Information entropy and particle production in branching processes

P. Brogueira and J. Dias de Deus
Department of Physics, IST, 1096 Lisboa Codex, Portugal

I. P. da Silva
Department of Mathematics, CMAF-UL, Av. Prof. Gama Pinto, 1699 Lisboa Codex, Portugal
(Received 20 June 1995)

We show that the information entropy S_i, where i is the number of steps, is a good parameter to characterize chaoticity in branching processes. The quantity $S_i-\ln\langle n\rangle_i$, where $\langle n\rangle_i$ is the average number of particles produced at step i, approaches $-\infty$ in Abelian processes and a finite constant in non-Abelian ones.

PACS number(s): 13.85.Hd, 05.45.+b, 12.40.Ee

In a branching process, particle production occurs by degradation of virtual mass and splitting of a particle into other particles. The particle that initiates the process may be a quark or a gluon, in QCD, or an electron, in QED. The non-Abelian nature of QCD, with self-interacting gluons, makes the gluon distribution of QCD very different from the photon distribution of QED. Branching processes in multiparticle production were previously used. See, for instance, [1]. For a discussion on QCD see [2].

Very recently, in [3], the question of finding ways of characterizing the time evolution of a branching process was discussed. By using a QCD Monte Carlo simulation model [4], with appropriate splitting functions, the authors of [3] were able to show that the quantity V_i,

$$V_i\equiv(\langle n^2\rangle_i-\langle n\rangle_i^2)/\langle n\rangle_i^2, \tag{1}$$

i being the number of steps in the branching process and n the number of produced particles in an event, could be used as an adequate parameter to measure chaotic behavior: a relatively i independent V_i, for large i, indicating chaos and V_i decreasing with i meaning the absence of chaos. According to [3], in QCD chaos occurs while in an Abelian theory chaos does not occur.

In this paper we start by presenting two very simple mathematical models explicitly showing the required properties for an Abelian model (with production of noninteracting particles) and for a non-Abelian model (with production of self-interacting particles), models I and II, respectively.

Next, we introduce what we believe to be a more natural and general parameter to measure chaoticity in the time evolution of a branching process, the information entropy S_i, defined as

$$S_i\equiv-\sum_n P_i(n)\ln P_i(n), \tag{2}$$

where $P_i(n)$ is the probability of having n particles produced at level i. The definition (2) was applied to particle production in [5].

As we limit ourselves here to studying the time evolution in the branching process we do not take into account momentum splitting functions and momentum distributions. The emphasis is on event topology. We always start with $P_1(1)=1$, $S_1=0$.

In model I, for QED, the electron has a probability $(1-\alpha)$ to remain an electron and a probability α to become an electron and a noninteracting photon. The generated probabilistic branching tree is shown in Fig. 1(a).

In model II, for pure QCD, the gluon has a probability $(1-\alpha)$ to remain a gluon and a probability α to become two gluons. The corresponding tree is shown in Fig. 1(b).

In model I it is easily shown that one obtains, for the probability $P_i(n)$, with $0<\alpha<1$,

$$P_i(n)=\binom{i-1}{n-1}(1-\alpha)^{i-n}\alpha^{n-1}, \tag{3}$$

$$n=1,2,\ldots,i. \tag{4}$$

This is, essentially, the binomial distribution with

$$\langle n\rangle_i=1+(i-1)\alpha, \tag{5}$$

$$\langle n^2\rangle_i=1+(i-1)3\alpha+(i-1)(i-2)\alpha^2. \tag{6}$$

The quantity V_i, Eq. (1), asymptotically, as $i\to\infty$, goes to zero (as obtained in [3]). In fact, in general,

$$C_{q,i}\equiv\frac{\langle n^q\rangle_i}{\langle n\rangle_i^q}\xrightarrow[i\to\infty]{} C_q=1. \tag{7}$$

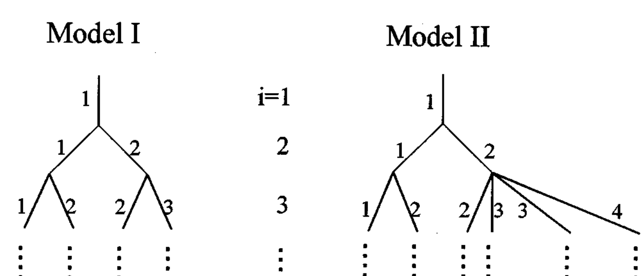

FIG. 1. Levels $i=1,2,3,\ldots$ in particle production of branching processes for models I and II. Along the tree lines the number n of produced particles is indicated. In the case of model I, QED, n means $(n-1)$ photons and 1 electron. In the case of model II, QCD, n is the number of gluons.

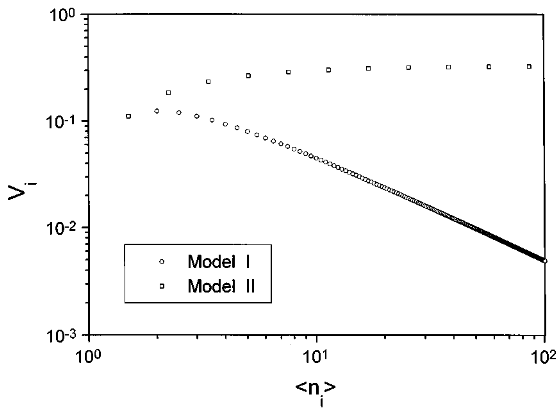

FIG. 2. Parameter V_i, as a function $\langle n\rangle_i$ for models I and II. The obtained behavior is in qualitative agreement with the results of Ref. [3]. See Fig. 1 of [3].

In other words, in model I the particle distribution, asymptotically, approaches a δ function. In the limit $\alpha \to 1$ the distribution (3) approaches first a Poisson distribution. For larger values of α the distribution becomes narrower.

In model II, the $P_i(n)$ can be obtained by iteration:

$$P_i(n)=\sum_{r=\lceil \frac{n}{2}\rceil,\dots,n} \binom{r}{n-r} \alpha^{n-r}(1-\alpha)^{2r-n}P_{i-1}(r), \quad (8)$$

where $\lceil n/2\rceil = n/2$ for n even and $\lceil n/2\rceil=(n+1)/2$ for n odd, and

$$n=1,2,\dots,2^{i-1}. \quad (9)$$

The first two moments are

$$\langle n\rangle_i=(1+\alpha)^{i-1} \quad (10)$$

$$\langle n^2\rangle_i=(1+\alpha)^{i-2}[2(1+\alpha)^{i-1}-(1-\alpha)]. \quad (11)$$

The quantity V_i, (1), asymptotically, as $i\to\infty$, approaches a finite constant, depending on α,

$$V_i \xrightarrow[i\to\infty]{} \frac{2}{1+\alpha}-1. \quad (12)$$

This result is similar to the result of the non-Abelian case of [3]. The V_i dependence on $\langle n\rangle_i$ for models I and II, with $\alpha=0.5$, is shown in Fig. 2.

In general, for model II, we have shown in [6] that

$$C_{q,i}\equiv \frac{\langle n^q\rangle_i}{\langle n\rangle_i^q} \xrightarrow[i\to\infty]{} C_q=\text{const}>1, \quad (13)$$

with

$$C_2=1+\frac{1-\alpha}{1+\alpha}, \quad C_3=1+6\frac{1-\alpha}{(1+\alpha)^2}-\frac{(1-\alpha)(2-\alpha)}{(1+\alpha)(2+\alpha)},\dots . \quad (14)$$

In the $\alpha\to 0$ limit the distribution approaches the exponential distribution.

We can see that the results of [3] are a particular case of the general result:

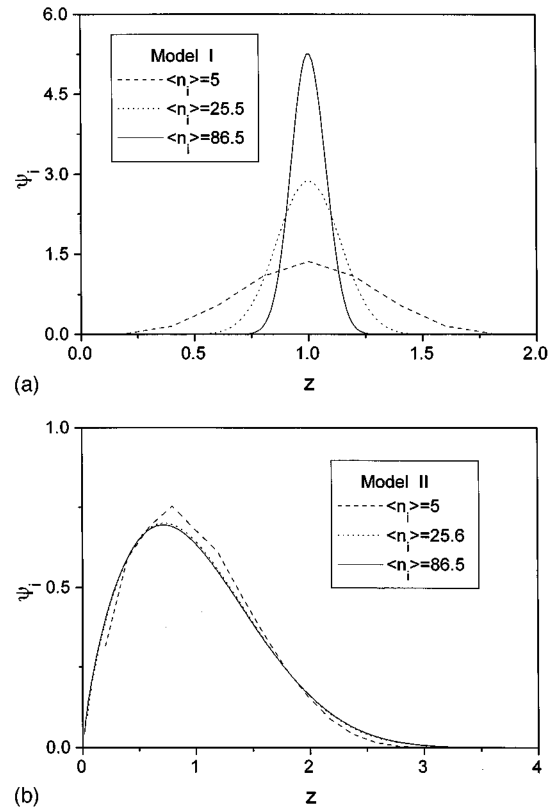

FIG. 3. The KNO function, $\Psi_i \equiv \langle n\rangle_i(n)$ as a function of $z\equiv n/\langle n\rangle$. (a) for model I. (b) for model II.

$$C_{q,i} \xrightarrow[i\to\infty]{} 1 \quad (\text{model I}) \quad (15)$$

$$C_{q,i} \xrightarrow[i\to\infty]{} \text{const}>1 \quad (\text{model II}). \quad (16)$$

We shall now reformulate (15) and (16) in terms of the information entropy.

Let us introduce the Koba-Nielsen-Olesen (KNO) function ψ_i [7]:

$$\langle n\rangle_i P_i(n)\equiv \psi_i(z), \quad (17)$$

with

$$z\equiv n/\langle n\rangle, \quad (18)$$

and take the large i, large $\langle n\rangle_i$, continuous approximation, such that

$$\int_0^\infty \psi_i(z)dz=\int_0^\infty z\psi_i(z)dz=1, \quad (19)$$

and

$$C_{q,i}=\int_0^\infty z^q\psi_i(z)dz. \quad (20)$$

From (15), (16), and (20) we see that in the case of model I (Abelian model), there is no KNO limiting function, ψ_i

indefinitely approaching a δ function (see [8] for a discussion). In the case of model II (non-Abelian model) a limiting KNO function exists:

$$\psi_i(z) \xrightarrow[i,\langle n\rangle_i\to\infty]{} \psi(z), \tag{21}$$

in agreement with the original QCD branching calculations [9]. The difference between model I and model II, concerning the KNO function $\psi_i(z)$, can be directly seen in Fig. 3.

We turn finally to the information entropy (2). In the continuous approximation one obtains

$$S_i-\ln\langle n\rangle_i=-\int \psi_i(z)\ln\psi_i(z)dz. \tag{22}$$

We propose to use the quantity

$$Q_i\equiv S_i-\ln\langle n\rangle_i \tag{23}$$

to measure chaoticity in the evolution of a branching process. If Q_i approaches a finite constant as $i\to\infty$, as in the non-Abelian case, the process is chaotic. If Q_i continuously decreases as $i\to\infty$ approaching $-\infty$, as in the Abelian case, the process is not chaotic. The situation is shown in Fig. 4 for models I and II, $\alpha=0.5$.

We would like to make some final remarks on the non-Abelian model II. We believe that model II has the correct mathematical structure of QCD as far as multiplicities are concerned. It satisfies KNO scaling and for $\alpha\simeq0.5$ the model well reproduces the QCD gluon jet multiplicity distribution.

As the range of applicability of model II varies between the δ-function distribution, for $\alpha\to1$, and the exponential distribution, for $\alpha\to0$, one could think that model II is similar to the negative binomial distribution, which has the same limits as the negative binomial parameter k varies between ∞ and 1. However this is not so. Model II does not belong to the negative binomial distribution family, in the sense that the higher-order cumulants K_q (the normalized correlation functions) are not all related to the second cumulant K_2 by the relation $K_q\equiv A_q(K_2)^{q-1}$ [10], $A_1=A_2=1$ and, in general, $A_q>0$. In model II,

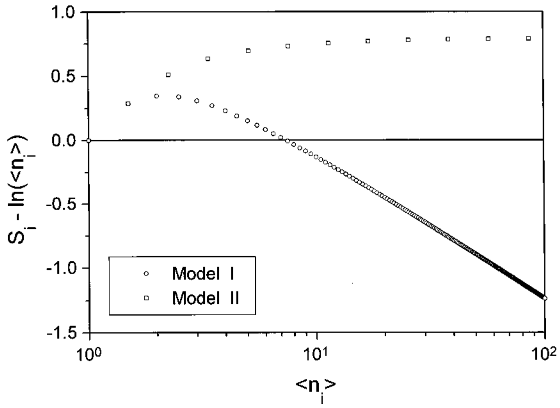

FIG. 4. The quantity $Q_i\equiv S_i-\ln\langle n\rangle_i$, as a function of $\langle n\rangle_i$ for models I and II.

$$K_2\equiv[\langle n(n-1)\rangle-\langle n\rangle^2]/\langle n\rangle^2=(C_2-1)-1/\langle n\rangle \tag{24}$$

asymptotically, as $\langle n\rangle\to\infty$, is always positive,

$$K_3\equiv\{\langle n(n-1)(n-2)\rangle-3[\langle n(n-1)\rangle-\langle n\rangle^2]\langle n\rangle -\langle n\rangle^3\}/\langle n\rangle^3 \tag{25}$$

may be either positive or negative, depending on the value of α. From (14) it is easy to see that asymptotically, $K_3<0$ for $\alpha>\sqrt{3}-1$. This kind of behavior, with the cumulants K_q changing sign, has been previously discussed, in the context of QCD, in [11].

Violations of scaling, for different initiating energies, may occur as a result of varying the parameter α. Asymptotic freedom suggests that, as the energy increases α, effectively, decreases.

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