Bulk viscous cosmology

Winfried Zimdahl*

Departament de Física, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Barcelona, Spain

The full causal Müller-Israel-Stewart (MIS) theory of dissipative processes in relativistic fluids is applied to a flat, homogeneous, and isotropic universe with bulk viscosity. It is clarified in which sense the so-called truncated version is a reasonable limiting case of the full theory. The possibility of bulk viscosity-driven inflationary solutions of the full theory is discussed. As long as the particle number is conserved, almost all these solutions exhibit an exponential increase of the temperature. Assuming that the bulk viscous pressure of the MIS theory may also be interpreted as an effective description for particle production processes, the thermodynamical behavior of the Universe changes considerably. In the latter case, the temperature increases at a lower rate or may remain constant during a hypothetical de Sitter stage, accompanied by a substantial growth of the comoving entropy.

PACS number(s): 98.80.Hw, 04.40.Nr, 05.70.Ln, 95.30.Tg

I. INTRODUCTION

Nonequilibrium thermodynamical processes are supposed to play a crucial role in the physics of the early Universe. Traditionally, for the description of these phenomena, the theories of Eckart [1] and Landau and Lifshitz [2] were used. Because of the work of Müller [3], Israel [4], Israel and Stewart [5,6], Pavón, Jou, and Casas-Vázquez [7], Hiscock and Lindblom [8] it became clear, however, that the Eckart-type theories suffer from serious drawbacks concerning causalty and stability. These difficulties could be traced back to their restriction to first-order deviations from equilibrium. If one includes higher-order deviations as well, the corresponding problems disappear. By now, it is generally agreed that any analysis of dissipative phenomena in relativity should be based on the theories of Müller, Israel, and Stewart (MIS), including at least second-order deviations from equilibrium, although, in specific cases, the latter might reproduce results of the Eckart theory [9]. Cosmological implications of second-order theories were first considered by Belinskii *et al.* [10]. In the realm of cosmology, especially bulk viscous phenomena have attracted considerable interest (see, e.g. [11]), since bulk viscosity is the only possible dissipative mechanism in homogeneous and isotropic spacetimes. While the coefficient of bulk viscosity vanishes both for pure relativistic and pure nonrelativistic equations of state, it may be important, e.g., for mixtures of radiation and matter [12]. On the other hand, it is well known [13–15] and widely used [17–25] that particle production processes in the expanding Universe may be phenomenologically described in terms of effective viscous pressures.

A major point of interest in the study of bulk viscous universes has been the question whether there are conditions under which a sufficiently large bulk viscous pressures could lead to an inflationary behavior. While some authors concluded that a bulk viscosity-driven inflation is impossible [26], others [27–29] found inflationary solutions of the cosmological evolution equations. Partially, these differences occur since different equations of state were used. But the results also depend on whether the investigations were performed within the full, causal second-order theory or in a truncated version of it.

One should be aware that discussing the issue of bulk viscous-driven inflationary solutions at all, implies in any case an extrapolation of nonequilibrium thermodynamical theories beyond the range for which their applicability was strictly justified [29]. Bulk viscous inflation, if it exists, is a far-from-equilibrium phenomenon, while even the full, causal second-order MIS theory is a theory for small deviations from equilibrium. Therefore, all theoretical conclusions are necessarily tentative. It is the hope that they nevertheless will provide an indication of the correct behavior far from equilibrium.

As to the relation between the full theory and the truncated version, to be discussed in some detail below, the following comment should be made from the outset. To decide whether a theory is truncated or not on apparently obvious formal grounds, i.e., from the appearance of the causal evolution equations may be misleading. The structure of the evolution equation depends on the choice of the basic thermodynamical variables. In most cases, the latter are equilibrium variables and the Gibbs equation has their familiar form. In the framework of ''extended irreversible thermodynamics'' (EIT), however, a generalized Gibbs equation is used which includes dissipative quantities as independent variables. Temperature and pressure in EIT are nonequilibrium quantities, different from their equilibrium counterparts. Written in terms of these nonequilibrium quantities, the causal evolution equation formally may look identical to the truncated theory written in terms of the more familiar equilibrium variables. These points have been clarified in a recent paper by Gariel and Le Denmat [30] who pointed out that the apparently (because of the use of nonequilibrium variables) truncated theory of Pavón *et al.* [7], in fact, is equivalent to the full theory of Israel [4]. The reader is also referred to corresponding comments in [28,29].

In the present paper the symbols T and p always denote

---
*Present address: Fakultät für Physik, Universität Konstanz, PF 5560 M678, D-78434 Konstanz, Germany.

the equilibrium temperature and the equilibrium pressure, respectively.

In Sec. II we reconsider the bulk viscous cosmological dynamics within the causal second-order theories. The evolution law for the temperature of the cosmic fluid is shown to be different in general from the *ad hoc* relationship used in previous treatments by Romano and Pavón [31], Zakari and Jou [28], and Maartens [29] (Sec. II A). In Sec. II B we discuss the conditions under which the truncated version yields results close or identical to those of the full theory. The general viscous fluid dynamics of the full MIS theory is presented in Sec. II C. Section II D investigates the conditions for viscous exponential inflation. Almost all corresponding solutions imply an exponential increase of the fluid temperature. As a consequence of this behavior, there is, in general, no substantial growth of the comoving entropy as was found previously [29]. The latter result corresponds to a very specific limiting case. In Sec. III the viscous pressure of the full causal theory is assumed to describe partially or fully the effect of particle creation taken into account by a nonvanishing source term in the particle number balance. In this setting, the backreaction of the viscous pressure on the temperature is different from the conventional viscous fluid case of Sec. II. In the limit that the viscous pressure is entirely because of particle production, there exist stable inflationary solutions for which both the particle number density and the temperature are constant and, moreover, the comoving entropy grows exponentially. Section IV summarizes the results of the paper. Units have been chosen so that $c = k_B = 1$.

II. BULK VISCOUS FLUID DYNAMICS

A. General relations

The energy momentum tensor of a relativistic fluid, with bulk viscosity as the only dissipative phenomenon, is

$$T^{ik} = \rho u^i u^k + (p + \pi) h^{ik}. \tag{1}$$

ρ is the energy density, u^i is the four-velocity, p is the equilibrium pressure, h^{ik} is the projection tensor $h^{ik} = g^{ik} + u^i u^k$, and π is the bulk viscous pressure. The particle flow vector N^a is given by

$$N^a = n u^a, \tag{2}$$

where n is the particle number density. Limiting ourselves to second-order deviations from equilibrium, the entropy flow vector S^a takes the form [5,26]

$$S^a = s N^a - \frac{\tau \pi^2}{2 \zeta T} u^a, \tag{3}$$

where s is the entropy per particle, τ is the relaxation time, T is the temperature, and ζ is the coefficient of bulk viscosity. The conservation laws

$$N^a_{;a} = 0, \tag{4}$$

and

$$T^{ab}_{;b} = 0, \tag{5}$$

imply

$$\dot{n} + \Theta n = 0, \tag{6}$$

and

$$\dot{\rho} = -\Theta(\rho + p + \pi), \tag{7}$$

respectively, where $\Theta \equiv u^a_{;a}$ is the fluid expansion and $\dot{n} \equiv n_{,a} u^a$ etc. Combining (6) and (7) with the Gibbs relation

$$T ds = d\frac{\rho}{n} + p d\frac{1}{n}, \tag{8}$$

we get

$$n T \dot{s} = -\Theta \pi. \tag{9}$$

From (3) and (6), we find

$$S^a_{;a} = -\frac{\pi}{T} \left[\Theta + \frac{\tau}{\zeta} \dot{\pi} + \frac{1}{2} \pi T \left(\frac{\tau}{\zeta T} u^a \right)_{;a} \right] \tag{10}$$

for the entropy production density $S^a_{;a}$. The simplest way to guarantee $S^a_{;a} \geq 0$ implies the evolution equation

$$\pi + \tau \dot{\pi} = -\zeta \Theta - \frac{1}{2} \pi \tau \left[\Theta + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} \right] \tag{11}$$

for π, leading to

$$S^a_{;a} = \frac{\pi^2}{\zeta T}. \tag{12}$$

For $\tau \to 0$, Eq. (11) reduces to the corresponding relation of the Eckart theory. The frequently used truncated version

$$\pi + \tau \dot{\pi} = -\zeta \Theta, \tag{13}$$

also known as Maxwell-Cattaneo equation, follows if the bracket term on the right-hand side (RHS) of (11) can be neglected compared with the viscosity term $-\zeta \Theta$. Below, we shall give explicit criteria for this approximation. If the brackets term vanishes identically, i.e., if the condition

$$\Theta + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} = 0 \tag{14}$$

is satisfied, the full and the truncated theories become identical. As we shall see, this is possible only in exceptional cases.

Let us assume equations of state in the general form

$$p = p(n, T) \tag{15}$$

and

$$\rho = \rho(n, T), \tag{16}$$

according to which the particle number density n and the temperature T are our basic thermodynamical variables. Differentiating the latter relation, using the balances (6) and (7) as well as the general thermodynamic relation

$$\frac{\partial \rho}{\partial n} = \frac{\rho + p}{n} - \frac{T}{n} \frac{\partial p}{\partial T}, \tag{17}$$

one finds the following evolution law for the temperature (cf. [22]):

$$\dot{T} = \Theta n \frac{\partial \rho / \partial n}{\partial \rho / \partial T} + \frac{\dot{\rho}}{\partial \rho / \partial T}, \tag{18}$$

or

$$\frac{\dot{T}}{T} = -\Theta \left[\frac{\partial p / \partial T}{\partial \rho / \partial T} + \frac{\pi}{T \partial \rho / \partial T} \right]. \tag{19}$$

For $\pi = 0$ and with $\Theta = 3\dot{R}/R$, where R is the scale factor of the Robertson-Walker metric, (19) reproduces the well known $T_r \sim R^{-1}$ behavior in a radiation-dominated Friedmann-Lemaître-Robertson-Walker (FLRW) universe, while for $\rho = nm + \frac{3}{2}nT$, $p = nT$, and $T \ll m$, we recover $T_m \sim R^{-2}$ in the matter-dominated case. For a viscous fluid, the behavior of the temperature depends on π. Since π is expected to be negative, the second term in the brackets on the RHS of (19) will counteract the first one. Close to equilibrium, i.e., for $|\pi| < p$, the existence of a bulk viscous pressure implies that in an expanding Universe the temperature decreases less rapidly than in the perfect fluid case.

B. The truncated version

While the truncated version was used in most of the earlier applications, more recently an increasing number of authors [26,28,31,29] has studied the full theory and compared the results of the latter wih those of the truncated version. In some cases, these results differ dramatically, which may be interpreted as a breakdown of the Maxwell-Cattaneo-type equations as a reasonable approximation to the full theory under the corresponding conditions. What seems to be missing, however, are general criteria according to which one may decide whether the truncated version is sensible and beyond which limits it fails to give an answer close to that of the full theory. Intuitively, one expects the coincidence to be better the closer one is to the equilibrium case. Below, we shall give an example showing that there are identical results in exceptional cases even far from equilibrium. In order to clarify the approximative character of the truncated theory, we assume, as usual, the relation $\zeta = \rho \tau$ that guarantees a finite propagation velocity of viscous pulses [10,27–29,31]. In the following section, we are going to generalize this relation. Using (7) and (19) in Eq. (11) with $\zeta = \rho \tau$, we find

$$\pi + \tau \dot{\pi} = -\rho \Theta \tau \left[1 + \frac{\pi}{2\rho}(1 + b + \gamma) + \frac{\pi^2}{2\rho^2}(a+1) \right], \tag{20}$$

with the abbreviations

$$a \equiv \frac{\rho}{nTc_v}, \quad c_v \equiv \frac{1}{n} \frac{\partial \rho}{\partial T}, \quad b \equiv \frac{\partial p / \partial T}{\partial \rho / \partial T} = \frac{1}{nc_v} \frac{\partial p}{\partial T},$$

$$\gamma = \frac{\rho + p}{\rho}. \tag{21}$$

Obviously, the truncated version is expected to be applicable for

$$\frac{|\pi|}{2\rho}(1 + b + \gamma) \ll 1, \quad \frac{\pi^2}{2\rho^2}(1 + a) \ll 1. \tag{22}$$

Since for ''ordinary'' matter b and γ lie in the ranges $1/3 \leq b \leq 2/3$ and $1 \leq \gamma \leq 4/3$, respectively, the first condition is roughly equivalent to $\pi \ll \rho$. For radiation with $\gamma = 4/3$, $b = 1/3$, and $a = 1$, the second condition is implied by the first one. For matter with $\gamma = 1$, $b = 2/3$, and $a = 2m/3T \gg 1$, however, the second condition has to be checked separately. The inequalities (22) may be regarded as criteria under which the truncated theory is a reasonable approximation to the full theory. Given equations of state (15) and (16), any solution π of the truncated theory may be tested according to (22) whether or not, and to which accuracy, it approximates the full theory.

But there is a different possibility, namely, the case

$$\frac{\pi}{2\rho}(1 + b + \gamma) + \frac{\pi^2}{2\rho^2}(a+1) = 0, \tag{23}$$

in which all the terms that distinguish the full from the truncated theory cancel among themselves. Relation (23) is identical to (14) for $\zeta = \rho \tau$, i.e., it is the condition under which the full theory is identical to the truncated one. Solving (23) for π / ρ, yields

$$\frac{\pi}{\rho} = -\frac{1 + \gamma + b}{1 + a}. \tag{24}$$

According to the above-mentioned parameter ranges for γ, b, and a, $|\pi| \ll \rho$ is only possible for $a \gg 1$, i.e., for massive particles. Using (24) in (7), yields

$$\frac{\dot{\rho}}{\rho} = -\Theta \frac{\gamma a - 1 - b}{1 + a}. \tag{25}$$

For radiation ($\gamma = 4/3$, $b = 1/3$, $a = 1$) we find $\dot{\rho} = 0$ and, consequently, $\Theta = \Theta_0 = $ const in a flat FLRW universe. The truncated and the full theories coincide in a specific bulk viscosity-driven inflationary universe. Since with (24), π is completely determined by ρ, provided, the equations of state are given, the remaining equation (13) is no longer a dynamical equation on its own, but may be used to calculate $\tau = \tau(\Theta_0)$. Since the solution is stationary, i.e., $\dot{\pi} = 0$, we find

$$\tau_0^r = \frac{4}{3} \Theta_0^{-1}. \tag{26}$$

In a radiation-dominated universe, the full theory and the truncated version admit a common, bulk viscosity-driven inflationary solution with a relaxation time of the order of the expansion time. This seems to be a new result. For the temperature dependence we find, from (19) and (24),

$$\frac{\dot{T}}{T} = -\Theta \frac{b - a(\gamma + 1)}{1 + a}, \tag{27}$$

yielding

$$\frac{\dot{T}}{T} = \Theta \qquad (28)$$

for radiation, or $T \sim R^3$. The temperature increases in an expanding Universe. This implication of the condition (14) was first noticed by Maartens [29]. While this might appear strange at the first glance, it is an unavoidable feature of any bulk viscosity-driven inflation as is obvious from (19). This point will be discussed in more detail in the following section.

C. The dynamics of the full second-order theory

In this section we shall investigate the full causal theory assuming the existence of general equations of state (15) and (16) and

$$\frac{\zeta}{\tau} \equiv f = f(\rho). \qquad (29)$$

Following Belinskii *et al.* [10], usually the relation $\zeta/\tau = \rho$ was used to guarantee that the propagation velocity of viscous pulses, which is expected to be of the order [28]

$$v \sim \left(\frac{\zeta}{\rho \tau} \right)^{1/2}, \qquad (30)$$

does not exceed the velocity of light. We shall not immediately specify to $f = \rho$ in order to admit a certain range for this propagation velocity. As we shall see below, this additional freedom, allowing, e.g., $f = \alpha \rho$ with $0 < \alpha \leqslant 1$, may be useful in characterizing a possible inflationary phase. In this case, one has $\dot{f} = f' \dot{\rho}$, where $f' \equiv df/d\rho$. Consequently, using (7),

$$\frac{\dot{f}}{f} = -\Theta \rho \frac{f'}{f} \left(\gamma + \frac{\pi}{\rho} \right). \qquad (31)$$

Together with (19), Eq. (11) may now be written as

$$\pi + \tau \dot{\pi} = -\rho \Theta \tau \left[\frac{f}{\rho} + \frac{\pi}{2\rho} \left(1 + b + \gamma \frac{\rho f'}{f} \right) + \frac{\pi^2}{2\rho^2} \left(a + \frac{\rho f'}{f} \right) \right]. \qquad (32)$$

It is obvious how the applicability conditions (22) of the truncated version have to be modified in this more general case. Restricting ourselves to a flat FLRW universe with

$$\frac{\Theta^2}{3} = \kappa \rho, \qquad (33)$$

where κ is Einstein's gravitational constant, and

$$\dot{\Theta} = -\frac{3\kappa}{2}(\rho + p + \pi), \qquad (34)$$

the latter equation may be used to eliminate π and $\dot{\pi}$, respectively. The bulk pressure π may be written as

$$\kappa \pi = -3\gamma H^2 - 2\dot{H}, \qquad (35)$$

where $3H = \Theta$ and $H \equiv \dot{R}/R$ or

$$\frac{\pi}{\rho} = -\gamma - \frac{2}{3} \frac{\dot{H}}{H^2}. \qquad (36)$$

From (15), using (6) and (19), one realizes

$$\dot{p} = v_s^2 \dot{\rho} + \Theta \pi (v_s^2 - b), \qquad (37)$$

with the sound velocity v_s given by

$$v_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_{\text{isentropic}} = \frac{n}{\rho + p} \frac{\partial p}{\partial n} + \frac{T}{\rho + p} \frac{(\partial p/\partial T)^2}{\partial \rho/\partial T}. \qquad (38)$$

Differentiating (34) and using (7) and (37), $\kappa \dot{\pi}$ on the left-hand side (LHS) of (32) may be replaced by

$$\kappa \dot{\pi} = -2\ddot{H} - 6H\dot{H}(1 + b) + 9H^3 \gamma (v_s^2 - b). \qquad (39)$$

Applying (35) and (39) in (32), we arrive at the evolution equation for H

$$\tau \ddot{H} - \frac{\dot{H}^2}{H} \tau \left(a + \frac{\rho}{f} f' \right) - 3\dot{H} H \tau \left[\gamma a + \frac{\gamma}{2} \frac{\rho}{f} f' - \frac{3}{2}(1 + b) \right] + \dot{H}$$
$$- \frac{9}{2} H^3 \tau \left[\frac{f}{\rho} + \frac{\gamma}{2}(\gamma a - 1 - b) + \gamma(v_s^2 - b) \right] + \frac{3}{2} \gamma H^2 = 0. \qquad (40)$$

It should be pointed out again that in arriving at this evolution equation, only equations of state (15), (16), and (29) were used. We think this set of equations of state to be more general than that used in previous papers [31,28,29]. Especially, no specific temperature law, such as $T = \beta \rho^r$ in [31], had to be postulated. There is no freedom to impose a separate temperature law. The behavior of the temperature is generally governed by (19). As it is obvious from (18), a relation $T \sim \rho^r$ is possible for $\partial \rho / \partial n = 0$. Even in this case, however, r is not arbitrary but determined by $r = \rho(T d\rho/dT)^{-1}$. In the specific case of a flat FRLW universe with

$$\frac{\dot{H}}{H} = \frac{1}{2} \frac{\dot{\rho}}{\rho}, \qquad (41)$$

Eq. (19) reduces to

$$\frac{\dot{T}}{T} = 3(\gamma a - b) \frac{\dot{R}}{R} + a \frac{\dot{\rho}}{\rho}. \qquad (42)$$

For constant values of a, b, and γ

$$T \sim \rho^a R^{3(\gamma a - b)} \qquad (43)$$

results, which in the radiation-dominated case with $a = 1$, $b = 1/3$, $\gamma = 4/3$, specifies to

$$T_r \sim \rho_r R^3. \qquad (44)$$

Obviously, with $\rho_r \sim R^{-4}$ for $\pi = 0$, one obtains the correct limiting case $T_r \sim R^{-1}$.

D. Viscous exponential inflation

It has been a matter of some debate whether a bulk viscous pressure is able to drive inflation [26–29,31]. While

Hiscock and Salmonson [26] have shown that, using the equations of state for a Boltzmann gas, the full causal theory does not admit an inflationary phase, different equations of state, such as those used by Romano and Pavón [31], Zakari and Jou [28], and Maartens [29], are compatible with a de Sitter phase. It should be stressed again that dealing with the question of a viscosity-driven inflation, one has to assume that the MIS theory is applicable far from equilibrium [29]. Under these premises, we are going to consider now the possibility of a solution of the evolution equation for H with $H = H_0 = $ const. In such a case, Eq. (40) yields

$$\tau_0^{-1} = 3H_0\left[\frac{f}{\gamma\rho} + \frac{1}{2}(\gamma a - 1 - b) + v_s^2 - b\right]. \quad (45)$$

In the radiation-dominated case, this reduces to

$$\tau_0^r = \frac{4}{9}\frac{\rho}{f}H_0^{-1}. \quad (46)$$

Keeping in mind that until now only $f = f(\rho)$ was used, one recognizes that the ratio between τ_0^r and H_0^{-1} crucially depends on f/ρ. If, as is usually assumed, $f = \rho$, we have $\tau_0^r < H_0^{-1}$. If, however, f is allowed to be smaller than ρ, we may have $\tau_0^r > H_0^{-1}$. It follows from (30) that a relation $f < \rho$ implies a lower propagation velocity compared with the case $f = \rho$. Provided $f = \rho$ is equivalent to a propagation velocity that coincides with the velocity of light, $f = \rho/3$, e.g., leads to a propagation with $v_s = 1/\sqrt{3}$, the velocity of sound (see Israel and Stewart [6]). In the latter case, the relation (46) specifies to

$$\tau_0^r = \frac{4}{3}H_0^{-1}, \quad (47)$$

implying $\tau_0^r > H_0^{-1}$. In this case, the nonequilibrium is "frozen in." The viscous inflation is "nonthermalizing" [29]. While for $\tau_0^r < H_0^{-1}$, there is a quick (compared with the expansion rate) relaxation to (local) equilibrium, this is no longer the case if τ_0^r becomes comparable to the expansion rate H_0^{-1}. Nonequilibrium situations such as this may occur, e.g., in grand unified theories (GUT's) close to the Planck time where the underlying microscopic process is the decay of heavy vector bosons [32]. According to Maartens [29], the condition $\tau_0 > H_0^{-1}$ may be regarded as a consistency criterion for causal viscous inflation. Clearly, for $\tau_0 < H_0^{-1}$, the Universe will relax to an equilibrium state in less than one expansion time. A successful inflation, however, has to last for many characteristic expansion times.

In the matter-dominated case with $\gamma = 1$, $b = 2/3$, $v_s^2 \ll 1$, $a = 2m/3T \gg 1$, we find

$$\tau_0^m \sim \frac{T}{m}H_0^{-1}, \quad (48)$$

i.e., we are always in the range $\tau_0^m \ll H_0^{-1}$. The time scale for the relaxation to equilibrium is much smaller than the expansion time scale.

Calculating the dependence $\tau_0(H_0)$ along the same lines that lead to (45) within the truncated version, provides us with

$$\tau_0^{-1} = 3\gamma^{-1}\left[\frac{f}{\rho} + \gamma(v_s^2 - b)\right]H_0. \quad (49)$$

Specializing to the radiation-dominated case, (49) reduces to (46), i.e., the truncated version yields exactly the same dependence as does the full theory. We have rediscovered the specific case, discussed in Sec. II B, in which the full theory coincides with the truncated version. For the matter-dominated case, however, the result of the truncated theory is different from (48). This is not unexpected after the considerations of Sec. II B.

We shall now look at the behavior of the basic thermodynamic quantities under the condition $H = H_0$. Because of (40), this implies $\rho = \rho_0 = $ const and (36) reduces to

$$\frac{\pi}{\rho} = -\gamma. \quad (50)$$

According to (6), the particle number density decreases exponentially. The temperature evolution law (19), together with (50), becomes

$$\frac{\dot{T}}{T} = -3H_0\left[\frac{\partial p/\partial T}{\partial \rho/\partial T} - \frac{\rho}{nTc_v}\gamma\right], \quad (51)$$

equivalent to

$$\dot{T} = \frac{3H_0}{c_v}\frac{\partial\rho}{\partial n}. \quad (52)$$

Generally, the temperature is not constant in the inflationary phase. Provided c_v is finite, a constant temperature $T = T_0$ is only possible for $\partial\rho/\partial n = 0$, corresponding to $T \sim \rho^r$ with $r = \rho(Td\rho/dT)^{-1}$ [see the discussion below (40)]. For any $\partial\rho/\partial n > 0$, the temperature T increases during inflation. Since $\partial\rho/\partial n > 0$ and $\partial\rho/\partial n = 0$, respectively, lead to dramatically different results for the entropy production during a bulk viscosity-driven inflationary phase (see below), this point deserves a detailed discussion. It is frequently used that in the equations of state for radiation in equilibrium, $p = nT$ and $\rho = 3nT$, which are specific cases of (15) and (16), the number density n may be eliminated according to $n \sim T^3$, resulting in $p = p(T)$ and $\rho = \rho(T)$ with $p = \rho/3$ and $\rho \sim T^4$. Then T is the only independent variable and $\partial\rho/\partial T \to d\rho/dT$. In order to check whether a corresponding procedure is possible for $\pi \neq 0$, we shall assume an arbitrary dependence $n = n(T)$ instead of the equilibrium relation $n \sim T^3$. With (19), we find

$$\dot{n} = -3HT\frac{dn}{dT}\left(b + a\frac{\pi}{\rho}\right). \quad (53)$$

Comparison with (6) leads to the general relation

$$\frac{dT}{T} = \left(b + a\frac{\pi}{\rho}\right)\frac{dn}{n}. \quad (54)$$

For $\pi = 0$, we recover $T \sim n^{1/3}$ for radiation ($b = 1/3$). For matter ($b = 2/3$), the correct result $T \sim n^{2/3} \sim R^{-2}$ is obtained as well.

With the condition (50) for bulk viscosity-driven exponential inflation and for $a = $ const equation (54) yields

$$T \sim n^{b - \gamma a}. \tag{55}$$

In the radiation-dominated case

$$n \sim T^{-1} \tag{56}$$

results. While in equilibrium n is an increasing function of T, n decreases with T in a de Sitter phase, characterized by (50), which is a far-from-equilibrium state. According to (56), the exponential decrease of the particle number density following from (6) is necessarily accompanied by a corresponding increase in the temperature. This is obviously incompatible with $T \sim \rho^{1/4}$ from $\partial \rho / \partial n = 0$. Equations of state with $\partial \rho / \partial n = 0$ imply that the exponential dilution of the particles of the out-of-equilibrium fluid according to (6) does not have any impact at all on the energy density of this fluid. We conclude that the previously used [31,28,29] *ad hoc* assumption $T \sim \rho^r$ is not consistent with (15), (16), and $n = n(T)$ if applied to a bulk pressure-driven inflationary phase.

While an increasing temperature during the de Sitter stage appears unfamiliar, this kind of behavior is not quite unexpected for an equation of state (16). Since n decreases exponentially, T must increase accordingly in order to guarantee $\dot{\rho} = 0$, as long as $\partial \rho / \partial n > 0$ and $\partial \rho / \partial T > 0$. Exactly this kind of behavior was found by Hiscock and Salmonson [26] for the truncated Israel-Stewart theory. As was demonstrated above, the full theory exhibits a corresponding feature as well. (Note, that according to [26] there does not exist a de Sitter phase in the full theory for the case of a Boltzmann gas.) If one does not use, however, a relationship $T \sim \rho^r$ and the behavior of T is governed by (52) with $\partial \rho / \partial n > 0$, this has important consequences for the entropy production during bulk viscous-driven inflation, discussed by Maartens [29]. The entropy in a comoving volume is $\Sigma = n s R^3$. With (6) and (9), the change of Σ is

$$\dot{\Sigma} = -\frac{\pi \Theta}{T} R^3. \tag{57}$$

In the de Sitter phase with (50), the latter expression reduces to

$$\dot{\Sigma}_{H = H_0} = 3 H_0 \gamma \frac{\rho_0 R^3}{T}. \tag{58}$$

This change of the comoving entropy depends crucially on the behavior of T. For $T = $ const which follows from $T \sim \rho^r$, equivalent to $\partial \rho / \partial n = 0$ [see (52)], we recover the exponential increase of $\dot{\Sigma}$ found by Maartens [29]. For equations of state (15) and (16), however, with $\partial \rho / \partial n > 0$, the temperature dependence in the case of radiation ($\gamma = 4/3$) is $T_r \sim \rho_0 R^3$ as follows from (44). $\dot{\Sigma}$ turns out to be constant rather than exponentially increasing. Integrating (58), one finds only a linear growth in Σ. Consequently, there is no way to generate a considerable amount of entropy during a bulk viscosity-driven de Sitter phase.

To investigate the stability of the solutions $H = H_0$, we probe the latter with small perturbations, i.e., we assume

$H = H_0 [1 + h(t)]$ with $|h| \ll 1$ in Eq. (40). Equation (40) is valid for spatially flat homogeneous and isotropic spacetimes, therefore the stability analysis is restricted to this case as well. We assume that the dimensionless quantities f/ρ, γ, a, b, and v_s remain unchanged for small deviations from $H = H_0$. Since we found the relaxation time τ_0 in the inflationary phase to be proportional to H_0^{-1}, it is natural to assume $\tau \sim H^{-1}$ generally and to fix the proportionality factor by (45). Retaining only terms linear in h, the resulting equation for h is

$$\ddot{h} + 3 H_0 K \dot{h} = 0, \tag{59}$$

with

$$K \equiv 1 + \frac{f}{\gamma \rho} + v_s^2 - \frac{\gamma}{2} \left(\frac{\rho}{f} f' + a \right). \tag{60}$$

There exists a solution $h = $ const that may be used to redefine H_0. The other solution is stable for $K > 0$. In the radiation-dominated case $\gamma = 4/3$, $v_s^2 = 1/3$, $a = 1$, $b = 1/3$ and with $f = \alpha \rho$ the solution is stable for any $\alpha > 0$. In the matter-dominated case $\gamma = 1$, $v_s^2 \ll 1$, $a \gg 1$, and $b = 2/3$ we find $K \approx -a/2 = -m/3T$, i.e., the solution is unstable.

We conclude that there exist stable viscosity-driven inflationary solutions as long as the equation of state is close to that for relativistic particles. There do not exist stable solutions for equations of state close to that for dust. Even for the stable solutions, however, the temperature increases exponentially during the de Sitter stage. Since the particle number density decreases exponentially, this unfamiliar behavior is unavoidable to guarantee $\rho = \rho_0$ as long as $\partial \rho / \partial n$ and $\partial \rho / \partial T$ are assumed to be positive. A temperature dependence such as this is not as strange as it might appear at the first glance. A nonvanishing bulk viscosity has always the tendency to heat up the Universe. Close to local equilibrium, this means that the decrease of T due to the expansion of the Universe is less than that without bulk viscosity. Applying the cosmological dynamics with bulk viscosity to situations far from equilibrium, in our case to a hypothetical inflationary phase, the π term just compensates the equilibrium terms in (7) and (34). It overcompensates them, however, in the case of the temperature (19).

In the following section we show how particle production processes may modify this behavior of the thermodynamic quantities during a de Sitter phase.

III. BULK VISCOUS PRESSURE AND PARTICLE PRODUCTION

A. Basic dynamics

Throughout this paper, the Universe is studied within a single fluid model. There is entropy production due to a non-vanishing bulk pressure. While both for pure radiation and pure dust the bulk viscosities tend to zero, considerable values of the latter are expected in mixtures of relativistic and nonrelativistic matters. Consequently, the single fluid universe with bulk viscosity may be regarded as a simplified description of a system of two (or more) interacting components. Usually, it is assumed that the interaction responsible for the existence of a nonvanishing viscous pressure does not

change the overall number of fluid particles; i.e., it is assumed that (4) holds. However, interactions with conserved particle numbers are only a special case, particularly at high energies. Moreover, at times of the order of the Planck time or during the scalar field decay in inflationary scenarios, particle or string production processes are supposed to affect the cosmological dynamics, leading to features such as ''deflationary universes'' and ''string-driven inflation'' [16–20,38].

In this section we generalize the previous formalism to the case that the existence of a nonvanishing bulk pressure is accompanied by an increase in the number of fluid particles. We start our investigations with (1), (2), (3), and (5) as well, but (4) is now replaced by

$$N^a_{;a} = n\Gamma, \tag{61}$$

yielding

$$\dot{n} + \Theta n = n\Gamma. \tag{62}$$

Γ is the particle production rate which has to be regarded as an input quantity in our phenomenological description. Instead of (9), now

$$nT\dot{s} = -\Theta\pi - (\rho+p)\Gamma \tag{63}$$

results. The entropy production density is given by

$$TS^a_{;a} = -n\mu\Gamma - \pi\left[\Theta + \frac{\tau}{\zeta}\dot{\pi} + \frac{1}{2}\pi T\left(\frac{\tau}{\zeta T}u^a\right)_{;a}\right], \tag{64}$$

where μ is the chemical potential

$$\mu = Ts - \frac{\rho+p}{n}. \tag{65}$$

A change in the number of particles, i.e., $\Gamma \neq 0$, is believed to be phenomenologically equivalent to an effective viscous pressure [13,15,17–21]. A discussion of this equivalence on the level of relativistic kinetic theory has been given recently [33]. In order to relate the present investigations to previous work, we shall in a first step focus on the Eckart theory, i.e., to the case $\tau = 0$. Equation (64) reduces to

$$TS^a_{E;a} = -n\mu\Gamma - \pi_E\Theta \tag{66}$$

in this case, where the subscript E stands for ''Eckart.'' Formal rewriting yields

$$TS^a_{E;a} = -\pi_E[\Theta + n\mu\pi_E^{-1}\Gamma]. \tag{67}$$

As usual, one has to guarantee $S^a_{E;a} \geq 0$. Since we expect the particle production to be effectively equivalent to a viscous pressure, we have to demand additionally that the entire entropy production is given in terms of π_E, i.e.,

$$S^a_{E;a} = \frac{\pi_E^2}{\zeta T}, \tag{68}$$

where π_E is determined by

$$\pi_E = -\zeta[\Theta + n\mu\pi_E^{-1}\Gamma], \tag{69}$$

or

$$\pi_E^2 + \zeta\pi_E\Theta = -\zeta n\mu\Gamma, \tag{70}$$

where ζ now is a generalized bulk viscosity coefficient. For $\Gamma \neq 0$, the latter inhomogeneous quadratic equation replaces the familiar linear relation $\pi_E = -\zeta\Theta$. Via (70), the particle production rate Γ influences the viscous pressure π. For $\Gamma = 0$ as well as for $\mu = 0$, we recover $\pi_E = -\zeta\Theta$. It is convenient to split π_E in the following way. Let λ be a not necessarily constant parameter lying in the range $0 \leq \lambda \leq 1$ such that the fraction $\lambda\pi_E$ of π_E describes a ''creation'' pressure, because of $\Gamma \neq 0$, while the fraction $(1-\lambda)\pi_E$ is connected with a conventional bulk viscosity. With this splitting, Eq. (63) may be written as

$$nT\dot{s}_E = -(1-\lambda)\Theta\pi_E - \lambda\Theta\pi_E - (\rho+p)\Gamma. \tag{71}$$

Frequently [22–25], the assumption was made that the creation process does not affect the entropy per particle. This is equivalent to the requirement that the terms in (71) due to the creation process cancel among themselves:

$$(\rho+p)\Gamma = -\lambda\pi_E\Theta. \tag{72}$$

Physically, this means that the particles are created with a fixed given entropy. Then, $\dot{s}_E$ is given by

$$nT\dot{s}_E = -(1-\lambda)\Theta\pi_E. \tag{73}$$

Only that fraction of π_E, that is not connected with a change in the particle number, contributes to $\dot{s}_E$. In the limit $\lambda = 1$, in which the entire viscous pressure is due to particle production, one has $\dot{s}_E = 0$ and $S^a_{E;a} = ns_E\Gamma$, i.e., there is entropy production because of the enlargement of the phase space. Using (72) in (69), yields

$$\pi_E = -\zeta\Theta\left[1 - \lambda\frac{n\mu}{\rho+p}\right]. \tag{74}$$

Combining the latter relation again with (72), we find that the part $\lambda\zeta$ of the generalized coefficient of bulk viscosity ζ that arises due to a nonvanishing particle production rate is

$$\lambda\zeta = \frac{\Gamma\Theta^{-2}}{\rho+p-\lambda n\mu}. \tag{75}$$

Of course, Eq. (74) with (68) is also obtained if the condition (72) is immediately introduced in (66).

Obviously, Eq. (68) with (69) is not the only way to guarantee $S^a_{;a} \geq 0$. For a different approach that treats Γ as a thermodynamic flux independent of π, the reader is referred to a paper by Gariel and Le Denmat [34].

Coming back now to the full second order theory again, we try to find a causal evolution equation for π that yields (12) in the case $\Gamma \neq 0$ as well. Equation (64) may be written as

$$TS^a_{;a} = -\pi\left[\Theta + \frac{\tau}{\zeta}\dot{\pi} + \frac{1}{2}\pi T\left(\frac{\tau}{\zeta T}u^a\right)_{;a} + \mu\frac{n\Gamma}{\pi}\right], \tag{76}$$

generalizing the relation (67) of the Eckart theory. The condition $S^a_{;a} \geq 0$ with (12) is now satisfied for

$$\pi^2 + \tau\pi\dot\pi + \frac{1}{2}\pi^2 T\left(\frac{\tau}{\zeta T}u^a\right)_{;a} + \zeta\pi\Theta = -\zeta\mu n\Gamma, \quad (77)$$

instead of (70). The viscous pressure π is determined by a nonlinear inhomogeneous differential equation. For $\mu = 0$, Eq. (77) formally reduces to (11). For the sake of generality, we shall however retain the μ term in the following considerations. A chemical potential may act, e.g., as an effective symmetry-breaking parameter in relativistic field theories [35–37].

In the Eckart theory, the general nonlinear relation (70) was reduced to the linear relation (74) by the requirement (72). Unfortunately, a corresponding simplification of (77) via a relation (72) is *not* possible in general. The essential physical difference between the noncausal and the causal theories is the appearance of a finite relaxation time within the latter. If a nonvanishing Γ is responsible for the occurrence of an effective viscous pressure, a causal theory has to include a finite relaxation time which is just the time interval during which the corresponding part of π decays to zero after Γ has been switched off. But the relation (72) is an Eckart-type relation due to which $\lambda\pi = 0$ is immediately implied by $\Gamma = 0$. Therefore, an approximation (72) cannot be used in a causal theory. If, however, the relation (72) which follows from the requirement that the creation process does not affect the entropy per particle has to be abandoned, this means that within the causal theory the creation process contributes to $\dot s$ in general. While in the Eckart theory, the particle production rate was rather simply related to the viscous pressure by (72), Γ enters the equation for π of the causal theory as an independent parameter in a less obvious way.

Our present approach assumes that the deviations from equilibrium may be characterized in terms of one single quantity π also for the case of a nonvanishing particle production rate Γ. This implies that there is only one (generalized) coefficient of bulk viscosity ζ and one relaxation time τ. The advantage of this assumption is that there exists only one causal evolution equation, namely, (77), taking into account the influence of Γ on π. (For an alternative proposal, see again [31].) Since the latter equation is nonlinear, there is no obvious separation into a conventional bulk pressure and a "creation" pressure. We shall assume further, that ζ and τ continue to be related by (29). The dependence of the relaxation time τ on Γ will be discussed below for specific cases.

For $\mu = 0$, Eq. (77) appears to be identical to the case $\Gamma = 0$, but there are differences in the behavior of the particle number density and the temperature. The former is determined by (62). Using (16), (17), (7), and (62) instead of (6), the temperature law (19) is replaced by (cf. [22])

$$\frac{\dot T}{T} = -\Theta\left[\frac{\partial p/\partial T}{\partial\rho/\partial T} + \frac{\pi}{T\partial\rho/\partial T}\right] + \Gamma\left[\frac{\partial p/\partial T}{\partial\rho/\partial T} - \frac{\rho + p}{T\partial\rho/\partial T}\right] \quad (78)$$

in the case of a nonvanishing particle production rate. The particle production affects the temperature not only through the effective viscous pressure π but there is an additional direct coupling as well. Alternatively, Eq. (78) may be written as

$$\frac{\dot T}{T} = -(\Theta - \Gamma)\frac{\partial p/\partial T}{\partial\rho/\partial T} - \frac{\pi\Theta + (\rho + p)\Gamma}{T\partial\rho/\partial T}$$

$$= -(\Theta - \Gamma)\frac{\partial p/\partial T}{\partial\rho/\partial T} + \frac{n\dot s}{\partial\rho/\partial T}. \quad (79)$$

With this evolution law for the temperature, different from (19), the evolution equation (77) for π becomes different from (32) even in the case $\mu = 0$, although (10) and (76) seem to coincide. The relation (37) for $\dot p$ is now replaced by

$$\dot p = \rho\left[(\Gamma - \Theta)\gamma v_s^2 - b\left(\gamma\Gamma + \Theta\frac{\pi}{\rho}\right)\right]. \quad (80)$$

Instead of (39), we have

$$\kappa\dot\pi = -2\ddot H - 6H\dot H(1 + b) + 9H^3\gamma(v_s^2 - b)$$
$$- 3H^2\Gamma\gamma(v_s^2 - b). \quad (81)$$

The evolution equation for π may be written

$$\pi + \tau\dot\pi = -\rho\tau\left\{\frac{f}{\rho}\left(\Theta + \frac{n\mu}{\rho + p}\frac{\gamma}{\pi/\rho}\Gamma\right)\right.$$
$$+ \frac{\pi}{2\rho}\left[\Theta\left(1 + b + \gamma\frac{\rho f'}{f}\right) - \Gamma(b - \gamma a)\right]$$
$$\left. + \frac{\pi^2}{2\rho^2}\Theta\left(a + \frac{\rho f'}{f}\right)\right\}, \quad (82)$$

generalizing (32). A procedure analogous to that leading to (40) in Sec. II provides us with an evolution equation for H in a flat, homogeneous, and isotropic universe with particle production:

$$\left\{\tau\ddot H - \frac{\dot H^2}{H}\tau\left(a + \frac{\rho}{f}f'\right) - 3\dot H H\tau\left[\gamma a + \frac{\gamma}{2}\frac{\rho}{f}f' - \frac{3}{2}(1 + b)\right]\right.$$
$$+ \dot H + \frac{1}{2}\Gamma\dot H\tau(\gamma a - b) - \frac{9}{2}H^3\tau\left[\frac{f}{\rho} + \frac{\gamma}{2}(\gamma a - 1 - b)\right.$$
$$\left. + \gamma(v_s^2 - b)\right] + \frac{3}{2}\tau H^2\Gamma\gamma\left(v_s^2 - b + \frac{\gamma a - b}{2}\right) + \left.\frac{3\gamma H^2}{2}\right\}$$
$$\times\left\{-\gamma - \frac{2}{3}\frac{\dot H}{H^2}\right\} - \frac{3}{2}H^2\Gamma\tau\frac{f}{\rho}\gamma\frac{n\mu}{\rho + p} = 0. \quad (83)$$

For an easier comparison with Eq. (40), we kept separate the common factor π/ρ [see (36)] on the LHS of this equation. Having only one evolution equation for H is a consequence of our previous assumption that it is possible to characterize the deviations from equilibrium by a single quantity π only.

Equation (83) simplifies considerably for $\mu = 0$

$$\tau\ddot{H} - \frac{\dot{H}^2}{H}\tau\left(a + \frac{\rho}{f}f'\right) - 3\dot{H}H\tau\left[\gamma a + \frac{\gamma}{2}\frac{\rho}{f}f' - \frac{3}{2}(1+b)\right] + \dot{H}$$

$$+ \frac{1}{2}\Gamma\dot{H}\tau(\gamma a - b) - \frac{9}{2}H^3\tau\left[\frac{f}{\rho} + \frac{\gamma}{2}(\gamma a - 1 - b) + \gamma(v_s^2\right.$$

$$\left. - b)\right] + \frac{3}{2}\tau H^2\Gamma\gamma\left(v_s^2 - b + \frac{\gamma a - b}{2}\right) + \frac{3}{2}\gamma H^2 \overset{(\mu=0)}{=} 0.$$

$$(84)$$

For $\Gamma = 0$, both equations reduce to (40). Eliminating π in the evolution equation (78) for T with the help of (36), we find

$$\frac{\dot{T}}{T} = 3(\gamma a - b)\frac{\dot{R}}{R} - (\gamma a - b)\Gamma + a\frac{\dot{\rho}}{\rho}, \qquad (85)$$

instead of (42). For constant values of a, b, and γ,

$$T \sim \rho^a\left(\frac{R^3}{N}\right)^{\gamma a - b} = \rho^a\left(\frac{1}{n}\right)^{\gamma a - b}. \qquad (86)$$

For $\gamma = 4/3$, $b = 1/3$, $a = 1$, this specifies to

$$T \sim \rho\frac{R^3}{N} = \frac{\rho}{n}. \qquad (87)$$

B. Inflationary solutions

Looking for solutions $H = H_0 = \text{const}$ of (83), we find the expression for τ_0 that generalizes (45):

$$\tau_0^{-1} = 3H_0\left[\frac{f}{\gamma\rho} + \frac{\gamma a - 1 - b}{2} + v_s^2 - b\right]$$

$$- \Gamma\left[\frac{n\mu}{\rho + p}\frac{f}{\gamma\rho} + \frac{\gamma a - b}{2} + v_s^2 - b\right]. \qquad (88)$$

If the particle production rate is comparable with the expansion rate, it may essentially influence the relaxation time τ_0. Especially interesting is the possibility that the resulting effect of the Γ terms in (88) is to enlarge τ_0. In this case, particle production may either enable or improve the fulfillment of the "freezing-in" condition $\tau_0 > H_0^{-1}$ for the nonequilibrium. In other words, the consistency criterion for inflation might be easier to satisfy in the case with particle production than without. Of course, these considerations make only sense as long as τ_0 remains finite. As we shall see below, there are parameter combinations for which τ_0 diverges.

If, during some time interval, the particle production rate Γ is proportional to the expansion rate and approximately constant as well, i.e., $\Gamma = 3\lambda H_0$, we get

$$\tau_0^{-1} = 3H_0\left[\frac{f}{\gamma\rho}\left(1 - \frac{\lambda n\mu}{\rho + p}\right) - \frac{1}{2}\right.$$

$$\left. + (1 - \lambda)\left(\frac{\gamma a - b}{2} + v_s^2 - b\right)\right]. \qquad (89)$$

For particles with $\mu = 0$ and an equation of state close to that for radiation, it is obvious that for any $\lambda > 0$, the relaxation time τ_0 is larger than that for $\lambda = 0$. In the limiting case $\lambda = 1$, the production rate of the particles coincides with the expansion rate. τ_0 then is given by

$$\tau_0^{-1} \overset{(\lambda=1)}{=} 3H_0\left[\frac{f}{\gamma\rho}\left(1 - \frac{n\mu}{\rho + p}\right) - \frac{1}{2}\right]. \qquad (90)$$

Specifying again to $\gamma = 4/3$ and restricting ourselves to $\mu = 0$,

$$\tau_0^r \overset{(\lambda=1)}{=} \frac{4}{9}\frac{H_0^{-1}}{\frac{f}{\rho} - \frac{1}{2}} \qquad (91)$$

results. For $f = \rho$, the usual choice in the literature [10,28,29,31], this expression for τ_0 is twice as large as the corresponding expression from (46), although $\tau_0^r < H_0^{-1}$ in both cases. For $f = 2\rho/3$, however, we find $\tau_0^r < H_0^{-1}$ from (46), while (91) yields $\tau_0^r > H_0^{-1}$. In the latter case, the nonequilibrium is "frozen in," in the former one it is not. This demonstrates explicitly that under certain circumstances, particle production may improve the conditions for bulk viscous inflation. On the other hand, Eq. (91) makes sense only for $f/\rho > 1/2$ since for $f = \rho/2$, the relaxation time diverges. Especially the case $f = \rho/3$, dealt with for $\Gamma = 0$ in Sec. II, is impossible for $\Gamma = 3H_0$.

The case $\Gamma = 3H_0$ is singled out in different respects as well. For any $\Gamma < 3H_0$, the particle number density n is decreasing according to (62), while according to (86), the temperature T increases correspondingly to guarantee $\dot{\rho}_0 = \dot{H}_0 = 0$. In the limit $\Gamma = 3H_0$, n becomes constant and it follows from (86) that the temperature is constant as well. Only in this extreme limiting case which corresponds to $\dot{s} = 0$ (this is only possible since we restricted ourselves to a time interval with $\Gamma = \text{const}$), the temperature may remain constant in a de Sitter phase driven by an effective bulk pressure. For $\Gamma < 3H_0$ with $\dot{s} \neq 0$, the temperature necessarily increases, although for $\Gamma \neq 0$ at a lower rate than that for $\Gamma = 0$.

C. Entropy production

Eliminating π from (63) with the help of (36) and (41), the time dependence of the entropy per particle is determined by

$$nT\dot{s} = (\rho + p)[3H - \Gamma] + \dot{\rho}, \qquad (92)$$

or

$$nT\dot{s} = \left(\frac{\rho R^{3\gamma}}{N^\gamma}\right)^{\cdot}\frac{N^\gamma}{R^{3\gamma}}. \qquad (93)$$

With (62) and

$$T = T_0\left(\frac{\rho}{\rho_0}\right)^a\left(\frac{R^3 N_0}{R_0^3 N}\right)^{\gamma a - b} \qquad (94)$$

from (86), where the subscript 0 refers to some initial time, one finds

$$\dot{s} = \frac{1}{n_0 T_0} \left(\frac{\rho_0}{\rho} \right)^a \left(\frac{R_0^3 N}{R^3 N_0} \right)^{\gamma a - 1 - b} \left(\frac{N}{R^3} \right)^\gamma \left(\frac{\rho R^{3\gamma}}{N^\gamma} \right)^{\cdot} . \quad (95)$$

In the inflationary phase with $\rho = \rho_0$ and $H = H_0$, we have

$$\dot{s} = \frac{\rho_0}{n_0 T_0} \gamma \left(\frac{R_0^3 N}{R^3 N_0} \right)^{\gamma a - 1 - b} [3H_0 - \Gamma], \quad (96)$$

which for radiation ($\gamma = 4/3$, $a = 1$, $b = 1/3$) reduces to

$$\dot{s} = \frac{4\rho_0}{n_0 T_0} \left[H_0 - \frac{1}{3} \Gamma \right]. \quad (97)$$

Integration of (97) yields

$$s = \frac{4\rho_0}{n_0 T_0} \left[H_0(t - t_0) - \frac{1}{3} \int_0^t \Gamma dt \right] + s(t_0). \quad (98)$$

The basic difference compared with the case $\Gamma = 0$ is that the change of the entropy in a comoving volume, $\Sigma = n s R^3$, is no longer determined by the change of s alone. With the equations of state for radiation, Σ is given by

$$\Sigma = \left\{ \frac{4\rho_0}{n_0 T_0} \left[H_0(t - t_0) - \frac{1}{3} \int_0^t \Gamma dt \right] + s(t_0) \right\} N_0 \exp \int_0^t \Gamma dt \quad (99)$$

in the inflationary phase. If Γ is again assumed to be approximately given by $\Gamma = 3\lambda H_0$, the expression (99) reduces to

$$\Sigma(t) = \left[\frac{4\rho_0}{n_0 T_0} (1 - \lambda) H_0(t - t_0) + s(t_0) \right] N(t_0)$$
$$\times \exp[3\lambda H_0(t - t_0)]. \quad (100)$$

It follows that for $\Gamma \neq 0$, we have an exponential increase of the comoving entropy Σ during the de Sitter phase. Starting from the latter expression for Σ, it is possible to apply Maartens' numerical estimations [29] concerning the entropy production during the inflationary phase. All his considerations of this point are valid in the present case for $\lambda \neq 0$, provided his H_0 is replaced by λH_0. His conclusion that it is possible ''to generate the right amount of entropy without reheating'' may hold in a universe with particle production. There is complete equivalence to Maartens' result for the entropy production for $\lambda = 1$. Consequently, a substantial production of entropy during a (bulk viscosity-induced) dissipational inflationary phase is possible if at least a part of the bulk viscous pressure is related to an increase in the particle number. In a sense we have reestablished Maartens' result, although within a different setting: An exponential growth of the comoving entropy is only possible if the viscous pressure π in (1) is, at least partially, a ''creation'' pressure connected with an increase in the number of fluid particles rather than an increase in the entropy per particle.

It remains to consider the stability of the inflationary solutions for $\Gamma \neq 0$. Proceeding as in Sec. II we have

$$\ddot{h} + (3H_0 K - \Gamma M)\dot{h} = 0 \quad (101)$$

instead of (59), where

$$M \equiv 2\frac{f}{\gamma\rho} \frac{n\mu}{\rho + p} + v_s^2 - b. \quad (102)$$

Consequently, the inflationary solutions are stable for $3H_0 K - \Gamma M > 0$. For $\mu = 0$ and the equations of state for radiation, one has $M = 0$ and the stability condition reduces to $K > 0$ again. The stability properties of the corresponding inflationary solutions are not affected at all by a nonvanishing Γ. With equations of state for matter, the inflationary solutions have the same instability as for $\Gamma = 0$.

IV. SUMMARY

Using particle number density and temperature as basic thermodynamical variables of the cosmic fluid we have studied the full Müller-Israel-Stewart theory for a spatially flat, homogeneous, and isotropic universe with bulk viscous pressure. We found general criteria for the applicability of the so-called truncated versions. It was shown that in exceptional cases, there exist common solutions of the full and the truncated theories far from equilibrium. The possibility of exponential inflationary solutions driven by bulk viscosity was investigated. Almost all corresponding solutions imply an exponential growth of the temperature during the de Sitter stage. Since the number density of the fluid particles decreases exponentially, only a corresponding increase in the temperature guarantees a constant energy density as long as the specific heat is positive and finite and the energy density increases with the particle number density. In the second part of the paper the bulk viscous pressure was allowed to account partially or fully for particle production processes. A nonvanishing particle production rate may enlarge the relaxation time for a viscous pressure that is supposed to drive inflation. It may help to ''freeze in'' the corresponding nonequilibrium, i.e., to improve the conditions for inflation. There exist stable, inflationary solutions for equations of state close to that for radiation. Because of the increase in the particle number, the comoving entropy increases exponentially in this period. Only in the limiting case where the entire inflation-driving viscous pressure is due to particle production, both the number density and the temperature remain constant during the de Sitter phase.

[1] C. Eckart, Phys. Rev. **58**, 919 (1940).

[2] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1958).

[3] I. Müller, Z. Phys. **198**, 329 (1967).

[4] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).

[5] W. Israel and J.M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).

[6] W. Israel and J.M. Stewart, Proc. R. Soc. London **A365**, 43 (1979).

[7] D. Pavón, D. Jou, and J. Casas-Vázquez, Ann. Inst. Henri Poincaré A **36**, 79 (1982).

[8] W.A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).

[9] W.A. Hiscock and L. Lindblom, Phys. Rev. D **35**, 3723 (1987).

[10] V.A. Belinskii, E.S. Nikomarov, and I.M. Khalatnikov, Sov. Phys. JETP **50**, 213 (1979).

[11] Ø. Grøn, Astrophys. Space Sci. **173**, 191 (1990).

[12] N. Udey and W. Israel, Mon. Not. R. Astron. Soc. **199**, 1137 (1982).

[13] Ya.B. Zel'dovich, JETP Lett. **12**, 307 (1970).

[14] G.L. Murphy, Phys. Rev. D **8**, 4231 (1973).

[15] B.L. Hu, Phys. Lett. **90A**, 375 (1982).

[16] N. Turok, Phys. Rev. Lett. **60**, 549 (1988).

[17] J.D. Barrow, Phys. Lett. B **180**, 335 (1986).

[18] J.D. Barrow, Phys. Lett. B **183**, 285 (1987).

[19] J.D. Barrow, Nucl. Phys. **B310**, 743 (1988).

[20] J.D. Barrow, in *The Formation and Evolution of Cosmic Strings*, edited by G. Gibbons, S.W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990), p. 449.

[21] I. Prigogine, J. Geheniau, E. Gunzig, and P. Nardone, Gen. Relativ. Gravit. **21**, 767 (1989).

[22] M.O. Calvão, J.A.S. Lima, and I. Waga, Phys. Lett. A **162**, 223 (1992).

[23] W. Zimdahl and D. Pavón, Phys. Lett. A **176**, 57 (1993).

[24] W. Zimdahl and D. Pavón, Mon. Not. R. Astron. Soc. **266**, 872 (1994).

[25] W. Zimdahl and D. Pavón, Gen. Relativ. Gravit. **26**, 1259 (1994).

[26] W.A. Hiscock and J. Salmonson, Phys. Rev. D **43**, 3249 (1991).

[27] D. Pavón, J. Bafaluy, and D. Jou, Class. Quantum Grav. **8**, 347 (1991).

[28] M. Zakari and D. Jou, Phys. Rev. D **48**, 1597 (1993).

[29] R. Maartens, Class. Quantum Grav. **12**, 1455 (1995).

[30] J. Gariel and G. Le Denmat, Phys. Rev. D **50**, 2560 (1994).

[31] V. Romano and D. Pavón, Phys. Rev. D **50**, 2572 (1994).

[32] D.V. Nanopoulos, Prog. Part. Nucl. Phys. **6**, 23 (1981); G. Börner, *The Early Universe* (Springer-Verlag, Berlin, 1993), Chap. 8.3.

[33] J. Triginer, W. Zimdahl, and D. Pavón, Class. Quantum Grav. **13**, 403 (1996).

[34] J. Gariel and G. Le Denmat, Phys. Lett. A **200**, 11 (1995).

[35] J.I. Kapusta, Phys. Rev. D **24**, 426 (1981).

[36] H.E. Haber and H.A. Weldon, Phys. Rev. D **25**, 502 (1982).

[37] J.A.S. Lima, R. Portugal, and I. Waga, Phys. Rev. D **37**, 2755 (1988).

[38] J.A.S. Lima and J.M.F. Maia, Phys. Rev. D **49**, 5597 (1994).