Quantum field theory constrains traversable wormhole geometries

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Recently a bound on negative energy densities in four-dimensional Minkowski spacetime was derived for a minimally coupled, quantized, massless, scalar field in an arbitrary quantum state. The bound has the form of an uncertainty-principle-type constraint on the magnitude and duration of the negative energy density seen by a timelike geodesic observer. When spacetime is curved and/or has boundaries, we argue that the bound should hold in regions small compared to the minimum local characteristic radius of curvature or the distance to any boundaries, since spacetime can be considered approximately Minkowski on these scales. We apply the bound to the stress-energy of static traversable wormhole spacetimes. Our analysis implies that either the wormhole must be only a little larger than Planck size or that there is a large discrepancy in the length scales which characterize the wormhole. In the latter case, the negative energy must typically be concentrated in a thin band many orders of magnitude smaller than the throat size. These results would seem to make the existence of macroscopic traversable wormholes very improbable.

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I. INTRODUCTION

In recent years there has been considerable interest in the topic of traversable wormholes, solutions of Einstein's equations which act as tunnels from one region of spacetime to another, through which an observer might freely pass [1–3]. Traversable wormhole spacetimes have the property that they must involve ''exotic matter,'' that is, a stress tensor which violates the weak energy condition. Thus the energy density must be negative in the frame of reference of at least some observers. Although classical forms of matter obey the weak energy condition, it is well known that quantum fields can generate locally negative energy densities, which may be arbitrarily large at a given point. A key issue in the study of wormholes is the nature and magnitude of the violations of the weak energy condition which are allowed by quantum field theory. One possible constraint upon such violations is given by averaged energy conditions [4]. In particular, the averaged null energy condition (ANEC) states that $\int T_{\mu\nu}k^{\mu}k^{\nu}d\lambda \geqslant 0$, where the integral is taken along a complete null geodesic with tangent vector k^{μ} and affine parameter λ. This condition must be violated in wormhole spacetimes [2]. Although the ANEC can be proved to hold in Minkowski spacetime, it is generally violated in curved spacetime [5,6]. The extent to which it can be violated is not yet well understood, but limits on the extent of ANEC violation will place constraints upon allowable wormhole geometries [7,8].

A second type of constraint upon violations of the weak energy condition are ''quantum inequalities'' (QI's), which limit the magnitude and spatial or temporal extent of negative energy [9–13]. These constraints are intermediate between pointwise conditions and the averaged energy conditions in that they give information about the distribution of negative energy in a finite neighborhood. For the most part, inequalities of this type have only been proved in flat spacetime. The main purpose of this paper will be to argue that restricted versions of the flat spacetime inequalities can be employed in curved spacetime, and that these inequalities place severe constraints upon wormhole geometries. We assume that the stress energy of the wormhole spacetime is a renormalized expectation value of the energy-momentum tensor operator in some quantum state, and ignore fluctuations in this expectation value [14,15].

In this paper, we restrict our attention to static, spherically symmetric wormholes. We will also assume that the spacetime contains no closed timelike curves. This latter assumption may not be necessary, but we make it in order to ensure that quantum field theory on the wormhole spacetime is well defined. In Sec. II, a flat spacetime quantum inequality is reviewed, and an argument is presented for the application of this inequality in small regions of a curved spacetime. In Sec. III, we briefly review some of the essential features of traversable (Morris-Thorne) wormholes. We next consider a number of particular wormhole models in Sec. IV, and argue that the quantum inequality places strong restrictions upon the dimensions of these wormholes. In Sec. V we formulate a more general bound upon the relative dimensions of an arbitrary Morris-Thorne wormhole. Finally, in Sec. VI we summarize and interpret our results. Our units are taken to be

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those in which $\hbar=G=c=1$, and our sign conventions are those of Ref. [1].

II. QUANTUM INEQUALITIES IN FLAT AND CURVED SPACETIME

In Ref. [13], an inequality was proved which limits the magnitude and duration of the negative energy density seen by an inertial observer in Minkowski spacetime (without boundaries). Let $\langle T_{\mu\nu}\rangle$ be the renormalized expectation value of the stress tensor for a free, massless, minimally coupled scalar field in an arbitrary quantum state. Let u^μ be the observer's four-velocity, so that $\langle T_{\mu\nu}u^\mu u^\nu\rangle$ is the expectation value of the local energy density in this observer's frame of reference. The inequality states that

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2}\geqslant-\frac{3}{32\pi^2\tau_0^4}, \tag{1}$$

for all τ_0, where τ is the observer's proper time. The Lorentzian function which appears in the integrand is a convenient choice for a sampling function, which samples the energy density in an interval of characteristic duration τ_0 centered around an arbitrary point on the observer's world line. The proper time coordinate has been chosen so that this point is at $\tau=0$. The physical content of Eq. (1) is that the more negative the energy density is in an interval, the shorter must be the duration of the interval. Consider, for example, a pocket of negative energy which our observer traverses in a proper time $\Delta\tau$. A natural choice of the sampling time is $\tau_0=\Delta\tau$, in which case we infer that the average value of the negative energy in this pocket is bounded below by $-3/[32\pi^2(\Delta\tau)^4]$. Because Eq. (1) holds for all τ_0, we must obtain a true statement with other choices. If we let $\tau_0<\Delta\tau$, then we obtain a weaker bound. If we let $\tau_0>\Delta\tau$, then we appear to obtain a stronger bound. However, now the range over which we are sampling extends beyond the boundaries of the pocket and may include positive energy contributions. Hence it is to be expected that the lower bound on the average energy density should be less negative.

The basic premise of this paper is that one may obtain a constraint upon the renormalized stress tensor in a curved spacetime using Eq. (1), provided that τ_0 is taken to be sufficiently small. The main purpose of this section is to explore the rationale for this premise. The basic idea is that a curved spacetime appears flat if restricted to a sufficiently small region. However, this idea is sufficiently subtle to require an extended discussion.

First, let us recall the situation in classical general relativity. The principle of equivalence has its mathematical expression in the fact that the geodesic equations involve the spacetime metric and the connection coefficients, but not the curvature tensor. Thus, if we go to a local inertial frame, the equations of motion for a point test particle take the flat space form. However, it is possible for equations of motion to contain curvature terms explicitly. An example is the equation of motion for a classical spinning test particle [16]. In this case, the principle of equivalence does not hold in its simplest form, and one can treat the system as being in locally flat spacetime only to the extent that the curvature terms are negligible.

In quantum field theory, we will be more interested in the extent to which solutions of wave equations can be approximated by the flat space forms. Consider, for example, the generalized Klein-Gordon equation

$$\Box\phi+m^2\phi+\xi R\phi=0\,, \tag{2}$$

where ξ is an arbitrary constant and R is the scalar curvature. The solutions of this equation will generally not be similar to the flat space solutions unless the curvature term is small compared to the other terms in the equation. However, this is still not sufficient to guarantee that a flat space mode is a solution of Eq. (2). It is also necessary to require that the modes have a wavelength that is small compared to the local radii of curvature of the spacetime. In this limit, it is possible to obtain WKB-type solutions to Eq. (2), which are approximately plane wave modes. For an illustration of this, see the work of Parker and Fulling [17] on adiabatic regularization. These authors give generalized WKB solutions of wave equations in an expanding spatially flat Robertson-Walker universe. In the limit that the wavelength of a mode is short compared to the expansion time scale (which is the spacetime radius of curvature in this case), the leading term, which is of the plane wave form, becomes a good approximation.

Our primary concern is when we may expect the inequality (1), which was derived from Minkowski space quantum field theory, to hold in a curved spacetime and/or one with boundaries. For a given τ_0, the dominant contribution to the right-hand side of this inequality arises from modes for which $\lambda\sim\tau_0$. In particular, modes for which $\lambda\gg\tau_0$ yield a small contribution. To see this more explicitly, note that the right-hand side of Eq. (1) arises from the integral $(4\pi^2)^{-1}\int_0^\infty d\omega\,\omega^3 e^{-2\omega\tau_0}$. [See Eq. (63) of Ref. [13].] Thus if the long wavelength modes ($\omega\ll\tau_0^{-1}$) were to be omitted or to be distorted by the presence of spacetime curvature or boundaries, the result would not change significantly. This suggests that we can apply the inequality in a curved spacetime as long as τ_0 is restricted to be small compared to the local proper radii of curvature and the proper distance to any boundaries in the spacetime. This is the criterion that the relevant modes be approximated by plane wave modes.

The specific example of the Casimir effect may be useful as an illustration. Here one has a constant negative energy density, which would not be possible if Eq. (1) holds for all τ_0. However, if we impose some restrictions on the allowable values of τ_0, then the inequality *does* in fact still apply. Let us consider a massless scalar field with periodicity of length L in the z direction. Let us also consider an observer moving with velocity v in the $+z$ direction. In the rest frame of this observer, the expectation value of the energy density is

$$\langle T_{\mu\nu}u^\mu u^\nu\rangle=-\frac{\pi^2}{45L^4}(1+3v^2)\gamma^2, \tag{3}$$

where $\gamma=(1-v^2)^{-1/2}$. Because this quantity is a constant, Eq. (1) becomes

$$-\frac{\pi^2}{45L^4}(1+3v^2)\gamma^2\geqslant-\frac{3}{32\pi^2\tau_0^4} \tag{4}$$

or, equivalently,

$$\tau_0 \leqslant \frac{3L}{2\pi}\left(\frac{5}{6}\right)^{1/4}[(1+3v^2)\gamma^2]^{-1/4}. \tag{5}$$

Thus for the special case of a static observer ($v=0$), we must have

$$\tau_0 \leqslant \frac{3L}{2\pi}\left(\frac{5}{6}\right)^{1/4} \approx 0.46\ L. \tag{6}$$

There are two relevant length scales in the observer's frame of reference. The first is the (Lorentz-contracted) periodicity length $l_1 = L/\gamma$, and the second is the proper time required to traverse this distance, $l_2 = L/(v\gamma)$. Here l_1 is the smaller of the two, and plays a role analogous to the minimum radius of curvature in a curved spacetime. Thus we should let

$$\tau_0 = f\ l_1 = \frac{fL}{\gamma}. \tag{7}$$

Equation (5) will be satisfied if

$$f \leqslant g(v) \equiv \frac{3}{2\pi}\left(\frac{5}{6}\right)^{1/4}[(1+3v^2)(1-v^2)]^{-1/4}. \tag{8}$$

The function $g(v)$ has its minimum value at $v = 1/\sqrt{3}$, at which point

$$g\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{2\pi}\left(\frac{5}{8}\right)^{1/4} \approx 0.42. \tag{9}$$

Thus if we restrict $\tau_0 < 0.42 l_1$, then the Minkowski space quantum inequality also holds in the compactified spacetime. Note that the constraint obtained by considering arbitrary v differs only slightly from that for static observers, Eq. (6).

The Casimir effect example contains some of the essential features that we encounter in a renormalized stress tensor on a curved background spacetime. However, on a curved spacetime $\langle T_{\mu\nu}\rangle$ is a sum of a state-dependent part and a state-independent geometrical part. The latter consists of terms which are either quadratic in the Riemann tensor or else linear in second derivatives of the Riemann tensor. One source of curvature dependence in $\langle T_{\mu\nu}\rangle$ is the well-known trace anomaly. For the case of the conformal ($\xi = 1/6$) scalar field, it is

$$\langle T^{\mu}_{\mu}\rangle = \frac{1}{2880\pi^2}(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma} - R_{\alpha\beta}R^{\alpha\beta} + \nabla_\rho \nabla^\rho R). \tag{10}$$

Other fields have trace anomalies with similar coefficients, i.e., with magnitudes of the order of 10^{-4}. Thus these terms will give a very small contribution to a quantum inequality of the form of Eq. (1) when $\tau_0 \ll l$, where l is the characteristic radius of curvature.

A related source of curvature dependence in the renormalized stress tensor is the possible presence of finite terms of the form of the quadratic counter-terms required to remove the logarithmic divergences in a curved spacetime. These terms are the tensors

$$H^{(1)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}[\sqrt{-g}R^2] = 2\nabla_\nu \nabla_\mu R - 2g_{\mu\nu}\nabla_\rho \nabla^\rho R + \frac{1}{2}g_{\mu\nu}R^2 - 2RR_{\mu\nu} \tag{11}$$

and

$$H^{(2)}_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}[\sqrt{-g}R_{\alpha\beta}R^{\alpha\beta}] = 2\nabla_\alpha \nabla_\nu R^{\alpha}_{\mu} + \nabla_\rho \nabla^\rho R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\nabla_\rho \nabla^\rho R + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - 2R^{\rho}_{\mu}R_{\rho\nu}. \tag{12}$$

There can be a term of the form $c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu}$ in $\langle T_{\mu\nu}\rangle$. More generally, there might be a term of the form $(c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu})\ln(R\mu^{-2})$, where μ is an arbitrary renormalization mass scale [18]. A shift in the value of μ adds a term proportional to $c_1 H^{(1)}_{\mu\nu} + c_2 H^{(2)}_{\mu\nu}$ to $\langle T_{\mu\nu}\rangle$. Visser [6] has recently discussed how terms of this form are likely to lead to violations of the ANEC in curved spacetime. The problem is that quantum field theory by itself is not able to predict the values of c_1 and c_2 or, equivalently, of μ. Thus very large values of these parameters are not, *a priori*, ruled out. The status of these terms in the semiclassical Einstein equations has been the subject of much discussion in the literature. They appear to give rise to unstable behavior [19], analogous to the runaway solutions of the Lorentz-Dirac equation of classical electron theory. More recently Simon [20] has suggested that it may be possible to reformulate the semiclassical theory to avoid unstable solutions.

If one ignores the possibility of runaway solutions, then if these terms are to produce a significant correction to the geometry of a spacetime whose curvature is far below Planck dimensions, then at least one of the dimensionless constants c_1 or c_2 must be extremely large. The Einstein tensor is of order l^{-2} and the $H^{(1)}_{\mu\nu}$ and $H^{(2)}_{\mu\nu}$ tensors are of order l^{-4}, in Planck units. The latter are negligible unless their coefficients are at least of order $(l/l_P)^2$, where l_P is the Planck length. Thus if the state-independent geometrical part of $\langle T_{\mu\nu}\rangle$ is to be the source of the exotic matter which generates the wormhole geometry, either the wormhole must be of Planck dimensions, or else one must accept large dimensionless coefficients. For example, unless one of these constants is at least of order 10^{70}, the quadratic curvature terms will be negligible for the discussion of a wormhole whose throat radius is of the order of 1 m. A value of c_1 or c_2 of 10^{70} could arise from a single quantum field or from 10^{70} fields, each giving a contribution of order unity [21]. Both possibilities seem equally unnatural.

An alternative is for the state-dependent part of $\langle T_{\mu\nu}\rangle$ to be the source of the exotic matter. A nonexotic stress tensor may be made arbitrarily large by increasing the particle content of the quantum state. One might naively expect that the same could be done for a stress tensor representing exotic matter. However, the essential content of the quantum inequality (1) is that arbitrarily extended distributions of arbitrarily negative energy are not possible in Minkowski spacetime. In this section we have argued that the bound should also be applicable in curved spacetimes for sampling times small compared to either the minimum local radius of curvature or the proper distance to any boundary.

Let us recall that Eq. (1) was proved for the specific case of a free massless, minimally coupled scalar field. It should be straightforward to generalize the arguments of Ref. [13] to the case of other massless fields, such as the electromagnetic field. Although this has not yet been done, it is unlikely that the result will be significantly different. Generalizations to massive fields may also be possible, although the results may be more complicated due to the presence of two length scales: τ_0 and the particle's Compton wavelength. However, it seems unlikely that adding a mass will make it easier to have large negative energy densities, as one now has to overcome the positive rest mass energy. Thus, one suspects that massive fields will satisfy inequalities which are more restrictive than Eq. (1). The effect of including interactions is the most difficult to assess. If an interacting theory were to allow regions of negative energy much more extensive than allowed in free theories, there would seem to be a danger of an instability where the system spontaneously makes a transition to a configuration with large negative energy density. However, this must be regarded as an open question.

III. MORRIS-THORNE WORMHOLES

The spacetime geometry for a Morris-Thorne (MT) traversable wormhole is described by the metric [1]

$$ds^2=-e^{2\Phi(r)}dt^2+\frac{dr^2}{1-b(r)/r}+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{13}$$

where the two adjustable functions $b(r)$ and $\Phi(r)$ are the ''shape function'' and the ''redshift function,'' respectively. The shape function $b(r)$ determines the shape of the wormhole as viewed, for example, in an embedding diagram. The metric (13) is spherically symmetric and static, with the proper circumference of a circle of fixed r being given by $2\pi r$. The coordinate r is nonmonotonic in that it decreases from $+\infty$ to a minimum value r_0, representing the location of the throat of the wormhole, where $b(r_0)=r_0$, and then it increases from r_0 to $+\infty$. Although there is a coordinate singularity at the throat, where the metric coefficient g_{rr} becomes divergent, the radial proper distance

$$l(r)=\pm\int_{r_0}^{r}\frac{dr}{[1-b(r)/r]^{1/2}} \tag{14}$$

is required to be finite everywhere. Note that because $0\leqslant 1-b(r)/r\leqslant 1$, the proper distance is greater than or equal to the coordinate distance: $|l(r)|\geqslant r-r_0$. The metric (13) may be written in terms of the proper radial distance as

$$ds^2=-e^{2\Phi(r)}dt^2+dl^2+r^2(l)(d\theta^2+\sin^2\theta d\phi^2). \tag{15}$$

The proper distance decreases from $l=+\infty$ to zero at the throat, and then from zero to $-\infty$ on the ''other side'' of the wormhole. For the wormhole to be traversable it must have no horizons, which implies that $g_{tt}=-e^{2\Phi(r)}$ must never be allowed to vanish, and hence $\Phi(r)$ must be everywhere finite.

The four-velocity of a static observer is $u^\mu=dx^\mu/d\tau=(u^t,0,0,0)=(e^{-\Phi(r)},0,0,0)$. The observer's four-acceleration is

$$a^\mu=\frac{Du^\mu}{d\tau}=u^\mu{}_{;\nu}u^\nu=(u^\mu{}_{,\nu}+\Gamma^\mu_{\beta\nu}u^\beta)u^\nu. \tag{16}$$

For the metric (13) we have

$$a^t=0,$$

$$a^r=\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2=\Phi'(1-b/r), \tag{17}$$

where $\Phi'=d\Phi/dr$. From the geodesic equation, a radially moving test particle which starts from rest initially has the equation of motion

$$\frac{d^2r}{d\tau^2}=-\Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2=-a^r. \tag{18}$$

Hence a^r is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant r,θ,ϕ. Note for future reference that from Eq. (17), *a static observer at the throat of any wormhole is a geodesic observer*. For $\Phi'(r)\neq 0$ wormholes, static observers are not geodesic (except at the throat), whereas for $\Phi'(r)=0$ wormholes they are. A wormhole is ''attractive'' if $a^r>0$ (observers must maintain an outward-directed radial acceleration to keep from being pulled into the wormhole) and ''repulsive'' if $a^r<0$ (observers must maintain an inward-directed radial acceleration to avoid being pushed away from the wormhole). From Eq. (17), this distinction depends on the sign of Φ'. For $a^r=0$, the wormhole is neither attractive nor repulsive.

Substitution of Eq. (13) into the Einstein equations gives the stress-energy tensor required to generate the wormhole geometry. It is often convenient to work in the static orthonormal frame given by the basis

$$\begin{aligned}
e_{\hat t}&=e^{-\Phi}e_t,\\
e_{\hat r}&=(1-b/r)^{1/2}e_r,\\
e_{\hat\theta}&=r^{-1}e_\theta,\\
e_{\hat\phi}&=(r\sin\theta)^{-1}e_\phi.
\end{aligned} \tag{19}$$

This basis represents the proper reference frame of an observer who is at rest relative to the wormhole. In this frame the stress tensor components are given by

$$T_{\hat t\hat t}=\rho=\frac{b'}{8\pi r^2}, \tag{20}$$

$$T_{\hat r\hat r}=p_r=-\frac{1}{8\pi}\left[\frac{b}{r^3}-\frac{2\Phi'}{r}\left(1-\frac{b}{r}\right)\right], \tag{21}$$

$$T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = P = \frac{1}{8\pi}\left[\frac{1}{2}\left(\frac{b}{r^3} - \frac{b'}{r^2}\right) + \frac{\Phi'}{r}\left(1 - \frac{b}{2r} - \frac{b'}{2}\right) + \left(1 - \frac{b}{r}\right)[\Phi'' + (\Phi')^2]\right]. \tag{22}$$

The quantities ρ, p_r, and P are the mass-energy density, radial pressure, and transverse pressure, respectively, as measured by a static observer [22]. At the throat of the wormhole, $r = r_0$, these reduce to

$$\rho_0 = \frac{b_0'}{8\pi r_0^2}, \tag{23}$$

$$p_0 = -\frac{1}{8\pi r_0^2}, \tag{24}$$

$$P_0 = \frac{1 - b_0'}{16\pi r_0}\left(\Phi_0' + \frac{1}{r_0}\right), \tag{25}$$

where $b_0' = b'(r_0)$ and $\Phi_0' = \Phi'(r_0)$.

The curvature tensor components are given by

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \left(1 - \frac{b}{r}\right)[\Phi'' + (\Phi')^2] + \frac{\Phi'}{2r^2}(b - b'r), \tag{26}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{\Phi'}{r}\left(1 - \frac{b}{r}\right), \tag{27}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{1}{2r^3}(b'r - b), \tag{28}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{b}{r^3}. \tag{29}$$

All other components of the curvature tensor vanish, except for those related to the above by symmetry. At the throat, these components reduce to

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}|_{r_0} = \frac{\Phi_0'}{2r_0}(1 - b_0'), \tag{30}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}|_{r_0} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}|_{r_0} = 0, \tag{31}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}|_{r_0} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}|_{r_0} = -\frac{1}{2r_0^2}(1 - b_0'), \tag{32}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}|_{r_0} = \frac{1}{r_0^2}. \tag{33}$$

Let us now define the following set of length scales:

$$\overline{r_0} = b, \quad r_1 = \left|\frac{b}{b'}\right|, \quad r_2 = \left|\frac{\Phi}{\Phi'}\right|, \quad r_3 = \left|\frac{\Phi'}{\Phi''}\right|. \tag{34}$$

The quantities r_1, r_2, and r_3 are a measure of the coordinate length scales over which b, Φ, and Φ', respectively, change. The number of length scales corresponds to the number of derivatives which appear in the curvature tensor, and b. It will prove convenient to absorb $|\Phi|$ into another length scale defined by

$$R_2 = \frac{r_2}{|\Phi|} = \frac{1}{|\Phi'|}. \tag{35}$$

The smallest of the above length scales is

$$r_m \equiv \min(\overline{r_0}, r_1, R_2, r_3). \tag{36}$$

As an aside, note that if $r_m = R_2$, then we can say that either r_2 is very small or $|\Phi|$ is very large (which implies that the redshift or blueshift $e^{\pm|\Phi|}$ is very large), or both. The curvature components may be written in terms of these length scales as

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \left(1 - \frac{b}{r}\right)\left[\pm\frac{1}{R_2 r_3} + \frac{1}{R_2^2}\right] \pm \frac{b}{r}\left(\pm\frac{1}{2r_1 R_2} - \frac{1}{2rR_2}\right), \tag{37}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \pm\left(1 - \frac{b}{r}\right)\frac{1}{rR_2}, \tag{38}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{b}{r}\left(\pm\frac{1}{2rr_1} - \frac{1}{2r^2}\right), \tag{39}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{b}{r^3}. \tag{40}$$

The choice of plus or minus signs in the various terms of the above equations will depend on the signs of the derivatives of b and Φ, which will in turn depend on the specific wormhole geometry.

Let the magnitude of the maximum curvature component be $R_{\max}$. Since the largest value of $(1 - b/r)$ and of b/r is 1, an examination of Eqs. (37)–(40) shows that $R_{\max} \lesssim 1/(r_m^2)$. Therefore the smallest proper radius of curvature (which is also the coordinate radius of curvature in an orthonormal frame) is

$$r_c \approx \frac{1}{\sqrt{R_{\max}}} \gtrsim r_m. \tag{41}$$

Our length scales at the throat become

$$\overline{r_0} = r_0, \quad r_1 = \left|\frac{r_0}{b_0'}\right|, \quad R_2 = \frac{r_2}{|\Phi_0|}, \quad r_3 = \left|\frac{\Phi_0'}{\Phi_0''}\right|. \tag{42}$$

At the throat of the wormhole Eqs. (37)–(40) simplify to

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}}|_{r_0} = \pm\frac{1}{2r_0 R_2} \pm \frac{1}{2r_1 R_2}, \tag{43}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}}|_{r_0} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}}|_{r_0} = 0, \tag{44}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}|_{r_0} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}|_{r_0} = \pm\frac{1}{2r_0 r_1} - \frac{1}{2r_0^2}, \tag{45}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}\big|_{r_0}=\frac{1}{r_0^2}. \tag{46}$$

(At the throat, the length scale r_3 does not explicitly appear in the curvature components.) Again, we see that $R_{\max}\lesssim 1/(r_m^2)$ and $r_c \gtrsim r_m$.

We wish to work in a small spacetime volume around the throat of the wormhole such that all dimensions of this volume are much smaller than r_c, the smallest proper radius of curvature anywhere in the region. Thus, in the absence of boundaries, spacetime can be considered to be approximately Minkowskian in this region, and we should be able to apply our QI bound.

IV. SPECIFIC EXAMPLES

To develop physical intuition for the general case, as well as to get a feeling for the magnitudes of the numbers involved, in this section we apply our bound to a series of specific examples.

A. $\Phi=0$, $b=r_0^2/r$ wormholes

This is a particularly simple wormhole which is discussed in box 2 and the bottom left-hand column of p. 400 of Ref. [1]. In terms of the proper radial distance $l(r)$, the metric is

$$ds^2=-dt^2+dl^2+(r_0^2+l^2)(d\theta^2+\sin^2\theta d\phi^2), \tag{47}$$

where $l=\pm(r^2-r_0^2)$. (Recall that $l=0$ at the throat.) The stress-tensor components are given by

$$\rho=p_r=-P=-\frac{r_0^2}{8\pi r^4}=-\frac{r_0^2}{8\pi(r_0^2+l^2)^2}. \tag{48}$$

The curvature components are

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}=-R_{\hat{l}\hat{\theta}\hat{l}\hat{\theta}}=-R_{\hat{l}\hat{\phi}\hat{l}\hat{\phi}}=\frac{r_0^2}{(r_0^2+l^2)^2}. \tag{49}$$

Note that all the curvature components are equal in magnitude, and have their maximum magnitude $1/(r_0^2)$ at the throat. The same holds true for the stress-tensor components. At the throat, our length scales are $\overline{r_0}=r_0=r_1$, and so $r_c=r_0$.

Let us apply our QI bound to a static observer at $r=r_0$. (Recall that such an observer is geodesic.) Since the energy density seen by this static observer is constant, we have

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2}=\rho_0\gtrsim-\frac{c}{\tau_0^4}, \tag{50}$$

where $c\equiv 3/(32\pi^2)$, τ is the observer's proper time, and τ_0 is the sampling time. Choose our sampling time to be $\tau_0=fr_m=fr_0\ll r_c$, with $f\ll 1$. Substitution into Eq. (50) yields

$$r_0\lesssim\frac{l_P}{2f^2}, \tag{51}$$

where l_P is the Planck length. Here it is fairly obvious that any reasonable choice of f gives a value of r_0 which is not much larger than l_P. For example, for $f\approx 0.01$, $r_0\lesssim 10^4\ l_P=10^{-31}$ m. Note from Eqs. (48) and (49) that if we choose our spacetime region to be such that $l\ll r_0$, then the curvature and stress-tensor components do not change very much.

B. $\Phi=0$, $b=r_0=$const wormholes

For this wormhole $\Phi=0$ and $b=$const, and so $b'=0$, and therefore $\rho=0$. This is a special case of ''zero density'' wormholes [23]. Here g_{tt} is the same as for Minkowski spacetime, while the spatial sections are the same as those of Schwarzschild. The energy density and radial pressure seen by a static observer are

$$\rho=0\,,\quad p_r=-\frac{r_0}{8\pi r^3}. \tag{52}$$

Since the energy density is zero in the static frame, to obtain a bound we boost to the frame of a radially moving geodesic observer. The energy density in the boosted frame is, by a Lorentz transformation,

$$T_{\hat{0}'\hat{0}'}=\rho'=\gamma^2(\rho+v^2p_r), \tag{53}$$

where v is the velocity of the boosted observer relative to the static frame, and $\gamma=(1-v^2)^{-1/2}$. In our case, we have

$$\rho'=-\frac{\gamma^2v^2r_0}{8\pi r^3}. \tag{54}$$

Note that in this case *any* nonzero v gives $\rho'<0$, in contrast to the discussion surrounding Eq. (57) of Ref. [1]. The nonzero curvature components in the static frame are

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}=R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}=-\frac{r_0}{2r^3},\quad R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}=\frac{r_0}{r^3}. \tag{55}$$

Here the only relevant length scale is r_0, since $\Phi=0$ and $b=r_0$ everywhere. For a general wormhole, when we boost to the radially moving frame, we have

$$R_{\hat{0}'\hat{1}'\hat{0}'\hat{1}'}=R_{\hat{t}\hat{r}\hat{t}\hat{r}},$$

$$R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}=R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}=\frac{\gamma^2}{2r^2}\left[v^2\left(b'-\frac{b}{r}\right)+2(r-b)\Phi'\right],$$

$$R_{\hat{2}'\hat{1}'\hat{2}'\hat{1}'}=R_{\hat{3}'\hat{1}'\hat{3}'\hat{1}'}=\frac{\gamma^2}{2r^2}\left[\left(b'-\frac{b}{r}\right)+2v^2(r-b)\Phi'\right],$$

$$R_{\hat{2}'\hat{0}'\hat{2}'\hat{1}'}=R_{\hat{3}'\hat{0}'\hat{3}'\hat{1}'}=\frac{\gamma^2v}{2r^2}\left[\left(b'-\frac{b}{r}\right)+2(r-b)\Phi'\right],$$

$$R_{\hat{2}'\hat{3}'\hat{2}'\hat{3}'}=R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}. \tag{56}$$

In the present case the nonzero components in the primed frame are

$$R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}=R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'}=-\frac{\gamma^2 v^2 r_0}{2r^3},$$

$$R_{\hat{2}'\hat{1}'\hat{2}'\hat{1}'}=R_{\hat{3}'\hat{1}'\hat{3}'\hat{1}'}=-\frac{\gamma^2 r_0}{2r^3},$$

$$R_{\hat{2}'\hat{0}'\hat{2}'\hat{1}'}=R_{\hat{3}'\hat{0}'\hat{3}'\hat{1}'}=-\frac{\gamma^2 v r_0}{2r^3},$$

$$R_{\hat{2}'\hat{3}'\hat{2}'\hat{3}'}=R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}=\frac{r_0}{r^3}. \tag{57}$$

In the vicinity of the throat, the magnitude of the maximum curvature component in the boosted frame is $R'_{\max}\lesssim \gamma^2/(r_0^2)$, and therefore the smallest local proper radius of curvature in that frame is $r'_c \gtrsim r_0/\gamma$. Apply our QI bound to the boosted observer and take $\tau_0 = f r_0/\gamma \ll r'_c$, for $f \ll 1$. Since the energy density does not change much over this time scale, we may write

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty}\frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2}\approx \rho'_0 \gtrsim -\frac{c}{\tau_0^4}, \tag{58}$$

which leads to

$$r_0 \lesssim \frac{\gamma}{2f^2 v}\, l_P\,. \tag{59}$$

In this case, any nonzero v gives us a bound, but we can find the optimum bound by minimizing γ/v, a procedure which yields $v=1/\gamma=1/\sqrt{2}$ and

$$r_0 \lesssim \frac{l_P}{f^2}. \tag{60}$$

Equation (60) is essentially the same as Eq. (51), which was the bound we obtained in the $\Phi=0$, $b=r_0^2/r$ case.

C. ‘‘Absurdly benign’’ wormholes

Classically, one can design a wormhole so that the exotic matter is confined to an arbitrarily small region around the throat. MT call this an ‘‘absurdly benign’’ wormhole. It is given by the choices $\Phi=0$ everywhere and

$$\begin{aligned} b(r)&=r_0[1-(r-r_0)/a_0]^2 \quad \text{for } r_0\le r\le r_0+a_0,\\ &=0 \quad \text{for } r\ge r_0+a_0. \end{aligned} \tag{61}$$

For $r_0 \le r < r_0+a_0$,

$$\rho=-\frac{r_0}{4\pi r^2 a_0^2}(a_0+r_0-r)<0\,, \tag{62}$$

$$p_r=-\frac{r_0}{8\pi r^3 a_0^2}(a_0+r_0-r)^2, \tag{63}$$

$$P=-\frac{1}{2}(\rho+p_r). \tag{64}$$

For $r \ge r_0+a_0$, the spacetime is Minkowskian, and $\rho=p_r=P=0$. The quantity a_0 represents the thickness in r (on one side of the wormhole) of the negative energy region. Evaluation of the curvature components using Eq. (61) shows that they have maximum magnitude at the throat where

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}|_{r_0}=R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}|_{r_0}=-\frac{1}{a_0 r_0}-\frac{1}{2r_0^2}, \tag{65}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}|_{r_0}=\frac{1}{r_0^2}. \tag{66}$$

At the throat, our length scales become

$$\bar{r}_0=r_0, \quad r_1=\frac{a_0}{2}, \tag{67}$$

and $r_m=\min(r_0,r_1)$. Again we see that $R_{\max}\lesssim 1/(r_m^2)$, and so the smallest local radius of curvature is $r_c \gtrsim r_m$.

Application of our QI bound to a static observer at the throat yields

$$\rho_0=-\frac{1}{4\pi a_0 r_0}\gtrsim -\frac{c}{\tau_0^4}. \tag{68}$$

Although this wormhole was designed for maximum confinement of the negative energy near the throat, i.e., $a_0 \ll r_0$, there is nothing in principle to keep us from choosing $a_0 \gtrsim r_0$. In what follows, we shall consider both situations. First assume $a_0 < r_0$. We then choose our sampling time to be $\tau_0 = f a_0$, where $f \ll 1$. Equation (68) then yields

$$a_0 \lesssim \left(\frac{r_0}{8f^4 l_P}\right)^{1/3} l_P\,. \tag{69}$$

A reasonable choice of f is $f\approx 0.01$. For a small ‘‘human-sized’’ wormhole with $r_0\approx 1$ m, our bound gives $a_0 \lesssim 10^{14}\, l_P \approx 10^{-21}$ m $\approx 10^{-6}$ F, or approximately $1/10^6$ of the proton radius. The situation does not improve much for larger wormholes. For $r_0 \approx 1$ light year, $a_0 \lesssim 2\times 10^{19}\, l_P$ ≈ 0.2 F. With $r_0 \approx 10^5$ light years, $a_0 \lesssim 10^{21}\, l_P \approx 10^{-14}$ m. So even with a throat radius the size of a galaxy, the negative energy must be distributed in a band no thicker than about ten proton radii. Now suppose that $r_0 < a_0/2$, so that $r_m = r_0$. In that case, we choose $\tau_0 = f r_0$, and our bound gives

$$r_0 \lesssim \left(\frac{a_0}{8f^4 l_P}\right)^{1/3} l_P\,; \tag{70}$$

i.e., a_0 and r_0 are simply interchanged. Therefore, the same numerical examples just discussed now apply to r_0, for given choices of a_0. For example, when $a_0 \approx 1$ light year, now the throat size is less than about 0.2 F, so that even for very large a_0, r_0 must be extremely small. When $a_0 \approx r_0$, the bound on r_0 is essentially Eq. (51). One might worry that since a_0 is the coordinate thickness in r of the negative energy density, it might not be a good measure of the proper radial thickness of the negative energy density band seen by the static observer. In fact, a detailed calculation shows that a_0 *is* the proper thickness in this case, to within factors of order unity.

D. ''Proximal Schwarzschild'' wormholes

Another special case of a zero density wormhole is the ''proximal Schwarzschild'' wormhole [24]. Here $b=r_0=\text{const}$, and g_{tt} is only slightly different from that of Schwarzschild. The metric in this case is

$$ds^2=-\left(1-\frac{r_0}{r}+\frac{\epsilon}{r^2}\right)dt^2+\frac{dr^2}{(1-r_0/r)}+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{71}$$

We recover the Schwarzschild solution for $\epsilon=0$; however, any $\epsilon>0$ gives us a wormhole. The energy density and radial pressure seen by a static observer are

$$\rho=0\,, \tag{72}$$

$$p_r=-\frac{\epsilon}{8\pi r^4}\frac{(2-r_0/r)}{(1-r_0/r+\epsilon/r^2)}. \tag{73}$$

We will assume that $\sqrt{\epsilon}\ll r_0$; hence, the radial pressure is highly peaked near the throat. Here the proper distance from $r=r_0$ to $r=r_0+\sqrt{\epsilon}$, corresponding to the coordinate thickness $\sqrt{\epsilon}$, is

$$\Delta l=\int_{r_0}^{r_0+\sqrt{\epsilon}}\frac{dr}{\sqrt{1-r_0/r}}\approx\int_{r_0}^{r_0+\sqrt{\epsilon}}\frac{\sqrt{r_0}dr}{\sqrt{r-r_0}}=2\sqrt{r_0\sqrt{\epsilon}}. \tag{74}$$

A disadvantage of this wormhole is that it entails extremely large redshifts. The metric coefficient g_{tt} is very close to that of Schwarzschild, and therefore this wormhole is very close to having a horizon at its throat.

In the region $r_0\leqslant r\leqslant r_0+\sqrt{\epsilon}$, the curvature components have their maximum magnitudes at the throat (except $R_{\hat t\hat\theta\hat t\hat\theta}=R_{\hat t\hat\phi\hat t\hat\phi}$, which vanish there). At $r=r_0$,

$$R_{\hat t\hat r\hat t\hat r}|_{r_0}\approx\frac{1}{4\epsilon}, \tag{75}$$

$$R_{\hat r\hat\theta\hat r\hat\theta}|_{r_0}=R_{\hat r\hat\phi\hat r\hat\phi}|_{r_0}=-\frac{1}{2r_0^2}, \tag{76}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi}|_{r_0}=\frac{1}{r_0^2}. \tag{77}$$

Our length scales at the throat become

$$\overline{r_0}=r_0,\quad r_1=\infty,\quad R_2\approx\frac{2\epsilon}{r_0},\quad r_3\approx\frac{\epsilon}{r_0}. \tag{78}$$

Note that $r_1=\infty$ is due to the fact that $b'=0$. Although we can write the curvature components in terms of these length scales, in this case the smallest radius of curvature in the static frame is $r_c\approx1/\sqrt{R_{\max}}\approx2\sqrt{\epsilon}$, which is larger than our smallest length scale $r_m=r_3$. Thus we will get a stronger bound if we frame our argument in terms of $r_c\approx2\sqrt{\epsilon}$.

Since the energy density is zero in the static frame, we must apply our bound in the frame of a boosted observer passing through the throat. The curvature tensor components in this frame are

$$R_{\hat 1'\hat 0'\hat 1'\hat 0'}|_{r_0}=R_{\hat t\hat r\hat t\hat r}|_{r_0}\approx\frac{1}{4\epsilon},$$

$$R_{\hat 2'\hat 0'\hat 2'\hat 0'}|_{r_0}=R_{\hat 3'\hat 0'\hat 3'\hat 0'}|_{r_0}=-\frac{\gamma^2v^2}{2r_0^2},$$

$$R_{\hat 2'\hat 1'\hat 2'\hat 1'}|_{r_0}=R_{\hat 3'\hat 1'\hat 3'\hat 1'}|_{r_0}=-\frac{\gamma^2}{2r_0^2},$$

$$R_{\hat 2'\hat 0'\hat 2'\hat 1'}|_{r_0}=R_{\hat 3'\hat 0'\hat 3'\hat 1'}|_{r_0}=-\frac{\gamma^2v}{2r_0^2},$$

$$R_{\hat 2'\hat 3'\hat 2'\hat 3'}|_{r_0}=R_{\hat\theta\hat\phi\hat\theta\hat\phi}|_{r_0}=\frac{1}{r_0^2}. \tag{79}$$

Which of these components has the maximum magnitude depends on whether $\sqrt{2\epsilon}$ is greater than or less than r_0/γ.

First consider the case $\sqrt{2\epsilon}<r_0/\gamma$. Then $R'_{\max}=R_{\max}\approx1/(4\epsilon)$, and $r'_c=r_c\approx2\sqrt{\epsilon}$. Take $\tau_0=fr'_c\approx2f\sqrt{\epsilon}$, with $f<<1$. The energy density in the boosted frame should be approximately constant over this sampling time. Therefore, our QI bound gives

$$\rho'_0=-\frac{\gamma^2v^2}{8\pi r_0^2}\gtrsim-\frac{c}{\tau_0^4}, \tag{80}$$

and hence

$$\frac{\sqrt{\epsilon}}{r_0}\lesssim\left(\frac{1-v^2}{64\,v^2f^4}\right)^{1/4}\left(\frac{l_P}{r_0}\right)^{1/2}. \tag{81}$$

By making v arbitrarily close to 1, we can make the right-hand side of the bound arbitrarily small.

It may be more appropriate to express the width of the band of exotic matter in the static frame in terms of proper length, rather than coordinate length. Using Eq. (74), our bound Eq. (81) can be rewritten as

$$\frac{\Delta l}{r_0}\lesssim\left(\frac{1-v^2}{v^2f^4}\right)^{1/8}\left(\frac{l_P}{r_0}\right)^{1/4}. \tag{82}$$

In this form, the bound is quite a bit weaker, due to the smaller powers on the right-hand side of the inequality. We can still in principle make the right-hand side arbitrarily small, albeit only by choosing v *exceedingly* close to 1. However, our bound must hold for *any* boosted observer. Consequently, for the case $\sqrt{2\epsilon}<r_0/\gamma$, proximal Schwarzschild wormholes with any finite value of Δl would seem to be physically excluded.

Next consider the case where $\sqrt{2\epsilon}>r_0/\gamma$. Then $R'_{\max}=\gamma^2/(2r_0^2)$, and the smallest local radius of curvature in the boosted frame is $r_c' \approx \sqrt{2} r_0/\gamma$. Take $\tau_0 = f r_c'$, with $f \ll 1$. Application of our bound in this case yields

$$r_0 \lesssim \frac{\sqrt{2\pi c}}{f^2}\left(\frac{\gamma}{v}\right). \tag{83}$$

We get the optimum bound by minimizing γ/v, i.e.,

$$r_0 \lesssim \frac{l_P}{2f^2}, \tag{84}$$

which is the same as the bound we obtained in the $\Phi=0$, $b=r_0^2/r$ case.

E. Morris-Thorne-Yurtsever wormhole

Morris, Thorne, and Yurtsever (MTY) [2] have discussed a wormhole consisting of an $r_0=Q=M$ Reissner-Nordström (RN) metric with a pair of spherical charged Casimir plates positioned on each side of the throat within a very small proper distance s of one another. That is, the spacetime is extreme RN from each plate out to $r=\infty$, and approximately flat between the plates. The Casimir energy density between the plates is negative, while the stress energy of the external classical electromagnetic field is ''near exotic,'' i.e., $(\rho_c+p_c)|_{\mathrm{EM}}=0$. For $r \geqslant r_0+\delta$, the metric has the extreme RN form

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{(1-M/r)^2} + r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{85}$$

MTY show that, for this wormhole,

$$p_0 = -\frac{1}{8\pi r_0^2} = p_{\mathrm{Casimir}}. \tag{86}$$

Then because $\rho_{\mathrm{Casimir}} = p_{\mathrm{Casimir}}/3$ and $\rho_{\mathrm{Casimir}}=\rho_0$, it follows that $\rho_0 = -(24\pi r_0^2)^{-1}$. If we apply our bound to a (very tiny) static observer at the throat, we obtain

$$-\frac{1}{24\pi r_0^2} \gtrsim -\frac{c}{\tau_0^4}. \tag{87}$$

Because spacetime is approximately flat between the plates, the constraint on the choice of τ_0 is that discussed in the Casimir effect example in Sec. II. Take $\tau_0 = fs$, whereupon we find

$$r_0 \gtrsim f^2 s^2. \tag{88}$$

A reasonable choice of f in this case would seem to be $f\approx 0.1$. For $s\approx 10^{-10}$ cm $\approx 10^{23}\, l_P$, one finds that $r_0 \gtrsim 10^{44}\, l_P \sim 0.01$ AU (astronomical unit). Since MTY calculate $r_0\approx 1$ AU for a plate separation of $s\approx 10^{-10}$ cm, this wormhole satisfies our bound.

However, this wormhole has a number of undesirable features. First, in order to traverse it, an observer must go through the plates. This implies that ''holes'' or ''trapdoors'' must be cut in the plates to allow passage. Second, because the plates are located at $r\approx r_0+\delta$, with $\delta\approx 10^{-10}$ cm (neglecting the thickness of the plates), this wormhole is extremely close to being a black hole, i.e., $|g_{tt}|_{r_0+\delta} = (1-M/r)^2 \approx \delta^2/M^2$. Infalling photons with frequency at infinity ω_∞ will have local frequency $\omega_{\mathrm{local}} \approx \omega_\infty (M/\delta)$, as measured by a static observer on the plate. For $\delta\approx 10^{-10}$ cm and $r_0 \approx M \approx 1$ AU $\approx 10^{13}$ cm, we have $\omega_{\mathrm{local}} \approx 10^{23}\omega_\infty$. A typical infalling 3 K photon in the cosmic microwave background radiation, upon arriving at one of the plates, would get blueshifted to a temperature $T_{\mathrm{local}} \approx 10^{23}$ K. A 0.1 MeV γ-ray photon would get blueshifted to $E_{\mathrm{local}} \approx 10^{19}$ GeV $\approx E_P$, where E_P is the Planck energy. Stray cosmic-ray particles, with typical energies of ~ 1 GeV, falling into the wormhole will have $E_{\mathrm{local}} \approx E_\infty (M/\delta) \approx 10^{23}$ GeV $\approx 10^4 E_P$. A static observer just outside the plates would likely be incinerated by infalling radiation. Similarly, the plates would have to be constructed out of material capable of withstanding these large energies. In addition, the large local impacts of infalling radiation and particles on the plates will tend to push them together, thereby upsetting the force balance, and hence will probably destabilize the wormhole. One *could* imagine elaborate radiation shielding constructed around a large region far away from, but enclosing, the wormhole. However, if the wormhole is unstable to infalling radiation, then infalling spaceships would seem like an even more remote possibility.

V. GENERAL BOUNDS FOR WORMHOLES

In this section QI bounds will be formulated on the relative size scales of arbitrary static, spherically symmetric, MT wormholes; i.e., no assumptions will be made about the specific forms of $\Phi(r)$ and $b(r)$. We work in the vicinity of the throat and analyze two general subcases: (1) $b_0'<0$ and (2) $b_0' \geqslant 0$. Let r_m be the smallest of the length scales: $\overline{r_0}, r_1, R_2, r_3$, in this region. We saw from Eqs. (37)–(40) in Sec. III that the magnitude of the maximum curvature component in this region is $R_{\max} \lesssim 1/(r_m^2)$. It follows that the smallest proper radius of curvature (in the static orthonormal frame) is $r_c \approx 1/\sqrt{R_{\max}} \gtrsim r_m$. Spacetime can be considered to be approximately flat in this region, and therefore our QI bound should be applicable.

A. Case (1): $b_0'<0$

Since $b_0'<0$, the energy density is negative at the throat, and so we can apply our bound to a static observer at the throat. This observer is geodesic, and the energy density is constant; so we have

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2} = \rho_0 = \frac{b_0'}{8\pi r_0^2} \gtrsim -\frac{c}{\tau_0^4}, \tag{89}$$

where again $c \equiv 3/(32\pi^2)$, τ is the observer's proper time, and τ_0 is the sampling time. Choose our sampling time to be $\tau_0 = f r_m \ll r_c$, with $f \ll 1$. Our QI then becomes

$$r_m \lesssim \left(\frac{8\pi c}{|b_0'|}\right)^{1/4} \frac{\sqrt{r_0}}{f}. \tag{90}$$

Since our observer is static at the throat, we may write $|b_0'| = r_0/r_1$, and use $8\pi c \approx 1/4$, to get

$$r_m \lesssim \frac{(r_0 r_1)^{1/4}}{f} \tag{91}$$

or, alternatively,

$$\frac{r_m}{r_0} \lesssim \frac{1}{f}\left(\frac{r_1 l_P^2}{r_0^3}\right)^{1/4}. \tag{92}$$

Now examine specific cases. For $r_m = r_0$, we have $r_0/r_1 \lesssim f^{-4/3}(l_P/r_1)^{2/3}$. As an example, if $r_1 \approx 1$ m and for $f \approx 0.01$, then $r_0/r_1 \lesssim 10^{-21}$. Even if we choose f to be very small, this large discrepancy in length scales will not change much, and only increases as r_1 increases. For $r_m = r_1$, $r_1/r_0 \lesssim f^{-4/3}(l_P/r_0)^{2/3}$, and so for $r_0 \approx 1$ m and $f \approx 0.01$, $r_1/r_0 \lesssim 10^{-21}$. Again the problem only gets worse as the throat size r_0 increases. When $r_m = r_0 = r_1$, $r_0 \lesssim l_P/f^2$; for $f \approx 0.01$, $r_0 \lesssim 10^4 l_P \sim 10^{-31}$ m. For $\Phi =$ const wormholes, the only relevant length scales are r_0 and r_1. The above results imply that when $b_0' < 0$ these wormholes are extremely unlikely, unless one is willing to accept a huge discrepancy in length scales. For $r_m = R_2$ and $R_2 \leqslant r_0 \leqslant r_1$, from Eq. (91) we have that $R_2/r_1 \lesssim (1/f)(r_0 l_P^2/r_1^3)^{1/4} \lesssim (1/f) \times (l_P/r_1)^{1/2}$. For $r_m = R_2$ and $R_2 \leqslant r_1 \leqslant r_0$, $R_2/r_0 \lesssim (1/f) \times (l_P/r_0)^{1/2}$. An identical argument yields similar inequalities for the case where $r_m = r_3$. Thus we find that if r_0 and/or r_1 are macroscopic, then the ratio of the minimum length scale to the macroscopic length scale must be very tiny.

If the minimum scale happens to be R_2, then our bounds imply that either r_2, the scale over which Φ changes, is very small or $|\Phi|$ is very large, or both. A situation in which $|\Phi|$ is very large near the throat, assuming $\Phi(\infty) = 0$, would not seem to be a desirable characteristic of a traversable wormhole, as it implies very large redshifts or blueshifts for a static observer at the throat. A large negative Φ_0 implies that the spacetime is close to having a horizon at the throat. A large positive Φ_0 implies that photons of moderate frequency fired outward by an observer at the throat would be blueshifted to very high frequencies upon reaching distant observers. In the latter case, observers must be shot inward with initially large kinetic energies in order to reach the throat.

If some of the wormhole parameters change over very short length scales, then it would seem from the ‘‘tidal force constraints’’ [see Eqs. (49) and (50) of MT] that tidal accelerations might also change over very short length scales. As a result, an observer traveling through the wormhole could encounter potentially wrenching tidal forces rather abruptly. None of these scenarios seem terribly convenient for wormhole engineering.

B. Case (2): $b_0' \geqslant 0$

When $b_0' \geqslant 0$, the energy density at the throat is nonnegative for static observers. To obtain a bound in this case, we Lorentz transform to the frame of a radially moving boosted observer at the throat. Since the maximum magnitude curvature component in the static frame is $R_{\max} \lesssim 1/(r_m^2)$, in the boosted frame the curvature component with the largest magnitude, $R'_{\max}$, can be *no larger than* about γ^2/r_m^2. Therefore, the smallest proper radius of curvature in the boosted frame is $r_c' \approx 1/\sqrt{R'_{\max}} \gtrsim r_m/\gamma$. Spacetime should be approximately flat in the boosted frame on scales much less than r_c'. Hence let us take our sampling time to be $\tau_0 = f r_m/\gamma \ll r_c'$, with $f \ll 1$. The energy density in this frame should not change much over the sampling time, and so the application of our bound gives

$$\frac{\tau_0}{\pi}\int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}u^\mu u^\nu\rangle d\tau}{\tau^2+\tau_0^2} \approx \langle T_{\mu\nu}u^\mu u^\nu\rangle = \rho_0' \gtrsim -\frac{c}{\tau_0^4}. \tag{93}$$

At the throat, from Eq. (53), the energy density in the boosted frame is

$$\rho_0' = \gamma^2(\rho_0 + v^2 p_0) = \frac{\gamma^2}{8\pi r_0^2}(b_0' - v^2). \tag{94}$$

In order for $\rho_0' < 0$, we must require $v^2 > b_0'$. After making the required substitutions, we obtain

$$\frac{r_m}{r_0} \lesssim \left(\frac{1}{v^2 - b_0'}\right)^{1/4} \frac{\sqrt{\gamma}}{f}\left(\frac{l_P}{r_0}\right)^{1/2}. \tag{95}$$

For $b_0' \geqslant 0$, $0 \leqslant b_0' \leqslant 1$, since $b_0' \leqslant 1$, which follows from the fact that at the throat we must have $\rho_0 + p_0 \leqslant 0$ [25]. The quantity b_0' is fixed by the wormhole geometry, whereas our choice of v^2 is arbitrary, subject to $b_0' < v^2 < 1$. Our bound, Eq. (95), is weakest when b_0' is extremely close to 1. However, this would seem to be a highly special case, $b_0' = 1$ corresponding to the maximum possible positive energy density at the throat and to $\rho_0 + p_0 = 0$; i.e., the null energy condition $T_{\mu\nu}K^\mu K^\nu \geqslant 0$, applied to radial null vectors, is barely satisfied at the throat. The latter implies that such a wormhole ‘‘flares outward’’ very slowly from the throat [see, for example, Eq. (56) of MT]. To see how close b_0' must be to 1 in order to significantly affect our bound, a numerical example is instructive. Let $b_0' = 1 - 10^{-8}$, $v^2 = 1 - 10^{-9}$, and $f \approx 0.01$. For $r_0 \approx 1$ m $\approx 10^{35}\, l_P$, we find that $r_m \lesssim 10^{-11}$ m. Even for $r_0 \approx 1$ AU$\approx 10^{46}\, l_P$, we obtain $r_m \lesssim 10^{-6}$ m. If we consider a more ‘‘typical’’ $b_0' \geqslant 0$ to be in about the middle of the allowed range, say, $b_0' \approx 1/2$, then if we choose $v^2 \approx 3/4$, it follows that $(v^2 - b_0')^{-1/4} \approx 1$. If we choose $f \approx 0.01$, then we find $r_m/r_0 \lesssim 100(l_P/r_0)^{1/2}$. Even a much smaller choice of f does not avoid the large discrepancy in wormhole length scales.

The bound (95) is a ‘‘safe’’ bound, but in specific cases it may not be the optimal bound, due to our rather conservative condition on $R'_{\max}$, i.e., $R'_{\max} \lesssim \gamma^2/r_m^2$, and hence on τ_0. For example, in cases where $R'_{\max} = |R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = |R_{\hat{t}\hat{r}\hat{t}\hat{r}}|$, such as the proximal Schwarzschild wormhole, we can get a stronger bound than that obtained from the inequality for the general case.

VI. CONCLUSIONS

In this paper, we used a bound on negative energy density derived in four-dimensional Minkowski spacetime to constrain static, spherically symmetric traversable wormhole geometries. In Sec. II, we argued that the bound should also be applicable in curved spacetime on scales which are much smaller than the minimum local radius of curvature and/or the distance to any boundaries in the spacetime. The upshot of our analysis is that either a wormhole must have a throat size which is only slightly larger than the Planck length l_P, or there must be large discrepancies in the length scales which characterize the geometry of the wormhole. These discrepancies are typically of order $(l_P/r_0)^n$, where r_0 is the throat radius and $n \lesssim 1$. They imply that generically the exotic matter is confined to an extremely thin band and/or that the wormhole geometry involves large redshifts (or blueshifts). The first feature would seem to be rather physically unnatural. Furthermore, wormholes in which the characteristics of the geometry change over short length scales and/or entail large redshifts would seem to present severe difficulties for traversability, such as large tidal forces.

There is a number of possible ways to circumvent our conclusions. The primary contributions to the exotic matter might come from the state-independent geometrical terms of $\langle T_{\mu\nu}\rangle$. However, as discussed in Sec. II, in this case the dimensionless coefficients of these terms would have to be enormous to generate a wormhole of macroscopic size. One possibility would be a model in which the effective values of these coefficients are governed by a new field ϕ in such a way that they are large only when ϕ is large. It may then be possible to find self-consistent solutions in which ϕ is large only in a very small region, and hence it is conceivable that one might be able to create thin bands of negative energy in this way [26]. Our bound was strictly derived only for a massless, minimally coupled scalar field, but we argued that similar bounds are likely to hold for other massless and massive quantum fields. Another possible circumvention of the bound might be to superpose the effects of many fields, each of which satisfies the bound [27]. For example, suppose we postulate N fields, each of which contribute approximately the same amount to our bound. Then the right-hand side of the inequality (1) would be replaced by $-Nc/\tau_0^4$. However, in practice N has to be extremely large in order to have a significant effect. For example, in the case of the $\Phi=0$, $b=$const wormholes discussed in Sec. IV, the constraint on the throat size becomes $r_0 \lesssim \sqrt{N}/(2f^2)$. For $f \approx 0.01$, $r_0 \lesssim \sqrt{N} 10^4 l_P \sim \sqrt{N} 10^{-31}$ m. Therefore, to get $r_0 \approx 1$ m, we would need either 10^{62} fields or a few fields for which the constant c is many orders of magnitude larger than $3/(32\pi^2)$. Neither of these possibilities seems very likely. Last, it may be that the semiclassical theory breaks down above the Planck scale, due to large stress-tensor fluctuations when the mean energy density is negative [14,15]. In that case, it becomes difficult to predict what happens. However, one might expect the time scale of such fluctuations to be of the order of the minimum radius of curvature. Since our sampling time is chosen to be much smaller than this, it may be that our analysis is unaffected by the fluctuations.

We showed that the Morris-Thorne-Yurtsever [2] wormhole was compatible with our bound. When this model was proposed some years ago, it was hoped that one might eventually be able to do better at spreading the exotic matter out over macroscopic dimensions. Our results indicate that this kind of wormhole might be the generically allowed case. However, as we pointed out, this wormhole has undesirable features, such as large redshifts near the throat which may pose problems for stability and traversability. It might seem that our conclusions imply that the most physically reasonable wormholes are the ''thin-shell'' type [28]. However, these models are constructed by ''cutting and pasting'' two copies of (for example) Minkowski or Schwarzschild spacetime, with a resulting δ-function layer of negative energy at the throat. (Note that in these wormholes, by construction, the throat is not located at $b=r_0$.) Physically one does not really expect infinitely thin layers of energy density and curvature in nature [29]. Such approximations are meant to be idealizations of situations in which the thickness of these layers are small compared to other relevant length scales. Our results can be construed as placing upper bounds on the actual allowed thicknesses of such layers of negative energy density. We conclude that, unless one is willing to accept fantastically large discrepancies in the length scales which characterize wormhole geometries, it seems unlikely that quantum field theory allows macroscopic static traversable wormholes [30].

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[24] See Ref. [3], Sec. 13.4.3. Note that the right-hand side of Eq. (13.88) of that reference is missing a factor of $(2 - r_0/r)$ in the numerator.

[25] See the argument in Ref. [3], pp. 104–105 and 109.

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