Time-symmetric initial data for multibody solutions in three dimensions

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Time-symmetric initial data for two-body solutions in three-dimensional anti–de Sitter gravity are found. The spatial geometry has a constant negative curvature and is constructed as a quotient of two-dimensional hyperbolic space. Apparent horizons correspond to closed geodesics. In an open universe, it is shown that two black holes cannot exist separately, but are necessarily enclosed by a third horizon. In a closed universe, two separate black holes can exist provided there is an additional image mass. [S0556-2821(96)05010-2]

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I. INTRODUCTION

Black holes in three spacetime dimensions [1,2] share many of the features of four-dimensional black holes. In this paper, the issue of constructing multibody and in particular multi-black-hole solutions is considered. In [3], it was shown that there are no static multi-black-hole solutions in 2+1 dimensions. Indeed, since there is a negative cosmological constant present, one would expect the black holes to attract and only time-dependent solutions to exist. In [4], it was shown that additional conical singularities will appear in the time-dependent solutions. To better understand these multibody solutions, we focus on the problem of constructing initial data for two bodies initially at rest. We take advantage of the fact that the space exterior to the sources has constant negative curvature and therefore can be constructed as a quotient of hyperbolic space.

II. TIME-SYMMETRIC INITIAL DATA IN THREE DIMENSIONS

Black holes in 2+1 dimensions are solutions to Einstein's equations with a negative cosmological constant Λ:

$$G_{\mu\nu}+\Lambda g_{\mu\nu}=8\pi G T_{\mu\nu}, \quad \Lambda<0. \tag{2.1}$$

The initial data constraints for (2.1) on an initial spacelike slice Σ with spatial metric h_{ij} and extrinsic curvature K_{ij} are given by

$$\frac{R}{2}+K_{ij}K^{ij}-K^2+\frac{1}{l^2}=8\pi G T_{\mu\nu}n^\mu n^\nu,$$

$$l^2=-\Lambda^{-1}, \quad K\equiv K_i^i,$$

$$\nabla_j K_i^j-\nabla_i K=8\pi G T_{i\mu}n^\mu, \tag{2.2}$$

where n^μ is the normal to Σ and R is the scalar curvature of h_{ij}. $i,j,\ldots$ refer to indices tangent to the spatial slice. In this paper, we are concerned with the case $K_{ij}=0$ corresponding to time-symmetric or momentarily static initial data. The momentum constraint is satisfied if $T_{i\mu}n^\mu=0$ while the Hamiltonian constraint becomes

$$\frac{R}{2}+\frac{1}{l^2}=8\pi G\rho, \quad \rho=T_{\mu\nu}n^\mu n^\nu. \tag{2.3}$$

On an apparent horizon, S, the convergence (or expansion) of outgoing null geodesics vanishes. In terms of h_{ij} and K_{ij}, this corresponds to the condition

$$H=(h^{ij}-\tilde{n}^i\tilde{n}^j)K_{ij}, \tag{2.4}$$

where H is the mean spatial curvature of S viewed as a surface embedded in the $(D-1)$-dimensional space Σ with metric h_{ij} and where $\tilde{n}^i$ is the normal to S in Σ. For time-symmetric initial data, S is an apparent horizon if $H=0$, i.e., if S is a minimal surface. A curve which is minimal is a geodesic. Hence, apparent horizons for time-symmetric initial data in 2+1 dimensions are closed geodesics.

III. INITIAL DATA FOR STATIC CIRCULARLY SYMMETRIC ONE-BODY SOLUTIONS

We first consider initial data for the static one-body solutions. The static circularly symmetric solutions to (2.1) are given by [1]

$$dS^2=-\left(\frac{r^2}{l^2}-8GM\right)dt^2+\left(\frac{r^2}{l^2}-8GM\right)^{-1}dr^2+r^2d\phi^2,$$

$$0<\phi<2\pi \tag{3.1}$$

where M is the total mass. For various ranges of M, (3.1) describes the following solutions: (1) $M>0$, black hole with event horizon located at $r_H=(8GM)^{1/2}l$ and singularity at $r=0$; (2) $M=0$, black hole vacuum; (3) $-1/8G<M<0$, one-particle solutions with a naked conical singularity at $r=0$ and no event horizon [5,6]; (4) $M=-1/8G$, three-dimensional anti–de Sitter space.

The $t=0$ initial spacelike slices of (3.1) are time symmetric and hence, from (2.3), have constant negative curvature. This implies that they can be obtained as quotients of two-dimensional hyperbolic space. We now review this construction.

A. Two-dimensional hyperbolic space

Two-dimensional hyperbolic space, H^2, can be described as the two-dimensional hypersurface

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$$T^2-X^2-Y^2=l^2 \tag{3.2}$$

in the flat three-dimensional space with metric of signature $(-++)$:

$$dS^2=-dT^2+dX^2+dY^2. \tag{3.3}$$

The Poincare disk model for hyperbolic space is

$$ds^2=\frac{4}{(1-z\overline{z}/l^2)^2}\,dz d\overline{z}, \quad 0\leqslant|z|<l, \;\; z=\rho e^{i\phi}, \tag{3.4}$$

and can be obtained by the stereographic projection of the hypersurface (3.2) through the point $(-1,0,0)$ in the (T,X,Y) space onto the disk of radius l in the X-Y plane. The boundary $|z|=l$ in (3.4) is spatial infinity. Geodesics on the Poincare disk are segments of circles or lines which intersect the boundary of the disk orthogonally. The isometry group of (3.4) is SU(1,1) with the action

$$z/l\to\frac{\alpha(z/l)+\beta}{\overline{\beta}(z/l)+\overline{\alpha}}, \quad |\alpha|^2-|\beta|^2=1. \tag{3.5}$$

Another representation of H^2 which will be useful is the Poincare metric on the upper half xy plane:

$$ds^2=l^2\left(\frac{dx^2+dy^2}{y^2}\right), \quad y>0. \tag{3.6}$$

This can be obtained from (3.3) by the embedding

$$T+Y=l^2/y, \quad T-Y=\frac{x^2+y^2}{y}, \quad X=\frac{x}{y}\,l. \tag{3.7}$$

It can be obtained from (3.4) by applying an inversion in a circle in the z plane given by $z/l\to(z/l-i)(-iz/l+1)^{-1}$ where $z=x+iy$. Geodesics on the upper-half plane are vertical lines or semicircles which intersect the real axis orthogonally. The isometry group of (3.6) is SL(2,R) with the action

$$z/l\to\frac{a(z/l)+b}{c(z/l)+d}, \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,R). \tag{3.8}$$

SL(2,R) and SU(1,1) isometries are related by conjugation:

$$\widetilde{S}=NSN^{-1},$$

$$N=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i\\ -i & 1\end{pmatrix},$$

$$S\in SL(2,R), \quad \widetilde{S}\in SU(1,1). \tag{3.9}$$

Finally, in terms of polar coordinates $r=2[\rho/(1-\rho^2/l^2)]$, (3.4) takes the form

$$ds^2=\frac{dr^2}{r^2/l^2+1}+r^2d\phi^2, \quad r\geqslant 0, \;\; 0\leqslant\phi<2\pi. \tag{3.10}$$

This is the $t=0$ spatial geometry of three-dimensional anti–de Sitter space (3.1) with $M=-1/(8G)$.

Let us briefly discuss the conjugacy classes of isometries of H^2. Two isometries are conjugate if and only if the traces of the corresponding SL(2,R) or SU(1,1) matrices, Γ, are equal. There are three types of conjugacy classes: *elliptic* $[(\mathrm{Tr}\,\Gamma)^2<4]$, *hyperbolic* $[(\mathrm{Tr}\,\Gamma)^2>4]$, or *parabolic* $[(\mathrm{Tr}\,\Gamma)^2=4]$. Elliptic isometries are conjugate to rotations about the origin in the Poincare disk. Hyperbolic isometries are conjugate to scalings in the upper-half plane. There is only one parabolic conjugacy class corresponding to isometries which are conjugate to a translation in the x direction in the upper-half plane. Elliptic isometries have one fixed point, hyperbolic isometries have two fixed points both at infinity, and parabolic isometries have one fixed point at infinity. The $t=0$ initial slices for the one-body solutions (3.1) can be obtained essentially by identifying H^2 periodically in an isometry generator. Elements conjugate to one another generate isometric quotients. As we shall see below, elliptic isometries generate the $M<0$ one-particle solutions, hyperbolic isometries $M>0$ black holes, and the parabolic conjugacy class generates the $M=0$ black hole vacuum.

B. $M>0$ black hole solutions

The spatial geometry of the black hole solution can be obtained by identifying H^2 periodically in a hyperbolic generator [1]. The $t=0$ spatial slice of the black hole spacetime (3.1) is given by

$$ds^2=\left(\frac{r^2}{l^2}-8GM\right)^{-1}dr^2+r^2d\phi^2, \quad 0<\phi<2\pi. \tag{3.11}$$

Defining the radial coordinate $x=(r^2-8GMl^2)^{1/2}$, one obtains

$$ds^2=\left(\frac{x^2}{l^2}+8GM\right)^{-1}dx^2+(x^2+8GMl^2)d\phi^2,$$

$$-\infty<x<\infty, \;\; 0<\phi<2\pi. \tag{3.12}$$

The complete spatial geometry has a wormhole structure with cylindrical topology, $S^1\times R$. It is similar to the Einstein-Rosen bridge in the Schwarzschild solution except it is not asymptotically flat. The wormhole mouth is at the horizon $r_H=(8GM)^{1/2}l$.

The simplest way to see how (3.11) can be obtained from a quotient of H^2 is using the coordinates $(\widetilde{r},\widetilde{\phi})$ defined by the embedding (3.2):

$$T=\widetilde{r}\cosh\widetilde{\phi}, \quad X=\sqrt{\widetilde{r}^{\,2}-l^2}, \quad Y=\widetilde{r}\sinh\widetilde{\phi},$$

$$\widetilde{\phi}\in(-\infty,\infty), \quad l<\widetilde{r}<\infty. \tag{3.13}$$

Note that $\widetilde{\phi}$ has infinite range because it parametrizes *boosts* in the T-Y plane. Inserting into (3.3), the metric becomes

$$ds^2=\left(\frac{\tilde{r}^2}{l^2}-1\right)^{-1}d\tilde{r}^2+\tilde{r}^2d\tilde{\phi}^2,\quad \tilde{\phi}\in(-\infty,\infty). \tag{3.14}$$

If one identifies $\tilde{\phi}$ with period $2(8GM)^{1/2}\pi$ where M is the mass of the black hole, and rescales the coordinates

$$\tilde{r}=r/(8GM)^{1/2},\quad \tilde{\phi}=(8GM)^{1/2}\phi, \tag{3.15}$$

one obtains the black hole spatial geometry (3.11).

In terms of the coordinates (3.7) (x,y) on the upper-half plane, the identification $\tilde{\phi}\sim\tilde{\phi}+2(8GM)^{1/2}\pi$ corresponds to the identification by the scaling $(x,y)\sim e^{-2(8GM)^{1/2}\pi}(x,y)$. The black hole spatial geometry is then the semiannulus in the upper-half plane with outer unit radius and inner radius $e^{-2(8GM)^{1/2}\pi}$ identified [7]. From (3.9), the corresponding SU(1,1) matrix generating the identifications is given by

$$\Gamma_M=\begin{pmatrix}\cosh[\pi(8GM)^{1/2}] & i\sinh[\pi(8GM)^{1/2}]\\ -i\sinh[\pi(8GM)^{1/2}] & \cosh[\pi(8GM)^{1/2}]\end{pmatrix}. \tag{3.16}$$

Hence, the $M>0$ black hole is obtained by identifying two geodesics which do not intersect and are not tangential to one another at infinity.

C. $-1/(8G)<M<0$ solutions as quotients of hyperbolic space

The $-1/(8G)<M<0$ solutions

$$ds^2=\left(\frac{r^2}{l^2}+(1-4Gm)^2\right)^{-1}dr^2+r^2d\phi^2,$$

$$m=\frac{1}{4G}-\left(\frac{-M}{2G}\right)^{1/2} \tag{3.17}$$

are the familiar anti–de Sitter conical spaces with a particle source [5] obtained by excising a wedge of deficit angle $8\pi Gm$ from the Poincare disk with vertex located at the source. The edges of the wedge are then identified by a rotation $z\rightarrow e^{i8\pi Gm}z$ which as an element of SU(1,1) is given by

$$\Gamma_m=\begin{pmatrix}e^{i4\pi Gm} & 0\\ 0 & e^{-i4\pi Gm}\end{pmatrix}. \tag{3.18}$$

m is the proper mass defined by $m=\int\rho dA$. The maximum allowable deficit angle of 2π corresponds to $m_c\equiv 1/4G$. For a particle located at a different point, the wedge excised consists of two geodesic edges with the same deficit angle since the model is conformal.

D. $M=0$ black hole vacuum

The $t=0$ spatial geometry of the $M=0$ black hole vacuum is given by

$$ds^2=l^2\,\frac{dr^2}{r^2}+r^2d\phi^2. \tag{3.19}$$

This is a surface of revolution of constant negative curvature known as a *tractroid*. Transforming to the new coordinates $y=l/r$, $x=\phi$, (3.19) becomes

$$ds^2=l^2\left(\frac{dx^2+dy^2}{y^2}\right),\quad y>0,\ 0\leqslant x<2\pi. \tag{3.20}$$

Comparing with (3.6), the $M=0$ spatial geometry is the upper-half plane with x identified periodically in 2π. Alternatively, using (3.9) it can be obtained as a quotient of the Poincaré disk by the SU(1,1) parabolic transformation

$$\Gamma_{M=0}=\begin{pmatrix}1+\pi i & \pi\\ \pi & 1-\pi i\end{pmatrix}\in \mathrm{SU}(1,1). \tag{3.21}$$

Identifying in x with different periods are conjugate transformations. Hence, the resulting spaces generated by identifying in x with different periods are all isometric to the $M=0$ solution. The $M=0$ solution is obtained by identifying two geodesics which are tangential to one another at ∞.

The generator for the $M=0$ solution (3.21) can be obtained by conjugating the generator for the $M<0$ conical solution (3.18) by a translation T_d by a distance d in the x direction in the Poincaré disk,

$$T_d=\begin{pmatrix}\cosh(d/2l) & \sinh(d/2l)\\ \sinh(d/2) & \cosh(d/2l)\end{pmatrix}, \tag{3.22}$$

and taking the simultaneous limit $m\rightarrow 0$, $d\rightarrow\infty$,

$$\Gamma_{M=0}=\lim_{m\rightarrow 0,d\rightarrow\infty}T_d\Gamma_m T_d^{-1}, \tag{3.23}$$

with $m\cosh d/l=m_c$ fixed. This limit is analogous to a contraction where a lightlike solution is obtained by taking a simultaneous $m\rightarrow 0$, $v\rightarrow c$ limit with the energy held fixed. The $M=0$ generator can be obtained in the same way from the $M>0$ black hole generator (3.16).

IV. TWO-BODY SOLUTIONS

In this section, we construct initial data for two-body solutions. As in the one-body case, we obtain these solutions by taking quotients of H^2. While the quotient group for the one-body solutions is generated by one element, for the multibody solutions more generators are required. The total mass of the system, M, can be obtained from the spatial metric at large distances. However, a simpler way is to obtain it from the generator for the system. The generator Γ for the total system is the composition of the generators for the individual bodies. Express Γ as an effective one-body generator up to a conjugation corresponding to an overall isometry

$$\Gamma=T\Gamma_M T^{-1}, \tag{4.1}$$

where Γ_M is given in (3.16). The total mass can then be obtained from the trace:

$$\cosh[\pi(8GM)^{1/2}]=\frac{1}{2}\mathrm{Tr}\Gamma. \tag{4.2}$$

We now consider three two-body systems: two particles, a particle and black hole, and two black holes. Depending on the masses and locations of the bodies, there are three qualitatively different kinds of solutions: the space exterior to the two bodies is open without horizons; the space exterior to the two bodies is open with the two bodies enclosed by an apparent horizon; or, the space exterior to the two bodies is closed with additional image masses. For some values of these parameters, there may be no solution (at least with positive-energy matter). The same kinds of solutions were found in the case of a ring of pressure-free dust [8].

A. Two-particle solutions

Consider two particles initially at rest with masses m, $\tilde{m}$ separated by a geodesic distance d. To construct the spatial geometry, it is convenient to use the Poincaré disk representation. As in the flat case [5], we excise a wedge for each of the particles with deficit angles $8\pi Gm$ and $8\pi G\tilde{m}$. For a particle of mass m translated by a distance d, the generator is

$$\Gamma = T_d \Gamma_m T_d^{-1}, \tag{4.3}$$

where Γ_m is the one-particle generator (3.18) and T_d is the translation (3.22). The effective one-body generator for the whole system is then the product

$$\Gamma = \Gamma_m T_d \Gamma_{\tilde{m}} T_d^{-1}. \tag{4.4}$$

From (4.2), the total mass of the system is given by

$$\cosh[\pi(8GM)^{1/2}] = \cosh(d/l)\sin(4\pi Gm)\sin(4\pi G\tilde{m}) - \cos(4\pi Gm)\cos(4\pi G\tilde{m}). \tag{4.5}$$

Depending on the values of the parameters $(m,\tilde{m},d)$, the space can be open without a horizon, open with the two particles enclosed by a horizon, or closed with an additional image mass. If $m+\tilde{m}<m_c$, the space is open with total mass given by (4.5). If $\tilde{m}=0$, we recover the one-particle mass $M=-(1-4Gm)^2/(8G)$ from (3.17). When $d=0$, we recover the flat space formula $m_{\text{total}}=m+\tilde{m}$ where $M=-(1-4Gm_{\text{total}})^2/(8G)$. M increases with d due to the attractive anti–de Sitter force, and increases with the masses $m,\tilde{m}$ due to the rest mass contribution to the total energy.

Now consider the case $m+\tilde{m}\geqslant m_c$. Recall that this condition means that the sum of the deficit angles of the particles exceeds 2π. In flat space, this would imply that the universe closes with an additional image mass appearing. In anti–de Sitter space, this is not necessarily the case. Consider the two subcases $m\leqslant m_c/2$, $\tilde{m}>m_c/2$ (or equivalently, $m>m_c/2$ and $\tilde{m}\leqslant m_c/2$) and $m,\tilde{m}>m_c/2$ separately.

For $m\leqslant m_c/2$ and $\tilde{m}>m_c/2$, the different kinds of solutions that can occur as d decreases are as follows. For d in the range

$$\cosh(d/l) > \frac{\tan(4\pi Gm)}{-\tan(4\pi G\tilde{m})}, \quad \text{open, no horizon,} \tag{4.6}$$

the space is open with total mass (4.5) and without any horizon. For d in the range

$$f_c(m,\tilde{m}) < \cosh(d/l) \leqslant \frac{\tan(4\pi Gm)}{-\tan(4\pi G\tilde{m})} \quad \text{black hole} \tag{4.7}$$

with

$$f_c(m,\tilde{m}) \equiv \frac{1+\cos(4\pi Gm)\cos(4\pi G\tilde{m})}{\sin(4\pi Gm)\sin(4\pi G\tilde{m})}, \tag{4.8}$$

the two particles are surrounded by a horizon. The size of the horizon is that of a black hole of mass M (4.5) as can be verified directly from the geometry. The space outside the horizon is of course the black hole spatial geometry. When $\cosh(d/l)=f_c(m,\tilde{m})$, (4.5) yields $M=0$, and the space outside the particles forms an infinite thin throat at infinity which is identical to the $M=0$ throat. For

$$\cosh(d/l) < f_c(m,\tilde{m}) \quad \text{closed space} \tag{4.9}$$

the space is closed with an additional image mass, m', appearing. The total mass, M, is negative with $m'=m_{\text{total}}=1/4G-(-M/2G)^{1/2}$.

For $m,\tilde{m}>m_c/2$, there are only two kinds of solutions. For d in the range

$$\cosh(d/l) > f_c(m,\tilde{m}) \quad \text{black hole} \tag{4.10}$$

the two particles are enclosed by a horizon. If $\cosh(d/l)=f_c(m,\tilde{m})$, the infinite throat forms at infinity. For d in the range, (4.9), the space is closed with an additional image mass. Since $f_c(m,\tilde{m})\geqslant 1$ and from (4.9), we recover that the space is closed in the $l\to\infty$ flat space limit.

B. A system consisting of one particle and one black hole

Consider a system consisting of a particle of mass m initially at rest located a geodesic distance R from the horizon of a black hole of mass M also initially at rest. The spatial geometry can again be obtained from a quotient of the Poincaré disk. Consider the fundamental region on the Poincaré disk associated with the one-black-hole solution. This is the region bound by two geodesics which do not intersect and are not tangential to one another at infinity. Now insert a point particle of mass m a distance R from the horizon of the black hole by excising a wedge of deficit angle $8\pi Gm$ with vertex at the point particle and identifying the two edges.

The total mass of the system can be found from the effective one-body generator Γ obtained from the composition of the generators for the black hole (3.16) and the inverse of the generator for the translated particle (4.3):

$$\Gamma = \Gamma_M (T_R \Gamma_m T_R^{-1})^{-1}. \tag{4.11}$$

The inverse is taken because the wedge is being *removed* from the space. From the trace (4.2), we find the total mass M_{total} of the system:

$$\cosh[\pi(8GM_{\text{total}})^{1/2}] = \cosh\beta_M \cos(4\pi Gm) + \sinh(R/l)\sinh\beta_M \sin(4\pi Gm), \tag{4.12}$$

where $\beta_M \equiv \pi(8GM)^{1/2}$. As $m \rightarrow 0$, we recover $M_{\rm total} \rightarrow M$, and similarly, as $M \rightarrow -1/8G$, we recover $M_{\rm total} = -(1-4Gm)^2/(8G)$.

Depending on the values of the parameters (m,M,R), the space exterior to the particle and black hole can be open without a horizon, open with the particle and black hole enclosed by a horizon, or closed with an additional image mass. Consider increasing values of m.

For $0 < \tan(4\pi Gm) < \sinh \beta_M$, the different kinds of solutions that can occur as R decreases are as follows. For R in the range

$$\sinh(R/l) > \frac{\tan(4\pi Gm)}{\tanh \beta_M} \quad \text{open, no horizon,} \tag{4.13}$$

the exterior space is open with total mass (4.12), and without any additional horizons. Again, the total mass increases with R and with m. For R in the range

$$\sinh(R/l) \leqslant \frac{\tan(4\pi Gm)}{\tanh \beta_M} \quad \text{black hole} \tag{4.14}$$

the particle and black hole are surrounded by a horizon with mass given by (4.12) as can be verified directly from the geometry.

For m in the range, $\sinh \beta_M \leqslant \tan(4\pi Gm) < \infty$, (4.13) continues to hold. A horizon forms for

$$\frac{1-\cosh \beta_M \cos(4\pi Gm)}{\sinh \beta_M \sin(4\pi Gm)} \leqslant \sinh(R/l) \leqslant \frac{\tan(4\pi Gm)}{\tanh \beta_M} \quad \text{black hole.} \tag{4.15}$$

For R in the range

$$\sinh(R/l) < \frac{1-\cosh \beta_M \cos(4\pi Gm)}{\sinh \beta_M \sin(4\pi Gm)} \quad \text{closed space,} \tag{4.16}$$

the space exterior to the particle and black hole closes with an additional image mass appearing.

Finally, for $m > m_c/2$ $[\tan(4\pi Gm) < 0]$, an open solution without a horizon is not possible. In addition, if R is small enough,

$$\sinh(R/l) < \frac{\tanh \beta_M}{\tan(4\pi Gm)} \quad \text{no solution,} \tag{4.17}$$

instead of the space closing, there is no solution.

We now consider solutions describing a point particle in the $M=0$ black hole vacuum (3.19). These solutions can be obtained from the solutions above by taking a limit in which $M \rightarrow 0$ and $R \rightarrow \infty$ simultaneously. The distance R to the horizon of the $M>0$ black hole from the particle at coordinate r is given by $R = l \cosh^{-1}(r/r_H)$. Now substituting this into the expression (4.12) and taking $M \rightarrow 0$ yields the total mass of the system:

$$\cosh[\pi(8GM_{\rm total})^{1/2}] = \cos(4\pi Gm) + \pi r/l \sin(4\pi Gm). \tag{4.18}$$

As $m \rightarrow 0$ we recover $M_{\rm total} = 0$. We find similar qualitative behavior to the finite M case. As m increases, a horizon forms at

$$\tan(4\pi Gm) = \pi r/l \tag{4.19}$$

For a particle far down the $M=0$ throat $(r \ll l)$, it requires an arbitrarily small amount of mass to form a horizon. For a particle far away, $r \gg l$, a horizon forms when $m \approx m_c/2$.

C. Two-black-hole solution

Consider two black holes of masses M_1 and M_2 initially at rest. Let the geodesic distance between the horizons be given by d. Depending on the values of the parameters (M_1, M_2, d), either the black holes are enclosed by a third horizon, or the space is closed with an additional image mass. For d in the range

$$\cosh(d/l) > g_c^+(M_1, M_2) \quad \text{black hole,} \tag{4.20}$$

with

$$g_c^{\pm}(M_1,M_2) \equiv \frac{\cosh\beta_{M_1} \cosh\beta_{M_2} \pm 1}{\sinh\beta_{M_1} \sinh\beta_{M_2}},$$

$$\beta_{M_{1,2}} \equiv \pi(8GM_{1,2})^{1/2}, \tag{4.21}$$

the two black holes are merged, i.e., surrounded by a horizon. The length of the horizon l_0 is determined from the geometry as follows. By cutting the space in half along geodesic seams connecting the horizons, one obtains two hexagonal regions with interior right angles. Using hyperbolic geometry, one finds

$$\cosh[l_0/(2l)] = \cosh(d/l) \sinh\beta_{M_1} \sinh \beta_{M_2} - \cosh\beta_{M_1} \cosh\beta_{M_2}. \tag{4.22}$$

For $\cosh(d/l) = g_c^+(M_1, M_2)$, (4.22) yields $l_0 = 0$, and the infinite $M=0$ throat forms at infinity. For d in the range

$$g_c^-(M_1,M_2) < \cosh(d/l) < g_c^+(M_1,M_2) \quad \text{closed space,} \tag{4.23}$$

the space exterior to the two black holes is closed with an additional image mass. Finally, for d in the range

$$\cosh(d/l) \leqslant g_c^-(M_1,M_2) \quad \text{no solutions,} \tag{4.24}$$

there are no solutions. For the case of equal masses, the right-hand side of (4.24) is unity and therefore, there are always solutions. In conclusion, we find the striking feature that when the space exterior to the two black holes is open, the black holes are necessarily merged, i.e., surrounded by an apparent horizon. However, if the space exterior to the two black holes is closed, the black holes can exist separately provided there is an additional point mass or a third black hole. This can be seen from the fact that the solution describing two black holes surrounded by a horizon can alternatively be viewed as a closed universe with three separate black holes. One can also find solutions in which one or both of the black holes is the $M=0$ black hole vacuum. As in the previous section, these solutions can be obtained by taking a simultaneous limit in which $M_{1,2}\rightarrow 0$ and $d\rightarrow\infty$. One finds the same qualitative features as for nonzero mass.

V. CONCLUSION

The next step is to evolve this initial data. An exact time-dependent solution describing the merging of two bodies to form a black hole should be possible. It would also be interesting to see whether critical behavior of the sort found by Choptuik [9] exists in this case. Since previous studies of critical behavior in gravitational collapse were restricted to cases with a high degree of symmetry, this would be an example of critical behavior under more general circumstances. In addition, the collision of two bodies likely produces a naked singularity for certain initial conditions. This is reminiscient of [10,11] where it was shown that extremal charged black holes in four dimensions in a theory with cosmological constant can collide and form naked singularities. Also, the extremal $M=0$ black hole vacuum is similar in some ways to the extremal charged (3+1)-dimensional black hole.

Note added. Upon completion of this paper, I received a draft of a paper from Dieter Brill (gr-qc/9511022) which included a discussion of initial data for multi-black-hole solutions.

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[1] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[2] For a review, see R. Mann (unpublished); S. Carlip, Class. Quantum Grav. **12**, 2853 (1995).

[3] O. Coussaert and M. Henneaux, in *Geometry of Constrained Dynamical Systems*, edited by John M. Charap (Cambridge University Press, Cambridge, England, 1995).

[4] G. Clement, Phys. Rev. D **50**, R7119 (1994).

[5] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).

[6] S. Deser and R. Jackiw, Ann. Phys. **153**, 405 (1984).

[7] S. Carlip and C. Teitelboim, Phys. Rev. D **51**, 622 (1995).

[8] Y. Peleg and A. Steif, Phys. Rev. D **51**, R3992 (1995).

[9] M. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[10] D. Kastor and J. Traschen, Phys. Rev. D **47**, 5370 (1993).

[11] D. Brill, G. Horowitz, D. Kastor, and J. Traschen, Phys. Rev. D **49**, 840 (1994).