

Coherence effects in neutrino oscillations

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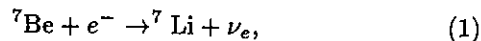
We study the effect of coherent and incoherent broadening on neutrino oscillations both in vacuum and in the presence of matter (the MSW effect). We show under very general assumptions that it is not possible to distinguish experimentally neutrinos produced in some region of space as wave packets from those produced in the same region of space as plane waves with the same energy distribution.

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I. INTRODUCTION

Neutrino oscillations have been the subject of intense theoretical and experimental research. To date there is no evidence for oscillations in terrestrial neutrino beams. The deficit of solar neutrinos can be explained by neutrino oscillations with a $(\text{mass})^2$ difference $\Delta m^2 \sim 10^{-6} \text{ eV}^2$ together with the enhancement of these oscillations as the neutrinos pass through the Sun by the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1]. There are also hints of possible neutrino oscillations with $\Delta m^2 \sim 10^{-2} \text{ eV}^2$ in atmospheric neutrino experiments [2].

Approximately 25 years ago an interesting suggestion was made for probing even lower values of Δm^2 using the 3% annual variation of the Earth-Sun distance [3]. In this case, rather than simply observing a net average decrease in the electron neutrino intensity by an amount $\sin^2(2\theta)/2$ one could observe the actual oscillations of the electron neutrino flux. The idea is to use the ν_e 's from e^- capture on Be,



which results in a neutrino energy $E_\nu \sim 0.86 \text{ MeV}$ with a small energy spread. Thus, if the neutrino oscillation length

$$L_{\text{osc}} = \frac{4\pi E_\nu}{\Delta m^2} \quad (2)$$

is within one or two orders of magnitude of the variation $\Delta R \sim 5 \times 10^{11} \text{ cm}$ of the Earth-Sun distance, then, depending on the value of $\sin^2(2\theta)$, it may be possible to see the neutrino oscillations provided Δm^2 is in the range 10^{-9} – 10^{-11} eV^2 .

One of the most essential ingredients in making the above scenario work is that the spread in energy ΔE of the neutrino "beam" is not too wide. This is especially true in this case since $R/\Delta R \gg 1$. If ΔE is too large, then by the time the neutrinos arrive at the Earth the oscillation patterns for neutrinos of different energies get sufficiently out of phase to wipe out any potentially observable oscillations. This results simply in a decrease of the total ν_e intensity by an amount $\sin^2(2\theta)/2$. A coherence length L_{max} is usually defined as the distance at which a neutrino of energy E has undergone one oscillation more than a neutrino of energy $E + \Delta E$. This coherence length is given by

$$L_{\text{max}} = \frac{4\pi E^2}{(\Delta m^2) \Delta E} = L_{\text{osc}} \left(\frac{E}{\Delta E} \right), \quad (3)$$

and the total number of complete oscillations will be

$$N_{\text{max}} = \frac{L_{\text{max}}}{L_{\text{osc}}}. \quad (4)$$

Thus when $\Delta E/E$ is larger than about $1/30$ we can no longer observe the oscillations and a narrow energy range ΔE is therefore required.

The argument above assumed that the energy spread of the neutrino beam is incoherent in origin in the sense that it is due to slightly different energies of various neutrinos. The main origin of this energy spread ΔE is that the continuum electrons which are captured by Be have an energy spread $\Delta E_e \sim kT$ which translates into a similar spread $\Delta E_\nu \sim kT$ of the emerging neutrino energies. Another slightly smaller contribution to $\Delta E_\nu \equiv \Delta E$ originates from the different Doppler shifts due to thermal nuclear velocities (relative to the line of sight), the ana-

logue of the well-known Doppler broadening in atomic spectroscopy [4].

“Coherent broadening,” namely, the quantum-mechanical spread δE of a *single* neutrino, can also lead to the loss of the oscillation pattern [5]. The well-known natural linewidth in atomic spectroscopy,

$$\delta E \sim \Gamma \sim (\tau_{\text{decay}})^{-1}, \quad (5)$$

is an example of coherent broadening. The finite lifetime τ_{decay} of the level interrupts the classical emission of the wave train and limits the size of the wave packet δx to

$$\delta x = c\tau_{\text{decay}}, \quad (6)$$

with the momentum ($\delta P = \delta E/c$) and the configuration space (δx) widths being inversely related to each other via a Fourier transform of a Lorentzian to an exponential.

Another example of coherent broadening is the collisional broadening (also known as the “pressure broadening”) of the neutrino line. It stems from the interruption of coherent emission by collisions of the emitting atoms. The corresponding wave packet size is given by an analogue of Eq. (6) but with τ_{decay} replaced by $t_{\text{collision}}$, the effective time interval between “relevant collisions.” This (nuclear) collisional broadening effect has been extensively studied as the major contributor to the loss of coherence in neutrino oscillation. There have been various estimates of the strength of the effect leading to estimates of the size of $\delta x = ct_{\text{collision}}$ [4–7].

A third contribution to the coherent broadening which we believe is likely to contribute even more to the energy spread δE of the neutrino wave packet is the small size of the wave packets of the captured electrons. Since the K electron ionization energy in berillium, $E_{\text{ion}} = Z^2 R_y = 16R_y \sim 220$ eV, is small in comparison with the thermal $kT \sim$ keV energy, the capture in reaction (1) is primarily that of continuum electrons. An electron wave packet of size δ_e will traverse the (pointlike) nucleus in a time

$$\delta t = \frac{\delta_e}{v_e}, \quad (7)$$

where v_e is the velocity of the electron. Because the weak interaction underlying the capture process (1) is local, the time available for the ν_e emission is δt and the size of the outgoing ν_e wave packet emitted with velocity c ($c = 1$ in our units) will be

$$\delta_\nu = \frac{\delta_e}{v_e}. \quad (8)$$

The thermal kinetic energy of a typical electron is $\frac{1}{2}m_e v_e^2 \sim \frac{3}{2}kT$. Thus

$$v_e = \sqrt{\frac{3kT}{m_e}} \sim 0.08. \quad (9)$$

It remains only to estimate the appropriate wave packet size δ_e to be used in Eq. (7). The electrons suffer many random collisions in the hot core which tend to localize the wave function and reduce the wave packet size. If

the only information available is that the electrons are in thermal equilibrium, then δ_e is expected to be of the order of the thermal wavelength:

$$\delta_e \sim \frac{2\pi}{m_e v_e} \sim \frac{2\pi}{\sqrt{3m_e kT}}, \quad (10)$$

which then leads to a neutrino wave packet size

$$\delta_\nu = \frac{2\pi}{\delta E_\nu} \sim 6 \times 10^{-8} \text{ cm}. \quad (11)$$

This δ_ν is *smaller* (and the corresponding incoherent broadening is *larger*) than all previous estimates.

The three mechanisms described above all lead to the conclusion that neutrinos are emitted in the Sun as wave packets with a rather small size δ_ν , corresponding to “coherent broadening” of the neutrino line by an amount $\delta E \sim 2\pi/\delta_\nu$. This coherent broadening also leads to the loss of the oscillation pattern [5] after a coherence length L_{coh} which is precisely equal to the coherence length L_{max} derived in Eq. (3). This result can be derived technically by decomposing the wave packet into plane waves of energy E with a probability distribution

$$P(E) = |\Psi(E)|^2 \quad (12)$$

and repeating the discussion leading to Eqs. (3) and (4). This leads to identical conclusions but with ΔE replaced by the energy spread δE given by

$$(\delta E)^2 = \int dE P(E) (E^2 - \bar{E}^2). \quad (13)$$

There is, however, a simple intuitive explanation for how the oscillations are lost in terms of the wave packet of the neutrino in configuration space [5]. Consider an electron neutrino wave packet which is emitted at $t = 0$ from the solar core. At $t = 0$ the ν_e can be written as a superposition of two wave packets with identical shape corresponding to the mass eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$:

$$|\nu_e(t=0)\rangle = \cos(\theta)|\nu_1\rangle + \sin(\theta)|\nu_2\rangle. \quad (14)$$

This initial wave packet will quickly spread in the directions (x, y) perpendicular to the direction of motion but the spreading in the direction of motion (z) is negligible due to Lorentz contraction effects. Because of the different mass of the ν_1 and ν_2 , their wave packets travel with a different (group) velocity

$$\Delta v = v_2 - v_1 = \frac{\Delta m^2}{2E^2}. \quad (15)$$

Thus after a time t has elapsed and the neutrino has traveled a distance $r \sim t$ from the source the two wave packets move with respect to each other by an amount

$$\Delta r = \Delta v t \sim \frac{\Delta m^2}{2E^2} r. \quad (16)$$

Neutrino oscillations are simply the “beating” of the two

wave packets as they slide relative to each other by $\Delta r = \lambda$ with

$$\lambda = \frac{2\pi}{E} \quad (17)$$

the wavelength of the neutrino. The oscillation length of Eq. (2) is then recovered as

$$L_{osc} = \{\text{value of } r \text{ for which } \Delta r = \lambda\} = \frac{\lambda}{\Delta v} = \frac{4\pi E}{\Delta m^2}, \quad (18)$$

The total number of possible neutrino oscillations is simply the total number of wavelengths within the wave packet, $N_{max} = \delta_\nu/\lambda = E/\delta E$. After this number of oscillations the two wave packets do not overlap at all and all oscillations are lost. Thus the coherence distance L_{coh} , which is the maximum distance over which we see oscillations, is given by

$$L_{coh} = N_{max}L_{osc} = \left(\frac{E}{\delta E}\right)L_{osc}, \quad (19)$$

which is precisely the result of Eqs. (3) and (4) for the case of incoherent energy broadening. Indeed, once Δr is greater than the size δ_ν of the wave packet, the ν_1 and the ν_2 will have completely separated spatially. We would thus expect that they will not interfere when interacting locally with an electron or nucleus in a detector.

The main aim of this paper is to study whether the two effects discussed above, namely, the incoherent versus the coherent broadening, can be distinguished. They are clearly distinct physical phenomena which can be controlled (at least in principle) at the *source*. In an atomic physics analogue the Doppler broadening can be controlled relative to the natural linewidth by adjusting the temperature of the system or by confining the atoms to a narrow channel transverse to the line of sight [8]. The more interesting question is the following: Can we distinguish these effects at the detector? In this paper we shall show that in all physically interesting situations the answer is "no." We shall discuss some simple cases in which this answer is clear and then we shall prove some general theorems which will show that under a wide variety of physically attainable situations these two effects cannot be distinguished.

II. COHERENT VERSUS INCOHERENT BROADENING

Our goal in this section is to see whether one can distinguish an incoherent ensemble of plane waves with a mean energy E and an energy spread ΔE from an ensemble of wave packets each with the same mean energy E and the same energy width $\delta E = \Delta E$. Before proceeding we should make one point clear. Even in the "incoherent" case in which we have an ensemble of plane waves these waves certainly do not have an infinite extent in the z direction (the direction of motion). In fact even if we took each "plane wave" (with an energy in the

MeV range) to have an energy uncertainty of the order of 10^{-5} eV (which is certainly a great underestimate for the solar neutrino case) the corresponding wave packet would still be only of the order of a cm in size. Thus when discussing "plane waves" we are in fact referring to wave packets which are much larger than those discussed in the case of coherent broadening but much smaller than any macroscopic scales in the problem.

A. Example

Our aim will be to show that the two broadening effects discussed above cannot be distinguished. We begin with a concrete suggestion for distinguishing these effects and we then show what goes wrong with this suggestion.

Let us suppose that we were able to measure the energy of a neutrino with a precision ϵ which is much better than $\delta E = \Delta E$. We then expect that for an incoherent beam of neutrinos with energies in a range ΔE about E we could recover the oscillations by measuring the neutrino energy to the precision $\epsilon \ll \Delta E$. By plotting the observed neutrino count as a function of

$$r' = r \frac{\bar{E}}{E}, \quad (20)$$

we should see oscillations up to a new distance

$$L_\epsilon = (E/\epsilon)L_{osc} > L_{max}, \quad (21)$$

with no loss of statistics. Note that ΔE is replaced by ϵ in Eq. (3).

If, on the other hand, we began with a wave packet with energy spread δE , then, at a distance larger than L_{coh} [Eq. (19)], the wave packet of the ν_1 and the ν_2 are completely separated and one might naively expect that there will be no oscillations even if the energy could be measured more accurately.

This argument turns out to be wrong and we can understand what goes wrong in a very intuitive way. If we choose to measure the energy very accurately (to an accuracy ϵ), we require a time $t \sim 1/\epsilon$ to make this measurement. If, during this time t , the second wave packet arrives at the detector, then we will once more see the oscillations. The condition for recovering the oscillations is therefore

$$t \sim 1/\epsilon > \Delta r \sim \frac{\Delta m^2}{2E^2} r, \quad (22)$$

where Δr is the distance between the wave packets and r is the distance from the source [Eq. (16)]. The oscillations thus persist up to a new distance L'_ϵ which is the value of r for which Eq. (22) breaks down:

$$L'_\epsilon = (E/\epsilon)L_{osc}, \quad (23)$$

which is precisely the same as the result (21) obtained for the incoherent neutrino beam.

This behavior of the coherent beam is analogous to what occurs for a high- Q oscillator hit by two successive

pulses. The first pulse (in our case the ν_1 beam) comes along and sets the oscillator in motion. It then continues to oscillate for a time $t \sim 1/\epsilon$, during which time the second pulse (in our case the ν_2 beam) arrives and causes the oscillator to be further excited. In this way coherence is maintained between the ν_1 and the ν_2 beams even when they are spatially separated. What happens is that the accurate measurement of the energy picks out the plane wave in the wave packet which has existed coherently through both pulses.

Our main goal will be to understand how general the above result is. In other words, to what extent is it true that an ensemble of plane waves will give the same result as wave packets? Although there were some initial attempts to distinguish these processes, it is now widely believed that they are indistinguishable. Our goal in this paper is to prove some theorems which clarify the conditions under which this is true and to show how general the result is.

B. Measuring observables which commute with momentum

Before discussing the most general situation we review here the proof that the coherently and incoherently broadened neutrino beams lead to the same total rate and energy distributions for both ν_e 's and ν_μ 's.

1. Oscillations in vacuum

Let us consider two cases representing two possible electron neutrino beams¹ leaving some region of the Sun at time $t = 0$. In case (a) we have an incoherent mixture of neutrinos, each of which is a nearly ideal plane wave (with some extremely small energy spread $\delta E_{pw} \ll \delta E$). In this mixture the probability of find-

ing a neutrino of energy E is given by some probability distribution $P(E)$ which is centered about some energy E_0 with a width ΔE . In case (b) all the neutrinos come with the same quantum state. This state is a wave packet with amplitude $\Psi(E)$ for a plane wave component of energy E . We choose this amplitude so that the probability distribution $|\Psi(E)|^2$ precisely matches the distribution $P(E)$ of case (a). Consequently the widths of the two distributions are also equal: $\delta E = \Delta E$. In this section, for simplicity, we shall treat the plane waves of case (a) as ideal plane waves with $\delta E_{pw} = 0$.

At $t = 0$ the wave function for case (b) is given by

$$|\psi(t=0)\rangle = \sum_{p,i} \alpha_{p,i} |p, i\rangle, \quad (24)$$

where the sum (which is actually an integral) is over momenta p in the z direction (the direction of motion) and over mass eigenstates $i = 1, 2$ and the $\alpha_{p,i}$ are chosen to give an electron neutrino with the appropriate wave function at $t = 0$. Since the $|p, i\rangle$ are eigenstates of the Hamiltonian, at a later time t , the wave function is given by

$$|\psi(t)\rangle = \sum_{p,i} \alpha_{p,i} e^{-i\epsilon_p^{(i)}t} |p, i\rangle, \quad (25)$$

where $\epsilon_p^{(i)} = \sqrt{p^2 + m_i^2}$ is the energy of ν_i with momentum p .

Suppose now that at time t we measure an observable Q which commutes with the momentum operator. Q may, for example, be the total number of electron neutrinos in some range of momenta. This is, in fact, the most common kind of measurement which can be made. In this case Q has only diagonal matrix elements in momentum space. Therefore the expectation value of Q at time t is given by

$$\langle \psi(t) | Q | \psi(t) \rangle = \sum_p \left[\left(\sum_j \alpha_{p,j}^* e^{i\epsilon_p^{(j)}t} \langle p, j | \right) Q \left(\sum_i \alpha_{p,i} e^{-i\epsilon_p^{(i)}t} | p, i \rangle \right) \right]. \quad (26)$$

The expression inside the square brackets is precisely the expression for the expectation value $\langle Q \rangle_p$ of Q for a plane wave which has a total weight (i.e., normalization) $|\alpha_{p,1}|^2 + |\alpha_{p,2}|^2$ and a relative amplitude $\alpha_{p,1}$ and $\alpha_{p,2}$ for ν_1 and ν_2 , respectively, at $t = 0$. Thus,

$$\langle Q \rangle = \sum_p \langle Q \rangle_p, \quad (27)$$

¹They could, of course, be any linear combination of electron and muon neutrinos. Electron neutrinos were chosen for definiteness only.

which is precisely the result one obtains for the incoherent beam of case (a). Thus the measurement of any observable which commutes with momentum yields the same result for case (a) and case (b).

Although this result may seem trivial, it is in fact rather powerful. From this result we can verify the result claimed in Sec. I that if we use the variation in the Earth-Sun distance to look for oscillations in the neutrinos from ⁷Be both the coherent and the incoherent neutrino beams give the same oscillation pattern. *A priori* the above theorem is not applicable since the experiment involves measuring the spatial dependence of the neutrino flux which involves the use of an operator which does not commute with momentum. This is, however, an example

for which the conversion of spatial to temporal dependence can be done reliably. Thus, although we measure the spatial variation in the neutrino flux, we can compute the temporal dependence of this flux by computing, for example, the total number of electron neutrinos with a given energy as a function of time. This estimate will be reliable since, as discussed previously, even the "plane wave" packet is still extremely small (certainly much less than 1 cm in size) relative to the relevant astronomical scales.

2. Oscillations in matter

The above proof, that an ensemble of plane waves cannot be distinguished, at the detector, from an ensemble of wave packets with the same energy distribution, can be extended to the case of neutrino oscillations in matter (the MSW effect) [9]. To this end imagine that at $t = 0$ an electron neutrino is produced (at the origin) in mat-

ter in which the density of electrons (along the direction of motion of the neutrino) is given by $\rho_e(z)$. (This is of course an approximation in which we neglect variations of the density in the transverse directions.) The "vacuum eigenstates" $|\nu_1\rangle$ and $|\nu_2\rangle$ are no longer eigenstates of this system. Instead one can find new eigenstates of the Hamiltonian which include the full spatial variation of the density. These eigenstates will of course no longer be momentum eigenstates. For relativistic neutrinos one should, in principle, solve the Dirac Equation but for the present discussion since spin is not a crucial variable it suffices to consider the Klein-Gordon equation [10]

$$\left[-\left(\frac{\partial}{\partial t} + iA_m\right)^2 + \nabla^2 \right] \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = M_0^2 \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}, \quad (28)$$

where M_0 is the vacuum mass matrix and the matrix A_m accounts for the effect of charged-current scattering of the ν_e off the electrons in the medium:

$$M_0^2 = \begin{pmatrix} m_1^2 \cos^2 \theta + m_2^2 \sin^2 \theta & (m_1^2 - m_2^2) \sin \theta \cos \theta \\ (m_1^2 - m_2^2) \sin \theta \cos \theta & m_1^2 \sin^2 \theta + m_2^2 \cos^2 \theta \end{pmatrix}, \quad A_m = \begin{pmatrix} \sqrt{2}G_F \rho_e(z) & 0 \\ 0 & 0 \end{pmatrix}. \quad (29)$$

The eigenstates of this system with energy E will no longer be eigenstates of p_z (p_x and p_y are assumed to be zero) but will be labeled by some other parameter which we call γ . We shall call these eigenstates

$$|\gamma, 1\rangle \quad \text{and} \quad |\gamma, 2\rangle. \quad (30)$$

In the regions of space where the density vanishes these eigenstates will behave as plane waves² with some momentum $p_{\text{vac}}(\gamma)$. They will correspond to vacuum mass eigenstates of the system. In a region of space in which the electron density is nonzero but nearly constant the eigenstates $|\gamma, i\rangle$ will again be nearly plane waves but now corresponding to the usual neutrino eigenstates *in matter*.

Suppose now that at $t = 0$ we prepare an electron neutrino in a state described by some rather narrow wave packet $|\psi(t=0)\rangle$. [This is case (b) of Sec. II B 1.] At $t = 0$, in analogy with Eq. (24), this state can be expanded in the eigenstates $|\gamma, i\rangle$ described above:

$$|\psi(t=0)\rangle = \sum_{\gamma, i} \alpha_{\gamma, i} |\gamma, i\rangle. \quad (31)$$

We now allow the state to propagate to a later time t . At this later time the state is given by

$$|\psi(t)\rangle = \sum_{\gamma, i} \alpha_{\gamma, i} e^{-i\epsilon_\gamma^{(i)} t} |\gamma, i\rangle, \quad (32)$$

where $\epsilon_\gamma^{(i)}$ is the energy of the state $|\gamma, i\rangle$. In any reasonable case the size of the wave packet at time t will be much smaller than the scale of variations in the electron density. (This is especially true if the measurement is made in vacuum.) Thus Eq. (32) amounts to an expansion in the momentum eigenstates of the neutrinos in matter with density ρ equal to the density at the location of the wave packet. Every γ corresponds to some momentum $p(\gamma)$ which depends on the density ρ . Thus

$$|\psi(t)\rangle \simeq \sum_{p, i} \hat{\alpha}_{p, i} |p, i; \rho\rangle, \quad (33)$$

where for any given value of p and the corresponding value of γ the coefficients $\hat{\alpha}_{p, 1}$ and $\hat{\alpha}_{p, 2}$ are linear combinations of $\alpha_{\gamma, 1}$ and $\alpha_{\gamma, 2}$.

Now suppose that at time t we measure some operator Q which commutes with (the z component of the) momentum. The off-diagonal matrix elements of Q vanish in the momentum basis. Thus Q will have only diagonal matrix elements between the various $|p, i; \rho\rangle$ in Eq. (33). Since each of these $|p, i; \rho\rangle$ corresponds to one of the energy eigenstates $|\gamma, i\rangle$, it follows that the expectation value of Q is given by

²It may in fact be a superposition of an incoming and an outgoing plane wave if there is reflection.

$$\langle \psi(t) | Q | \psi(t) \rangle \sim \sum_{\gamma} \left[\left(\sum_j \alpha_{\gamma,j}^* e^{i\epsilon_{\gamma}^{(j)} t} \langle \gamma, j | \right) Q \left(\sum_i \alpha_{\gamma,i} e^{-i\epsilon_{\gamma}^{(i)} t} | \gamma, i \rangle \right) \right]. \quad (34)$$

This expression is analogous to Eq. (26) in Sec. II B 1. Each term in the sum is precisely the result which we would have obtained for the expectation value $\langle Q \rangle_{\gamma}$ of Q for a state which was initially in an approximate momentum eigenstate corresponding to γ but with total weight $|\alpha_{\gamma,1}|^2 + |\alpha_{\gamma,2}|^2$ and a *relative* amplitude $\alpha_{\gamma,1}$ and $\alpha_{\gamma,2}$ for ν_1 and ν_2 , respectively. (Recall that the realistic plane waves are actually extremely narrow on the scale of the density variations.) Thus,

$$\langle Q \rangle = \sum_{\gamma} \langle Q \rangle_{\gamma}, \quad (35)$$

which is precisely the result one obtains for the incoherent beam of case (a). There is thus no difference between the wave packet [case (b)] and the plane wave [case (a)] ensemble *even in matter* when only operators which commute with momentum are measured.

C. Unrealistic measurements which can identify wave packets

From the above proof it seems that a keen measurement which combines a measurement of both position and momentum information might be able to distinguish an ensemble of wave packets from an ensemble of plane waves. The simplest way of doing this would, however, require precise knowledge of the point of origin and the time of origin of the wave packet. Suppose, for example, that we *knew* that all the wave packets in our ensemble [case (b)] were centered at the origin ($z = 0$) precisely at time $t = 0$. Suppose also that in the alternative scenario [case (a)] we also knew that each (nearly ideal) plane wave [which is still a wave packet but with a much larger spatial extent than that of case (b)] in the ensemble was centered at the origin at $t = 0$. Under these assumptions about our previous knowledge and by a careful timing measurement at the Earth to determine the duration of the neutrino pulse we *could* distinguish the two cases. [In fact in case (b) there may be two separated pulses.] This scenario is, of course, totally unrealistic and we shall see below that if we allow for an uncertainty in the location of the initial packets it again becomes impossible to distinguish the two cases by *any* measurement at the earth.

There is another scenario under which it is clearly possible to distinguish the two cases. Suppose we have a detailed theory for the production mechanisms of the two cases which lead to some different observable at the source. Suppose, for example, that the position or momentum distributions for the two cases are expected to differ. Then clearly such information can be used to decide which mechanism is producing the neutrinos (or, more realistically, which mechanism dominates). However, in the case of level broadening we have no such

information. Both the energy and the position distributions are expected to be roughly the same. The question which we are asking is the following: Assume we are given two “sources” of neutrinos (or production mechanisms) with the *same* position (z) and momentum (p) distributions. Is it possible to tell by measurements at the detector which of the two “sources” produced these neutrinos?

D. General theorem

This question can be set up more precisely as follows: Consider the following two modifications of the scenarios case (a) and case (b) discussed above.

In case (A) we have a nearly ideal plane wave which is actually a wave packet of a fairly large size Δz . (Recall that Δz will typically be much less than a cm.) We imagine an ensemble of such “plane waves,” each of which has a nearly precise momentum (in the z direction) centered about p_0 with a spread δp . Assume that each plane wave has exactly the same spatial location. [This is precisely case (a) above.]

In case (B) we have an ensemble of wave packets. Each wave packet has a spatial size δz which is much smaller than Δz and a corresponding momentum spread $\delta p = 1/\delta z$ which is precisely equal to the δp of case (A). Up to this point this looks exactly like case (b) above except we now allow each wave packet in our ensemble to be, at $t = 0$, at a different spatial location. We assume that the wave packets are produced in precisely the same region Δz in which the neutrinos of case (A) are produced with precisely the same z distribution.³ The two cases are shown pictorially in Fig. 1.

All the above information is given to the experimenter together with the additional information that the z and p distributions for both cases are equal at $t = 0$. The question is the following: With only this information can the experimenter distinguish with any experiment the cases (A) and (B) above?

Intuitively one might guess that the answer is “yes.” There should be some way to tell if we are dealing with wave packets or with (almost) plane waves. But in fact

³In a realistic situation both the “plane waves” of case (a) and the wave packets of case (b) will be distributed over a region of space much larger than Δz . In both cases this excess spread is incoherent. It is thus sufficient to prove our result for the case when the wave packet is distributed in z by the size Δz of the plane wave of case (a).

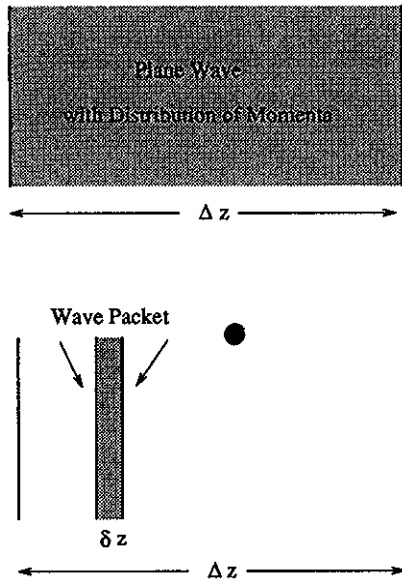


FIG. 1. Pictorial representation of case (a) and case (b) described in Sec. IID.

the answer is “no.” No experiment can distinguish the above two cases.

The most general proof of this statement would proceed as follows.

Step 0. Choose values for $\delta z = 1/\delta p \ll \Delta z$ and for the mean momentum p_0 which were defined above.

Step 1. Begin with an arbitrary (smooth) but fixed expression for the wave function of the nearly ideal plane wave of case (A). The only constraints on this wave function will be that it is centered (say) at the origin and that its spread in position is (a fairly large) Δz with a correspondingly tiny spread in momentum about some momentum p . Then consider an ensemble of such states each with a different momentum p . Choose an arbitrary but fixed distribution for these momenta. The constraint on this distribution is that it is centered about the momentum p_0 with the given width δp .

Step 2. Construct the density matrix for the ensemble described in step 1 above.

Step 3. One must now prove that it is always possible to construct the following, seemingly completely different ensemble, which, nonetheless, yields a density matrix identical to the one obtained in step 2 above. We first construct a wave packet which is centered at some location z . We are free to choose the form of the wave function with the only constraint that its spread in position be approximately equal to $\delta z \ll \Delta z$ with a corresponding momentum spread $\delta p = 1/\delta z$. We then construct an ensemble of such wave packets and choose a distribution of locations z with the only constraint that this distribution be centered at the origin with a spread in position approximately equal to Δz .

The claim is that we can *always* choose the distributions in Step 3 so that the density matrix for Step 3 is identical to that of Step 2. This then implies that any

measurement at all which is done on the two ensembles *at any time* t gives the same result. We also claim the converse of this theorem, namely, that given a “wave packet” ensemble constructed as in step 3 it is always possible to find a “plane wave” ensemble as constructed in step 1 with the same density matrix.

Note how the mass eigenstates ν_1 and ν_2 appear nowhere in the above discussion. The reason for this and, in our opinion, the power of this proof is that it relies entirely on properties of the system at $t = 0$ at which time the state is a pure ν_e state.

1. Illustration in the simplest case

We can show the essence of the proof by the following simple example. We model the wave packet [of case (B)] by a superposition of only two momentum eigenstates $|p_1\rangle$ and $|p_2\rangle$. In step 1 above we imagine having the state $|p_1\rangle$ with probability $|\alpha|^2$ and the state $|p_2\rangle$ with probability $|\beta|^2$; ($|\alpha|^2 + |\beta|^2 = 1$). The density matrix for this system is simply

$$|\alpha|^2 |p_1\rangle\langle p_1| + |\beta|^2 |p_2\rangle\langle p_2|. \quad (36)$$

For implementing step 3 we may construct an analogue of wave packets at two different locations as an ensemble consisting of these two states with equal probability:

$$|\psi_{\pm}\rangle = \alpha |p_1\rangle \pm \beta |p_2\rangle. \quad (37)$$

The density matrix in this case,

$$|\psi_+\rangle\langle\psi_+| + |\psi_-\rangle\langle\psi_-|, \quad (38)$$

is precisely the same as the density matrix for case (A) in Eq. (36).

This completes the proof in this simple case.

2. Gaussian distributions

One case in which steps 0–3 above can be carried out explicitly is when all distributions are Gaussian. Thus in step 1 we choose the “plane wave” of momentum p to have a wave function

$$|p; \text{plane}\rangle = \frac{1}{(\sqrt{2\pi}\sigma)^{\frac{1}{2}}} \int dl \exp\left(-\frac{(l-p)^2}{4\sigma^2}\right) |l\rangle, \quad (39)$$

where $\sigma \sim 1/\Delta z$. We then consider an ensemble of these states with a Gaussian distribution of momenta p :

$$\frac{1}{(\sqrt{2\pi}\sigma_p)} \exp\left(-\frac{(p-p_0)^2}{2\sigma_p^2}\right), \quad (40)$$

where $\sigma_p \sim \delta p = 1/\delta z \gg \sigma$. The density matrix for this case [case (A)] is given by

$$\rho_A = \frac{1}{(\sqrt{2\pi}\sigma_p)} \frac{1}{(\sqrt{2\pi}\sigma)} \int dl dl' dp \exp\left(-\frac{(p-p_0)^2}{2\sigma_p^2} - \frac{(l-p)^2}{4\sigma^2} - \frac{(l'-p)^2}{4\sigma^2}\right) |l\rangle\langle l'|. \quad (41)$$

We now proceed with step 3 corresponding to case (B). Consider a wave packet with mean momentum p_0 centered at some location z :

$$|z; \text{packet}\rangle = \frac{1}{(\sqrt{2\pi}\hat{\sigma}_p)^{\frac{1}{2}}} \int dl e^{-ilz} \exp\left(-\frac{(l-p_0)^2}{4\hat{\sigma}_p^2}\right) |l\rangle. \quad (42)$$

We shall soon see that the correct choice for $\hat{\sigma}_p$ is

$$\hat{\sigma}_p^2 = \sigma_p^2 + \sigma^2, \quad (43)$$

which is approximately equal to $\sigma_p = 1/\delta z$ as required. We then consider an ensemble of these states with a Gaussian distribution of positions z centered at the origin of the form

$$\frac{1}{(\sqrt{2\pi}\sigma_z)} \int dz \exp\left(-\frac{2z^2}{\sigma_z^2}\right). \quad (44)$$

The correct choice for spread in position, σ_z , will turn out to be

$$\sigma_z^2 = \frac{\sigma_p^2}{\sigma^2(\sigma^2 + \sigma_p^2)}. \quad (45)$$

This σ_z is approximately equal to $1/\sigma = \Delta z$ as required. The density matrix for this situation [case (B)] is given by

$$\rho_B = \frac{1}{(\sqrt{2\pi}\hat{\sigma}_p)} \frac{2}{(\sqrt{2\pi}\sigma_z)} \int dl dl' dz \left[\exp\left(-\frac{2z^2}{\sigma_z^2} - \frac{(l-p_0)^2}{4\hat{\sigma}_p^2} - \frac{(l'-p_0)^2}{4\hat{\sigma}_p^2}\right) \right] e^{-i(l-l')z} |l\rangle\langle l'|. \quad (46)$$

With the choices we have made for $\hat{\sigma}_p$ and σ_z in Eqs. (43) and (45) it turns out that the density matrices ρ_A and ρ_B are *precisely equal*. The calculation is straightforward and most easily done by computing the matrix elements $\langle l|\rho_{A,B}|l'\rangle$. In order to compute the matrix elements of ρ_A only the integration over p must be done. This is a Gaussian integral. For ρ_B only the integral over z must be done. This is simply the Fourier transform of a Gaussian. The result is the same for ρ_A and ρ_B and is given by

$$\langle l|\rho_{A,B}|l'\rangle = \frac{1}{(\sqrt{2\pi}\hat{\sigma}_p)} \exp\left(-\frac{(l-p_0)^2}{4\hat{\sigma}_p^2} - \frac{(l'-p_0)^2}{4\hat{\sigma}_p^2} - \frac{(l-l')^2\sigma_p^2}{8\sigma^2\hat{\sigma}_p^2}\right). \quad (47)$$

We thus establish, for the Gaussian case, that the two ensembles are identical.

3. General proof

In the Gaussian case described above we did not use the fact that $\delta z \ll \Delta z$. In the case of a more general shape for the “plane wave” and the wave packet we shall present a proof which does rely on this approximation. We conjecture that it is possible to slightly modify the theorem⁴ so that it will be valid for general values of δz and Δz but we do not have a proof at this time.

⁴The modification we have in mind is to relax the unnecessary restriction that the shape of the wave packet be independent of z . It is reasonable to consider an ensemble of wave packets, all of which have the same width but with slightly different shapes. The same could be done for the “nearly plane waves.”

We begin again with step 1 for which we choose a “plane wave” of momentum p to have a wave function

$$|p; \text{plane}\rangle = \int dl f_\sigma(l-p) |l\rangle, \quad (48)$$

where the function $f_\sigma(l-p)$ has a width $\sigma \sim 1/\Delta z$. We then consider an ensemble of these states with a distribution of momenta p given by some function $g_{\sigma_p}(p-p_0)$ with a width $\sigma_p \sim \delta p = 1/\delta z \gg \sigma$. The density matrix for this case [case (A)] is given by

$$\rho_A = \int dl dl' dp [f_\sigma^*(l'-p) f_\sigma(l-p) g_{\sigma_p}(p-p_0)] |l\rangle\langle l'|. \quad (49)$$

We now proceed with step 3 corresponding to case (B). Consider a wave packet with mean momentum p_0 centered at some location z :

$$|z; \text{packet}\rangle = \int dl e^{-ilz} \alpha_{\sigma_p}(l - p_0) |l\rangle, \quad (50)$$

which has *approximately* a width σ_p . We then consider an

ensemble of these states with a distribution of positions z centered at the origin given by some function $h_{\sigma_z}(z)$ with a width σ_z which is approximately equal to $1/\sigma = \Delta z$. The density matrix for this situation [case (B)] is given by

$$\begin{aligned} \rho_B &= \int dl dl' dz \left[\alpha_{\sigma_p}^*(l' - p_0) \alpha_{\sigma_p}(l - p_0) h_{\sigma_z}(z) \right] e^{-i(l-l')z} |l\rangle \langle l'| \\ &= \int dl dl' \left[\alpha_{\sigma_p}^*(l' - p_0) \alpha_{\sigma_p}(l - p_0) \tilde{h}_\sigma(l - l') \right] |l\rangle \langle l'|, \end{aligned} \quad (51)$$

where $\tilde{h}_\sigma(l - l')$ is the Fourier transform of $h_{\sigma_z}(z)$ which has a width approximately equal to σ .

The requirement that the two density matrices be equal is now simply stated as

$$\alpha_{\sigma_p}^*(l' - p_0) \alpha_{\sigma_p}(l - p_0) \tilde{h}_\sigma(l - l') = \int dp f_\sigma^*(l' - p) f_\sigma(l - p) g_{\sigma_p}(p - p_0). \quad (52)$$

[It is now clear why the theorem, as stated, cannot be true in general. Given arbitrary smooth functions g_{σ_p} and f_σ with the restrictions described previously, it is certainly not possible, in general, to find functions α_{σ_p} and \tilde{h}_σ which satisfy Eq. (52) since the integral in (52) will not always factorize in the required form.] The result is, however, valid when the width σ_p of g_{σ_p} is much larger than the width σ of f_σ . If $\sigma \ll \sigma_p$ and if the function g_{σ_p} is sufficiently smooth,

$$f_\sigma^*(l' - p) f_\sigma(l - p) g_{\sigma_p}(p - p_0) = f_\sigma^*(l' - p) f_\sigma(l - p) \sqrt{g_{\sigma_p}(l - p_0)} \sqrt{g_{\sigma_p}(l' - p_0)} + O\left(\frac{\sigma}{\sigma_p}\right). \quad (53)$$

Thus,

$$\int dp f_\sigma^*(l' - p) f_\sigma(l - p) g_{\sigma_p}(p - p_0) \sim \sqrt{g_{\sigma_p}(l - p_0)} \sqrt{g_{\sigma_p}(l' - p_0)} \int dq f_\sigma^*(q) f_\sigma(l - l' + q). \quad (54)$$

Thus if we identify the function α_{σ_p} with the square root of g_{σ_p} and the function $h_{\sigma_z}(z)$ with the square of the Fourier transform of $f_\sigma(p)$, then the equality in Eq. (52) is satisfied to order $\delta z/\Delta z$ as required.

E. Consequences

Although the result proven above is not entirely general, it is sufficient for all cases of practical interest. The reason for this is that we have actually proven three things. The result that measurements which commute with momentum could not distinguish coherent from incoherent broadening was completely general and did not depend on the shape of the wave packet or on its width. Second the proof that for Gaussian wave packets the two effects could not be distinguished with *any* measurement was also general and it did not depend on the width of the Gaussians. Third, our extension of the proof to arbitrary wave packet shapes was possible in the limit $\delta z \ll \Delta z$. A practical attempt to distinguish the two mechanisms of broadening would likely begin with a theoretical calculation which assumes Gaussian wave packets for simplicity. Furthermore, it would likely compare the wave packets to actual plane waves for which $\Delta z \rightarrow \infty$. We have *shown* that any such attempt is doomed to failure. We

conjecture that the result is more general so that for an arbitrary shape of wave packet it is possible to find an ensemble of nearly plane waves which mimic its behavior exactly.

An interesting corollary to the result proved in the previous section is that one cannot tell, on an event-by-event basis, whether one has a wave packet or a "plane wave." The proof is as follows. Suppose it were possible, on an event-by-event basis, to distinguish a wave packet from a plane wave. It would then be trivially possible to distinguish the cases (A) and (B) above since in one case we are presented with a plane wave and in the other case with a wave packet. In fact in just one event we would know with which case we are dealing. But as we saw in the previous section, we cannot do this since the density matrices for the two cases are identical. It follows that no such determination can be made on an event-by-event basis.

This result does not contradict the recent work of several authors [11] on the ability to measure the wave function of a single particle via a "protective measurement." There are at least two requirements for such a measurement to be possible. The first is that the system needs an energy gap so that successive (soft) measurements keep the particle in the same state. The second requirement is that it is known *a priori* that the system is in an eigen-

state of the Hamiltonian. Thus, for example, it is in principle possible to measure the ground state wave function of a typical atom even on a single atom, but if we do not know whether the atom is in an eigenstate of the Hamiltonian or in a superposition of eigenstates, then this cannot be determined on a single atom. An argument very similar to that in the previous paragraph can be used to prove this result. In our case neither of these conditions is satisfied. We do not have a gap and we certainly are not in an eigenstate of the Hamiltonian when we are dealing with a wave packet and/or we start with a pure flavor state such as a $|\nu_e\rangle$.

The theorem presented in the previous section also provides a general tool for understanding how the size of a quantum-mechanical wave packet affects physical results in various circumstances. In fact the style of our proof which relies on the *initial* properties of the system rather than on the details of its time evolution is extremely useful. There have been several instances in which either careless approximations or faulty logic have led to conclusions which disagree with our very general result. To illustrate this point imagine, instead of using our general proof, that we evolve each of the two ensembles to a later time t and *then* compared them. We must, by our theorem, get the same density matrix for each ensemble. But in doing this calculation we might make several approximations to simplify the calculation. We might, for example, neglect the longitudinal spreading of the wave packet. It turns out that even when this spreading is negligible compared to the size of the wave packet it has a significant effect on the final density matrices and we would find significant differences between the two ensembles. We know from our theorem that this cannot be the case. Indeed when the effect of longitudinal spreading is included all results computed with ρ_A and ρ_B agree.

III. SUMMARY AND CONCLUSIONS

The main focus of this paper was the question of our ability experimentally (even in principle) to distinguish incoherent broadening of a neutrino line (such as the ${}^7\text{Be}$ solar neutrino line) from coherent broadening of such a line. Of particular interest was whether these two types of broadening would have different effects on neutrino oscillations and the MSW effect. We began by identifying processes which contribute to these mechanisms of broadening. Coherent broadening results from several processes including the natural width of the emitting nucleus, pressure broadening caused by collisions of this nucleus and the finite size of the wave packet of the captured electron. We argued that this last process leads to the smallest estimate for the spatial size of the neutrino wave packet ($\sim 6 \times 10^{-8}$ cm). Incoherent broadening results mainly from the thermal energy spread of the captured electron as well as from the Doppler shift due to the thermal motion of the emitting nucleus.

We then began to present our argument that although

the two forms of broadening were distinct physical processes which could be controlled at the source they could not be distinguished at the detector. We first showed that if the detector had an excellent energy resolution, not only could oscillations due to an incoherent ensemble of (nearly) monoenergetic neutrinos be restored, but oscillations of a coherent neutrino beam could also be restored despite the physical separation of the ν_1 and the ν_2 at the detector. We then proved that the measurement of any operator which commuted with momentum could never distinguish a wave packet from a plane wave. We extended the proof of this result to the case in which the neutrino propagates in matter (the MSW effect).

The next stage was to show that if we had no *a priori* knowledge of any difference in the properties of the coherent versus the incoherent neutrino "beams" there was no measurement which could distinguish them. Our method was to show that it was possible to construct two ensembles, one corresponding to "nearly plane waves" and the other to wave packets which had the same density matrix at $t = 0$. This would imply that the density matrices were equal at all later times and that no measurements could distinguish the two cases. We presented a complete proof in the case of Gaussian wave packets by showing that the density matrix at the source for an ensemble of plane waves with a given (Gaussian) energy distribution was *equal* to that of an ensemble of wave packets each with a much narrower z distribution but distributed, incoherently, over the same range of positions as the "incoherent" ensemble. We extended this proof to the case of non-Gaussian wave packets in the limit that the spatial size of the wave packet was much smaller than the spatial size of the "nearly plane wave." We conjectured that the result is even more general and that given any ensemble of "nearly plane waves" with a given energy and position distribution we can construct an ensemble of wave packets which has precisely the same density matrix.

There have been claims in the literature that wave packets could give different results than plane waves with the same momentum distribution. These differences show up either when the neutrinos are nearly non-relativistic or when their momentum distribution is extremely broad so that $\delta p \sim p$. This of course implies that some of the components of the neutrino wave function are nonrelativistic and that some of the neutrinos are moving "backwards." In all these cases it is essential to include the longitudinal spreading of the neutrino wave packet and to remember that if one calculates the number of neutrinos which should be observed at some location z one must compute the flux of neutrinos which involves the neutrino velocity. If these cautions are kept in mind one confirms the results of our theorem, that there are no differences between the two scenarios.

Although we have chosen to focus this paper on neutrinos and neutrino oscillations, it is clear that the result is much more general. It applies to any particle for which the question of the distinguishability of a wave packet from plane waves is relevant. Some examples include neutral Kaon oscillations and the effect of wave packets in scattering theory.

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