}$, Eq. (2.20b), depend only on this contraction. Similarly, the interaction between three bodies is described by the “ V ” and “ T ” diagrams of Fig. 7, and we see that it depends not only on contractions like $(\alpha_A \alpha_B)_0$ [cf. the second V diagram and the first T diagram], but also on $(\alpha_A \beta_B \alpha_C)_0$ [cf. the first V diagram, involving only scalar propagators]. Here again, we understand very simply why the three-body interaction Lagrangian $L_{BC}^{(3)A}$ of Eq. (2.20c) depends only on these two types of contractions, involved in G_{AB} and $\bar{\beta}_{BC}^A$.

The new parameters appearing at order G^3 can now be found by examining the diagrams connecting four bodies in Fig. 7. While the H and F diagrams depend only on the previous contractions of $\alpha_{A,B,\dots}^a$ and $\beta_{A,B,\dots}^{ab}$, three new contractions occur in the first ϵ , Z , and X diagrams. Indeed, they involve respectively the contractions $(\beta'^{abc} \alpha_B^a \alpha_C^b \alpha_D^c)_0$, $(\alpha^a \beta_{ab}^B \beta_C^{bc} \alpha_D^d)_0$, and $(R_{abcd} \alpha_A^a \alpha_B^b \alpha_C^c \alpha_D^d)_0$, which are independent from $(\alpha_A \alpha_B)_0$ and $(\alpha_A \beta_B \alpha_C)_0$ in generic tensor-multiscalar theories. It is convenient to introduce compact notations for these parameters; we define

$$\epsilon_{BCD}^A \equiv \frac{(\beta'^{abc} \alpha_B^a \alpha_C^b \alpha_D^c)_0}{[1 + (\alpha_A \alpha_B)_0][1 + (\alpha_A \alpha_C)_0][1 + (\alpha_A \alpha_D)_0]}, \quad (3.18a)$$

$$\zeta_{ABCD} \equiv \frac{(\alpha_A \beta_B \beta_C \alpha_D)_0}{[1 + (\alpha_A \alpha_B)_0][1 + (\alpha_B \alpha_C)_0][1 + (\alpha_C \alpha_D)_0]}, \quad (3.18b)$$

$$\chi_{ABCD} \equiv (R_{abcd} \alpha_A^a \alpha_B^b \alpha_C^c \alpha_D^d)_0, \quad (3.18c)$$

where the choice of the carrier letters ϵ , ζ , χ is made by pictorial analogy with the corresponding diagrams ϵ , Z , and X . The introduction of factors $[1 + (\alpha_A \alpha_B)_0]^{-1}$ is made to simplify some equations below, where we shall factorize the effective gravitational constant G_{AB} , Eq. (2.21a).

The main conclusion of our diagrammatic analysis of tensor-multiscalar theories is therefore that only *three* new self-gravity-dependent parameters appear at the G^3 level. By contrast, previous studies in the literature suggested the need for a much larger number of parameters in the general framework of Lorentz-invariant theories of gravity [21,22]. Let us note that if we had restricted ourselves to theories involving only one scalar field, only two self-gravity-dependent parameters ϵ_{BCD}^A and ζ_{ABCD} would have showed up, the Riemann tensor of the scalar manifold being then identically zero.⁷ We are going to see that the parameter χ_{ABCD} disappears anyway when considering the weak self-gravity limit.

⁶Those internal contractions make it very easy to work with n scalar fields. Actually, once one is used to the notation, it is easier to see which new 2PN parameters can occur in the multiscalar case rather than in the monoscalar one.

⁷In the one scalar case, we could also express formally ζ_{ABCD} in terms of 1PK parameters, using $(\alpha_A \beta_B \beta_C \alpha_D) = (\alpha_A \beta_B \alpha_C) (\alpha_B \beta_C \alpha_D) / (\alpha_B \alpha_C)$, but this expression is singular when α_B or $\alpha_C \rightarrow 0$.

B. 2PK approximation

The diagrammatic results of the previous subsection gave the structure of the nonlinearity expansion, in powers of Gm/rc^2 , of the N -body Lagrangian, keeping unexpanded all powers of v/c and of the self-gravity s . The second post-Keplerian limit of tensor-multiscalar theories can now be obtained by expanding the results above in powers of v^2/c^2 (still keeping s unexpanded). First of all, the Minkowski proper time τ_A involved in (3.16) is obviously expanded as

$$d\tau_A = dt_A \sqrt{1 - \mathbf{v}_A^2/c^2} = dt_A \left[1 - \frac{1}{2} \frac{v_A^2}{c^2} - \frac{1}{8} \left(\frac{v_A^2}{c^2} \right)^2 - \frac{1}{16} \left(\frac{v_A^2}{c^2} \right)^3 + \mathcal{O} \left(\left(\frac{v_A^2}{c^2} \right)^4 \right) \right]. \quad (3.19)$$

The $(v_A^2/c^2)^3$ term is necessary to write $S_0[\sigma]$ at the 2PK order, but the expansion (3.19) can be truncated to order $(v_A^2/c^2)^2$ for the two-body diagram “ I ,” and to order (v_A^2/c^2) for the three-body diagrams V and T . In the 2PK diagrams ϵ , Z , F , H , X , it is enough to replace the proper time τ_A by the coordinate time t_A .

Similarly, one must expand in powers of v/c the unit four-velocity $u_A^\mu = dx_A^\mu/cd\tau_A = (1, v_A^i/c)/\sqrt{1 - \mathbf{v}_A^2/c^2}$, and in particular the contraction $2(f_{\mu\nu} u_A^\mu u_B^\nu)^2 - 1$ which appears for each graviton propagator connecting two bodies (see Appendix A). One gets easily

$$2(u_A u_B)^2 - 1 = 1 + 2 \frac{(\mathbf{v}_A - \mathbf{v}_B)^2}{c^2} + 2 \frac{(\mathbf{v}_A - \mathbf{v}_B)^2 (\mathbf{v}_A^2 + \mathbf{v}_B^2) - (\mathbf{v}_A \times \mathbf{v}_B)^2}{c^4} + \mathcal{O} \left(\frac{1}{c^6} \right), \quad (3.20)$$

where $\mathbf{v}_A \times \mathbf{v}_B$ denotes the usual vector skew product. Finally, the Green function (3.17) can be expanded as

$$\begin{aligned} c\mathcal{G}_{\text{sym}}(x_A - x_B) &= [\delta(t_A - t_B - r_{AB}/c) + \delta(t_A - t_B + r_{AB}/c)]/2r_{AB} = \sum_{n=0}^{\infty} \frac{|\mathbf{x}_A - \mathbf{x}_B(t_B)|^{2n-1}}{(2n)!c^{2n}} \frac{\partial^{2n}}{(\partial t_B)^{2n}} \delta(t_A - t_B) \\ &= \frac{\delta(t_A - t_B)}{r_{AB}} + \frac{|\mathbf{x}_A - \mathbf{x}_B(t_B)|}{2c^2} \frac{\partial^2 \delta(t_A - t_B)}{\partial t_B^2} + \frac{|\mathbf{x}_A - \mathbf{x}_B(t_B)|^3}{24c^4} \frac{\partial^4 \delta(t_A - t_B)}{\partial t_B^4} + \mathcal{O} \left(\frac{1}{c^6} \right). \end{aligned} \quad (3.21)$$

The three terms in (3.21) are needed for the “ I ” diagram, while only the first two are needed for the V and T diagrams, and only the instantaneous Green function $\delta(t_A - t_B)/r_{AB}$ for the 2PK diagrams ϵ , Z , F , H , X . The time derivatives of δ functions in (3.21) imply that the 2PK Lagrangian depends not only on the positions \mathbf{x}_A and the velocities \mathbf{v}_A of the bodies, but also on their accelerations \mathbf{a}_A . (Higher derivatives can be eliminated, at 2PK order, by means of suitable integrations by parts.) As discussed in [17], this is a consequence of our choice of gauge for the Einstein-Hilbert action $S_{\text{spin } 2}$. Indeed, all the accelerations can be eliminated by choosing for instance the Arnowitt-Deser-Misner (ADM) gauge [10], instead of the harmonic one we use in Appendix A below to simplify our calculations. (Actually, the simplest practical way of going from the harmonic gauge to a higher-derivatives-free gauge is to “wrongly” eliminate the higher derivatives by using the equations of motion in the Lagrangian [17].)

The 2PK Lagrangian describing the motion of N self-gravitating bodies is hence obtained straightforwardly from the action (3.14). We refer the reader to Appendix A below for its explicit derivation, and quote here only its general structure. It can be written as a sum of i -body interaction terms ($1 \leq i \leq 4$)

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)} + \mathcal{O}(1/c^6), \quad (3.22)$$

where $L^{(1)} = \sum_A L_A^{(1)} = -\sum_A m_A^0 c^2 \sqrt{1 - \mathbf{v}_A^2/c^2}$ is the Lagrangian describing free bodies. Its 2PK expansion, generalizing

(2.20a) to order $m_A c^2 (v_A^2/c^2)^3$, is given by (3.19). The two-body interaction Lagrangian $L^{(2)}$, given by the “ I ” diagrams of Fig. 7, has the structure

$$\begin{aligned} L^{(2)} &= \frac{1}{2} \sum_{A \neq B} \left[\frac{G_{AB}}{G_*} L_{AB}^{(2)\text{GR}} + \frac{G_{AB} m_A^0 m_B^0 \bar{\gamma}_{AB}}{c^2} \left(\frac{(\mathbf{v}_A - \mathbf{v}_B)^2}{r_{AB}} \right. \right. \\ &\quad \left. \left. + \frac{f_1(\mathbf{r}_{AB}, \mathbf{v}_A, \mathbf{v}_B, \mathbf{a}_A, \mathbf{a}_B)}{c^2} \right) \right], \end{aligned} \quad (3.23)$$

where $L_{AB}^{(2)\text{GR}}$ is the expression obtained in general relativity (pure spin-2 interaction):

$$\begin{aligned} L_{AB}^{(2)\text{GR}} &= G_* m_A^0 m_B^0 \left[\frac{1}{r_{AB}} + \frac{f_2(\mathbf{r}_{AB}, \mathbf{v}_A, \mathbf{v}_B)}{c^2} \right. \\ &\quad \left. + \frac{f_3(\mathbf{r}_{AB}, \mathbf{v}_A, \mathbf{v}_B, \mathbf{a}_A, \mathbf{a}_B)}{c^4} \right], \end{aligned} \quad (3.24)$$

and where the expression of the function f_1 entering the $\bar{\gamma}_{AB}/c^4$ term in (3.23) is given in Eqs. (A17) and (A18).

In other words, the two-body interaction Lagrangian $L^{(2)}$ in tensor-multiscalar theories presents two main differences with respect to the general relativistic result: (i) the bare gravitational constant G_* is replaced by the effective one G_{AB} , Eq. (2.21a); (ii) a correcting term proportional to $\bar{\gamma}_{AB}/c^2$ must be added.

The three-body interaction Lagrangian, corresponding to the V and T diagrams of Fig. 7, has the structure

$$L^{(3)} = \frac{1}{2} \sum_{B \neq A \neq C} \frac{G_{AB} G_{AC} m_A^0 m_B^0 m_C^0}{c^2} \left[-\frac{(1+2\bar{\beta}_{BC}^A)}{r_{AB} r_{AC}} + \frac{f_4}{c^2} + \frac{\bar{\beta}_{BC}^A f_5}{c^2} \right] + \sum_{A,B,C} \frac{O(\bar{\gamma}_{AB}, \bar{\gamma}_{AC}, \bar{\gamma}_{BC})}{c^4}, \quad (3.25)$$

where the functions f_4 and f_5 depend only on the positions and the velocities of the three bodies. The last term in Eq.(3.25) denotes a sum of terms which are at least linear in the indicated $\bar{\gamma}_{AB}$, and which depend also on positions and velocities.

Finally, the four-body interaction Lagrangian, corresponding to the ϵ , Z , F , H , and X diagrams of Fig. 7, has the structure

$$\begin{aligned} L^{(4)} = & \sum_{A,B,C,D} L_{ABCD}^{(4)\text{RG}} + \frac{1}{6} \sum_{A \neq (B,C,D)} \frac{G_{AB} G_{AC} G_{AD} m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{AC} r_{AD} c^4} \epsilon_{BCD}^A + \frac{1}{2} \sum_{A \neq B \neq C \neq D} \frac{G_{AB} G_{BC} G_{CD} m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{BC} r_{CD} c^4} \zeta_{ABCD} \\ & + \frac{1}{24\pi} \sum_{A,B,C,D} \frac{G_*^3 m_A^0 m_B^0 m_C^0 m_D^0}{c^4} \chi_{ABCD} \int d^3 \mathbf{x} \frac{(\mathbf{x} - \mathbf{x}_A) \cdot (\mathbf{x} - \mathbf{x}_C)}{|\mathbf{x} - \mathbf{x}_A|^3 |\mathbf{x} - \mathbf{x}_B| |\mathbf{x} - \mathbf{x}_C|^3 |\mathbf{x} - \mathbf{x}_D|} + \sum_{A,B,C,D} \frac{O(\bar{\gamma}_{AB}, \bar{\gamma}_{AC}, \dots)}{c^4} \\ & + \sum_{A,B,C,D} \frac{O(\bar{\beta}_{BC}^A, \bar{\beta}_{AD}^B, \dots)}{c^4}, \end{aligned} \quad (3.26)$$

where $L_{ABCD}^{(4)\text{RG}} \propto G_*^3 m_A^0 m_B^0 m_C^0 m_D^0$ is the general relativistic result, and where the 2PK parameters ϵ_{BCD}^A , ζ_{ABCD} , and χ_{ABCD} have been defined in Eqs. (3.18) above. Note that the notation $A \neq B \neq C \neq D$ excludes only the equalities $A=B$, $B=C$, or $C=D$.

C. 2PN approximation

The expression of the second post-Newtonian approximation can now be obtained by expanding the results of the previous subsection in powers of the compactnesses $s \sim Gm/Rc^2$ of the bodies. To do this, we make use of results derived in [24]. Following section 8 of this reference, we define

$$c_A \equiv -2 \frac{\partial \ln \tilde{m}_A}{\partial \ln \tilde{G}} = O(s), \quad (3.27a)$$

$$c'_A \equiv \frac{\partial c_A}{\partial \ln \tilde{G}} = O(s), \quad (3.27b)$$

$$a_A \equiv -4 \frac{\partial \ln \tilde{m}_A}{\partial \tilde{\gamma}} = O(s^2), \quad (3.27c)$$

$$b_A \equiv \frac{\partial \ln \tilde{m}_A}{\partial \tilde{\beta}} = O(s^2). \quad (3.27d)$$

Using the results (3.4) of [24] and neglecting the rotational kinetic energy terms with respect to the pressure (as is appropriate for solar system bodies), we can write these compactness parameters explicitly at 2PN order:

$$c_A = \frac{1}{c^2} \langle \tilde{U} \rangle_A + \frac{2}{c^4} \left\langle 3 \tilde{\gamma} \frac{\tilde{p}}{\tilde{\sigma}} \tilde{U} - 2 \tilde{\beta} \tilde{U}^2 \right\rangle_A + O\left(\frac{1}{c^6}\right), \quad (3.28a)$$

$$c'_A \sim \frac{1}{c^2} \langle \tilde{U} \rangle_A + O\left(\frac{1}{c^4}\right), \quad (3.28b)$$

$$a_A = \frac{12}{c^4} \left\langle \frac{\tilde{p}}{\tilde{\sigma}} \tilde{U} \right\rangle_A + O\left(\frac{1}{c^6}\right), \quad (3.28c)$$

$$b_A = \frac{1}{c^4} \langle \tilde{U}^2 \rangle_A + O\left(\frac{1}{c^6}\right). \quad (3.28d)$$

Here $\tilde{\sigma} \equiv (\tilde{T}^{00} + \tilde{T}^{ii})/c^2$ denotes the mass density, \tilde{p} the pressure, \tilde{U} the potential satisfying $\tilde{\Delta} \tilde{U} = -4\pi \tilde{G} \tilde{\sigma}$, and the angular brackets denote an average weighted by $\tilde{\sigma}$: $\langle f \rangle \equiv \int \tilde{\sigma} f d^3 \tilde{\mathbf{x}} / \int \tilde{\sigma} d^3 \tilde{\mathbf{x}}$. The tilde decorating the various quantities means that they are measured in physical units and expressed in terms of physically rescaled local coordinates $\tilde{x}^\mu = A(\varphi(x_A)) x^\mu$ adapted to describing the neighborhood of body A . The mass density $\tilde{\sigma}$ can be expressed in terms of the coordinate-conserved baryonic-rest-mass density, say $\tilde{\mu}$, as $\tilde{\sigma} = \tilde{\mu} + (\tilde{\mu} \tilde{h} + 2\tilde{p} - \tilde{\mu} \tilde{U})/c^2 + O(1/c^4)$, where \tilde{h} denotes the enthalpy. Note that Eq. (3.28a) is more accurate than the usually employed 1PN expression for the compactness:⁸ $c_A = \langle \tilde{U} \rangle_A / c^2 + O(1/c^4) = -2 \tilde{E}_A^{\text{grav}} / \tilde{m}_A c^2 + O(1/c^4)$. A precise value of c'_A is not needed for the following as it will only appear through the combination $(4\tilde{\beta} - \tilde{\gamma})^2 c'_A$.

As shown in section 8 of [24], the body-dependent parameters (2.21) can then be expanded as

⁸Note also that it simplifies very much in the general relativistic case where the $1/c^4$ correction vanishes.

$$G_{AB}/G = \tilde{G}_{AB}/\tilde{G} = 1 - \frac{\eta}{2}(c_A + c_B) + \left(\zeta + 4\bar{\beta} - \frac{\bar{\gamma}}{2} \right) c_A c_B + \bar{\beta}(2 + \bar{\gamma})(a_A + a_B) + \left(\frac{\varepsilon}{2} + \zeta - 8\bar{\beta}^2 \right) (b_A + b_B) + O(s^3), \quad (3.29a)$$

$$\bar{\gamma}_{AB} = \bar{\gamma} + \eta \left(1 + \frac{\bar{\gamma}}{2} \right) (c_A + c_B) + O(s^2), \quad (3.29b)$$

$$\bar{\beta}_{BC}^A = \bar{\beta} - \left(\frac{\varepsilon}{2} + \frac{\zeta}{2} + \bar{\beta} - 8\bar{\beta}^2 + \bar{\beta}\bar{\gamma} \right) c_A - \left(\frac{\zeta}{2} + \bar{\beta} - \frac{\eta\bar{\beta}}{2} \right) (c_B + c_C) - \frac{\eta^2}{4} c'_A + O(s^2). \quad (3.29c)$$

Here we have set $G \equiv G_*(1 + \alpha_0^2)$ for the Einstein-frame effective gravitational constant, and, as usual, $\eta \equiv 4\bar{\beta} - \bar{\gamma}$. We have also introduced the notation

$$\varepsilon \equiv \frac{(\beta'_{abc} \alpha^a \alpha^b \alpha^c)_0}{(1 + \alpha_0^2)^3}, \quad (3.30a)$$

$$\zeta \equiv \frac{(\alpha_a \beta_b^a \beta_c^b \alpha^c)_0}{(1 + \alpha_0^2)^3}. \quad (3.30b)$$

These parameters are the weak-self-gravity limits of the parameters (3.18a) and (3.18b) corresponding to the diagrams ε and Z :

$$\varepsilon_{BCD}^A = \varepsilon + O(s), \quad \zeta_{ABCD} = \zeta + O(s). \quad (3.31)$$

The parameter χ_{ABCD} , Eq. (3.18c), vanishes in the weak-self-gravity limit because of the antisymmetry of the Riemann tensor $R_{abcd}(\varphi) = -R_{bacd}(\varphi)$:

$$\chi_{ABCD} = (R_{abcd} \alpha^a \alpha^b \alpha^c \alpha^d)_0 + O(s) = 0 + O(s^2). \quad (3.32)$$

Indeed, for the same reason, the first correction of order $O(s)$ is easily seen to vanish too.

Our final conclusion is therefore that the 2PN limit of tensor-multiscalar theories involves only the *two* new parameters⁹ ε and ζ , Eqs. (3.30), besides the usual 1PN Edington parameters $\bar{\beta}$ and $\bar{\gamma}$. These parameters consistently enter into several 2PN effects. First, they parametrize two new independent contributions to the four-body interaction Lagrangian:

$$L_{2PN}^{(4)}(\varepsilon, \zeta) = \frac{\varepsilon}{6} \sum_{A \neq (B,C,D)} \frac{G^3 m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{AC} r_{AD} c^4} + \frac{\zeta}{2} \sum_{A \neq B \neq C \neq D} \frac{G^3 m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{BC} r_{CD} c^4}. \quad (3.33)$$

Second, they parametrize the dependence upon self-gravity effects of the effective gravitational constant and of the 1PK parameters $\bar{\gamma}_{AB}$ and $\bar{\beta}_{BC}^A$. Discarding from Eqs. (3.29) the 2PN corrections proportional to the already experimentally constrained 1PN parameters $\bar{\beta}$ and $\bar{\gamma}$, we can rewrite them in the simpler form

$$G_{AB}/G = 1 + \eta \left(\frac{E_A^{\text{grav}}}{m_A c^2} + \frac{E_B^{\text{grav}}}{m_B c^2} \right) + 4\zeta \left(\frac{E_A^{\text{grav}}}{m_A c^2} \right) \left(\frac{E_B^{\text{grav}}}{m_B c^2} \right) + \left(\frac{\varepsilon}{2} + \zeta \right) \frac{\langle U^2 \rangle_A + \langle U^2 \rangle_B}{c^4} + O(\bar{\beta}, \bar{\gamma}) O(s^2) + O(s^3), \quad (3.34a)$$

$$\bar{\gamma}_{AB} = \bar{\gamma} + O(\bar{\beta}, \bar{\gamma}) O(s) + O(s^2), \quad (3.34b)$$

$$\bar{\beta}_{BC}^A = \bar{\beta} + (\varepsilon + \zeta) \frac{E_A^{\text{grav}}}{m_A c^2} + \zeta \left(\frac{E_B^{\text{grav}}}{m_B c^2} + \frac{E_C^{\text{grav}}}{m_C c^2} \right) + O(\bar{\beta}, \bar{\gamma}) O(s) + O(s^2). \quad (3.34c)$$

Equation (3.34a) shows that ε and ζ are two independent 2PN generalizations of the well-known Nordtvedt 1PN modification of the gravitational coupling [second term of the right-hand side of (3.34a), $\eta(E_A^{\text{grav}}/m_A c^2 + E_B^{\text{grav}}/m_B c^2)$]. In Eqs. (3.34), we have simplified the writing by dropping all tildes in dimensionless ratios such as $E_A^{\text{grav}}/m_A c^2$ or $\langle U^2 \rangle_A/c^4$. We shall continue doing so in the following each time this does not lead to any ambiguity.

In fact the two roles (3.33), (3.34) of ε and ζ are deeply connected. Indeed, the Lagrangian (3.33) represents the interaction between an arbitrary number of non-self-gravitating mass points. From it we can formally reconstruct the Lagrangian representing the interaction between N weakly-self-gravitating bodies, $\mathcal{A}, \mathcal{B}, \dots$ by considering each body as a collection of mass

⁹Note, as in footnote 7 above, that in the one scalar case ζ can be formally expressed in terms of 1PN parameters: $\zeta = -8\bar{\beta}^2/\bar{\gamma}$. However, this expression is singular when $\bar{\gamma} \rightarrow 0$.

points m_A, m_B, \dots (held together by a slow orbital motion within the volume \mathcal{A}). Let us denote by $\sigma(\mathbf{x})$ the average mass density within \mathcal{A} , such that $\int_{\mathcal{A}} d^3\mathbf{x}\sigma(\mathbf{x}) = m_A^0 = \sum_{A \in \mathcal{A}} m_A^0$ gives the total Einstein mass of body \mathcal{A} . Then, if A and B label two point masses belonging to the same body \mathcal{A} , one can rewrite the sum $\sum_{A \neq B} Gm_A^0 m_B^0 / r_{AB}$ as an integral

$$\int_{\mathcal{A}} \frac{G\sigma(\mathbf{x}_1)\sigma(\mathbf{x}_2)d^3\mathbf{x}_1d^3\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} = m_{\mathcal{A}}^0 \langle U \rangle_{\mathcal{A}} + O\left(\frac{1}{c^2}\right), \quad (3.35)$$

where $U(\mathbf{x}) \equiv \int d^3\mathbf{x}' G\sigma(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$ and where the angular brackets denote an average weighted by σ . Similarly, if A, B , and C belong to the same body \mathcal{A} , the sum $\sum_{A \neq B \neq C} G^2 m_A^0 m_B^0 m_C^0 / r_{AB} r_{BC}$ can be rewritten as an integral:

$$\int_{\mathcal{A}} \frac{G^2 \sigma(\mathbf{x}_1)\sigma(\mathbf{x}_2)\sigma(\mathbf{x}_3)d^3\mathbf{x}_1d^3\mathbf{x}_2d^3\mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_2||\mathbf{x}_2 - \mathbf{x}_3|} = m_{\mathcal{A}}^0 \langle U^2 \rangle_{\mathcal{A}} + O\left(\frac{1}{c^2}\right). \quad (3.36)$$

Using such results, the interaction Lagrangian (3.33) leads to

$$\begin{aligned} L(\varepsilon, \zeta) = & \sum_{A \neq (B,C,D)} \frac{G^3 m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{AC} r_{AD} c^4} \frac{\varepsilon}{6} + \sum_{A \neq B \neq C \neq D} \frac{G^3 m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB} r_{BC} r_{CD} c^4} \frac{\zeta}{2} + \sum_{B \neq A \neq C} \frac{G^2 m_A^0 m_B^0 m_C^0}{r_{AB} r_{AC} c^4} \left[\left(3 \times \frac{\varepsilon}{6} + \frac{\zeta}{2} \right) \langle U \rangle_{\mathcal{A}} + \frac{\zeta}{2} \langle U \rangle_B \right. \\ & \left. + \langle U \rangle_C \right] + \sum_{A \neq B} \frac{G m_A^0 m_B^0}{r_{AB} c^4} \left[\frac{\zeta}{2} \langle U \rangle_A \langle U \rangle_B + \left(\frac{3}{2} \times \frac{\varepsilon}{6} + \frac{\zeta}{2} \right) \langle U^2 \rangle_{\mathcal{A}} + \langle U^2 \rangle_B \right], \end{aligned} \quad (3.37)$$

where we have taken into account the different ways to choose infinitesimal elements in the same body to compute the correct multiplicities. [In pictorial language, if we denote the first two terms by $(\varepsilon_{BCD}/3)$ and $(Z_{ABCD}/2)$, respectively,¹⁰ the following ones correspond to $3 \times (\varepsilon_{ABC}/3) + (Z_{BAAc}/2)$, $(Z_{BBAC}/2 + Z_{BACC}/2)$, $(Z_{AAAB}/2)$, $\frac{3}{2} \times (\varepsilon_{AAB}/3 + \varepsilon_{BBAA}/3) + (Z_{AAAB}/2 + Z_{ABBB}/2)$, where a repeated index means infinitesimal volume elements inside the same body.]

The results (3.34) can be directly read off the Lagrangian (3.37). Indeed, the last sum in (3.37) leads to the ε and ζ renormalizations of the effective gravitational constant (3.34a), while the three-body sum $\sum_{B \neq A \neq C}$ yields the renormalizations (3.34c) of the body-dependent Eddington parameter $\bar{\beta}_{BC}^A$.

IV. SECOND POST-NEWTONIAN ORDER METRIC

A. N -body physical metric

As is well known, the N -body Lagrangian contains enough information to derive the physical metric $\tilde{g}_{\mu\nu}$ at any spacetime point x^λ outside the bodies generating $\tilde{g}_{\mu\nu}$. Deriving this metric is important because it allows one to compute 2PN effects on, for instance, clock comparison, light deflection or time-delay experiments. To compute $\tilde{g}_{\mu\nu}(x^\lambda)$,

one introduces a test particle of negligible mass¹¹ \tilde{m}_0 located at $x^\lambda = (x^0, x^i)$, with an arbitrary four-velocity $v^\mu = (c, v^i)$. We can now write the $(N+1)$ -body Lagrangian describing the N massive bodies and this test particle, and identify its \tilde{m}_0 -dependent part with the individual Lagrangian of the latter: namely,

$$\begin{aligned} L^{\text{test particle}}(x^\lambda, v^i) &= -\tilde{m}_0 c \sqrt{-\tilde{g}_{\mu\nu}(x^\lambda) v^\mu v^\nu} \\ &= -\tilde{m}_0 c^2 \left[-\tilde{g}_{00}(x^\lambda) - 2\tilde{g}_{0i}(x^\lambda) v^i/c \right. \\ & \quad \left. - \tilde{g}_{ij}(x^\lambda) v^i v^j / c^2 \right]^{1/2}. \end{aligned} \quad (4.1)$$

This identification therefore allows us to compute $\tilde{g}_{\mu\nu}(x^\lambda)$ up to order $1/c^6$ included for \tilde{g}_{00} [i.e., up to $O((Gm/rc^2)^3)$ included], $1/c^5$ for \tilde{g}_{0i} , and $1/c^4$ for \tilde{g}_{ij} .

If we are concerned only by the dependence upon the new parameters ε and ζ appearing at the 2PN order, Eq. (3.37), we can consider a particle at rest, since the ε and ζ terms are not velocity dependent. We therefore conclude immediately that \tilde{g}_{0i} and \tilde{g}_{ij} do not involve the parameters ε and ζ at the 2PN approximation (i.e., at order $1/c^5$ for \tilde{g}_{0i} and $1/c^4$ for \tilde{g}_{ij}), and that $\tilde{g}_{00} = -1 + \delta\tilde{g}_{00}$ can be deduced from the Lagrangian

¹¹The index 0 of this test mass should not be confused with the time component x^0 . It has been chosen so that the body-dependent quantities $\alpha_A^a, \beta_A^{ab}, \dots$ of Eqs. (2.17) reduce to their background values $\alpha_0^a \equiv \alpha^a(\varphi_0)$, $\beta_0^{ab} \equiv \beta^{ab}(\varphi_0)$, \dots for the test mass \tilde{m}_0 .

¹⁰The normalization of the ε term is chosen for consistency with Appendix A.

$$\begin{aligned}
L^{\text{test particle}}(x^\lambda) &= -\tilde{m}_0 c^2 \sqrt{1 - \delta\tilde{g}_{00}(x^\lambda)} \\
&= -\tilde{m}_0 c^2 + \frac{1}{2} \tilde{m}_0 c^2 \delta\tilde{g}_{00} + O((\delta\tilde{g}_{00})^2).
\end{aligned} \tag{4.2}$$

Comparing (4.2) with (3.37) yields

$$\begin{aligned}
\delta_{2\text{PN}}\tilde{g}_{00}(\varepsilon, \zeta) &= \frac{\varepsilon}{3} \sum_{A,B,C} \frac{G^3 m_A m_B m_C}{r_{0A} r_{0B} r_{0C} c^6} + \varepsilon \sum_{A \neq (B,C)} \frac{G^3 m_A m_B m_C}{r_{0A} r_{AB} r_{AC} c^6} + 2\varepsilon \sum_{A \neq B} \frac{G^2 m_A m_B}{r_{0A} r_{AB} c^6} \langle U \rangle_A + 2\zeta \sum_A \sum_{B \neq C} \frac{G^3 m_A m_B m_C}{r_{0A} r_{0B} r_{BC} c^6} \\
&+ 2\zeta \sum_{A \neq B \neq C} \frac{G^3 m_A m_B m_C}{r_{0A} r_{AB} r_{BC} c^6} + \zeta \sum_{A,B} \frac{G^2 m_A m_B}{r_{0A} r_{0B} c^6} (\langle U \rangle_A + \langle U \rangle_B) + 2\zeta \sum_{A \neq B} \frac{G^2 m_A m_B}{r_{0A} r_{AB} c^6} (\langle U \rangle_A + \langle U \rangle_B) \\
&+ (\varepsilon + 2\zeta) \sum_A \frac{G m_A}{r_{0A} c^6} \langle U^2 \rangle_A,
\end{aligned} \tag{4.3}$$

where $r_{0A} \equiv |\mathbf{x} - \mathbf{x}_A|$. [All the masses entering this equation are evaluated at $\varphi = \varphi_0$, although we have dropped their index 0 for easier reading.] When a continuous description of the source bodies is used, this takes the simpler form

$$\begin{aligned}
\delta_{2\text{PN}}\tilde{g}_{00}(\varepsilon, \zeta) &= \frac{\varepsilon}{3c^6} U^3(\mathbf{x}) + \frac{\varepsilon}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}') U^2(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\
&+ \frac{2\zeta}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \int d^3\mathbf{x}'' \frac{G\sigma(\mathbf{x}'') U(\mathbf{x}'')}{|\mathbf{x}' - \mathbf{x}''|} \\
&+ \frac{2\zeta}{c^6} U(\mathbf{x}) \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}') U(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.
\end{aligned} \tag{4.4}$$

Using a self-explanatory notation, the four terms of (4.4) correspond to the diagrams ϵ_{AAA}^0 , ϵ_{AA0}^A , Z_{AAA0} , and Z_{AA0A} , respectively, the index A meaning here any infinitesimal volume of matter, and the index 0 referring to the spacetime point x^λ where the metric is computed. It should be noted that formulas (4.3) and (4.4) are valid in the g^* -harmonic gauge used to derive the N -body Lagrangian in Sec. III A. However, it is easy to see that the corrections proportional to ε and ζ are the same in all usual coordinate systems (ADM gauge, \tilde{g} -harmonic, ...), since these parameters appear for the first time at order $1/c^6$ in the time-time component of the metric.¹² In particular, the fact that \tilde{g}_{0i} and \tilde{g}_{ij} do not involve ε and ζ at the 2PN order is valid in any of these coordinate systems.

Let us mention for completeness that the presence of ε and ζ modifies the total energy of a gravitating system by the amount

$$\begin{aligned}
\delta_{2\text{PN}}\tilde{E}(\varepsilon, \zeta) &= -\frac{\varepsilon}{6c^4} \int d^3\tilde{\mathbf{x}} \tilde{\sigma}(\tilde{\mathbf{x}}) \tilde{U}^3(\tilde{\mathbf{x}}) - \frac{\zeta}{2c^4} \int \int d^3\tilde{\mathbf{x}} d^3\tilde{\mathbf{x}}' \\
&\times \frac{\tilde{\sigma}(\tilde{\mathbf{x}}) \tilde{U}(\tilde{\mathbf{x}}) \tilde{\sigma}(\tilde{\mathbf{x}}') \tilde{U}(\tilde{\mathbf{x}}')}{|\tilde{\mathbf{x}} - \tilde{\mathbf{x}}'|},
\end{aligned} \tag{4.5}$$

where the tilde decorating the various quantities means as before that they are measured in the physical units (2.6).

B. Exact one-body metric

In the present subsection, we verify and complement the above results by using the exact static and spherically symmetric solution of the field equations (2.8a) and (2.8b) derived in section 2 of [24]. In the coordinate system introduced by Just [38,39],

$$ds_*^2 = -e^\nu c^2 dt^2 + e^{-\nu} [dr^2 + e^\lambda (d\theta^2 + \sin^2\theta d\phi^2)], \tag{4.6}$$

we found the solution

$$e^\lambda = r^2 - ar, \tag{4.7a}$$

$$e^\nu = (1 - a/r)^{b/a}, \tag{4.7b}$$

where

$$b \equiv \frac{2G_* m_A}{c^2} = \frac{2G_*}{c^4} \int \sqrt{g_*} d^3\mathbf{x} (-T_{*0}^0 + T_{*i}^i) \tag{4.8}$$

is the Einstein-frame mass parameter, and a is a constant of integration. Introducing the parameter $p \equiv (1/a) \ln(1 - a/r)$, the equations controlling the radial variation of the scalar fields can be expressed by saying that $\varphi^a(p)$ follows a geodesic of the metric $\gamma_{ab}(\varphi^c)$, i.e., $\delta \int \gamma_{ab}(\varphi^c(p)) (d\varphi^a/dp)(d\varphi^b/dp) dp = 0$. In particular, p being an affine parameter, the norm of $d\varphi^a/dp$ is constant and given by

¹²Of course, it is always possible to introduce by hand a spurious dependence on these parameters, by redefining for instance the spatial variables as $x'^i = x^i + O(\varepsilon, \zeta)/c^4$, whereas any usual coordinate transformation involves ε and ζ at order $1/c^6$.

$$4\gamma_{ab}(\varphi)\frac{d\varphi^a}{dp}\frac{d\varphi^b}{dp}=a^2-b^2=\text{const.} \quad (4.9)$$

The actual geodesic traced out by $\varphi^a(p)$ in the internal scalar space is uniquely determined by the initial values at $p=0$ (i.e., at spatial infinity): $\varphi^a(p=0)=\varphi^a(r=\infty)=\varphi_0^a$ and $(d\varphi^a/dp)(p=0)=k^a$, say. Assuming φ_0^a and k^a given, we can choose field variables in the scalar space so that the metric $\gamma_{ab}(\varphi)$ reduces to the constant δ_{ab} all along the line $\varphi(r)_{r\in[0,\infty]}$ (Fermi coordinate system in the φ space). This allows us to write the solution of (4.9) in the simple form

$$\varphi^a=\varphi_0^a+pk^a=\varphi_0^a+\frac{k^a}{a}\ln\left(1-\frac{a}{r}\right). \quad (4.10)$$

Using then the results of [24], we can relate the n integration constants k^a to the matter distribution:

$$k^a=\frac{G_*}{c^4}\int\sqrt{g_*}d^3\mathbf{x}\alpha^a(\varphi)(-T_{*0}^0-T_{*i}^i). \quad (4.11)$$

Note that (4.11) is valid only if the φ^a coordinates are ‘‘Fermi’’ all along the solution $\varphi^a(r)$. When using generic scalar variables, the quantity $\alpha^a(\varphi)$ appearing on the right-hand side of (4.11) must be replaced by the parallel-transported value of the vector $\alpha^a(\varphi)$ up to the point φ_0^a . To simplify, the reader can think that we work from the beginning with a *flat* σ -model metric $\gamma_{ab}(\varphi)=\delta_{ab}$. The only 2PN contribution that we would forget in that case would derive from the first X diagram of Fig. 7, and would involve the contraction $R_{abcd}(\varphi)\alpha_A^a\alpha_A^b\alpha_A^c\alpha_0^d$ which vanishes identically because of the antisymmetry of the Riemann curvature tensor $R_{abcd}(\varphi)=-R_{bacd}(\varphi)$.

By comparing the behavior at infinity of the solution (4.10),

$$\varphi^a=\varphi_0^a-\frac{k^a}{r}+O\left(\frac{1}{r^2}\right), \quad (4.12)$$

with the one obtained for an isolated pointlike body A [see Eq. (2.18)],

$$\varphi^a=\varphi_0^a-\frac{G_*\alpha_A^a m_A}{rc^2}+O\left(\frac{1}{r^2}\right)=\varphi_0^a-\alpha_A^a\frac{b}{2r}+O\left(\frac{1}{r^2}\right), \quad (4.13)$$

we can deduce that $k^a=G_*m_A\alpha_A^a/c^2$ so that, using (4.8) and (4.11), the actual expression of the parameter α_A^a for extended bodies reads

$$\alpha_A^a=\frac{2k^a}{b}=\frac{\int\sqrt{g_*}d^3\mathbf{x}\alpha^a(\varphi)(-T_{*0}^0-T_{*i}^i)}{\int\sqrt{g_*}d^3\mathbf{x}(-T_{*0}^0+T_{*i}^i)}. \quad (4.14)$$

Note the change of sign of the pressure term T_{*i}^i between the numerator and the denominator, which makes α_A^a differ from α_0^a at order $1/c^2$ even if the function $\alpha^a(\varphi)$ is a constant, as in the Jordan-Fierz-Brans-Dicke theory. In the latter case, one finds $\alpha_A^a=\alpha_0^a\times(1-2\int\sqrt{g_*}d^3\mathbf{x}T_{*i}^i/m_Ac^2)$. When comparing this result with $\alpha_A^a=\alpha_0^a+(\partial\tilde{m}_A/\partial\ln\tilde{G})(\partial\ln\tilde{G}/\partial\varphi^a)=\alpha_0^a\times(1-c_A)$, we get the following expression for the compactness in Jordan-Fierz-Brans-Dicke theory:

$$c_A=2\int\sqrt{g_*}d^3\mathbf{x}T_{*i}^i/m_Ac^2. \quad (4.15)$$

This result is an (exact) generalization of the Newtonian virial theorem $\int d^3\mathbf{x}\sigma U=6\int d^3\mathbf{x}p+O(1/c^2)$ which is valid in the Jordan-Fierz-Brans-Dicke theory and therefore also in general relativity (in the limit $\alpha^a\rightarrow 0$).

The exact solution (4.7)–(4.10) can now be expanded *à la* Eddington in powers of $1/r$. Although the coordinate system (4.6) is as good as any other to verify the 2PN terms involving the new parameters ε and ζ , it is convenient to express our results in isotropic coordinates, to make better contact with the general relativistic result. Let us then define a new radial coordinate ρ such that $\rho(1+a/4\rho)^2=r$. The Einstein line element (4.6) now reads [39]

$$\begin{aligned} ds_*^2 &= -\left(1-\frac{a}{4\rho}\right)^{2b/a}\left(1+\frac{a}{4\rho}\right)^{-2b/a}c^2dt^2+\left(1-\frac{a}{4\rho}\right)^{2-2b/a}\left(1+\frac{a}{4\rho}\right)^{2+2b/a}[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\phi^2)] \\ &= -c^2dt^2\left[1-\frac{b}{\rho}+\frac{b^2}{2\rho^2}-\frac{b^3}{6\rho^3}\left(1+\frac{1+\alpha_A^2}{8}\right)+O\left(\frac{1}{\rho^4}\right)\right] \\ &\quad +[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\phi^2)]\left[1+\frac{b}{\rho}+\frac{b^2}{2\rho^2}\left(1-\frac{1+\alpha_A^2}{4}\right)+O\left(\frac{1}{\rho^3}\right)\right], \end{aligned} \quad (4.16)$$

where we have used Eqs. (4.9) and (4.14) to replace the constant a^2 in terms of b^2 and $\alpha_A^2\equiv(\alpha_A^a\gamma_{ab}\alpha_A^b)_{\varphi=\varphi_0}$. The metric which determines the dynamics of test particles in the vicinity of A is not the Einstein metric $g_{\mu\nu}^*$, but the physical one $\tilde{g}_{\mu\nu}=A^2(\varphi)g_{\mu\nu}^*$. We must therefore expand also $A^2(\varphi)$ in powers of $1/\rho$, by using the exact solution (4.10) of the scalar fields. We get

$$\begin{aligned} \frac{A^2(\varphi)}{A^2(\varphi_0)} &= 1 - (\alpha_A \alpha_0) \frac{b}{\rho} + \left[(\alpha_A \alpha_0)^2 + \frac{(\alpha_A \beta_0 \alpha_A)}{2} \right] \frac{b^2}{2\rho^2} \\ &\quad - \left[(\beta'_{AAA})^0 + 6(\alpha_A \alpha_0)(\alpha_A \beta_0 \alpha_A) + 4(\alpha_A \alpha_0)^3 + \frac{(\alpha_A \alpha_0)(1 + \alpha_A^2)}{2} \right] \frac{b^3}{24\rho^3} + O\left(\frac{1}{\rho^4}\right), \end{aligned} \quad (4.17)$$

where we have set $(\alpha_A \alpha_0) \equiv (\alpha_A^a \gamma_{ab} \alpha^b)_{\varphi=\varphi_0}$, $(\alpha_A \beta_0 \alpha_A) \equiv (\alpha_A^a \beta_{ab} \alpha_A^b)_{\varphi=\varphi_0}$, $\beta'_{AAA} \equiv (\beta'_{abc} \alpha_A^a \alpha_A^b \alpha_A^c)_{\varphi=\varphi_0}$, and where the σ -model tensors α^a , β_{ab} , and β'_{abc} have been defined in Eqs. (2.10)–(2.12) above. Finally, when using the physical coordinates $\tilde{\rho} \equiv A(\varphi_0)\rho$ and $\tilde{t} \equiv A(\varphi_0)t$ [see Eq. (2.6)], we find that the physical line element $d\tilde{s}^2 = A^2(\varphi)ds_{*}^2$ reads

$$\begin{aligned} d\tilde{s}^2 &= -c^2 d\tilde{t}^2 \left[1 - 2\frac{\tilde{\mu}_A}{\tilde{\rho}c^2} + 2\left(\frac{\tilde{\mu}_A}{\tilde{\rho}c^2}\right)^2 (1 + \bar{\beta}_{AA}^0) - 3\left(\frac{\tilde{\mu}_A}{\tilde{\rho}c^2}\right)^3 (1 + \bar{\mathbf{B}}) + O\left(\frac{1}{\tilde{\rho}^4}\right) \right] \\ &\quad + [d\tilde{\rho}^2 + \tilde{\rho}^2(d\theta^2 + \sin^2\theta d\phi^2)] \left[1 + 2\frac{\tilde{\mu}_A}{\tilde{\rho}c^2}(1 + \bar{\gamma}_{A0}) + \frac{3}{2}\left(\frac{\tilde{\mu}_A}{\tilde{\rho}c^2}\right)^2 (1 + \bar{\Gamma}) + O\left(\frac{1}{\tilde{\rho}^3}\right) \right], \end{aligned} \quad (4.18)$$

where

$$\bar{\mathbf{B}} \equiv \frac{2}{9} \varepsilon_{AAA}^0 + \frac{8}{3} \bar{\beta}_{AA}^0 + (4 + \bar{\gamma}_{A0}) \frac{\bar{\gamma}_{A0}}{36} - \frac{(1 + \bar{\gamma}_{A0}/2)^2}{(1 + \bar{\gamma}_{AA}/2)} \frac{\bar{\gamma}_{AA}}{18}, \quad (4.19a)$$

$$\bar{\Gamma} \equiv \frac{4}{3} \bar{\beta}_{AA}^0 + (28 + 15\bar{\gamma}_{A0}) \frac{\bar{\gamma}_{A0}}{12} + \frac{(1 + \bar{\gamma}_{A0}/2)^2}{(1 + \bar{\gamma}_{AA}/2)} \frac{\bar{\gamma}_{AA}}{6}, \quad (4.19b)$$

and where we have set

$$\tilde{\mu}_A \equiv \mu_A / A(\varphi_0) \equiv \tilde{G}_{A0} \tilde{m}_A = G_* A^2(\varphi_0) [1 + (\alpha_A \alpha_0)] m_A, \quad (4.20a)$$

$$\bar{\gamma}_{A0} \equiv -2 \frac{(\alpha_A \alpha_0)}{1 + (\alpha_A \alpha_0)}, \quad \bar{\gamma}_{AA} \equiv -2 \frac{\alpha_A^2}{1 + \alpha_A^2}, \quad (4.20b)$$

$$\bar{\beta}_{AA}^0 \equiv \frac{1}{2} (\alpha_A \beta_0 \alpha_A) / [1 + (\alpha_A \alpha_0)]^2, \quad (4.20c)$$

$$\varepsilon_{AAA}^0 \equiv (\beta'_{abc} \alpha_A^a \alpha_A^b \alpha_A^c) / [1 + (\alpha_A \alpha_0)]^3. \quad (4.20d)$$

As a check of the consistency of our method, it is useful to compare the 2PK Lagrangian of Sec. III B with the test-particle Lagrangian (4.1) written for the metric (4.18). Let us first rewrite this metric in g^* -harmonic coordinates. We find

$$-\tilde{g}_{00} = 1 - 2\frac{\mu_A}{rc^2} + 2\left(\frac{\mu_A}{rc^2}\right)^2 (1 + \bar{\beta}_{AA}^0) - 2\left(\frac{\mu_A}{rc^2}\right)^3 \left(1 + \frac{3\bar{\mathbf{B}}}{4} + \frac{\bar{\gamma}_{A0}(4 + \bar{\gamma}_{A0})}{8(2 + \bar{\gamma}_{AA})} - \frac{\bar{\gamma}_{AA}}{4(2 + \bar{\gamma}_{AA})}\right) + O\left(\frac{1}{c^8}\right), \quad (4.21a)$$

$$\tilde{g}_{ij} = \delta_{ij} \left[1 + 2\frac{\mu_A}{rc^2} (1 + \bar{\gamma}_{A0}) + \left(\frac{\mu_A}{rc^2}\right)^2 \left(1 + 2\bar{\beta}_{AA}^0 + 3\bar{\gamma}_{A0} + \frac{7}{4}(\bar{\gamma}_{A0})^2 + \frac{\bar{\gamma}_{AA}(2 + \bar{\gamma}_{A0})^2}{4(2 + \bar{\gamma}_{AA})}\right) \right] + \frac{x^i x^j}{r^2} \left(\frac{\mu_A}{rc^2}\right)^2 \frac{(2 + \bar{\gamma}_{A0})^2}{2(2 + \bar{\gamma}_{AA})} + O\left(\frac{1}{c^6}\right), \quad (4.21b)$$

where $r = \sqrt{\delta_{ij} x^i x^j}$ denotes the g^* -harmonic radius, related to the isotropic radius ρ by

$$r = \rho \left[1 + \left(\frac{\mu_A}{2\rho c^2}\right)^2 \frac{(1 + \bar{\gamma}_{A0}/2)^2}{1 + \bar{\gamma}_{AA}/2} + O\left(\frac{1}{c^6}\right) \right]. \quad (4.22)$$

[This g^* -harmonic radius r should not be confused with the Just radius used in Eqs. (4.6)–(4.13) above, also denoted by r .] The test-particle Lagrangian deduced from the metric (4.21) agrees with the Lagrangian (3.22)–(3.26), written in the particular case of two bodies m_0 , m_A . It is notably easy to check the $\frac{1}{6} \varepsilon_{AAA}^0$ contribution in (3.26), which comes from the $\frac{3}{4} \bar{\mathbf{B}}$ term of (4.21a). [The ζ_{A0A0} and χ_{A0A0} contributions in (3.26) cannot be checked, because they involve the square of the test mass m_0 .] To ease the reading, we have used a slightly inconsistent notation in expressing the metric coefficient $\tilde{g}_{\mu\nu}$

(corresponding to the coordinate system \tilde{x}) in terms of the original (Einstein-frame) coordinates x . We use here, as we did above, the simplifying fact that dimensionless ratios such as μ/rc^2 are numerically equal to their physical-units counterparts $\tilde{\mu}/\tilde{r}c^2$.

The 2PN (weak-self-gravity) limit of the one-body metric (4.18) [or (4.21)] can be obtained by expanding the body-dependent parameters in powers of the compactness of body A , as in Eqs. (3.29) above:

$$\mu_A = Gm_A \left[1 - \frac{\eta}{2} c_A + \bar{\beta}(2 + \bar{\gamma}) a_A + \left(\frac{\varepsilon}{2} + \zeta - 8\bar{\beta}^2 \right) b_A \right] + O(s^3), \quad (4.23a)$$

$$\bar{\beta}_{AA}^0 = \bar{\beta} - (\zeta + 2\bar{\beta} - \eta\bar{\beta}) c_A + O(s^2) \Leftrightarrow \mu_A^2 (1 + \bar{\beta}_{AA}^0) = (Gm_A)^2 [1 + \bar{\beta} - (\zeta + 6\bar{\beta} - \bar{\gamma}) c_A + O(s^2)], \quad (4.23b)$$

$$\bar{B} = \frac{2}{9} \varepsilon + \frac{8}{3} \bar{\beta} + \frac{\bar{\gamma}}{18} + O(s), \quad (4.23c)$$

$$\bar{\gamma}_{A0} = \bar{\gamma} + \eta \left(1 + \frac{\bar{\gamma}}{2} \right) c_A + O(s^2) \Leftrightarrow \mu_A (1 + \bar{\gamma}_{A0}) = Gm_A \left(1 + \bar{\gamma} + \frac{\eta}{2} c_A + O(s^2) \right), \quad (4.23d)$$

$$\bar{\Gamma} = \frac{4}{3} \bar{\beta} + \frac{15 + 8\bar{\gamma}}{6} \bar{\gamma} + O(s). \quad (4.23e)$$

We thus recover that the spatial metric \tilde{g}_{ij} does not depend on the parameters ε and ζ at the 2PN order, as shown by Eqs. (4.23d) and (4.23e). On the other hand, the ε and ζ contributions to \tilde{g}_{00} can easily be deduced from the above results,

$$\delta_{2\text{PN}} \tilde{g}_{00}(\varepsilon, \zeta) = \frac{(\varepsilon + 2\zeta)}{c^6} \frac{Gm_A}{\rho} \langle U^2 \rangle_A + \frac{2\zeta}{c^6} \left(\frac{Gm_A}{\rho} \right)^2 \langle U \rangle_A + \frac{\varepsilon}{3c^6} \left(\frac{Gm_A}{\rho} \right)^3 + O(\bar{\beta}, \bar{\gamma}) + O\left(\frac{1}{c^8}\right), \quad (4.24)$$

consistently with Eqs. (4.3) and (4.4).

V. 2PN EXPERIMENTS

A. Constraints from 1PN experiments

Solar-system experiments impose tight bounds on the Eddington parameters $\bar{\gamma}$ and $\bar{\beta}$ [28,29]. Using Eqs. (2.15), they can be written as

$$|\bar{\gamma}| < 2 \times 10^{-3} \Leftrightarrow -10^{-3} < -\frac{\bar{\gamma}}{2 + \bar{\gamma}} = (\alpha^a \gamma_{ab} \alpha^b)_0 < 10^{-3}, \quad (5.1a)$$

$$|\bar{\beta}| < 6 \times 10^{-4} \Leftrightarrow -1.2 \times 10^{-3} < \frac{8\bar{\beta}}{(2 + \bar{\gamma})^2} = (\alpha^a \beta_{ab} \alpha^b)_0 < 1.2 \times 10^{-3}. \quad (5.1b)$$

Let us first discuss the lessons one can draw from the bounds (5.1) if one interprets them within what we consider a most natural theoretical framework. Since we assume that the (σ -model) metric γ_{ab} is positive definite, so that the theory does not incorporate any negative-energy excitation, Eq. (5.1a) constrains the magnitude of all the components of α_0^a . 1PN experiments thereby constrain the linear interaction of the scalar fields to matter to be small. On the other hand, our diagrammatic analysis of Sec. III shows that any observable deviation from general relativity involves at least two factors α_0^a , to fill the end blobs connected to scalar propagators, in diagrams such as Fig. 7 or any higher order ones. Moreover, theoretical ‘‘naturalness’’ leads us to expect the coupling function $A(\varphi)$ entering Eq. (2.3) not to involve any large dimensionless number, so that the successive derivatives of $\ln A(\varphi)$ [such as β_{ab}^0 , $\beta_{abc}^{\prime 0}$, ...] are *a priori* expected to be of order unity. The conclusion is that we anticipate the new parameters entering any post-Newtonian order

to be at most of the same order of magnitude as $\bar{\gamma}$. In particular, the 2PN parameters (3.30), which involve respectively 3 and 2 factors α_0^a , are expected to be of order $|\varepsilon| \sim |\alpha_0|^3 \lesssim 3 \times 10^{-5}$ and $|\zeta| \sim \alpha_0^2 \lesssim 10^{-3}$. The corresponding 2PN deviations from general relativity are thus generically constrained at the level $\lesssim 10^{-3} \times (Gm/Rc^2)^2 \sim 10^{-15}$, too small to be detectable in the solar system, even in future high-precision experiments. From this point of view, our conclusion is therefore that solar-system tests cannot probe the 2PN structure of gravity. However, they can give a clean access to the Eddington parameter $\bar{\gamma}$, which is of greatest significance as it measures the basic coupling strength between matter and the scalar fields, and constrains all the other PN parameters. Within this viewpoint, the only meaningful testing ground for 2PN and higher-order effects are binary-pulsar experiments. Indeed, since the self-energy Gm/Rc^2 of a neutron star is typically of order 0.2, the 2PN effects $\lesssim 10^{-3} \times (Gm/Rc^2)^2$ can yield significant deviations

on timing data. Moreover, in such strong-field conditions, the sum of the series of all higher post-Newtonian orders may be large enough to compensate the global α_0^2 factor of all the parameters. We have indeed shown in [40] that nonperturbative strong-field effects can compensate even a vanishingly small $\alpha_0^2 \approx -\frac{1}{2}\bar{\gamma}$. The overall conclusion is that two different regimes of gravity must be distinguished: (i) the weak-field conditions of the solar system can give a clean access to the fundamental parameter $\bar{\gamma}$; (ii) higher post-Newtonian effects can be tested in the strong-field regime of binary pulsars.

These conclusions, derived from an assumption of ‘‘naturalness’’ of the coupling function $A(\varphi)$, must be qualified if we adopt (still within the context of tensor-scalar theories) a more ‘‘phenomenological’’ point of view, i.e., if $A(\varphi)$ is permitted to involve large dimensionless numbers. In such a case, the 2PN parameters (3.30) can be of order unity in spite of the bounds (5.1a) on $\bar{\gamma}$. This is particularly true of the parameter ζ , which can be of order unity even if $A(\varphi)$ does not involve numbers larger than ~ 30 . For instance, in a model with two scalar fields and a flat σ -model metric $\gamma_{ab} = \delta_{ab}$, the values

$$\alpha_1^2 \approx 10^{-3}, \quad \alpha_2 = 0, \quad \beta_{11} \approx 1, \quad \beta_{12} \approx 30, \quad (5.2)$$

give $(\alpha^2)_0 = \alpha_1^2 \approx 10^{-3}$ and $(\alpha\beta\alpha)_0 = \alpha_1\beta_{11}\alpha_1 \approx 10^{-3}$, consistently with (5.1), but $\zeta \approx (\alpha\beta\beta\alpha)_0 = \alpha_1\beta_{11}^2\alpha_1 + \alpha_1\beta_{12}^2\alpha_1 \approx 1$. By contrast, getting $\varepsilon \sim 1$ seems less natural, as some components of $\beta_{abc}^{\prime 0}$ would have to be as large as 3×10^4 to compensate the $\alpha_0^a \alpha_0^b \alpha_0^c$ factor.

More generally, it is easy to see that ζ is *a priori* unconstrained by the solar-system bounds (5.1). Indeed, from the positive definiteness of the metric γ_{ab} , we can use the Cauchy-Schwarz inequality to obtain a lower bound¹³ for ζ :

$$(\alpha\beta\beta\alpha)_0 \geq [(\alpha\beta\alpha)_0]^2 / \alpha_0^2 \Leftrightarrow \zeta \geq 8\bar{\beta}^2 / |\bar{\gamma}|. \quad (5.3)$$

On the other hand, ζ has *a priori* no upper bound, and we can have $\zeta \geq 8\bar{\beta}^2 / |\bar{\gamma}|$ if the $(\varphi$ -space) vector $(\beta^a{}_b \alpha^b)_0 \equiv (\beta\alpha)_0^a$ is almost orthogonal to α_0^a , as in the example (5.2) above.

In Ref. [24], we adopted an even more ‘‘phenomenological’’ point of view, by assuming that the σ -model metric γ_{ab} could have *a priori* any signature (i.e., that the theory could involve negative-energy scalar fields). In that case, the bounds (5.1a) on $\bar{\gamma}$ do not constrain the magnitude of α_0^a , but only its direction: It should be almost a null vector in the φ space. Then, it is easy to see that ε is not constrained at all by the experimental limits (5.1), and that ζ is not constrained either if the theory involves at least three scalar fields. However, from a theoretical point of view, it seems more plausible that the gravitational interaction involves only a small number of massless scalar fields. If it involves only two of them (or less), the Cayley-Hamilton theorem yields

¹³The fact that we are able to derive relations between 1PN and 2PN parameters is a further illustration of the power of our field-theory approach.

$$(\alpha\beta^2\alpha) = (\text{Tr}\beta)(\alpha\beta\alpha) - (\det\beta)(\alpha^2), \quad (5.4)$$

where the trace and determinant are taken for the 2×2 matrix $(\beta^a{}_b)$. This relation shows that even in the case of a hyperbolic metric γ_{ab} , the parameter ζ is constrained to vanish with $\bar{\beta}$ and $\bar{\gamma}$.

On the other hand, one can make more definite assertions if we assume that there is only one scalar field. In this monoscalar case, the phenomenology of theoretically expected deviations depends upon the magnitude of the curvature parameter $\beta_0 = \partial^2 \ln A(\varphi_0) / \partial \varphi_0^2$. Indeed, we have then the following links between 1PN and 2PN parameters:

$$\bar{\beta} = \frac{1}{2} \beta_0 \alpha_0^2 / (1 + \alpha_0^2)^2 \approx -\frac{1}{4} \beta_0 \bar{\gamma}, \quad (5.5a)$$

$$\zeta = \beta_0^2 \alpha_0^2 / (1 + \alpha_0^2)^3 \approx -\frac{1}{2} \beta_0^2 \bar{\gamma}. \quad (5.5b)$$

Therefore, if β_0 is somehow known (or assumed) to be rather large¹⁴ (say $|\beta_0| > 5$), we could conclude that (i) among weak-field experiments, the most sensitive probes of possible deviations from general relativity are high-precision measurements of $\bar{\beta}$ and (ii) pulsar measurements (discussed below) of the parameter ζ appear, at face value, to be even more sensitive probes of non-Einsteinian physics. However, the latter conclusion is not quite justified because the approximate analytical treatment given here is inappropriate when $|\beta_0| > 5$, and must be replaced by a complete numerical treatment to correctly take into account the nonperturbative effects discovered in [40] (more about this below).

The conclusion of this lengthy discussion is therefore that the 1PN experimental limits (5.1) most plausibly constrain the 2PN parameters ε and ζ to be very small, but that there is no theoretical impossibility that the latter be of order unity. In the following, we will adopt a phenomenological point of view, and consider that these parameters are *a priori* unconstrained to discuss the possible experiments which may measure them. Let us underline, however, that this ‘‘phenomenological’’ attitude is still in the framework of our field-theory approach, and should not be confused with previous studies in the literature [20–22] which aimed at describing any conceivable deviation from general relativity.

B. Solar-system experiments

1. Light deflection and time delay

As discussed in the previous subsection, the most important parameter to determine experimentally is $\bar{\gamma}$, as it measures the basic coupling strength and constrains *a priori* all the possible deviations from general relativity. It has been shown in [27,26] that the cosmological evolution of tensor-

¹⁴Let us note in this respect that in the string-derived model of Ref. [26], $\beta_0 \approx 40\kappa$ where the more fundamental parameter κ is expected to be of order unity if the considered scalar field is the dilaton proper. On the other hand, in the case of a modulus field, κ can be expected to involve a small ‘‘loop factor’’ $\sim (4\pi^2)^{-1}$ which can naturally compensate the factor 40.

multiscalar theories generically drives $\bar{\gamma}$ towards 0, and that its present value is expected to be $\lesssim 10^{-7}$. The 1PN deviations proportional to $\bar{\gamma}$, entering light-deflection and time-delay experiments, are thus expected to be comparable to 2PN effects, if we adopt the phenomenological point of view that the 2PN parameters ε and ζ are *a priori* unconstrained (and that $\bar{\beta}$ may be much larger than $\bar{\gamma}$). It is therefore important to determine whether these effects can be distinguished from each other. We shall see here that light-deflection and time-delay experiments are ideally suited to an accurate determination of $\bar{\gamma}$ in the solar system.

First of all, it is immediate to see that these experiments do not depend on ε and ζ at the 2PN level. Indeed, our results of Sec. III and IV show that these parameters appear for the first time at order $O(1/c^6)$, in the time-time component of the metric \bar{g}_{00} . Therefore, they do not enter the 2PN geodesic equation satisfied by light, which involves only the $O(1/c^2)$ and $O(1/c^4)$ orders of the metric. Consequently, any experiment probing the propagation of light is independent from ε and ζ at the 2PN level.

We compute in Appendix C the 2PN light deflection by a massive body and find

$$\Delta\phi = \frac{2\mu_A(2 + \bar{\gamma}_{A0})}{\rho_0 c^2} \left[1 - \frac{\mu_A(2 + \bar{\gamma})}{\rho_0 c^2} \right] + \frac{\pi}{4} \left(\frac{\mu_A}{\rho_0 c^2} \right)^2 \left[15 + \frac{31}{2} \bar{\gamma} + 4 \bar{\gamma}^2 \right] + O\left(\frac{1}{c^6}\right). \quad (5.6)$$

Here ρ_0 denotes the minimal distance between the light ray and the center of body A measured in isotropic coordinates. The body-dependent parameter $\bar{\gamma}_{A0}$ entering the first term of (5.6) can be expanded in powers of the self-gravity of body A , as in Eq. (4.23d) above:

$$\bar{\gamma}_{A0} = \bar{\gamma} - (4\bar{\beta} - \bar{\gamma})(2 + \bar{\gamma}) \frac{E_A^{\text{grav}}}{m_A c^2} + O\left(\frac{1}{c^4}\right). \quad (5.7)$$

Note that $\bar{\beta}$ appears in the self-energy corrections to the 1PN effects, whereas it is absent from the $O((Gm/rc^2)^2)$ contributions.

Contrary to the hopes of Refs. [12,13,15,16], a conclusion of our approach is that improved light-deflection (or time-delay) experiments do not give access to any theoretically significant 2PN parameter. In fact, the formal 2PN generalization of the Eddington γ parameter introduced in Refs. [12,13] (under the different notations ϵ and δ ; see also [15] where it is denoted Λ) is equal to $1 + \bar{\Gamma}$, where $\bar{\Gamma}$ is the function of 1PN parameters given in Eq. (4.23e) above:

$$\begin{aligned} \epsilon_{\text{Epstein-Shapiro}} &= \delta_{\text{Fischbach-Freeman}} = \Lambda_{\text{Richter-Matzner}} \\ &= 1 + \frac{4}{3} \bar{\beta} + \frac{15 + 8 \bar{\gamma}}{6} \bar{\gamma}. \end{aligned} \quad (5.8)$$

Our conclusion concerning the impossibility to probe significant 2PN deviations from general relativity in light-deflection and time-delay experiments should not be viewed only negatively. The positive aspect is that these experiments can give a clean access to the fundamental parameter $\bar{\gamma}$, i.e.,

that no 2PN deviation from general relativity can complicate their interpretation in the case where the data are too scarce to allow a clean separation of $1/r$ and $1/r^2$ effects. Indeed, at the level $\bar{\gamma} \lesssim 10^{-6}$ that these experiments aim for, the 2PN effects proportional to this parameter in Eq. (5.6) are totally irrelevant.¹⁵ On the other hand, one remarks that the $\bar{\gamma}$ parameter which will be measured by these experiments is the body-dependent quantity $\bar{\gamma}_{A0}$ rather than its weak-field limit $\bar{\gamma}$. However, the self-gravity renormalization of $\bar{\gamma}$ displayed in Eq. (5.7) is already strongly constrained by the LLR (lunar laser ranging) bounds on $\eta \equiv 4\bar{\beta} - \bar{\gamma}$ ($|\eta| < 2 \times 10^{-3}$ [29]). Using $|E_{\odot}^{\text{grav}}|/m_{\odot} c^2 \sim 2 \times 10^{-6}$ for the Sun, one gets

$$(4\bar{\beta} - \bar{\gamma})(2 + \bar{\gamma}) |E_{\odot}^{\text{grav}}|/m_{\odot} c^2 < 10^{-8}. \quad (5.9)$$

Therefore, an experimental determination of $\bar{\gamma}_{\odot 0}$ at the 10^{-6} or 10^{-7} level, which is expected to be reachable in missions presently considered by the European Space Agency such as GAIA (Global Astrometric Interferometer for Astrophysics) or SORT (Solar Orbit Relativity Test), will indeed give a clean measurement of $\bar{\gamma}$. The same conclusion *a fortiori* applies to the expected measurement, at the 10^{-5} level, of $\bar{\gamma}_{\oplus 0}$ by NASA's Gravity Probe B mission. In this case, the relevant inequality is $(4\bar{\beta} - \bar{\gamma})(2 + \bar{\gamma}) |E_{\oplus}^{\text{grav}}|/m_{\oplus} c^2 < 2 \times 10^{-12}$.

Finally, let us remark that the value of $\bar{\gamma}$ we are talking about in this work is the one corresponding to the value of the scalar-field background φ_0^a around the solar system. As remarked in [22], it differs from, say, the cosmological average of $\bar{\gamma}$ (which is the one discussed in Refs. [27,26]) by the effect of the spatial fluctuation ΔU of the gravitational potential. However, we show in Appendix B that the corresponding change in $\bar{\gamma}$ is given by $4\bar{\beta}(2 + \bar{\gamma})\Delta U/c^2$ and is therefore expected to represent only a very small *fractional* change of $\bar{\gamma}$ of order¹⁶ $2|\beta^a_b|\Delta U/c^2$, where $|\beta^a_b|$ is the norm of the matrix β^a_b (say the modulus of its largest eigenvalue).

2. Other possible 2PN experiments

The previous subsection showed that experiments on the propagation of light in the solar system cannot give access (within our framework) to any post-Newtonian parameter but $\bar{\gamma}$. The question that we will address now is whether the 2PN parameters ε and ζ can be measured in the solar system, or at least if it is possible to constrain their magnitudes at an interesting level.

¹⁵At the qualitative level, it remains that checking the coefficient $15\pi/4$ appearing in the second term of (5.6) will be a new confirmation of the nonlinear structure of general relativity, even if it does not constrain any plausible theoretical alternatives.

¹⁶Even if $|\beta^a_b| \sim 30$, the fact that $\Delta U_{\text{cosmo}}/c^2 \sim 10^{-5}$ indicates a negligible fractional change $\delta\bar{\gamma}/\bar{\gamma} \sim 6 \times 10^{-4}$. Independently of this argument, we note that the present observational limits on $\bar{\beta}$ give $|\delta_{\text{cosmo}}\bar{\gamma}| \lesssim 5 \times 10^{-8}$.

One of the most famous tests of post-Newtonian gravity is the perihelion shift of Mercury, and more generally tests obtained through a global fit to the orbital motion of the planets. Before any calculation, it is clear that we cannot hope to measure the Mercury perihelion advance by traditional means to an accuracy of 2PN significance. Indeed, the 2PN effect is smaller than the 1PN prediction by a factor $\sim(Gm_\odot/rc^2)$, and corresponds to an advance of $\sim 10^{-6}$ arc sec per century. However, it may be, one day, possible to reach this level by using an artificial satellite orbiting around Mercury and tracked with very high accuracy from Earth. To increase the parameter Gm_\odot/rc^2 , one could also observe the perihelion shift of a drag-free satellite in close elliptical orbit around the Sun, but the construction of such a satellite seems unrealistic with present technology. Anyway, several reasons show that such experiments could not constrain ε and ζ at an interesting level. Indeed, we compute in Appendix C below the perihelion shift per orbit for a test mass $m_0 \ll m_\odot$, and we get in isotropic coordinates

$$\Delta\phi = \frac{6\pi Gm_\odot}{a(1-e^2)c^2} \left[1 + \frac{2\bar{\gamma} - \bar{\beta} + \zeta c_\odot}{3} + \frac{Gm_\odot}{ac^2} \left(\frac{7}{2} + \frac{\varepsilon}{6} + O(e^2) \right) \right] + \frac{O(\bar{\beta}, \bar{\gamma})}{c^4} + O\left(\frac{1}{c^6}\right), \quad (5.10)$$

where a is the coordinate semimajor axis of the orbit, e its coordinate eccentricity, and $c_\odot \sim 4 \times 10^{-6}$ is the compactness of the Sun. Although ζ enters this expression, it cannot be distinguished from a small contribution of $2\bar{\gamma} - \bar{\beta}$ at the 1PN level. In fact, it comes from the expansion (4.23b) of the body-dependent parameter $\bar{\beta}_{\odot\odot}^0 = \bar{\beta} - \zeta c_\odot + O(\bar{\beta}, \bar{\gamma})/c^2 + O(1/c^4)$, and is thus a mere renormalization of $\bar{\beta}$. On the contrary, the contribution proportional to ε can in principle be distinguished from the 1PN effects, since it involves a different power of (Gm/a) . Numerically, we find that Mercury's perihelion is deviated from the general relativistic prediction by $\approx 0.5\varepsilon$ mm per year, too small to be of observational significance. In the totally unrealistic situation of a drag-free satellite grazing the surface of the Sun, this 2PN deviation would be of order $\approx 30\varepsilon$ m yr $^{-1}$, which is *a priori* much easier to detect. However, to distinguish the $O(Gm/ac^2)$ and $O((Gm/ac^2)^2)$ contributions, it would be necessary to compare several satellites at different distances from the Sun, or alternatively to look at periodic effects on a given orbit, which may be much more difficult to observe than secular effects. Moreover, one will have the difficulty of separating the 2PN contribution from the Newtonian contribution due to the quadrupole moment of the Sun, which has the same dependence¹⁷ on a (not to mention the huge thermal and electromagnetic effects of the Sun on a close satellite). In conclusion, perihelion-shift experiments in the solar system can in principle give access to the 2PN parameter ε , but present technology does not give the hope of measuring it to any significant level.

¹⁷The peculiar anisotropy of multipolar effects will, however, help in this respect.

We have seen in Eqs. (3.29a) and (3.34a) above that the effective gravitational constant G_{AB} depends on the self-energies of the bodies A and B . In particular, the Earth (\oplus) and the Moon (\ominus) do not fall with the same acceleration towards the Sun, since $G_{\oplus\oplus} \neq G_{\ominus\oplus}$. This violation of the strong equivalence principle implies a polarization of the Moon's orbit (Nordtvedt effect), that can be tested in the lunar laser ranging data. The deviations from general relativity are proportional to the ratio

$$\frac{G_{\oplus\oplus} - G_{\oplus\ominus}}{G} = -\frac{\eta}{2}(c_\oplus - c_\ominus) + [\zeta + O(\bar{\beta}, \bar{\gamma})]c_\odot(c_\oplus - c_\ominus) + O(a_\oplus, b_\oplus, a_\ominus, b_\ominus), \quad (5.11)$$

where $c_\ominus \ll c_\oplus \ll c_\odot$ are the compactnesses of the three bodies, and where we have neglected $a_\oplus \sim b_\oplus \sim c_\oplus^2$ (as well as a_\ominus and b_\ominus) with respect to $c_\odot c_\oplus$. The first term, involving the parameter $\eta = 4\bar{\beta} - \bar{\gamma}$, is the standard 1PN deviation which has been constrained at the level $|\eta| < 2 \times 10^{-3}$ by LLR data. The dominant 2PN contribution, $\zeta c_\odot(c_\oplus - c_\ominus)$, is equivalent to a renormalization of η into $(\eta - 2\zeta c_\odot) \approx (\eta - 10^{-5}\zeta)$. From a theory-based viewpoint, such a renormalization is of no consequence: as η and ζ are both proportional to the basic coupling strength $\bar{\gamma}$, we can always neglect the *fractionally* small correction $10^{-5}\zeta$ to η (in the same way, as explained above, that one does not have to worry about ‘‘cosmic variance’’ effects). However, from a phenomenological viewpoint, ζ is an independent quantity which would complicate the interpretation of a high-precision LLR experiment reaching the $\eta \sim 10^{-5}$ level. [This level would correspond to measuring the Earth-Moon distance with 0.1 millimeter accuracy.] In other words, the phenomenological point of view obliges us to look for other independent experiments allowing one to separate the effects of η and ζ .

The polarization of Mercury's orbit around the Sun due to the presence of Jupiter (i.e., the Nordtvedt effect for Mercury) can give access to a different combination of η and the 2PN parameters ε , ζ . The corresponding deviation from general relativity is proportional to

$$\frac{G_{J\oplus} - G_{JM}}{G} = -\frac{\eta}{2}c_\odot + \left(\frac{\varepsilon}{2} + \zeta \right) b_\odot + \zeta \times O(c_\odot c_J) + O(\bar{\beta}, \bar{\gamma})O(c_M, c_\odot^2), \quad (5.12)$$

where $c_M \ll c_J \ll c_\odot$ are respectively the compactnesses of Mercury, Jupiter, and the Sun, and where we have neglected $c_\odot c_J$ with respect to $b_\odot \sim c_\odot^2$. The dominant 2PN contribution plays again the role of a renormalization of η , but this time into $[\eta - (\varepsilon + 2\zeta)b_\odot/c_\odot] \approx [\eta - 10^{-5}(\varepsilon/2 + \zeta)]$. Note that ε enters now this expression, whereas it was absent from the corresponding lunar result. This complicates the problem of separating the contributions of η and ζ . It would be necessary to dispose of a third experiment giving access to ε , for instance by using the perihelion shift (5.10). However, we have seen that such a measure is not likely to be performed with current (or foreseeable) technology.

Another attempt to determine ε could be to compare ultrastable clocks: one located on Earth and another one some-

where close to the Sun. Indeed, the Einstein effect gives access to $\sqrt{-\bar{g}_{00}}$, where $-\bar{g}_{00}$ is given by Eq. (4.21a) in g^* -harmonic coordinates and by the first bracket of Eq. (4.18) in isotropic coordinates. The general relativistic prediction for the rates of clocks is thus multiplied by a factor $[1 + (Gm_\odot/rc^2)^2 \bar{\beta}_{\odot\odot}^0 - \frac{1}{6}(Gm_\odot/rc^2)^3 \varepsilon + O(\bar{\beta}, \bar{\gamma})/c^6]$. This gives (relative to a clock on Earth) effects of order $\approx 1.6 \times 10^{-18} \varepsilon$ for a clock located at the surface of the Sun, and $\approx 3 \times 10^{-24} \varepsilon$ for a clock on Mercury. With foreseeable technology (10^{-18} stability), one might barely be able to constrain ε at the $O(1)$ level. As for the perihelion shift (5.10), it should be noted that the parameter ζ contained in $\bar{\beta}_{\odot\odot}^0 \approx \bar{\beta} - \zeta c_\odot$ is a mere renormalization of $\bar{\beta}$ and cannot be accessed independently.

A possible way to access this ζ contribution in $\bar{\beta}_{AA}^0$ would be to measure both $\bar{\beta}_{\odot\odot}^0 \approx \bar{\beta} - \zeta c_\odot$ and $\bar{\beta}_{\oplus\oplus}^0 \approx \bar{\beta} - \zeta c_\oplus \approx \bar{\beta}$ at a level $< c_\odot \sim 4 \times 10^{-6}$. This would necessitate the tracking of Mercury with few cm accuracy, and the observation of a drag-free artificial Earth satellite at the 10^{-2} cm level [41]. This second condition is two orders of magnitude smaller than what can be done presently.

The conclusion of the above discussion is that the 2PN parameters ε and ζ are extremely difficult to measure in the solar system. Within a phenomenological approach, the only role of these parameters is negative: they complicate the interpretation of high-precision 1PN experiments. Hence the solar system appears not to be an appropriate testing ground for probing the 2PN structure of gravity. On the contrary, it is perfectly suited for measuring the fundamental parameter $\bar{\gamma}$, as underlined in Sec. V B 1.

C. Binary-pulsar experiments

Let us now consider binary-pulsar experiments, which will turn out to be much better testing grounds for the 2PN structure of gravity. Since the self-energy of a neutron star is typically of order 0.2 (as compared to $\sim 10^{-6}$ for the Sun), the 2PN and higher-order effects play an important role in the behavior of binary pulsars. In the present subsection, we will show that the combined analysis of several binary-pulsar data has the capability of constraining both ε and ζ at an interesting level.¹⁸ Our aim is not to perform a full statistical analysis, but rather to illustrate the different types of constraints that can be obtained. We will consider four binary pulsars, for which different observable quantities can be measured.

The Hulse-Taylor binary pulsar PSR 1913+16 has been continuously observed since its discovery in 1974. Besides the Keplerian orbital parameters P (orbital period) and e (eccentricity), three ‘‘post-Keplerian’’ [31,32] observables have been measured with great accuracy: the periastron advance $\dot{\omega}$, the secular change of the orbital period \dot{P} , and the time-dilation parameter γ_{timing} which describes both the second-order Doppler effect due to the velocity of the pulsar and the redshift due to the presence of its companion. [This last parameter should not be confused with the metric γ_{ab} ,

the Eddington parameter $\bar{\gamma}$, nor its body-dependent generalization $\bar{\gamma}_{AB}$.] Within a given theory of gravity, these three timing observables¹⁹ can be predicted in terms of the masses m_A and m_B of the pulsar and its companion, which are *a priori* unknown. The theory is consistent with experiment if there exists a pair of masses (m_A, m_B) giving the correct observed values for the three quantities \dot{P} , $\dot{\omega}$, and γ_{timing} . This is the so called ‘‘ \dot{P} - $\dot{\omega}$ - γ ’’ test, that general relativity passes with flying colors, and which establishes the reality of gravitational waves. We give below the experimental values quoted in [43]:

$$P = 27906.9807804(6) \text{ s}, \quad (5.13a)$$

$$e = 0.6171308(4), \quad (5.13b)$$

$$\dot{P} = -2.422(6) \times 10^{-12}, \quad (5.13c)$$

$$\dot{\omega} = 4.226621(11)^\circ \text{ yr}^{-1}, \quad (5.13d)$$

$$\gamma_{\text{timing}} = 4.295(2) \times 10^{-3} \text{ s}, \quad (5.13e)$$

where figures in parentheses represent 1σ uncertainties in the last quoted digits. In fact the determination of \dot{P} is so precise that it is necessary to take into account the small variable Doppler effect due to the acceleration of the binary pulsar towards the center of the Galaxy [44]. This induces an extra contribution to \dot{P} , which takes the value $\dot{P}_{\text{gal}} = -0.0124(64) \times 10^{-12}$ in general relativity. The intrinsic variation of the orbital period (due to gravitational radiation damping) is thus given by

$$\dot{P}_{\text{observed}} - \dot{P}_{\text{gal}} = -2.4101(85) \times 10^{-12}. \quad (5.14)$$

In tensor-scalar theories of gravity, \dot{P}_{gal} is modified by a small contribution, $\delta\dot{P}_{\text{gal}}$, that we will take into account in our calculations below.

The binary pulsar PSR 1534+12 has been observed only since 1991, and the present experimental accuracy on its parameter \dot{P} is not comparable with the one of Eq. (5.13c) above. However, this pulsar is much closer to the Earth than PSR 1913+16, and it has been possible to measure $\dot{\omega}$ and γ_{timing} (with very good precision), as well as two new timing observables, r and s , measuring the amplitude and the shape of the Shapiro time-delay caused by the companion. Out of these two parameters, only s is measured with precision. [Note that, geometrically, $s \equiv \sin i$ is the sine of the angle between the orbit and the plane of the sky.] As above, the three quantities $\dot{\omega}$, γ_{timing} , and s can be predicted as functions of the masses m_A , m_B within a given theory of gravity. One can therefore test if this theory agrees with experiment by looking for a pair of masses (m_A, m_B) consistent with the three observed values of $\dot{\omega}$, γ_{timing} , and s (‘‘ $\dot{\omega}$ - γ - s ’’ test). We shall make use of the latest experimental data discussed in [45]:

¹⁸In the language of [32,42], we perform a combined theory-dependent analysis of several independent pulsar data.

¹⁹In order not to create any confusion with our use of the word post-Keplerian in the present paper, we refer to these quantities as ‘‘timing observables’’ instead of ‘‘post-Keplerian’’ parameters as used in [31,32].

$$P = 36351.70267(2) \text{ s}, \quad (5.15a)$$

$$e = 0.2736771(4), \quad (5.15b)$$

$$x = 3.729458(2) \text{ s}, \quad (5.15c)$$

$$\dot{\omega} = 1.75573(4)^\circ \text{ yr}^{-1}, \quad (5.15d)$$

$$\gamma_{\text{timing}} = 2.081(16) \times 10^{-3}, \quad (5.15e)$$

$$s = 0.981(8). \quad (5.15f)$$

Here $x = a_1 s/c$ is the projection of the semimajor axis (a_1) of the pulsar orbit on the line of sight (in light seconds). A precision should be given concerning the quoted experimental uncertainties. Reference [45] gives (for the more reliable 1.4 GHz data) the $\Delta\chi^2 = 2.30$ and $\Delta\chi^2 = 6.17$ contours in the (r, c) plane with $c \equiv \sqrt{1-s^2}$. We deduced from these the $\Delta\chi^2 = 4$ contour which defines, when projected onto the c axis, a $2\sigma_{\text{stat}}$ interval for c considered alone. We use the corresponding $2\sigma_{\text{stat}}$ interval for $s = \sqrt{1-c^2}$ as a realistic 1σ interval (in other words, we double the statistical 1σ interval for s to take into account possible systematic effects). We similarly doubled the statistical 1σ uncertainties obtained in [45] for $\dot{\omega}$ and γ_{timing} .

The binary pulsar PSR 0655+64 is composed of a neutron star of mass $\approx 1.4m_\odot$, and a light companion of mass $\approx 0.8m_\odot$, which is probably a white dwarf. The gravitational waves emitted by such a dissymmetrical system involve a large dipolar contribution of order $1/c^3$ in tensor-scalar theories, whereas the dominant radiation in general relativity is quadrupolar and of order $1/c^5$. The fact that the observed value of \dot{P} is very small (and consistent with zero) constrains therefore the existence of scalar fields, or more precisely the magnitude of their interaction with matter. We will see below that this system imposes a tight bound on the 2PN parameter ζ . The experimental data that we will need for our analysis are taken from [45]:

$$P = 88877.06194(4) \text{ s}, \quad (5.16a)$$

$$e < 3 \times 10^{-5}, \quad (5.16b)$$

$$\dot{P} = (1 \pm 4) \times 10^{-13}. \quad (5.16c)$$

The masses of the pulsar and its companion are not known independently; several pairs (m_A, m_B) are thus *a priori* possible, such as $(1.30m_\odot, 0.7m_\odot)$, $(1.35m_\odot, 0.8m_\odot)$ or $(1.40m_\odot, 0.9m_\odot)$. In our calculations below, we will choose the mass pair which gives the most conservative bounds on the 2PN parameters, namely $m_A = 1.30m_\odot$, $m_B = 0.7m_\odot$. [Smaller masses could be consistent with experimental data, but current lore favors neutron star masses close to $1.4m_\odot$.]

The fourth and last binary pulsar that we will consider, PSR 1800–27, is also a dissymmetrical system, involving a neutron star of mass $m_A \approx 1.4m_\odot$ and a light companion of negligible self-energy. The acceleration of the pulsar towards

the center of the Galaxy is therefore proportional to the self-gravity-modified effective gravitational constant \tilde{G}_{A0} [cf. Eqs. (4.20a) and (4.23a) above], whereas the companion is accelerated by a force proportional to the weak-field gravitational constant $\tilde{G}_{00} = \tilde{G}$, Eq. (2.14). As shown in [46], this violation of the strong equivalence principle causes a ‘‘gravitational Stark effect’’ on the orbit of the system, polarizing its eccentricity in a particular direction. A highly circular orbit, like the one of PSR 1800–27, is therefore very improbable. A statistical study can thus be performed to constrain the magnitude of the matter-scalar field interaction. Following the method of [46], Ref. [45] has obtained the bound

$$|\delta_A| < 1.4 \times 10^{-3} \quad (5.17)$$

on the parameter δ_A characterizing this gravitational Stark effect. This bound corresponds to the 90% confidence level, which plays the role of an ‘‘effective 1σ level’’ for the non-Gaussian statistics of this test. [Twice this value gives the 95% C.L., i.e., the standard 2σ level. Note that this ‘‘effective 1σ level’’ is more secure than the standard 1σ (68%) level.] We note, without making use of it, that Ref. [47] recently advocated a more conservative bound $|\delta_A| < 4 \times 10^{-3}$ at the 90% C.L. (deduced from the analysis of seven older binary pulsars) on the ground that the binary system PSR 1800–27 might be too young to provide a fully secure test.

The analytic expressions of all the observable quantities discussed above have been derived in [46,32,24]. The theoretical prediction for the observed time derivative of the orbital period has the form

$$\begin{aligned} \dot{P} = & \dot{P}_{\text{spin } 0}^{\text{monopole}} + \dot{P}_{\text{spin } 0}^{\text{dipole}} + \dot{P}_{\text{spin } 0}^{\text{quadrupole}} + \dot{P}_{\text{spin } 2}^{\text{quadrupole}} + \dot{P}_{\text{gal}} + \delta\dot{P}_{\text{gal}} \\ & + O\left(\frac{1}{c^7}\right), \end{aligned} \quad (5.18)$$

where the different contributions are given in Eqs. (6.52) and (9.22) of [24]. The same reference gives also the expressions of the periastron shift $\dot{\omega}$ and of the time-dilation parameter γ_{timing} in Eqs. (9.20), as well as the ‘‘Stark’’ parameter δ_A in Eq. (9.16). Finally, the theoretical prediction for the timing observable s is given in Eq. (3.15) of [32]. Introducing the notation

$$\begin{aligned} M & \equiv m_A + m_B, & X_A & \equiv m_A/M, \\ X_B & \equiv m_B/M, & n & \equiv 2\pi/P, \end{aligned} \quad (5.19)$$

it can be written as

$$\frac{x}{s} = \frac{X_B}{n} (G_{AB} M n / c^3)^{1/3}. \quad (5.20)$$

These theoretical predictions give (when written out in detail) the various timing observables as theory-dependent functions of the masses of the bodies. In general, because of possible nonperturbative strong-field effects [40], the latter functions should be considered as *functionals* of the $1 + n(n-1)/2$ arbitrary functions entering the definition of a tensor-scalar theory [notably $A(\varphi)$]. However, if we assume the absence of genuine nonperturbative effects, we can expand the functions giving the observables in powers of the compactnesses c_A, c_B of the bodies. This has the effect of reducing the functional dependence of the observables to a dependence upon a finite number of theory parameters. At the lowest orders, there appear the 1PN Eddington parameters $\bar{\beta}$ and $\bar{\gamma}$. As these are already tightly constrained by solar-system data, Eqs. (5.1), we will neglect them in the following, and investigate the limits that can be set on further theory parameters. The 2PN parameters ε and ζ appear precisely at the next significant order of the expansion in compactnesses [24]. The deeper layers of theory parameters introduced in [24] always appear with higher powers of the compactnesses. [As shown in Table 3 of [24], this is true even for the different contributions of the gravitational radiation, in spite of their rather complicated structure.]

Here we shall estimate what constraints are imposed by binary-pulsar data on ε and ζ , when restricting our attention to the lowest-order terms in compactnesses involving them. From a numerical point of view, our approximation is rather rough, as the next orders that we neglect are only $O(c_A) \sim 0.3$ smaller than the terms we will consider. A way of justifying our approach is to say that we assume that there is no fine-tuned compensation between the ε and ζ -dependent terms we retain and the higher-order terms which involve new (3PN, 4PN, ...) parameters. We believe that our simplified analysis will give at least the right order of magnitude for the constraints on ε and ζ that would follow from a more complete theory-dependent analysis.

To write in closed form the truncated expressions of the different observables, let us introduce the notation²⁰

$$\mathcal{V} \equiv (G_* M n)^{1/3}, \quad (5.21)$$

where M and n have been defined in Eq. (5.19) above, and where G_* is the bare gravitational constant. Note that since we neglect here $\alpha_0^2 \bar{\gamma}$, the bare constant G_* can be identified with the gravitational constant $G = G_*(1 + \alpha_0^2)$ measured in weak-field conditions. The timing observables can now be written as

$$\dot{P}_{\text{spin } 0}^{\text{monopole}} = -12\pi \frac{\mathcal{V}^5}{c^5} X_A X_B \zeta \frac{e^2(1+e^2/4)}{(1-e^2)^{7/2}} + O(s), \quad (5.22a)$$

$$\dot{P}_{\text{spin } 0}^{\text{dipole}} = -2\pi \frac{\mathcal{V}^3}{c^3} X_A X_B \left[(c_A - c_B)^2 \zeta \frac{1+e^2/2}{(1-e^2)^{5/2}} + O(s^3) \right] + 4\pi \frac{\mathcal{V}^5}{c^5} X_A X_B \left[(X_A - X_B)(c_A - c_B) \zeta \frac{1+3e^2+3e^4/8}{(1-e^2)^{7/2}} + O(s^2) \right], \quad (5.22b)$$

$$\dot{P}_{\text{spin } 0}^{\text{quadrupole}} = -\frac{32\pi}{5} \frac{\mathcal{V}^5}{c^5} X_A X_B [(c_B X_A + c_A X_B)^2 \zeta + (b_B X_A + b_A X_B)(\varepsilon + 2\zeta) + O(s^3)] \frac{1+73e^2/24+37e^4/96}{(1-e^2)^{7/2}}, \quad (5.22c)$$

$$\dot{P}_{\text{spin } 2}^{\text{quadrupole}} = -\frac{192\pi}{5} \frac{\mathcal{V}^5}{c^5} X_A X_B \left[1 + \frac{2}{3} c_A c_B \zeta + \frac{1}{3} (b_A + b_B)(\varepsilon + 2\zeta) + O(s^3) \right] \frac{1+73e^2/24+37e^4/96}{(1-e^2)^{7/2}}, \quad (5.22d)$$

$$\delta \dot{P}_{\text{gal}}^{1913+16} = -\frac{1.22 \times 10^{-18}}{n} (b_A X_A + b_B X_B)(\varepsilon + 2\zeta) + O(s^3), \quad (5.22e)$$

$$\dot{\omega} = \frac{\mathcal{V}^2}{c^2} \frac{n}{1-e^2} \left[3 + (c_A X_A + c_B X_B) \zeta + \frac{1}{2} (c_B X_A + c_A X_B)(\varepsilon + \zeta) + O(s^2) \right], \quad (5.22f)$$

$$\gamma_{\text{timing}} = \frac{\mathcal{V}^2}{c^2} \frac{e X_B}{n} [1 + X_B - 2\kappa_A c_B \zeta + O(s^2)], \quad (5.22g)$$

$$\delta_A = b_A \left(\frac{\varepsilon}{2} + \zeta \right) + O(s^3), \quad (5.22h)$$

$$\frac{x}{s} = \frac{\mathcal{V} X_B}{c n} \left[1 + \frac{1}{3} c_A c_B \zeta + \frac{1}{3} (b_A + b_B) \left(\frac{\varepsilon}{2} + \zeta \right) + O(s^3) \right]. \quad (5.22i)$$

As in Eqs. (3.27) above, s in the error terms on the right-hand side is a global notation for the compactnesses of the bodies, which should not be confused with the timing observable s on the left-hand side of (5.22i). The $O(\mathcal{V}^5/c^5)$ contribution in $\dot{P}_{\text{spin } 0}^{\text{dipole}}$ can of course be neglected with respect to its $O(\mathcal{V}^3/c^3)$ term. On the other hand, the monopolar and quadrupolar contributions should not be neglected in spite of their being also of order $O(\mathcal{V}^5/c^5)$. Indeed, the dipolar term (5.22b) can become very small if the pulsar A and its companion B are almost identical, and the monopolar and quadrupolar contribution then dominate. The galactic contribution to \dot{P} given in Eq. (5.22e) corresponds to the case of PSR 1913+16, and should not be used for other pulsars. The parameter κ_A entering (5.22g) is (minus) the logarithmic derivative of the inertia moment I_A of the pulsar with respect to the gravitational constant: $\kappa_A \equiv -\partial \ln I_A / \partial \ln G$. It has been estimated in Ref. [48] to range between 0.5 and 1.7, depending on the nuclear equation of state used to describe the neutron star matter. The “ \dot{P} - $\dot{\omega}$ - γ ” test in PSR 1913+16 is almost insensitive²¹ to the value chosen for κ_A . On the contrary, the “ $\dot{\omega}$ - γ - s ” test in PSR 1534+12 gives slightly tighter bounds on ε and ζ for $\kappa_A = 1.7$ than for $\kappa_A = 0.5$. In order to derive conservative bounds for these parameters, we will use the value $\kappa_A = 0.5$ in the following.

In all the equations (5.22), the compactnesses of the bodies can be estimated by using the results of Appendix B of [24]. For a realistic equation of state of matter inside a neutron star, we found in this reference

$$c_A \approx 0.21 m_A / m_\odot, \quad (5.23a)$$

$$a_A \approx 2.16 c_A^2, \quad (5.23b)$$

$$b_A \approx 1.03 c_A^2. \quad (5.23c)$$

The coefficient 0.21 of Eq. (5.23a) can be lowered to ≈ 0.15 for a stiff equation of state, and increased to ≈ 0.30 for a soft equation of state. We will choose the central value 0.21 in our calculations. When the companion of the pulsar is not itself a neutron star, but a weakly self-gravitating body like a white dwarf, its compactness $c_B \ll c_A$ can be neglected. This is the case for PSR 0655+64 and PSR 1800-27.

The predictions (5.22) of tensor-scalar theories can now be confronted to the experimental data (5.13)–(5.17). Since we are working at the first order in ε and ζ , we can replace the masses m_A, m_B of the bodies (as well as the corresponding compactnesses $c_A, c_B, a_A, a_B, b_A, b_B$) by their general relativistic predictions in all the terms involving one of these parameters. By contrast, the dominant contributions in $\dot{P}_{\text{spin } 2}^{\text{quadrupole}}$, $\dot{\omega}$, γ_{timing} , and x/s should be considered as functions of two *a priori* unknown masses m_A, m_B . For each value of ε and ζ , we can thus determine if the above four tests can be passed. More precisely, in the cases of PSR 0655+64 or PSR 1800-27, the measurement of the observ-

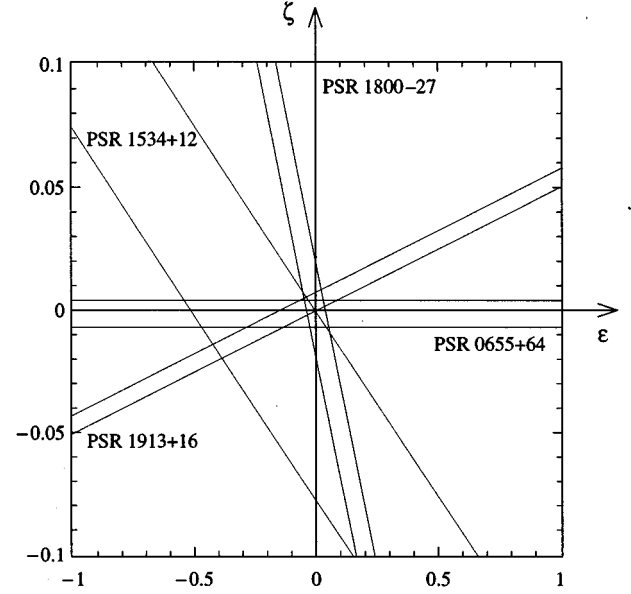


FIG. 8. Constraints imposed by four different binary-pulsar data on the 2PN parameters ε, ζ .

ables \dot{P} or δ_A directly defines (when assuming the above indicated values of the masses) a 1σ constraint on some linear combination of ζ and ε . In the cases of PSR 1913+16 or PSR 1534+12, the simultaneous measurement of three observables (\dot{P} - $\dot{\omega}$ - γ or $\dot{\omega}$ - γ - s), which are predicted to be some functions of the four quantities m_A, m_B, ζ , and ε , defines (by eliminating the two unknown masses between the three equations and by adding in quadrature the 1σ errors on the three observables) a 1σ constraint²² on some linear combination of ζ and ε . Summarizing: each set of pulsar data leads to a reduced χ^2 of the form $\chi_P^2(\zeta, \varepsilon) = (\zeta + \lambda_P \varepsilon - \mu_P)^2 / \sigma_P^2$, equivalent to the 1σ constraint $-\sigma_P < \zeta + \lambda_P \varepsilon - \mu_P < \sigma_P$. We find the bounds (at the 1σ level for the first three tests, and at the 90% C.L. for PSR 1800-27)

$$\text{PSR 1913+16: } -4 \times 10^{-4} < \zeta - 5 \times 10^{-2} \varepsilon < 7 \times 10^{-3}, \quad (5.24a)$$

$$\text{PSR 1534+12: } -8 \times 10^{-2} < \zeta + 0.15 \varepsilon < -10^{-4}, \quad (5.24b)$$

$$\text{PSR 0655+64: } -7 \times 10^{-3} < \zeta < 4 \times 10^{-3}, \quad (5.24c)$$

$$\text{PSR 1800-27: } |\zeta + \varepsilon/2| < 1.5 \times 10^{-2}. \quad (5.24d)$$

These four allowed regions of the ε - ζ plane are displayed in Fig. 8. Clearly pulsar data favor only a small neighborhood of the origin $\varepsilon = \zeta = 0$, i.e., of general relativity. To combine

²⁰The notation \mathcal{V} is a reminder of the fact that this quantity measures some mean orbital velocity.

²¹This follows from the near parallelism of the \dot{P} and $\dot{\omega}$ curves in the mass plane.

²²An alternative method is to start from the full $\chi^2(m_A, m_B, \zeta, \varepsilon) \equiv \sum_\alpha [q_\alpha^{\text{obs}} - q_\alpha^{\text{th}}(m_A, m_B, \zeta, \varepsilon)]^2 / \sigma_\alpha^2$ associated with the three measurements $q_\alpha = q_\alpha^{\text{obs}} \pm \sigma_\alpha$, and to reduce it to a function of ζ and ε by minimizing over m_A and m_B . Note that we neglect the correlations between the three observables q_α .

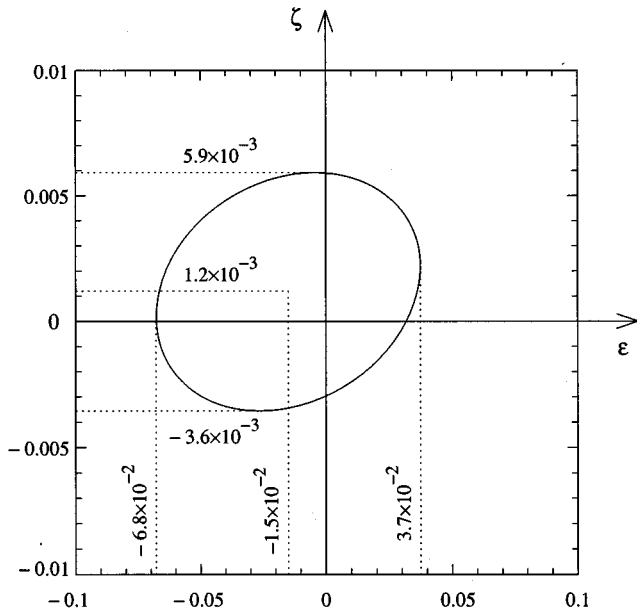


FIG. 9. Region of the ε - ζ plane allowed at the 1σ level by the four tests of Fig. 8.

the constraints on ε and ζ coming from different pulsar experiments, we have added their individual χ^2 (as defined above) as if they were part of a total experiment with uncorrelated Gaussian errors: $\chi_{\text{tot}}^2(\varepsilon, \zeta) = \chi_{1913+16}^2(\varepsilon, \zeta) + \chi_{1534+12}^2(\varepsilon, \zeta) + \chi_{0655+64}^2(\varepsilon, \zeta) + \chi_{1800-27}^2(\varepsilon, \zeta)$. [In spite of the non-Gaussian statistics of the gravitational Stark effect in PSR 1800–27, we used the bound (5.17) as an ‘‘effective 1σ level.’’] In the approximation explained above, each χ^2 is quadratic in ε and ζ . Therefore, the sum $\chi_{\text{tot}}^2(\varepsilon, \zeta)$ is a quadratic form in ε and ζ . The contour level $\Delta\chi_{\text{tot}}^2(\varepsilon, \zeta) = 2.3$, where²³ $\Delta\chi_{\text{tot}}^2 \equiv \chi_{\text{tot}}^2 - (\chi_{\text{tot}}^2)_{\text{min}}$, defines for two degrees of freedom the 68% C.L. (1σ level) ellipse represented in Fig. 9.

In conclusion, our analysis of these four binary-pulsar tests yields the bounds (at the combined 68% C.L.)

$$|\varepsilon| < 7 \times 10^{-2}, \quad |\zeta| < 6 \times 10^{-3}. \quad (5.25)$$

Because of our truncation of the observables (5.22) to their lowest order term in ε and ζ , these values should be considered only as estimates of the constraints that binary-pulsar data can provide. They show nevertheless that possible 2PN deviations from general relativity can be tested with great accuracy in binary-pulsar experiments, whereas we saw in Sec. V B that they are almost impossible to detect in the solar system.

It should be noted that though the inequalities (5.25) look numerically less stringent than the solar-system constraints, $|\bar{\gamma}| < 2 \times 10^{-3}$, $|\bar{\beta}| < 6 \times 10^{-4}$, they may, in some theoretical

²³The nice global consistency of the independent pulsar tests, proven by the overlapping of the strips in Fig. 8, means that the overall goodness-of-fit criterion associated with $(\chi_{\text{tot}}^2)_{\text{min}}$ is satisfied. This entitles us to use the variation of χ_{tot}^2 above $(\chi_{\text{tot}}^2)_{\text{min}}$ to define meaningful error levels on ε and ζ .

models, yield stronger bounds on the basic coupling parameters. For instance, in the single scalar case, we deduce from Eqs. (5.5) and (5.25) that pulsars give the limit $\alpha_0^2 < (6/\beta_0^2) \times 10^{-3}$, while solar tests give $\alpha_0^2 < \inf(1, 1.2/|\beta_0|) \times 10^{-3}$. Therefore, when $|\beta_0| > 5$ pulsar tests seem more restrictive. This conclusion is, however, not quite justified within our present approximate analytical treatment, because when $|\beta_0| > 5$ nonperturbative effects [40] become important so that one cannot neglect the higher-order terms $O(s^n)$ in Eqs. (5.22). For a detailed study of the regions in the (α_0, β_0) plane actually allowed when taking into account nonperturbative effects, see [49,50].

VI. CONCLUSIONS

We proposed in this paper a theory-based framework for conceiving and interpreting experimental tests of relativistic gravity. Previous frameworks were characterized by a phenomenological attitude. Eddington [1] initiated such an approach by assuming that the (static) spherically symmetric one-body solution in a generalized relativistic theory of gravity could differ from the Schwarzschild solution in having arbitrary coefficients in front of the different powers of the small parameter $Gm/\rho c^2$. Namely, he wrote

$$-g_{00} = 1 - 2\alpha \frac{Gm}{\rho c^2} + 2\beta \left(\frac{Gm}{\rho c^2} \right)^2 + \dots, \quad (6.1a)$$

$$g_{ij} = \delta_{ij} \left[1 + 2\gamma \frac{Gm}{\rho c^2} + \dots \right], \quad (6.1b)$$

thereby introducing at the first post-Newtonian (1PN) level two independent phenomenological parameters β and γ with values one in general relativity [after having remarked that the Newtonian level parameter α can be conventionally fixed to unity]. The Eddington approach has been extended in several directions. Following some pioneering work of Schiff [2] and Baierlein [4], Nordtvedt [5] and Will [6] introduced ten independent phenomenological parameters, β , γ , ξ , α_1 , α_2 , α_3 , ζ_1 , ζ_2 , ζ_3 , ζ_4 , to describe the most general N -body metric at the 1PN level. Their subsequent work delineated the various symmetries (e.g., boost symmetry, local position invariance) broken by the inclusion of some subset of these ten ‘‘PPN’’ parameters. In particular, their work made it clear that β and γ play a privileged role in respecting more symmetries than the other parameters. Epstein and Shapiro [12], and Fischbach and Freeman [13], extended the original Eddington expansion (6.1) of the *one-body* metric by introducing a parameter $\epsilon_{ES} \equiv \delta_{FF}$ describing the *second post-Newtonian* (2PN) order contribution to the (isotropic) spatial metric:

$$g_{ij} = \delta_{ij} \left[1 + 2\gamma \frac{Gm}{\rho c^2} + \frac{3}{2} \epsilon_{ES} \left(\frac{Gm}{\rho c^2} \right)^2 + \dots \right]. \quad (6.2)$$

Benacquista and Nordtvedt [20–22] tried to extend directly the N -body PPN formalism to the *second post-Newtonian* order by introducing a large number of *a priori* independent parameters. Finally, in a somewhat different vein, Damour and Taylor [31,32] introduced a phenomenological approach

specifically adapted to extracting the maximum possible number of relativistic gravity tests from binary-pulsar data (“parametrized post-Keplerian” formalism).

By contrast with such phenomenological approaches, we have systematically adopted in the present paper (which is an extension of our previous work [24]) a *theory-based* approach. Instead of trying to parametrize any conceivable phenomenological deviation from general relativity, we work within the simplest and best motivated class of non-Einsteinian theories: the tensor-multiscalar theories in which gravity is mediated by a tensor field ($g_{\mu\nu}^*$) together with one or several scalar fields (φ^a ; $a=1,2,\dots,n$). These theories are, in our opinion, preferred for three types of reasons: (i) massless scalars naturally appear as partners of the graviton in most unified theories (from Kaluza-Klein to string theory);

(ii) this is the only known class of theories respecting the basic tenets of field theory (notably the absence of negative-energy excitations) in which very high precision tests of the equivalence principle can naturally be compatible with post-Newtonian deviations at a measurable level;²⁴ (iii) they naturally “explain” the key role played by the original Eddington parameters β and γ at the 1PN level, and have a simple enough structure (in spite of their great generality²⁵) to allow one to work out in detail their observational consequences at the 2PN level.

Our main results are the following. Two, and only two, new parameters (beyond the non-Einsteinian 1PN parameters $\bar{\beta}\equiv\beta-1$, $\bar{\gamma}\equiv\gamma-1$) quantify possible *non-Einsteinian* effects at the 2PN level: ε and ζ . The role of these 2PN parameters is threefold.

(i) They parametrize 2PN deviations from the general relativistic (N -body) physical metric tensor:

$$\begin{aligned} \delta g_{00}(x) = & \frac{\varepsilon}{3c^6} U^3(\mathbf{x}) + \frac{\varepsilon}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')U^2(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \frac{2\zeta}{c^6} \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \int d^3\mathbf{x}'' \frac{G\sigma(\mathbf{x}'')U(\mathbf{x}'')}{|\mathbf{x}'-\mathbf{x}''|} \\ & + \frac{2\zeta}{c^6} U(\mathbf{x}) \int d^3\mathbf{x}' \frac{G\sigma(\mathbf{x}')U(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + O\left(\frac{\bar{\beta}}{c^6}, \frac{\bar{\gamma}}{c^6}\right) + O\left(\frac{1}{c^8}\right), \end{aligned} \quad (6.3a)$$

$$\delta g_{0i}(x) = O\left(\frac{\bar{\beta}}{c^5}, \frac{\bar{\gamma}}{c^5}\right) + O\left(\frac{1}{c^7}\right), \quad (6.3b)$$

$$\delta g_{ij}(x) = O\left(\frac{\bar{\beta}}{c^4}, \frac{\bar{\gamma}}{c^4}\right) + O\left(\frac{1}{c^6}\right). \quad (6.3c)$$

(ii) They determine the renormalizations, due to *self-gravity* effects, of the various effective gravitational coupling parameters between massive bodies:

$$\begin{aligned} G_{AB}/G = & 1 + \eta \left(\frac{E_A^{\text{grav}}}{m_A c^2} + \frac{E_B^{\text{grav}}}{m_B c^2} \right) + 4\zeta \left(\frac{E_A^{\text{grav}}}{m_A c^2} \right) \left(\frac{E_B^{\text{grav}}}{m_B c^2} \right) \\ & + \left(\frac{\varepsilon}{2} + \zeta \right) \frac{\langle U^2 \rangle_A + \langle U^2 \rangle_B}{c^4} + O(\bar{\beta}, \bar{\gamma}) O(s^2) \\ & + O(s^3), \end{aligned} \quad (6.4a)$$

$$\bar{\gamma}_{AB} = \bar{\gamma} + O(\bar{\beta}, \bar{\gamma}) O(s) + O(s^2), \quad (6.4b)$$

$$\begin{aligned} \bar{\beta}_{BC}^A = & \bar{\beta} + (\varepsilon + \zeta) \frac{E_A^{\text{grav}}}{m_A c^2} + \zeta \left(\frac{E_B^{\text{grav}}}{m_B c^2} + \frac{E_C^{\text{grav}}}{m_C c^2} \right) \\ & + O(\bar{\beta}, \bar{\gamma}) O(s) + O(s^2). \end{aligned} \quad (6.4c)$$

Here, G_{AB} denotes the effective Newtonian constant measuring the strength of the $O(m_A m_B / r_{AB})$ leading gravitational coupling between A and B , while $\bar{\gamma}_{AB}$ and $\bar{\beta}_{BC}^A$ denote effective Eddington parameters measuring the strength of the $O(m_A m_B (\mathbf{v}_A - \mathbf{v}_B)^2 / r_{AB} c^2)$ and $O(m_A m_B m_C / (r_{AB} r_{AC} c^2))$ post-Newtonian couplings. Moreover, η stands for the combination $4\bar{\beta} - \bar{\gamma} \equiv 4\beta - \gamma - 3$, while $s \sim E^{\text{grav}}/mc^2$ denotes the strength of self-gravity effects. The more complete version of our results on self-gravity renormalizations is given in Eqs. (3.29).

(iii) They determine the renormalizations of the locally measured coupling parameters (e.g., the local gravitational

²⁴For instance, in all positive-energy vector theories coupled (at the linear level) to some current, the existing equivalence principle tests at the 10^{-12} level necessarily constrain all post-Newtonian deviations at the $\sim 10^{-9}$ level. Though current unified models suggest the existence of massless scalar fields with composition-dependent couplings, they leave open the possibility of a very small parameter relating post-Newtonian deviations to equivalence-principle tests (e.g., a factor $\sim 10^{-5}$ was found in string-inspired models [26]).

²⁵Indeed, we consider the most general n -scalar models described by $1+n(n-1)/2$ arbitrary functions of n variables.

constant in physical units \tilde{G}_{loc} due to the presence of distant ‘‘spectator’’ matter (say, a mass m_S at a distance D) around the considered gravitating system:

$$\tilde{G}_{\text{loc}} = \tilde{G}_{\infty} \left[1 - \frac{Gm_S}{Dc^2} \eta \right] + O\left(\frac{1}{D^2}\right), \quad (6.5a)$$

$$\bar{\gamma}_{\text{loc}} = \bar{\gamma}_{\infty} + 4 \frac{Gm_S}{Dc^2} \bar{\beta} (2 + \bar{\gamma}) + O\left(\frac{1}{D^2}\right), \quad (6.5b)$$

$$\bar{\beta}_{\text{loc}} = \bar{\beta}_{\infty} - \frac{Gm_S}{Dc^2} \left(\frac{\varepsilon}{2} + \zeta - 8\bar{\beta}^2 \right) + O\left(\frac{1}{D^2}\right). \quad (6.5c)$$

On the other hand, we find that ε and ζ *do not* enter light-deflection nor time-delay experiments at the 2PN level (see Appendix C). In particular, we find that the ‘‘post-Eddington’’ formal 2PN parameter introduced in Refs. [12,13], as recalled in Eq. (6.2) above, is the following function of the 1PN parameters $\bar{\beta} \equiv \beta - 1$, $\bar{\gamma} \equiv \gamma - 1$:

$$\epsilon_{ES} = 1 + \frac{4}{3} \bar{\beta} + \frac{15 + 8\bar{\gamma}}{6} \bar{\gamma}. \quad (6.6)$$

This shows that second-order light-deflection (or time-delay) experiments do not probe any theoretically-motivated 2PN deviations from general relativity. More generally, after discussing observable effects linked to $\varepsilon \neq 0$ or $\zeta \neq 0$ (in planetary perihelia, lunar laser ranging, or other strong-equivalence-principle tests, and clock experiments), we conclude that the solar system is not an appropriate testing ground for probing possible 2PN deviations from general relativity. More precisely, we find that in the best cases, foreseeable technology might barely be able to constrain ε and ζ at the order unity level, while, in the worst cases, these parameters might complicate the interpretation of 1PN experiments by contaminating some observables.

This seemingly negative conclusion is, however, to be tempered by the following two positive conclusions of ours.

(a) We identified binary pulsar experiments as an excellent testing ground for the 2PN structure of relativistic gravity. By a simplified (linearized) analysis of existing data on the binary systems PSR 1913+16, PSR 1534+12, PSR 0655+64, and PSR 1800–27, we were in fact able to constrain already ε and ζ at the level

$$|\varepsilon| < 7 \times 10^{-2}, \quad |\zeta| < 6 \times 10^{-3}. \quad (6.7)$$

(b) We stressed that solar-system experiments are well suited to measuring with high precision the 1PN parameter $\bar{\gamma}$ which is of greatest significance among all post-Einstein parameters. Indeed, our theory-based analysis shows very clearly that $\bar{\gamma}$ is a direct measure of the coupling strength of matter to the scalar fields. All the other post-Einstein parameters ($\bar{\beta}$, ε , ζ , ...) necessarily tend to zero with $\bar{\gamma}$ in all theories having only positive-energy excitations (more about this below). From this (theory-based) point of view, the most important solar-system experiments would be high-precision light-deflection or time-delay experiments reaching the level, say, $\bar{\gamma} \approx 3 \times 10^{-7}$, which comes out naturally from mechanisms of cosmological attraction of tensor-scalar models to-

ward general relativity [27,26].²⁶ This conclusion should not be interpreted as meaning that high-precision measurements of $\bar{\beta}$ are uninteresting. As we saw above, they can, in some cases (e.g., a large β_0 in the single scalar case) be more sensitive probes of possible non-Einsteinian effects. However, in absence of any *a priori* information on the magnitude of β_0 (or, more generally, β^a_b), it remains that $\bar{\gamma}$ is the only securely direct measurement of the scalar coupling.

Let us end by stressing again some of the conceptual and technical differences between the theory-based framework introduced here, and the usual Eddington-Nordtvedt-Will PPN framework (and its extensions). The natural range of values of the PPN parameters is uniformly supposed to be of order unity, independently of the post-Newtonian order at which they appear. In our approach, there is one basic set of (experimentally small) coupling parameters,

$$\alpha_a = \frac{\partial \ln m(\varphi)}{\partial \varphi^a}, \quad (6.8)$$

where $m(\varphi)$ is the mass of a particle in the Einstein conformal frame. (This way of writing the definition of α_a is of great generality as it encompasses both self-gravity effects [24] and possible composition-dependent effects [26].)

All the phenomenological parameters $\bar{\gamma}$, $\bar{\beta}$, ζ , ε , ... measuring the 1PN, 2PN, etc. deviations from general relativity (see [24] for a list of the 3PN and higher parameters) are explicitly constructed by contracting at least two of the basic α_a 's with objects built from successive covariant field derivatives of the α_a 's: $\beta_{ab} \equiv D_a \alpha_b$, $\beta'_{abc} \equiv D_a D_b \alpha_c$, etc. (see Sec. II B for the definition of D_a). In particular, internal indices in the scalar-field space being raised and lowered by means of the positive-definite metric $\gamma_{ab}(\varphi^c)$ defining the kinetic energy of the scalar fields,

$$\bar{\gamma} = -2 \frac{\alpha_a \alpha^a}{1 + \alpha_a \alpha^a}, \quad (6.9a)$$

$$\bar{\beta} = \frac{1}{2} \frac{\alpha^a \beta_{ab} \alpha^b}{(1 + \alpha_a \alpha^a)^2}, \quad (6.9b)$$

$$\zeta = \frac{\alpha_a \beta^a_b \beta^b_c \alpha^c}{(1 + \alpha_a \alpha^a)^3}, \quad (6.9c)$$

$$\varepsilon = \frac{\beta'_{abc} \alpha^a \alpha^b \alpha^c}{(1 + \alpha_a \alpha^a)^3}. \quad (6.9d)$$

Therefore, contrary to the PPN philosophy where, say, the 1PN parameters $\bar{\gamma}$ and $\bar{\beta}$ could be accidentally small and the 2PN ones ζ and ε of order unity, our approach suggests that all of them tend to zero with $\bar{\gamma}$, or more precisely with $\alpha^2 = -\bar{\gamma}/(2 + \bar{\gamma})$ which is a positive measure of the total coupling strength of matter to the scalar field. [The ones, like

²⁶We note that the space mission concept SORT (Solar Orbit Relativity Test), proposed to the European Space Agency, aims at the level $\bar{\gamma} \sim 10^{-7}$.

ε , which are cubic in the α 's are even expected to be much smaller than the quadratically small ones²⁷ $\bar{\gamma}$, $\bar{\beta}$, ζ .]

Finally, let us note that, as far as we are aware, the present work is the first one to use in an effective way a direct classical diagrammatic approach to the relativistic N -body problem.²⁸ We hope to have convinced the reader of the technical power of this method. Indeed, once one is used to the notation, our results (6.9) on the complete set of parameters entering the first two post-Newtonian levels can be obtained in drawing just half a page of simple diagrams.

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APPENDIX A: EXPLICIT DIAGRAMMATIC CALCULATIONS

In order to compute explicitly the diagrams of Fig. 7, we must first derive the expressions of the propagators and of the different vertices of Fig. 4. As discussed in Sec. III A, we need to expand the gauged-fixed action of the theory up to the fourth order in $h_{\mu\nu} \equiv g_{\mu\nu}^* - f_{\mu\nu}$ and $\varphi^a - \varphi_0^a$. We need only the terms quadratic in $h_{\mu\nu}$ in the Einstein-Hilbert action (2.4) to define the graviton propagator (see, e.g., [51] for higher-order terms). We get easily

$$S_{\text{spin } 2} = \frac{c^3}{16\pi G_*} \int d^4x \left[-\frac{1}{2} (\partial_\mu h_{\alpha\beta}) Q^{\alpha\beta\gamma\delta} (\partial^\mu h_{\gamma\delta}) + \frac{1}{2} \left(\partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h_\nu^\nu \right)^2 + O(h^3) \right], \quad (\text{A1})$$

where the indices are raised with the flat metric $f^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and where $Q^{\alpha\beta\gamma\delta} \equiv \frac{1}{4} (f^{\alpha\gamma} f^{\beta\delta} + f^{\alpha\delta} f^{\beta\gamma} - f^{\alpha\beta} f^{\gamma\delta})$. To invert this kinetic term, it is necessary to fix the gauge. The most convenient choice for our calculations will be the harmonic gauge, corresponding to a gauge-fixing term $-(c^3/32\pi G_*) \int d^4x (\partial_\nu h_\mu^\nu - \frac{1}{2} \partial_\mu h_\nu^\nu)^2 + O(h^3)$ (see, e.g., [17] for the exact harmonic-gauge-fixing term). This cancels exactly the second term of (A1), and we get, after an integration by parts,

$$S_{\text{spin } 2}^{\text{GF}} = \frac{c^3}{16\pi G_*} \int d^4x \frac{1}{2} h_{\alpha\beta} Q^{\alpha\beta\gamma\delta} \square_f h_{\gamma\delta} + \text{tot. div.} + O(h^3), \quad (\text{A2})$$

²⁷Note that the expected extra smallness of ε gives a mnemonic rule for remembering its definition.

²⁸Bertotti and Plebanski [37] mainly discussed general features of a classical diagrammatic expansion, while previous work by the Japanese school [10,51] had used quantum diagrams and then converted S -matrix elements into an effective N -body potential.

where $\square_f \equiv f^{\mu\nu} \partial_\mu \partial_\nu$ is the flat d'Alembertian. This kinetic term has therefore the form $\int d^4x (-\frac{1}{2} h \mathcal{P}_h^{-1} h)$, where \mathcal{P}_h defines the graviton propagator, represented as a curly line in Fig. 7. In terms of the Green function (3.17), satisfying $\square_f \mathcal{G}(x) = -4\pi \delta^{(4)}(x)$, we thus get

$$\mathcal{P}_{\alpha\beta\gamma\delta}^h(x, y) = \frac{4G_*}{c^3} P_{\alpha\beta\gamma\delta} \mathcal{G}(x - y), \quad (\text{A3})$$

where $P_{\alpha\beta\gamma\delta} \equiv f_{\alpha\gamma} f_{\beta\delta} + f_{\alpha\delta} f_{\beta\gamma} - f_{\alpha\beta} f_{\gamma\delta}$ is the inverse of $Q^{\alpha\beta\gamma\delta}$.

Let us now expand the action (2.5) of the scalar fields up to the fourth order in $\varphi^a - \varphi_0^a$ or $h_{\mu\nu}$. To simplify the expressions, it is convenient to choose Riemann normal coordinates at φ_0 in the internal scalar space, so that the metric γ_{ab} can be expanded as in Eq. (3.15). It is also useful to define the origin of this scalar space at φ_0 , i.e., to choose the coordinates φ^a of this space so that $\varphi_0^a = 0$. Then, the scalar-field action (2.5) can be expanded as

$$S_{\text{spin } 0} = -\frac{c^3}{4\pi G_*} \int d^4x \frac{1}{2} \partial_\mu \varphi^a \partial_\nu \varphi^b \times (f^{\mu\nu} - h^{\mu\nu} + h_\rho^\mu h^{\rho\nu} + O(h^3)) \times \left(1 + \frac{1}{2} h_\alpha^\alpha - \frac{1}{2} h_{\alpha\beta} Q^{\alpha\beta\gamma\delta} h_{\gamma\delta} + O(h^3) \right) \times \left(\gamma_{ab}(\varphi_0) - \frac{1}{3} R_{abcd}(\varphi_0) \varphi^c \varphi^d + O(\varphi^3) \right), \quad (\text{A4})$$

where the terms inside parentheses are respectively the expansions of $g_{**}^{\mu\nu}$, $\sqrt{g_{**}}$, and $\gamma_{ab}(\varphi)$. The kinetic term of the scalar fields reads therefore

$$(c^3/4\pi G_*) \int d^4x \frac{1}{2} \varphi^a \gamma_{ab}(\varphi_0) \square_f \varphi^b + \text{tot. div.} = \int d^4x \left(-\frac{1}{2} \varphi \mathcal{P}_\varphi^{-1} \varphi \right),$$

where

$$\mathcal{P}_\varphi^{ab}(x, y) = \frac{G_*}{c^3} \gamma^{ab}(\varphi_0) \mathcal{G}(x - y) \quad (\text{A5})$$

is the scalar propagator, represented as a straight line in Fig. 7. [As before, γ^{ab} denotes the inverse of γ_{ab} .]

We can also derive from (A4) the expressions of the vertices connecting scalar fields and gravitons. Our conventions for defining vertices are the following: (i) We first define some formal ‘‘global’’ vertices $V_i \equiv iS_i$ when considering (as in Figs. 3 and 4) the gravitational sector as a whole, $\Phi = (\varphi, h)$. (ii) We then define the individual vertices $V_i(\varphi, h)$ by formally expanding the global multilinear forms $V_i(\Phi)$ as if Φ were equal to the sum $\varphi + h$, e.g.,

$$V_3(\varphi + h, \varphi + h, \varphi + h) = V_3(\varphi, \varphi, \varphi) + 3V_3(\varphi, \varphi, h) + 3V_3(\varphi, h, h) + V_3(h, h, h). \quad (\text{A6})$$

This convention allows us to use directly the simple global diagrams of Fig. 6 with the intuitive replacements of Fig. 7. For instance, the vertex of order $O(\varphi\varphi h)$, entering the first T diagram of Fig. 7 as well as the first four F diagrams and the first three H diagrams, reads

$$T_{\varphi\varphi h} = \frac{c^3}{4\pi G_*} \int d^4x \gamma_{ab}(\varphi_0) Q^{\mu\nu\alpha\beta} \partial_\mu \varphi^a \partial_\nu \varphi^b h_{\alpha\beta}, \quad (\text{A7})$$

where $Q^{\mu\nu\alpha\beta}$ is the same tensor as in (A1) above. In other words, the kernel defining the $T_{\varphi\varphi h}$ vertex is

$$(T_{\varphi\varphi h})_{ab}^{\alpha\beta}(x_1, x_2, x_3) = \frac{c^3}{4\pi G_*} \gamma_{ab}(\varphi_0) Q^{\mu\nu\alpha\beta} \times \frac{\partial}{\partial x_3^\mu} \delta(x_3 - x_1) \frac{\partial}{\partial x_3^\nu} \delta(x_3 - x_2). \quad (\text{A8})$$

Note that it is symmetric in x_1, x_2 (scalar lines), but not in all variables since they represent physically inequivalent lines. The vertex connecting two scalar lines and two graviton lines, entering the second X diagram of Fig. 7, reads

$$X_{\varphi\varphi hh} = -\frac{c^3}{12\pi G_*} \int d^4x \gamma_{ab}(\varphi_0) \times \left(2f^{\mu\alpha} Q^{\nu\beta\gamma\delta} - \frac{1}{2} f^{\mu\nu} Q^{\alpha\beta\gamma\delta} \right) \partial_\mu \varphi^a \partial_\nu \varphi^b h_{\alpha\beta} h_{\gamma\delta}. \quad (\text{A9})$$

Finally, the vertex connecting four scalar lines, entering the first X diagram of Fig. 7, reads

$$X_{\varphi^4} = \frac{c^3}{6\pi G_*} \int d^4x R_{abcd}(\varphi_0) \varphi^c \varphi^d \partial_\mu \varphi^a \partial^\mu \varphi^b, \quad (\text{A10})$$

where the first bracket comes from the expansion of $m_A(\varphi)$ around φ_0 , and the second one from the expansion of $d\mathcal{S}_A^*$ around the flat metric. All the fields appearing in (A11) must be evaluated on the world line $x^\mu = x^\mu(\tau_A)$. As above, $d\tau_A \equiv (1 - \mathbf{v}_A^2/c^2)^{1/2} dt$ denotes the Minkowski proper time of body A , $u_A^\mu \equiv dx_A^\mu/c d\tau_A$ denotes its (Minkowski) unit four-velocity, and the index 0 means that the corresponding quantities are evaluated at $\varphi^a = \varphi_0^a (=0)$. The different vertices connecting material bodies to gravitons or scalar fields can thus be read directly from (A11), taking into account both the factors 1, 2, or 3 entering their definition in Fig. 4 and the

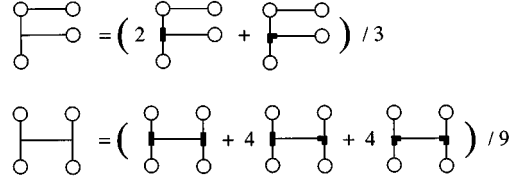


FIG. 10. Decomposition of the F and H diagrams of Fig. 6, showing in bold the lines involving derivatives in the vertex $\Phi\partial\Phi\partial\Phi$. Such a decomposition is useful to compute the last two F and H diagrams of Fig. 7 without symmetrizing the three-graviton vertex they contain.

where R_{abcd} is the Riemann curvature tensor of γ_{ab} . It should be noted that the four scalar lines of this vertex are not equivalent, since two of them involve the derivatives $\partial_\mu \varphi^a$ of the fields. The same remark holds also for the multi-graviton vertices of order $O(h^3)$ and $O(h^4)$ in (A1). One should therefore symmetrize these vertices [i.e., write the distributional kernel $X_{\varphi^4}(x_1, x_2, x_3, x_4)$ read off (A10) as a symmetric function of x_1, x_2, x_3, x_4] before computing the Fokker action (3.14). Alternatively, one can also use their nonsymmetric form, but take into account the different ways to choose the lines involving derivatives in the diagrams of Fig. 7. This does not change anything for the X diagrams, since the Lagrangian is anyway symmetrized over the four bodies A, B, C, D , but this leads to the numerical weights displayed in Fig. 10 for the F and H diagrams.

Let us now consider the matter action (2.16), describing N self-gravitating bodies. As shown in Sec. III A, we need to expand it only up to the third order in $h_{\mu\nu}$ and φ^a . Still using Riemann normal coordinates at $\varphi_0=0$ in the φ space, we get easily

$$S_m = - \sum_{A=1}^N \int m_A^0 c^2 d\tau_A \left[1 + (\alpha_a^A)_0 \varphi^a + \frac{1}{2} (\beta_{ab}^A + \alpha_a^A \alpha_b^A)_0 \varphi^a \varphi^b + \frac{1}{6} (\beta_{abc}^A + \beta_{ab}^A \alpha_c^A + \beta_{bc}^A \alpha_a^A + \beta_{ca}^A \alpha_b^A + \alpha_a^A \alpha_b^A \alpha_c^A)_0 \varphi^a \varphi^b \varphi^c + O(\varphi^4) \right] \left[1 - \frac{1}{2} h_{\mu\nu} u_A^\mu u_A^\nu - \frac{1}{8} (h_{\mu\nu} u_A^\mu u_A^\nu)^2 - \frac{1}{16} (h_{\mu\nu} u_A^\mu u_A^\nu)^3 + O(h^4) \right], \quad (\text{A11})$$

binomial factors coming from our convention (ii) above; cf. Eq. (A6). The linear interaction terms read

$$I_\varphi = - \sum_A \int d\tau_A m_A^0 c^2 (\alpha_a^A)_0 \varphi^a(x_A), \quad (\text{A12a})$$

$$I_h = \sum_A \int d\tau_A m_A^0 c^2 \frac{1}{2} u_A^\alpha u_A^\beta h_{\alpha\beta}(x_A). \quad (\text{A12b})$$

The corresponding spacetime sources (white blobs) read thus explicitly

$$\sigma_a(x) = - \sum_A \int d\tau_A m_A^0 c^2 (\alpha_a^A)_0 \delta^{(4)}(x - x_A(\tau)), \quad (A13a)$$

$$\sigma^{\alpha\beta}(x) = \frac{1}{2} \sum_A \int d\tau_A m_A^0 c^2 u_A^\alpha u_A^\beta \delta^{(4)}(x - x_A(\tau)). \quad (A13b)$$

As for the nonlinear interaction with the source (vertices connecting matter to several field lines), they read, when omitting a common $\sum_A \int d\tau_A m_A^0 c^2 \delta^{(4)}(x - x_A(\tau))$ in front,

$$V_{\varphi\varphi} : -(\beta_{ab}^A + \alpha_a^A \alpha_b^A), \quad (A14a)$$

$$V_{\varphi h} : \frac{1}{2} \alpha_a^A u_A^\alpha u_A^\beta, \quad (A14b)$$

$$V_{hh} : \frac{1}{4} u_A^\alpha u_A^\beta u_A^\gamma u_A^\delta, \quad (A14c)$$

$$\epsilon_{\varphi\varphi\varphi} : -\frac{1}{2} (\beta_{abc}^{\prime A} + 3\beta_{(ab}^A \alpha_c^A) + \alpha_a^A \alpha_b^A \alpha_c^A), \quad (A14d)$$

$$\epsilon_{\varphi\varphi h} : \frac{1}{4} (\beta_{ab}^A + \alpha_a^A \alpha_b^A) u_A^\alpha u_A^\beta, \quad (A14e)$$

$$\epsilon_{\varphi hh} : \frac{1}{8} \alpha_a^A u_A^\alpha u_A^\beta u_A^\gamma u_A^\delta, \quad (A14f)$$

$$\epsilon_{hhh} : \frac{3}{16} u_A^\alpha u_A^\beta u_A^\gamma u_A^\delta u_A^\xi u_A^\zeta. \quad (A14g)$$

The calculation of the different diagrams of Fig. 7 can now be performed straightforwardly. Let us summarize our diagrammatic rules: The Fokker action is a sum of contributions, each of which is represented by a diagram endowed with a numerical coefficient obtained by combining the factors indicated in Fig. 6 and in Fig. 7. Each (bare) diagram is computed by the following rules: (i) replace the white blobs by Eqs. (A13) if they have only one incident line, or by Eqs. (A14) [with an extra $\sum_A \int d\tau_A m_A^0 c^2 \delta^{(4)}(x - x_A(\tau))$] if they have several incident lines; (ii) replace each internal line by the appropriate propagator $\mathcal{P}(x, y)$; (iii) replace each field vertex by a suitably symmetrized distributional kernel; (iv) integrate over all the spacetime points. An additional rule is that infinite self-interactions are discarded. (As they have the same structure as in general relativity, this rule can probably be justified by methods similar to the ones used in deriving the 2PN Lagrangian in Einstein's theory [14].) In order to save space, we will not compute each of the diagrams in the present Appendix, but rather show the technique on the most important of them.

Let us start with the two-body interaction term $\frac{1}{2}I$ of Eq. (3.14). We get

$$\begin{aligned} \frac{1}{2}I &= \frac{1}{2} \int dx dy [\sigma_a(x) \mathcal{P}_\varphi^{ab}(x, y) \sigma_b(y) + \sigma^{\alpha\beta}(x) \mathcal{P}_{\alpha\beta\gamma\delta}^h(x, y) \sigma^{\gamma\delta}(y)] \\ &= \frac{1}{2} \sum_{A \neq B} \int d\tau_A \int d\tau_B (m_A^0 c^2) (m_B^0 c^2) \frac{G}{c^3} \mathcal{G}(x_A - x_B) [(\alpha_A \alpha_B)_0 + 2(u_A u_B)^2 - 1], \end{aligned} \quad (A15)$$

where $(\alpha_A \alpha_B) = \alpha_a^A \gamma^{ab} \alpha_b^B$ comes from the scalar propagator (first I diagram of Fig. 7), and $2(u_A u_B)^2 - 1 = u_A^\alpha u_A^\beta P_{\alpha\beta\gamma\delta} u_B^\gamma u_B^\delta$ comes from the graviton propagator (second I diagram). Introducing the notation $G_{AB} \equiv G_* [1 + (\alpha_A \alpha_B)_0]$ and $2 + \bar{\gamma}_{AB} \equiv 2[1 + (\alpha_A \alpha_B)_0]$ as in Eqs. (2.21) above, we thus get

$$\frac{1}{2}I = \frac{1}{2} \sum_{A \neq B} \int d\tau_A \int d\tau_B G_{AB} m_A^0 m_B^0 [1 + (2 + \bar{\gamma}_{AB})(u_A u_B)^2 - 1] c \mathcal{G}(x_A - x_B). \quad (A16)$$

This action describes the two-body interaction of self-gravitating bodies having arbitrary velocities. The second post-Keplerian approximation can now be obtained by expanding this result in powers of v/c , using Eqs. (3.19)–(3.21) above. We get an expression of the form (3.23), where the corrections proportional to $\bar{\gamma}_{AB}/c^4$ involve the function

$$f_1(\mathbf{r}_{AB}, \mathbf{v}_A, \mathbf{v}_B, \mathbf{a}_A, \mathbf{a}_B, \dot{\mathbf{a}}_B) = [(\mathbf{v}_A - \mathbf{v}_B)^2 (\mathbf{v}_A^2 + \mathbf{v}_B^2) / 2 - (\mathbf{v}_A \times \mathbf{v}_B)^2] / r_{AB} + \frac{1}{2} \frac{\partial^2}{\partial t_B^2} [|\mathbf{x}_A - \mathbf{x}_B(t_B)| (\mathbf{v}_A - \mathbf{v}_B(t_B))^2]_{t_B=t}. \quad (A17)$$

Note that only \mathbf{x}_B and \mathbf{v}_B are time differentiated in the second term, before setting $t_B = t$. The derivative of \mathbf{a}_B can be eliminated by an integration by parts, and up to a total derivative, we finally get

$$\begin{aligned} f_1(\mathbf{r}_{AB}, \mathbf{v}_A, \mathbf{v}_B, \mathbf{a}_A, \mathbf{a}_B) &= \frac{\mathbf{v}_{AB}^2}{2r_{AB}} [\mathbf{v}_A^2 + \mathbf{v}_B^2 + \mathbf{v}_A \cdot \mathbf{v}_B - (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})] - \frac{(\mathbf{v}_A \times \mathbf{v}_B)^2}{r_{AB}} + (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{a}_B \cdot \mathbf{v}_{AB}) + (\mathbf{v}_B \cdot \mathbf{n}_{AB})(\mathbf{a}_A \cdot \mathbf{v}_{AB}) \\ &\quad + (\mathbf{a}_A \cdot \mathbf{a}_B) r_{AB}, \end{aligned} \quad (A18)$$

where $\mathbf{v}_{AB} \equiv \mathbf{v}_A - \mathbf{v}_B$ and $\mathbf{n}_{AB} \equiv (\mathbf{x}_A - \mathbf{x}_B)/r_{AB}$.

Let us now consider the three-body interaction, written as $\frac{1}{2}V + \frac{1}{3}T$ in (3.14) and Fig. 6. The three V diagrams of Fig. 7 give the contribution

$$\begin{aligned} \frac{1}{2}V = & \frac{1}{2} \sum_{B \neq A \neq C} \int d\tau_A \int d\tau_B \int d\tau_C (m_A^0 c^2)(m_B^0 c^2)(m_C^0 c^2) \frac{G_*^2}{c^6} \mathcal{G}(x_A - x_B) \mathcal{G}(x_A - x_C) \\ & \times [(2(u_A u_B)^2 - 1 - (\alpha_A \alpha_B)_0)(2(u_A u_C)^2 - 1 - (\alpha_A \alpha_C)_0) - (\alpha_B \beta_A \alpha_C)_0 - 2(\alpha_A \alpha_B)_0(\alpha_A \alpha_C)_0]. \end{aligned} \quad (\text{A19})$$

Introducing as in Eqs. (2.21) the notation G_{AB} , $\bar{\gamma}_{AB}$, and $\bar{\beta}_{BC}^A$, this contribution can therefore be written as

$$\begin{aligned} \frac{1}{2}V = & \sum_{B \neq A \neq C} \int d\tau_A \int d\tau_B \int d\tau_C \frac{G_{AB} G_{AC} m_A^0 m_B^0 m_C^0}{c^2} c \mathcal{G}(x_A - x_B) c \mathcal{G}(x_A - x_C) \\ & \times \left\{ [1 + \bar{\gamma}_{AB} + (2 + \bar{\gamma}_{AB})(u_A u_B)^2 - 1] [1 + \bar{\gamma}_{AC} + (2 + \bar{\gamma}_{AC})(u_A u_C)^2 - 1] - 2 \bar{\beta}_{BC}^A - \frac{\bar{\gamma}_{AB} \bar{\gamma}_{AC}}{2} \right\}. \end{aligned} \quad (\text{A20})$$

The 2PK approximation can now be obtained easily by expanding this expression in powers of v/c . Note that the Newtonian approximation of the Green function (3.21), $c \mathcal{G}(x_A - x_B) = \delta(t_A - t_B)/r_{AB} + O(1/c^2)$, is sufficient for the terms proportional to $(u_A u_B)^2 - 1 = (\mathbf{v}_A - \mathbf{v}_B)^2/c^2 + O(1/c^4)$, or $(u_A u_C)^2 - 1$. The accelerations introduced by these Green functions in the other terms can be eliminated by suitable integrations by parts. For instance, one can write

$$\frac{1}{r_{AB}} \times \frac{1}{2c^2} \frac{\partial^2}{\partial t_C^2} |\mathbf{x}_A - \mathbf{x}_C(t_C)| = \text{tot. div.} - \frac{(\mathbf{v}_{AB} \cdot \mathbf{n}_{AB})(\mathbf{v}_C \cdot \mathbf{n}_{AC})}{2r_{AB}^2 c^2} + \frac{\mathbf{v}_A \cdot \mathbf{v}_C - (\mathbf{v}_A \cdot \mathbf{n}_{AC})(\mathbf{v}_C \cdot \mathbf{n}_{AC})}{2r_{AB} r_{AC} c^2}. \quad (\text{A21})$$

The T diagrams of Fig. 7 are a little more subtle to compute, because the central vertex is not located on a material body, and also because it involves derivatives of the fields. For instance, the first T diagram, involving one graviton and two scalar lines, gives the contribution

$$T_1 = \sum_{A,B,C} \int d\tau_A \int d\tau_B \int d\tau_C \int d^4x \frac{G_*^2 m_A^0 m_B^0 m_C^0}{2\pi} (\alpha_A \alpha_B)_0 u_C^\mu u_C^\nu \frac{\partial \mathcal{G}(x - x_A)}{\partial x^\mu} \frac{\partial \mathcal{G}(x - x_B)}{\partial x^\nu} \mathcal{G}(x - x_C), \quad (\text{A22})$$

where x^μ is the arbitrary spacetime location of the vertex. The lowest order term of the post-Keplerian approximation reads, thus,

$$\begin{aligned} T_1 = & \sum_{A,B,C} \int dt_A \int dt_B \int dt_C \int dt \int d^3\mathbf{x} \frac{G_*^2 m_A^0 m_B^0 m_C^0}{2\pi c^2} (\alpha_A \alpha_B)_0 u_C^\mu u_C^\nu \frac{\partial}{\partial x^\mu} \left[\frac{\delta(t - t_A)}{r_{xA}} \right] \frac{\partial}{\partial x^\nu} \left[\frac{\delta(t - t_B)}{r_{xB}} \right] \frac{\delta(t - t_C)}{r_{xC}} \times \left[1 + O\left(\frac{1}{c^2}\right) \right] \\ = & - \sum_{A,B,C} \int dt \int d^3\mathbf{x} \frac{G_* G_{AB} \bar{\gamma}_{AB} m_A^0 m_B^0 m_C^0}{4\pi c^4} \frac{(\mathbf{v}_{AC} \cdot \mathbf{n}_{xA})(\mathbf{v}_{BC} \cdot \mathbf{n}_{xB})}{r_{xA}^2 r_{xB}^2 r_{xC}} + O\left(\frac{1}{c^6}\right), \end{aligned} \quad (\text{A23})$$

where we have used $u_C^\mu \partial_\mu [\delta(t - t_A)/r_{xA}] = \delta(t - t_A) \mathbf{v}_{AC} \cdot \mathbf{n}_{xA} / r_{xA}^2 c + O(1/c^3)$. Therefore, although this T diagram is of the same formal order $G_*^2 m^3 / r^2 c^2$ as the V diagrams in the nonlinearity expansion (arbitrarily large velocities), it reduces to order $O(1/c^4)$ in the post-Keplerian approximation ($|v/c| \ll 1$). This is due to the particular form of the $T_{\varphi\phi h}$ vertex, Eq. (A7), which involves a specific combination of derivatives of the fields with the inverse of the tensor $P_{\alpha\beta\gamma\delta}$ entering the graviton propagator (A3). It should be noted that the presence of derivatives is not sufficient to conclude that the diagram is reduced by a factor $(v/c)^2$. For instance, the second T diagram of Fig. 7, involving three graviton lines, does contribute at the first post-Keplerian order,

$$\frac{1}{3}T_2 = - \sum_{B \neq A \neq C} \int dt \frac{G_*^2 m_A^0 m_B^0 m_C^0}{r_{AB} r_{AC} c^2} + O\left(\frac{1}{c^4}\right), \quad (\text{A24})$$

although the three-graviton vertex is also of the form $h \partial h \partial h$. Indeed, the dominant contribution to this diagram is proportional to the contraction $\partial_\mu \mathcal{G}(x - x_A) \partial^\mu \mathcal{G}(x - x_B) = \delta(t - t_A) \delta(t - t_B) \mathbf{n}_{xA} \cdot \mathbf{n}_{xB} / r_{xA}^2 r_{xB}^2 c^2 + O(1/c^4)$, which starts at order $1/c^2$. [An integration by parts reduces one of the Green functions to $\square_f \mathcal{G}(x) = -4\pi \delta^{(4)}(x)$, and one gets Eq. (A24) easily.] The sum of (A24) with the lowest order term of (A20) gives the 1PK contribution to the three-body interaction Lagrangian, displayed in Eqs. (2.20c) or (3.25).

Let us turn now to the most important four-body interaction terms, namely the ϵ , Z , and X diagrams of Fig. 7, which involve the new parameters (3.18). At the second post-Keplerian approximation, it is sufficient to use the instantaneous Green function $c \mathcal{G}(x_A - x_B) = \delta(t_A - t_B)/r_{AB}$ in these diagrams, and one easily gets

$$\begin{aligned} \frac{1}{3}\epsilon = & \frac{1}{2} \sum_{A \neq (B,C,D)} \int dt \frac{G_{AB}G_{AC}G_{AD}m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB}r_{AC}r_{AD}c^4} \left[1 + \frac{1}{3}\epsilon_{BCD}^A + \frac{2}{3}(\bar{\beta}_{BC}^A + \bar{\beta}_{BD}^A + \bar{\beta}_{CD}^A) + \frac{2}{3}(\bar{\gamma}_{AB} + \bar{\gamma}_{AC} + \bar{\gamma}_{AD}) \right. \\ & \left. + \frac{1}{2}(\bar{\gamma}_{AB}\bar{\gamma}_{AC} + \bar{\gamma}_{AB}\bar{\gamma}_{AD} + \bar{\gamma}_{AC}\bar{\gamma}_{AD}) + \frac{1}{3}\bar{\gamma}_{AB}\bar{\gamma}_{AC}\bar{\gamma}_{AD} \right] + O\left(\frac{1}{c^6}\right), \end{aligned} \quad (\text{A25})$$

$$\begin{aligned} \frac{1}{2}Z = & \frac{1}{2} \sum_{A \neq B \neq C \neq D} \int dt \frac{G_{AB}G_{BC}G_{CD}m_A^0 m_B^0 m_C^0 m_D^0}{r_{AB}r_{BC}r_{CD}c^4} \left[1 + \zeta_{ABCD} + 2(\bar{\beta}_{AC}^B + \bar{\beta}_{BD}^C) + \frac{1}{2}(2 + \bar{\gamma}_{BC})(\bar{\gamma}_{AB} + \bar{\gamma}_{CD} + \bar{\gamma}_{AB}\bar{\gamma}_{CD}) \right] \\ & + O\left(\frac{1}{c^6}\right). \end{aligned} \quad (\text{A26})$$

[Note that the ϵ contribution to the ϵ diagram has a nontrivial normalization: $\epsilon \sim \frac{1}{2}\epsilon G^3 m^4 / r^3 c^4$, while $Z \sim \zeta G^3 m^4 / r^3 c^4$.] The contribution of the first X diagram of Fig. 7 reads

$$\begin{aligned} \frac{1}{4}X_1 = & \frac{1}{4} \sum_{A,B,C,D} \int dt_A \int dt_B \int dt_C \int dt_D \int d^3\mathbf{x} \frac{G_*^3 m_A^0 m_B^0 m_C^0 m_D^0}{6\pi c^4} (R_{abcd} \alpha_A^a \alpha_B^b \alpha_C^c \alpha_D^d)_0 \\ & \times \frac{\delta(t-t_C)}{r_{xC}} \frac{\delta(t-t_D)}{r_{xD}} \partial_\mu \left[\frac{\delta(t-t_A)}{r_{xA}} \right] \partial^\mu \left[\frac{\delta(t-t_B)}{r_{xB}} \right] + O\left(\frac{1}{c^6}\right) \\ = & \frac{1}{24\pi} \sum_{A,B,C,D} \int dt \int d^3\mathbf{x} \frac{G_*^3 m_A^0 m_B^0 m_C^0 m_D^0 (\mathbf{n}_{xA} \cdot \mathbf{n}_{xB})}{r_{xA}^2 r_{xB}^2 r_{xC} r_{xD}} \chi_{ACBD} + O\left(\frac{1}{c^6}\right). \end{aligned} \quad (\text{A27})$$

Note that although χ_{ACBD} is antisymmetric in A, C (and B, D), and although the sum is taken over all possible choices of A, B, C, D , this contribution does not vanish identically since the integrand $(\mathbf{n}_{xA} \cdot \mathbf{n}_{xB}) / r_{xA}^2 r_{xB}^2 r_{xC} r_{xD}$ is not symmetric in A, C . This contribution vanishes nevertheless in the 2PN approximation, cf. (3.32), or if the theory involves only one scalar field [$R_{abcd}(\varphi) = 0$]. The contributions of the two other X diagrams of Fig. 7 are computed in the same way. The last one gives the usual contribution obtained in general relativity, and we find for the second X diagram

$$\frac{6}{4}X_2 = -\frac{1}{8\pi} \sum_{A,B,C,D} \int dt \int d^3\mathbf{x} \frac{G_*^2 G_{AB} \bar{\gamma}_{AB} m_A^0 m_B^0 m_C^0 m_D^0}{c^4} \frac{(\mathbf{n}_{xA} \cdot \mathbf{n}_{xB})}{r_{xA}^2 r_{xB}^2 r_{xC} r_{xD}} + O\left(\frac{1}{c^6}\right). \quad (\text{A28})$$

Let us end this Appendix by a brief discussion of the F and H diagrams of Fig. 7, which are less important since the deviations from general relativity they give at 2PN order are proportional to $\bar{\gamma}$, $\bar{\gamma}^2$, or $\bar{\beta}$, and therefore already tightly constrained by present experimental data, Eqs. (5.1). It is easy to see that the first four F diagrams do not contribute at the second post-Keplerian level, because of the presence of the $T_{\varphi_{ph}}$ vertex whose graviton is directly coupled to a material body. Indeed, the calculation is similar to that of the first T diagram, Eqs. (A22) and (A23), and the factor $(u^\mu \partial_\mu \mathcal{G})^2 \propto v^2$ reduces these F diagrams to the order $G^3 m^4 v^2 / r^3 c^6$. Similarly, the second H diagram is even of order $G^3 m^4 v^4 / r^3 c^8$, and contributes only at the fourth post-Keplerian level. On the contrary, the first and third H diagrams do contribute at the 2PK level, because the graviton of their $T_{\varphi_{ph}}$ vertices is connected to a second vertex, and not directly to a material body. These diagrams involve therefore contractions of the form $\partial_\mu \mathcal{G} \partial^\mu \mathcal{G}$, like in (A24), and are indeed of order $G^3 m^4 / r^3 c^4$. Note that the four material bodies of these H diagrams are not supposed to be necessarily different from each other, since they are not directly connected by a propagator (as opposed to the I , V , ϵ , or Z diagrams). Together with the first T diagram (A23) and the second X diagram (A28), they thus yield contributions proportional to

$$\frac{\bar{\gamma}_{AA}}{1 + \bar{\gamma}_{AA}/2} = \frac{G_{AA}}{G_*} \bar{\gamma}_{AA} = -2(\alpha_A \alpha_A)_0. \quad (\text{A29})$$

This explains the presence of this factor in the static and spherically symmetric solution discussed in Sec. IV, Eq. (4.21). [The isotropic form (4.18) contains extra α_A^2 terms due to the change of coordinates (4.22).] In fact, the contribution of the first H diagram to this one-body metric is proportional to $(\alpha_A \alpha_0)(\alpha_A^2)$: One of the white blobs, involving the background value $\alpha(\varphi_0) \equiv \alpha_0$, corresponds to the point where the metric is computed, and the three other blobs correspond to body A . Note also that the first T diagram, Eq. (A23), does not contribute to the α_A^2 terms of \bar{g}_{00} , since it vanishes in the static case. Finally, let us mention that the last F and H diagrams are the usual contributions obtained in general relativity, and that the fifth F diagram is obviously proportional to the second T diagram:

$$F_5 = \sum_{D \neq C} \frac{1}{3} T_2^{(A,B,C)} \times \frac{3G_{CD} \bar{\gamma}_{CD} m_D^0}{2r_{CD} c^2} + O\left(\frac{1}{c^6}\right), \quad (\text{A30})$$

where $\frac{1}{3} T_2^{(A,B,C)}$ is given by (A24) above, symmetrized over the three bodies A, B, C .

APPENDIX B: 2PN RENORMALIZATIONS OF COUPLING PARAMETERS, THE STRONG EQUIVALENCE PRINCIPLE, AND ALL THAT

As emphasized by Nordtvedt [22], the coupling parameters (\tilde{G} , \tilde{m} , $\tilde{\beta}$, $\tilde{\gamma}$, ...) of Lorentz-invariant gravitational theories are influenced not only by the self-gravity of the interacting bodies, but also by the presence of distant ‘‘spectator’’ matter around the system. We have already studied the self-gravity renormalizations in the main text. In this Appendix, we relate our self-gravity results to the ones of Ref. [22] and show how, within the context to tensor-multiscalar theories, one can derive very easily the influence of external matter.

Let us first mention that, from the 1PK Lagrangian (2.19), one can easily give explicit expressions for all the mass parameters $M(G), M(\gamma), M(\beta), \dots$ introduced in [22]. For the convenience of the reader, we give in Table I a translation of Nordtvedt’s notation in terms of our body-dependent parameters \tilde{G}_{AB} , $\tilde{\gamma}_{AB}$, and $\tilde{\beta}_{BC}^A$ defined in Eqs. (2.21) above. As before an index 0 in one of these parameters corresponds to a non-self-gravitating body, so that the σ -model tensors α_A^a, β_A^{ab} of Eqs. (2.17) should be replaced by their weak-field counterparts (2.10)–(2.12). For instance, $\tilde{\beta}_{00}^A = \frac{1}{2}(\alpha\beta_A\alpha)_0/[1+(\alpha_A\alpha)_0]^2$ instead of (2.21c). The 2PN renormalizations of \tilde{G}_{AB} , $\tilde{\gamma}_{AB}$, and $\tilde{\beta}_{BC}^A$ due to the self-energy of the bodies have been obtained in Eqs. (3.29) above. Using them in the translation of notation of Table I, we recover the corresponding (less complete) results of [22].

Let us now turn to the study of the influence of external matter. The effect of a distant spectator body on local gravitational physics can be analyzed straightforwardly in the context of tensor-multiscalar theories. Indeed, if this spectator S is located at a distance D from a local system, the local background values of the scalar fields are changed from their values φ_0^a far from the spectator to

$$\varphi_{\text{loc}}^a = \varphi_0^a - \frac{G_* \alpha_S^a m_S}{Dc^2} + O\left(\frac{1}{D^2}\right). \quad (\text{B1})$$

All the physical quantities f which depend on these background values are therefore renormalized (compared to the value they would be measured to have at infinity) by the presence of the spectator as

$$f \rightarrow f_{\text{loc}} = f - \frac{G_* m_S}{Dc^2} \alpha_S^a \frac{\partial f}{\partial \varphi_0^a} + O\left(\frac{1}{D^2}\right). \quad (\text{B2})$$

In particular, if the spectator body is supposed to have a negligible self-energy ($\alpha_S^a = \alpha_0^a$), we get, for the effective gravitational constant (2.14) and the Eddington parameters (2.15),

$$\tilde{G} \rightarrow \tilde{G} \left[1 - \frac{Gm_S}{Dc^2} \eta \right] + O\left(\frac{1}{D^2}\right), \quad (\text{B3a})$$

$$\tilde{\gamma} \rightarrow \tilde{\gamma} + 4 \frac{Gm_S}{Dc^2} \tilde{\beta}(2 + \tilde{\gamma}) + O\left(\frac{1}{D^2}\right), \quad (\text{B3b})$$

TABLE I. Expression of Nordtvedt’s mass parameters in tensor-multiscalar theories of gravity.

Nordtvedt’s parameters	Tensor-multiscalar theories
$M(I)$	\tilde{m}_A
$\Gamma_{ij}/M(I)_i M(I)_j$	\tilde{G}_{AB}
$\gamma^{\Theta}_{ij}/\Gamma_{ij}$	$1 + \tilde{\gamma}_{AB}$
$(2\beta - 1)\Gamma_{ijk}/M(I)_i M(I)_j M(I)_k$	$(1 + 2\tilde{\beta}_{BC}^A)\tilde{G}_{AB}\tilde{G}_{AC}$
$M(G)/M(I)$	\tilde{G}_{A0}/\tilde{G}
$M(\gamma)/M(I)$	$\tilde{G}_{A0}(1 + \tilde{\gamma}_{A0})/\tilde{G}(1 + \tilde{\gamma})$
$M(\beta)/M(I)$	$\tilde{G}_{A0}(1 + 2\tilde{\beta}_{A0}^0)/\tilde{G}(1 + 2\tilde{\beta})$
$M(\beta')/M(I)$	$\tilde{G}_{A0}^2(1 + 2\tilde{\beta}_{00}^A)/\tilde{G}^2(1 + 2\tilde{\beta})$

$$\tilde{\beta} \rightarrow \tilde{\beta} - \frac{Gm_S}{Dc^2} \left(\frac{\varepsilon}{2} + \zeta - 8\tilde{\beta}^2 \right) + O\left(\frac{1}{D^2}\right), \quad (\text{B3c})$$

where $\eta \equiv 4\tilde{\beta} - \tilde{\gamma}$ as before, and where the dimensionless ratio Gm_S/Dc^2 may be replaced by its expression in physical units $\tilde{G}\tilde{m}_S/\tilde{D}c^2$. Equation (B3a) is the well-known renormalization of the gravitational constant derived in [8]. The renormalizations of $\tilde{\gamma}$ and $\tilde{\beta}$ have also been studied in Eqs. (5.15) and (3.7) of [22], but they were expressed in terms of more 2PN parameters than the simple results (B3b) and (B3c). Note that the analysis of Ref. [22] is done in a different context within which it has not (yet) been possible to relate the 2PN renormalization of $\tilde{\gamma}$ to $\tilde{\beta}$. In our context, Eq. (B3b) shows that the renormalization of $\tilde{\gamma}$ is already constrained by the 1PN experimental bounds on $|\tilde{\beta}| < 6 \times 10^{-4}$.

Of course, the renormalizations (B3) can be generalized straightforwardly to the strong-field regime, by considering the body-dependent parameters $\tilde{G}_{AB}(\varphi_0)$, $\tilde{\gamma}_{AB}(\varphi_0)$, $\tilde{\beta}_{BC}^A(\varphi_0)$, and a compact spectator body ($\alpha_S^a \neq \alpha_0^a$). This has been done in section 7.2 of Ref. [24], where we analyzed the consequences of the strong equivalence principle (SEP) in tensor-multiscalar theories. This principle states that local gravitational physics is totally independent of the presence of spectator bodies. In other words, the renormalizations (B3) and their strong-field analogs are supposed to vanish. Let us prove that the only tensor-multiscalar theories containing only positive-energy excitations ($\gamma_{ab} > 0$) and satisfying the strong equivalence principle are perturbatively equivalent to general relativity (to all orders). Indeed, we showed in [24], using Eqs. (B3), that any such theory must satisfy $\tilde{\gamma} = 0$, among other relations. From Eq. (2.15a) and the positivity of γ_{ab} , this implies $\alpha_0^a = 0$. Using now the diagrammatically evident²⁹ fact that any observable deviation from general relativity must involve at least two factors α_0^a (to fill the end blobs connected to scalar propagators, in diagrams such as Fig. 7 or any higher-order ones), we find that all non-Einsteinian terms necessarily vanish. It should be noted that this result cannot be extended to the pure scalar theories we

²⁹Note that this would not be true if we were considering quantum (loop) diagrams.

also considered in section 7.2 of [24], in which the gravitational interaction is mediated by one or several scalar fields without any tensorial contribution. We showed that the SEP implies in that case $\bar{G}_{AB} = \bar{G}$, $\bar{\gamma}_{AB} = \bar{\gamma} = -2$, and $\bar{\beta}_{BC}^A = \bar{\beta} = -1/2$, i.e., that the theory is equivalent to Nordström's theory at the 1PK level. The assumption that the σ -model metric is positive does not allow us to complement this result to higher orders.

We can also prove that some assumptions made in Ref. [21] are inconsistent within the tensor-scalar framework. Indeed, Ref. [21] assumed one could work within a class of theories containing no dipolar radiation and still differing from general relativity at 2PN order. However, we can use the fact that the dominant dipolar radiation emitted by a binary system is proportional to $\gamma_{ab}(\alpha_A^a - \alpha_B^a)(\alpha_A^b - \alpha_B^b)$. Let us first assume (as we generally do) that γ_{ab} is positive definite. Then the assumption that the dipolar radiation vanishes implies that $\alpha_A^a = \alpha_B^a$ for any body A and B , and in particular that $\alpha_A^a = \alpha^a$ if we choose a non-self-gravitating body B ($\bar{m}_B = \text{const}$). Using now the expansion of α_A^a in powers of the compactness of body A , that we derived in Eqs. (8.3) and (8.4) of [24], and the fact that $\alpha_A^a = \alpha^a$ must be verified for any body A , we can conclude that $\alpha^a = 0$. In other words, the scalar fields are totally decoupled from matter, and the theory is strictly equivalent to general relativity. Even if we drop the assumption of a positive definite γ_{ab} , i.e., if we phenomenologically allow the presence of scalar fields with negative energy (ghost modes), then the same Eqs. (8.3) and (8.4) of [24] can be used to show that the theory is equivalent to general relativity up to the 2PN order included, but not beyond (which means that it can differ from it at the 1PK order, as illustrated in section 9 of [24]).

The ease with which we derived the results of this Appendix illustrates again the power of our field-theoretical approach.

APPENDIX C: EXPLICIT EXPRESSIONS OF THE 2PN LIGHT DEFLECTION AND PERIHELION SHIFT

Before computing the explicit expression of the light-deflection angle for the one-body metric (4.18), let us show how our diagrammatic approach allows one to get, without any calculations, the structure of the final result. More precisely, let us show that the light-deflection angle for a self-gravitating body has the structure

$$O(\bar{\gamma}_{A0})(G_{A0}m_A/rc^2) + O(\bar{\gamma}_{A0}, \bar{\gamma}_{AA})(G_{A0}m_A/rc^2)^2 + O(1/c^6), \quad (C1)$$

in which neither $\bar{\beta}_{BC}^A$ nor the new 2PK parameters we introduced above enter. Indeed, the action describing the electromagnetic field minimally coupled to the physical metric $\bar{g}_{\mu\nu}$ is a conformal invariant, and can thus be written as

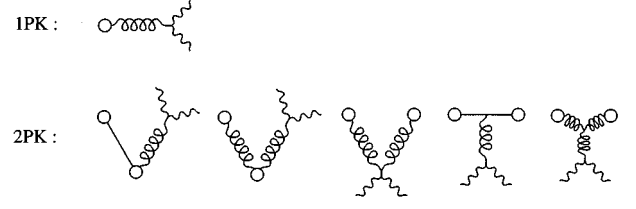


FIG. 11. Diagrams describing the interaction of light (wavy lines) with the gravitational field generated by material bodies, at orders $O(Gm/rc^2)$ –1PK– and $O((Gm/rc^2)^2)$ –2PK–.

$$S_{EM} = -\frac{1}{4} \int \frac{d^4x}{c} \sqrt{\bar{g}} \bar{g}^{\lambda\nu} \bar{g}^{\mu\rho} F_{\lambda\mu} F_{\nu\rho} \\ = -\frac{1}{4} \int \frac{d^4x}{c} \sqrt{g_*} g_*^{\lambda\nu} g_*^{\mu\rho} F_{\lambda\mu} F_{\nu\rho}. \quad (C2)$$

The second expression, involving the Einstein metric $g_{\mu\nu}^*$, shows that photons are coupled to the graviton $h_{\mu\nu} \equiv g_{\mu\nu}^* - f_{\mu\nu}$, but not to the scalar fields $\varphi^a - \varphi_0^a$. Therefore, the only diagrams describing the 2PK interaction of light with the gravitational field generated by material bodies are those of Fig. 11. [The ϵ , Z , F , H , and X diagrams enter the metric at order $O(1/c^6)$, and influence the propagation of light at the third post-Keplerian level.] The dominant contribution does not involve any scalar field, and is thus proportional to $G_* m_A / rc^2$. However, we do not have a direct experimental access to G_* , and this contribution should be rewritten in terms of the Newtonian potential $G_{A0} m_A / r$ felt by a test mass in the vicinity of body A . This can be done thanks to the identity $G_* \equiv G_{A0}(1 + \bar{\gamma}_{A0}/2)$, which derives from the definitions (2.21a) and (2.21b). We have thus recovered without any explicit calculation that the deflection of light and its time delay are both proportional to $2 + \bar{\gamma}_{A0}$ at the 1PK level, and to $2 + \bar{\gamma}$ at the 1PN level. [Remember that $\bar{\gamma}$ is usually denoted as $\gamma - 1$ in the literature, so that $2 + \bar{\gamma}$ is the usual factor $1 + \gamma$.] The corrections appearing at order $O((Gm/rc^2)^2)$ are due to the five remaining diagrams of Fig. 11. Three of them do not involve any scalar field, and are thus proportional to $G_*^2 = G_{A0} G_{B0}(1 + \bar{\gamma}_{A0}/2)(1 + \bar{\gamma}_{B0}/2)$. The other two involve a scalar line between two material bodies, and yield therefore contributions proportional to $G_*^2 (\alpha_A \alpha_B)_0 = -G_*^2 \bar{\gamma}_{AB} / (2 + \bar{\gamma}_{AB})$, where G_*^2 should be rewritten as above in terms of effective gravitational constants. In the case of a single material body A , it should be noted that the first two V diagrams of Fig. 11 do not contribute, since the same body cannot be directly connected by a propagator. On the contrary, the first T diagram of this figure does contribute, because the two blobs representing the same body A are connected to a scalar-scalar-graviton vertex, and not directly to each other. Hence this diagram yields a contribution proportional to $\bar{\gamma}_{AA} / (2 + \bar{\gamma}_{AA})$ in second-order light-deflection and time-delay experiments. In conclusion, our diagrammatic approach has allowed us to prove in a streamlined way that these experiments do not depend on $\bar{\beta}_{BC}^A$ at the 2PK level, and more precisely that they can differ from general relativ-

ity only by terms of the form indicated in Eq. (C1) in the case of a single material body A .

Let us now give explicit expressions for the 2PN light deflection and perihelion advance. Following Weinberg [52], we use here Schwarzschild-like coordinates (i.e., an ‘‘area radius’’ r) and write the integral giving the polar angle ϕ in the plane of the trajectory as

$$\phi = \pm \int \frac{\mathcal{J}\sqrt{A}\mathcal{B}dr/r^2}{[\mathcal{E}^2/c^2 - (m_0^2c^2 + \mathcal{J}^2/r^2)\mathcal{B}]^{1/2}}, \quad (\text{C3})$$

where \mathcal{E} denotes the conserved energy (including the rest mass contribution) of a test particle of mass m_0 , and \mathcal{J} its

angular momentum. [Note that these conserved quantities are related in a nonobvious way to the constants E_W and J_W used by Weinberg in Eq. (8.4.30) of [52]: $E_W \equiv (m_0c^2/\mathcal{E})^2$; $J_W \equiv \mathcal{J}c/\mathcal{E}$.] The area radius r should not be confused with the Just radius nor the g^* -harmonic radius, both denoted also by r in Sec. IV B. To rewrite the metric (4.18) in Schwarzschild coordinates

$$d\bar{s}^{-2}/A^2(\varphi_0) = -\mathcal{B}(r)c^2dt^2 + \mathcal{A}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (\text{C4})$$

we need to express the area radius r in terms of the isotropic one ρ . We find easily

$$r = \rho \left[1 + \frac{\mu_A}{\rho c^2} (1 + \bar{\gamma}_{A0}) + \frac{1}{4} \left(\frac{\mu_A}{\rho c^2} \right)^2 (1 + 3\bar{\Gamma} - 4\bar{\gamma}_{A0} - 2\bar{\gamma}_{A0}^2) + \mathcal{O}\left(\frac{1}{c^6}\right) \right], \quad (\text{C5})$$

where $\mu_A \equiv G_{A0}m_A$ is the Keplerian mass of the attracting body A , and $\bar{\gamma}_{A0}$, $\bar{\Gamma}$ are the deviations from general relativity in the spatial isotropic metric (4.18). Equations (4.19) and (4.20) give the expressions of these coefficients in tensor-multiscalar theories. Note, however, that our present calculation is valid for any static and spherically symmetric metric of the form (4.18), not necessarily the one predicted by tensor-scalar theories. The replacement of (C5) in (4.18) yields

$$\mathcal{A}(r) = 1 + 2\frac{\mu_A}{rc^2}(1 + \bar{\gamma}_{A0}) + 4\left(\frac{\mu_A}{rc^2}\right)^2 \left(1 + \frac{3}{4}\bar{\Gamma} + \frac{1}{2}\bar{\gamma}_{A0} + \frac{1}{4}\bar{\gamma}_{A0}^2 \right) + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (\text{C6a})$$

$$\mathcal{B}(r) = 1 - 2\frac{\mu_A}{rc^2} + 2\left(\frac{\mu_A}{rc^2}\right)^2 (\bar{\beta}_{AA}^0 - \bar{\gamma}_{A0}) - \frac{3}{2}\left(\frac{\mu_A}{rc^2}\right)^3 \left(\bar{B} + \bar{\Gamma} - \frac{8}{3}(1 + \bar{\gamma}_{A0})\bar{\beta}_{AA}^0 - \frac{2}{3}(2 - \bar{\gamma}_{A0})\bar{\gamma}_{A0} \right) + \mathcal{O}\left(\frac{1}{c^8}\right). \quad (\text{C6b})$$

The polar angle (C3) can now be obtained by a straightforward integration. The case of light corresponds to $m_0 = 0$, and the deflection angle $\Delta\phi$ is found to have the form

$$\Delta\phi = \Delta\phi_1 + \Delta\phi_2 + \mathcal{O}(1/c^6), \quad (\text{C7})$$

where

$$\Delta\phi_1 = \frac{2\mu_A(2 + \bar{\gamma}_{A0})}{\rho_0c^2} \left[1 - \frac{\mu_A(2 + \bar{\gamma}_{A0})}{\rho_0c^2} \right] \quad (\text{C8})$$

is the 1PK result up to a global correcting factor, and

$$\Delta\phi_2 = \frac{\pi}{4} \left(\frac{\mu_A}{\rho_0c^2} \right)^2 [15 + 3\bar{\Gamma} - 4\bar{\beta}_{AA}^0 + 8\bar{\gamma}_{A0}] \quad (\text{C9})$$

is the actual 2PK contribution. Note that, after having performed the integral (C3) in Schwarzschild coordinates, we have expressed the results in terms of ρ_0 , the minimal distance between the light ray and the center of body A , measured in *isotropic* coordinates. [Note that ρ_0 differs from the impact parameter.] The results are unchanged if one uses

g^* -harmonic coordinate to measure this minimal distance, since the transformation (4.22) introduces corrections only at order $\mathcal{O}(1/c^6)$. By contrast, the use of Schwarzschild coordinates transforms (C8) into $\Delta\phi_1 = [2\mu_A(2 + \bar{\gamma}_{A0})/r_0c^2][1 - (\mu_A/r_0c^2)]$, while (C9) remains unchanged. This second-order light deflection (C7)–(C9) agrees with previous calculations in the literature [12,13,15]. It can now be particularized to the case of tensor-multiscalar theories by using the expression (4.19b) for $\bar{\Gamma}$. We find that the term $\frac{4}{3}\bar{\beta}_{AA}^0$ in $\bar{\Gamma}$ cancels exactly the $-4\bar{\beta}_{AA}^0$ contribution in (C9), and we get

$$\Delta\phi_2 = \frac{\pi}{16} \left(\frac{\mu_A(2 + \bar{\gamma}_{A0})}{\rho_0c^2} \right)^2 \left[15 + \frac{\bar{\gamma}_{AA}}{2 + \bar{\gamma}_{AA}} \right]. \quad (\text{C10})$$

Together with Eq. (C8), this result confirms therefore the conclusion (C1) of our diagrammatic analysis: The first order deviation from general relativity is proportional to $\bar{\gamma}_{A0}$, while the second order contributions involve both $\bar{\gamma}_{A0}$ and $\bar{\gamma}_{AA}$, but not $\bar{\beta}_{AA}^0$. Note also the appearance of the same

coefficient $\mu_A(2 + \bar{\gamma}_{A0}) \equiv 2G_* m_A$ in (C8) and (C10), which is in fact a mere rewriting of the bare gravitational constant G_* in terms of observable quantities. The coefficient $\bar{\gamma}_{AA}/(2 + \bar{\gamma}_{AA}) \equiv -(\alpha_A \alpha_A)_0$ entering (C10) is due to the first T diagram of Fig. 11, where both blobs represent body A.

Let us now take the second post-Newtonian limit of the above 2PK result, as appropriate for interpreting future high-precision experiments in the solar system. The coefficient $\bar{\gamma}_{A0}$ of the leading term in (C8) should therefore be expanded as in Eq. (5.7), together with the renormalization (B3b) due to the gravitational potential of external masses, while the 2PK contribution (C9) becomes

$$\Delta\phi_2^{2\text{PN}} = \frac{\pi}{4} \left(\frac{\mu_A}{\rho_0 c^2} \right)^2 \left[15 + \frac{31}{2} \bar{\gamma} + 4 \bar{\gamma}^2 \right]. \quad (\text{C11})$$

$$\Delta\phi = \frac{6\pi}{h^2 c^2} \left[\frac{3 - \bar{\beta}_{AA}^0 + 2 \bar{\gamma}_{A0}}{3} + \frac{1}{h^2 c^2} \left(\frac{35}{4} + \frac{3}{4} \bar{\text{B}} + \frac{3}{2} \bar{\Gamma} - 9 \bar{\beta}_{AA}^0 + \frac{1}{2} (\bar{\beta}_{AA}^0)^2 + 8 \bar{\gamma}_{A0} + 2 \bar{\gamma}_{A0}^2 - 4 \bar{\beta}_{AA}^0 \bar{\gamma}_{A0} \right) + \frac{E}{c^2} \left(\frac{5}{2} + \frac{1}{2} \bar{\Gamma} - \frac{2}{3} \bar{\beta}_{AA}^0 + \frac{4}{3} \bar{\gamma}_{A0} \right) \right] + \mathcal{O}\left(\frac{1}{c^6}\right). \quad (\text{C12})$$

This expression agrees with the general relativistic result derived in [11,9]. Note that contrary to the light-deflection and time-delay formulas, the periastron advance involves not only $\bar{\beta}_{AA}^0$ but also the 2PK parameter ε_{AAA}^0 , entering $\bar{\text{B}} = \frac{2}{9} \varepsilon_{AAA}^0 + \mathcal{O}(\bar{\beta}, \bar{\gamma})$; cf. Eq. (4.19a). The 2PN limit of Eq. (C12) can be obtained easily by using the expansions (4.23) of the different body-dependent parameters.

The result (C12) is coordinate independent since it is expressed in terms of the conserved quantities E and h . It can nevertheless be helpful to rewrite it in a particular coordinate system. Both isotropic and g^* -harmonic coordinates give the same result at the 2PK order. Let us denote by $a(1-e)$ the coordinate periastron radius, and by $a(1+e)$ the coordinate apoastron radius (in isotropic or, equivalently, g^* -harmonic coordinates). The conserved quantities can then be rewritten as

$$E = -\frac{\mu_A}{2a} + \frac{1}{8c^2} \left(\frac{\mu_A}{a} \right)^2 (7 + 4 \bar{\gamma}_{A0}) + \mathcal{O}\left(\frac{1}{c^4}\right), \quad (\text{C13a})$$

$$\frac{1}{h^2} = \frac{\mu_A}{a(1-e^2)} - \frac{4}{c^2} \left(\frac{\mu_A}{a(1-e^2)} \right)^2 \left(1 + \frac{e^2}{2} - \frac{1}{2} \bar{\beta}_{AA}^0 + \frac{3}{4} \bar{\gamma}_{A0} + \frac{e^2}{4} \bar{\gamma}_{A0} \right) + \mathcal{O}\left(\frac{1}{c^4}\right). \quad (\text{C13b})$$

We recover in particular the standard 1PN formula, $\Delta\phi = 2\pi\mu_A(3 - \bar{\beta} + 2\bar{\gamma})/a(1-e^2)c^2 + \mathcal{O}(1/c^4)$, in terms of the *semilatus rectum* $a(1-e^2)$. The 2PN result takes the form displayed in Eq. (5.10) above.

The explicit expression of the second-order time delay has been derived in [16] for a general metric of the form (4.18). This result confirms also the diagrammatic analysis discussed at the beginning of the present Appendix: The parameter $\bar{\beta}_{AA}^0$ appears again in the combination $3\bar{\Gamma} - 4\bar{\beta}_{AA}^0$, like in (C9) above, and it vanishes therefore in the case of tensor-multiscalar theories, for which $\bar{\Gamma} = \frac{4}{3} \bar{\beta}_{AA}^0 + \mathcal{O}(\bar{\gamma}_{A0}, \bar{\gamma}_{AA})$.

The integral (C3) can also be used to derive the periastron advance of a test mass $m_0 \ll m_A$. Let us introduce as in Ref. [9] the notation $E \equiv (\mathcal{E} - m_0 c^2)/m_0$ for the specific conserved energy minus rest mass, and $h \equiv \mathcal{J}/m_0 \mu_A$ for the reduced conserved angular momentum. A straightforward integration yields the 2PK periastron shift per orbit:

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