Unitarity corrections to current algebra versus chiral perturbation calculations in kaon-pion scattering

J. Sá Borges*
Instituto de Física, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Maracanã, Rio de Janeiro, Brazil

F. R. A. Simão†
Centro Brasileiro de Pesquisas Físicas, Rua Xavier Sigaud 150, Rio de Janeiro, Brazil

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We compare chiral perturbation theory (ChPT) and the unitarization program of current algebra. In this paper, we compare the analytic structure these methods imply for low-energy, kaon-pion scattering and discuss their similarities. We also reproduce in this article a three-parameter fit of experimental kaon-pion S- and P-wave phase shifts with a current algebra quasiunitarized amplitude, published long ago, and we make a comparison with the recent results of ChPT.

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I. INTRODUCTION

The low energy structure of quantum chromodynamics (QCD) is a basic problem for meson physics. We will consider two methods that aim to help in its understanding.

The method called chiral perturbation theory (ChPT) consists in expanding the Green's functions of QCD in powers of momenta and of quark masses. As chiral symmetry implies a set of Ward identities which link the various Green's functions, it is possible to interrelate the expansion coefficients. To analyze the low-energy structure of QCD, ChPT considers the unique effective Lagrangian at lowest order, namely, the nonlinear σ model coupled with external fields [1].

The other method was invented in the early 1960s. Even ignoring the underlying theory, the chiral current algebra implies a set of Ward identities and the method consists in solving the system of Ward identities under suitable assumptions, such as saturation of axial divergences with meson poles [2].

It has been shown that tree-level ChPT calculations are equivalent to the well-known current algebra low-energy theorems. Unitarity corrections to current algebra soft-meson amplitudes allow one to go beyond threshold for meson processes and to access the resonance region for meson-meson scattering. On the other hand, loop diagrams in ChPT give quite large corrections to leading current algebra results even at threshold. As both of them follow from chiral symmetric Ward identities, it is interesting to compare the results obtained by these two methods.

Consider, for instance, pion-pion scattering. One of us compared [3] the analysis made in ChPT context [4] with the result of current algebra unitarization method proposed in Ref. [5]. The conclusion is that the amplitudes coming from these methods have the same structure in terms of s,t, and u variables. The differences are that our quasiunitarized amplitude has three parameters whereas ChPT has only two free constants and our amplitude does not have nonanalytic quark mass-dependent terms.

In this paper we study kaon-pion scattering. We will conclude in Sec. II that, apart from the error of missing t-channel η particle contribution, the current algebra quasiunitarized result, published long ago [6], keeps similarities with the kaon-pion ChPT scattering amplitude derived by Bernard *et al.* [7]. Moreover, current algebra amplitude do not have nonanalytic terms dependent on quark masses that characterizes ChPT calculation.

II. COMPARISON BETWEEN THE TWO METHODS

A. Current algebra quasiunitarized amplitude

We applied in Ref. [6] our current algebra unitarization method to kaon-pion scattering. The starting point in our derivation was an exact expression for the correlation function of four currents, with the quantum numbers of kaon and pion, in terms of three- and two-point functions.

From this expression, by using vertex and propagator estimates, we could reobtain the so-called soft meson total isospin I Weinberg amplitudes [8]: namely,

$$T^{\mathrm{ca}}_{I=3/2}=-\frac{1}{2F^2}(s-M^2), \tag{2.1a}$$

$$T^{\mathrm{ca}}_{I=1/2}=\frac{1}{4F^2}(s+2M^2-3u), \tag{2.1b}$$

where $M^2=m_K^2+m_\pi^2$, $m^2=m_K^2-m_\pi^2$, and $s+t+u=2M^2$. The remaining of the $K_q^\alpha\pi_p^\beta\to K_{-q'}^\gamma,\pi_{-p'}^\delta$ amplitude reflects the difference between soft- and hard-meson result and is the equation that follows from Eq. (2.14) of Ref. [6]: namely,

*Electronic address: SABORGES@VMESA.UERJ.BR
†Electronic address: SIMAO@CAT.CBPF.BR

$$
\begin{aligned}
\overline{T}_{\alpha\beta\gamma\delta}(s,t,u) = & -\frac{C_{K_A}^2 C_{A_1}^2}{F^4} t^c_{\alpha\beta\gamma\delta}(s,t,u) + d_{\alpha\gamma\epsilon} d_{\epsilon\beta\delta} f_K^\sigma(t)\Delta_\sigma^{-1}(t) f_\pi^\sigma(t) + f_{\alpha\gamma\epsilon} f_{\epsilon\beta\delta}(s-u)\Bigg[\left(\frac{1}{2F^2}S+1-f_K^v(t)\right)\Delta_\rho^{-1}(t) \\
& \times\left(\frac{1}{2F^2}+1-f^{\,v}_{\pi}(t)\right) - \frac{1}{4F^4}S\Bigg] + \Bigg\{ -f_{\alpha\beta\epsilon}\, f_{\epsilon'\gamma\delta}\Bigg[(q-p)_\tau\left(\frac{1}{2F^2}S+1-f_+(s)\right) \\
& + (q+p)_\tau\left(f_-(s)+\frac{D^\kappa}{\Delta^\kappa}\tilde f^{\,0}(s) - \frac{1}{2F^2}m^2\tilde D^\kappa(s)\right)\Bigg]\frac{1}{\overline{\Delta}^{\tau\tau'}_{K^*\,\epsilon\epsilon'}(s)}\Bigg[(q'-p')_{\tau'}\left(\frac{1}{2F^2}S+1-f_+(s)\right) \\
& + (q'+p')_{\tau'}\left(f_-(s)+\frac{D^\kappa}{\Delta^\kappa}\tilde f^{\,0}(s) - \frac{1}{2F^2}m^2\tilde D^\kappa(s)\right)\Bigg] \\
& + \frac{1}{4F^4} f_{\alpha\beta\epsilon} f_{\epsilon\gamma\delta}[(u-t)S+m^4D^\kappa(s)] + \tilde f^{\,0}_{\alpha\beta\epsilon}(s)\Delta^{-1}_{\kappa\;\epsilon\epsilon'}(s)\tilde f^{\,0}_{\epsilon'\gamma\delta}(s) + (s\leftrightarrow u)\Bigg\},
\end{aligned} \tag{2.2}
$$

where f^+_- are the K_{l3} form factors, $\tilde f^{\,0}$ and $\tilde f^\sigma_{K,\pi}$ are the scalar form factors of the kaon and pion, Δ_m are meson propagators, S is the Schwinger term, F is the pion decay constant, here considered equal to the kaon one, and $f_{\alpha\beta\gamma}$ ($d_{\alpha\beta\gamma}$) are the anti-symmetric (symmmetric) SU(3) structure constants.

Let us remember the main points of the unitarization program of current algebra.

The program, proposed by one of us and applied to pion-pion [5] and to kaon-pion [6] scattering, consists in estimating the behavior of form factors and propagators at low energies. In this way we have assumed that f_+ and electromagnetic form factors are, near threshold, of the same order of magnitude as current algebra amplitudes while other functions are comparably smaller at low energies.

For example, Eq. (2.15) of Ref. [6] establishes this assumption, for we write the K_{l3} form factors as

$$f_+(x)\simeq 1+f_+^{(1)}(x) \quad\text{and}\quad f_-(x)\simeq f_-^{(1)}(x)$$

for $x\simeq(m_K+m_\pi)^2$.

All functions denoted by a superscript (1) are of the order $(m_K+m_\pi)^2/X^2$, X being of the order of magnitude of the vector meson mass in a vector dominance approximation and so, at low energies, they can be considered as corrections to the soft-meson limit.

Let us explain how we were led to construct unitarized amplitudes.

Current algebra gives real amplitudes. The unitarization method must provide an imaginary part to the corrected partial wave. Thus at the first order of the calculation, by the optical theorem, one must have

$$\mathrm{Im}T^{(1)}_{\ell I}(s)=\frac{1}{16\pi}\rho(s)T^{\mathrm{ca}}_{\ell I}(s)^2,$$

where $T^{\mathrm{ca}}_{\ell I}$ is the soft-meson current algebra ℓ partial wave, with isospin I Weinberg amplitude obtained from Eqs. (2.1).

To construct unitarized amplitudes, we work with Eq. (2.2) using the implications of elastic unitarity for form factors and propagators in a peculiar way. For instance,

$$\mathrm{Im}f_+^{(1)}(x)=\frac{1}{16\pi}\rho(x)T^{\mathrm{ca}}_{1\;\frac{1}{2}}(x),$$

where

$$\rho(x)=\frac{1}{x}[x-(m_K+m_\pi)^2]^{1/2}[x-(m_K-m_\pi)^2]^{1/2}$$

and $T^{\mathrm{ca}}_{1\;1/2}$ is the current algebra isospin 1/2 P-wave $K\pi$ amplitude: namely,

$$T^{\mathrm{ca}}_{1\;\frac{1}{2}}(x)=\frac{1}{8F^2x}(x^2-2M^2x+m^4).$$

Considering the known imaginary part of each function entering into the amplitude, the method consists in obtaining their real parts by the dispersion relation technique. To converge, dispersion integrals need subtraction that we have fixed to zero. The final expression for the $I=3/2$ amplitude, presented in the appendix of Ref. [6], is

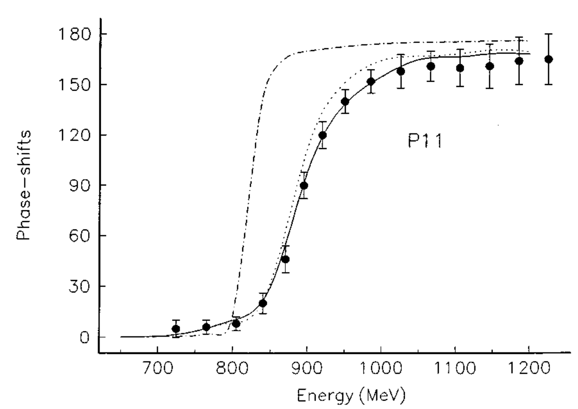

FIG. 1. Isospin 1/2 P-wave phase shifts. Solid line is our result, dot-dash line is our result without any contribution to t-channel, and short-dash line corresponds to the inclusion in our amplitude of t-channel exchange of Ref. [7]. The experimental points are from Ref. [12]. The parameters used in order to fit the $K^*(892)$ resonance are: $\xi_1=-0.0531$, $\xi_2=0.0837$, and $\xi_3=0.0188$.

$$T_{3/2}(s,t,u)=\frac{1}{2F^2}(M^2-s)+\frac{1}{4F^4}(s-M^2)\left[(s-M^2)G(s)+\frac{s}{32\pi^2}\frac{M^4}{m^4}\right]+\frac{1}{F^4}\xi_1(t-2m_K^2)(t-2m_\pi^2)$$
$$+\frac{1}{F^4}(\xi_2-\xi_3)(s-M^2)^2+\frac{1}{F^4}(\xi_2+\xi_3)(u-M^2)^2$$
$$+\frac{1}{4F^4}(u-s)\left[(2m_K^2-t)g_K(t)-\frac{t}{96\pi^2}\right]+\frac{1}{12F^4}(u-M^2)\left[(u-M^2)G(u)+\frac{u}{32\pi^2}\frac{M^4}{m^4}\right]$$
$$+\frac{1}{96\pi^2}\left(5u-2M^2-3\frac{m^4}{u}\right)\left[\left(5u-2M^2-3\frac{m^4}{u}\right)G(u)+\frac{u}{32\pi^2}\left(\frac{7}{2}\frac{M^4}{m^4}-1\right)+\frac{3}{32\pi^2}M^2\right]$$
$$+\frac{1}{32F^4}\left(t-s+\frac{m^4}{u}\right)\left[\left(u-2M^2+\frac{m^4}{u}\right)G(u)+\frac{u}{32\pi^2}\left(\frac{3}{2}\frac{M^4}{m^4}+\frac{1}{3}\right)-\frac{M^2}{32\pi^2}\right]. \tag{2.3}$$

In this expression ξ_1, ξ_2, and ξ_3 are "seagull"-free parameters,

$$16\pi^2G(x)=-\rho(x)\ln\frac{x-M^2+x\rho(x)}{2m_Km_\pi}+\left(\frac{M^2}{m^2}-\frac{m^2}{x}\right)\ln\frac{m_K}{m_\pi}+1+i\pi\rho(x),$$

and $g_K(t)$ is the equal mass limit of $G(t)$.

B. Chiral perturbation theory

Meson-meson transition amplitudes to second-order in the momenta and quark masses can be evaluated by expanding nonlinear σ-model Lagrangian L_σ in powers of the fields. The tree diagrams derived in this way give rise to the current algebra predictions up to order $O(p^2,m_m^2)$, with p denoting an external momentum and m_m the meson masses.

This approximation coincides with the result Weinberg obtained from current algebra, expressed in Eqs. (2.1).

To go further, as required by unitarity, $O(p^4,p^2m_m^2,m_m^4)$-corrected amplitudes are to be found. They came from tadpole graphs and loop diagrams with vertices from L_σ as well as higher-order derivative terms.

The arbitrary coupling constants coming from higher-order derivative terms allow one to absorb all divergences of one-loop diagrams. This is a very important point, first conjectured by Weinberg [9].

The application of ChPT to kaon-pion scattering performed in Ref. [7] follows exactly these lines. The corrections to current algebra come from loop diagrams, and tadpole and higher-order coupling tree graphs. The six renormalized coupling constants, denoted by L^r, depend on a renormalization scale μ and tadpole contributes introducing also scale dependent parameters.

The four-meson T matrix calculated from the effective action can be written in terms of physical masses and of physical decay constants.

The isospin 3/2 $K\pi$ amplitude obtained in Ref. [7] can be compared with expression (2.3) by using the relations given in Ref. [1]: namely,

$$L_{a,b}(x)=\frac{(m_a^2-m_b^2)^2}{4x}\overline{J}_{a,b}(x),$$

$$K_{a,b}(x)=\frac{(m_a^2-m_b^2)}{2x}\overline{J}_{a,b}(x),$$

$$J^r_{a,b}(x)=\overline{J}_{a,b}(x)-2k_{a,b},$$

$$k_{a,b}=F^2\frac{\mu_a-\mu_b}{m_a^2-m_b^2},$$

$$M^r_{a,b}(x)=\frac{1}{12x}\left(x-2(m_a^2+m_b^2)+4\frac{(m_a^2-m_b^2)^2}{x}\right)\overline{J}_{a,b}(x)-\frac{1}{48\pi^2}\frac{m_a^2m_b^2}{(m_a^2-m_b^2)x}\ln\frac{m_b^2}{m_a^2}-\frac{1}{96\pi^2}\left(\frac{(m_a^2+m_b^2)}{x}-\frac{1}{3}\right)-\frac{k_{a,b}}{6}.$$

Using $M^2=m_\pi^2+m_K^2$, $m^2=m_\pi^2-m_K^2$, $\Sigma^2=m_K^2+m_\eta^2$, and $\Delta^2=m_K^2-m_\eta^2$, the result is

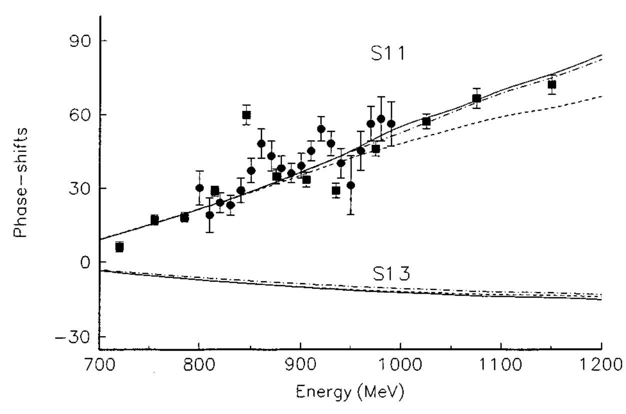

FIG. 2. Isospins 1/2 and 3/2 S-wave phase shifts with the same conventions as in Fig. 1 for line drawing. The experimental data are indicated by: boxes (Ref. [13]), circles (Ref. [14]).

$$
\begin{aligned}
T_{3/2}(s,t,u)=&\frac{1}{2F^2}(M^2-s)+\frac{1}{4F^4}(s-M^2)^2\overline{J}_{\pi K}(s)+\frac{1}{24F^4}[(u-s)(t-4m_\pi^2)+3t(2t-m_\pi^2)]\overline{J}_{\pi\pi}(t)\\
&+\frac{1}{48F^4}[(u-s)(t-4m_K^2)+9t^2]\overline{J}_{KK}(t)+\frac{m_\pi^2}{8F^4}\left(t-\frac{8}{9}m_K^2\right)\overline{J}_{\eta\eta}(t)+\frac{1}{32F^4}\left[\left(t-s+\frac{m^4}{u}\right)\left(u-2M^2+\frac{m^4}{u}\right)\right.\\
&\left.-\left(10u-4M^2-3\frac{m^4}{u}\right)\frac{m^4}{u}+11u^2-12M^2u+4M^4\right]\overline{J}_{\pi K}(u)+\frac{2}{F^4}(4L_1^r+L_3^r)(t-2m_K^2)(t-2m_\pi^2)\\
&+\frac{1}{F^4}4L_2^r(s-M^2)^2+\frac{2}{F^4}(2L_2^r+L_3^r)(u-M^2)^2+\frac{1}{32F^4}\left[\left(t-s+\frac{m^4}{u}\right)\left(u-2\Sigma^2+\frac{\Delta^4}{u}\right)\right.\\
&\left.+3\left(u^2-m^2\Delta^2\left(2-\frac{m^2\Delta^2}{u^2}\right)\right)-4M^2\left(u-\frac{m^2\Delta^2}{u}-\frac{1}{3}M^2\right)\right]\overline{J}_{K\eta}(u)+\frac{1}{4F^4}\frac{1}{32\pi^2}\\
&\times\left[\frac{1}{6}tu+\frac{1}{6}st-\frac{1}{3}su+\frac{1}{3}m^4-\frac{7}{2}t^2+\frac{8}{9}m_\pi^2m_K^2-\left(t-s+\frac{m^4}{u}\right)\left(\frac{m_K^2m_\eta^2}{\Delta^2}\ln\frac{m_\eta^2}{m_K^2}+\frac{m_\pi^2m_K^2}{m^2}\ln\frac{m_K^2}{m_\pi^2}+\frac{1}{2}(M^2+\Sigma^2)\right)\right]\\
&+\frac{1}{4F^2}\left[\frac{(\mu_K-\mu_\pi)}{4m^2}(tu-su+m^4+8(s-M^2)^2+11u^2-12uM^2+4M^4)\right.\\
&\left.+\frac{(\mu_\eta-\mu_K)}{4\Delta^2}\left(tu-su+m^4+3\left(u-\frac{2}{3}M^2\right)^2\right)\right]\\
&+\frac{1}{16F^2}[\mu_\pi(10s-10M^2+3m^2)-\mu_K(4s-4M^2+2m^2)-\mu_\eta(6s-6M^2+m^2)]\\
&+\frac{1}{4F^2}\left[\frac{\mu_\pi}{m_\pi^2}\left(\frac{1}{3}t(s-u)-t(2t-m_\pi^2)\right)+\frac{\mu_K}{m_K^2}\left(\frac{1}{6}t(s-u)-\frac{3}{2}t^2\right)+\frac{\mu_\eta}{m_\eta^2}m_\pi^2\left(\frac{8}{9}m_K^2-t\right)\right]\\
&+\frac{1}{F^4}[8L_4^rM^2t-2L_5^r(s-m^2)(M^2+m^2)+4(2L_6^r+L_8^r-2L_4^r)(M^4-m^4)].
\end{aligned}
\tag{2.4}
$$

The comparison of the last expression with Eq. (2.3) leads us to make the following observations. (i) We can identify $\overline{J}(x)$ with $G(x)$. (ii) The coefficients of $\overline{J}_{\pi K}(s)$ and $G(s)$, in the expressions (2.4) and (2.3), are the same. This gives the expected imaginary part for the amplitude in the physical region. (iii) The coefficients of $\overline{J}_{\pi K}(u)$ and $G(u)$ in these expressions are the same. This is a consequence of the correct crossing properties of amplitudes derived from Ward identities. (iv) We can relate the low-energy parameters of the two amplitudes: $\xi_1=2(4L_1^r+L_3^r)$, $\xi_2=4L_2^r+L_3^r$ and $\xi_3=L_3^r$. (v) Expression (2.3) does not have the terms equivalent to loop diagrams of $\eta\eta$ and $K\eta$ intermediate states. This is an error in the unitarization program to be cured. Nevertheless, as checked in Ref. [7], these terms introduce small corrections of order $\simeq$ 1%–3%. (vi) The number of free parameters of ChPT exceeds that of quasiunitarized amplitude in (iii). In fact, this comes because, as stated before, we have chosen some subtraction constants in the dispersion relations to be zero. (vii) Expression (2.4) corrects a misprinted factor 3/2 instead of 2/3 in the line before the last line of expression (3.16) of Ref. [7]. (viii) One can verify that the scale dependence in $1/16\pi^2\ln\mu$ coming from the renormalized ChPT parameters L_1^r, L_2^r, L_4^r, L_5^r, L_6^r, and L_8^r (which are, respectively, 3/32, 3/16, 1/8, 3/8, 11/144, and 5/48) [1] cancels with that coming from μ_π, μ_K, and μ_η, by using the Gell-Mann–Okubo relation $3m_\eta^2=4m_K^2-m_\pi^2$. (ix) In our final expression, the t-channel exchanges are somewhat incorrect because we did not include the pion electromagnetic form factors that are explicit in formula (2.2). In fact, it is easy to calculate that t-channel contribution for the 3/2 amplitude would be

$$\frac{1}{4F^4}(u-s)\left[(t-u)g_\pi(t)-\frac{t}{96\pi^2}\right],$$

to be added to the expression already included in formula (2.3): namely,

$$\frac{1}{4F^4}(u-s)\left[(2m_K^2-t)g_K(t)-\frac{t}{96\pi^2}\right].$$

This error, in the original work [6] also implies violating unitarity relation in crossed channel. However, we claim that this error has minor consequence on experimental data fitting. For this, we present in the Figs. 1 and 2 the results including t-channel exchanges of Ref. [7] and the results corresponding to no t-channel exchanges at all.

Finally, we would like to comment about phase-shift definition.

As elastic unitarity is not satisfied in either of the amplitudes, the definition of partial-wave phase shifts is arbitrary.

Nevertheless, one can make suitable definitions. We have adopted the definition $\tan\delta_{\ell I}=\mathrm{Im}T_{\ell I}/\mathrm{Re}T_{\ell I}$, knowing that $\mathrm{Im}T_{\ell I}=(\rho/16\pi)|T^{\mathrm{ca}}_{\ell I}|^2$. The authors of Ref. [7] preferred adopting the definition $\delta_{\ell I}=(\rho/16\pi)\mathrm{Re}T_{\ell I}$, which is valid for small δ.

Another method to implement elastic unitarity uses Padé approximation. By this way, Dobado and Peláez [10] have recently used ChPT amplitude (2.4) as the starting point to obtain a good fit of S and P $K-\pi$ phase shifts.

Our program, restricted to its first order, can be considered an alternative for their calculation. In effect, two-loop calculation in ChPT context is a very difficult task; however, one of us [11] gave the tools to go to the next-order approximation within the current algebra unitarization program.

III. CONCLUSIONS

In this paper we have extended to kaon-pion scattering the comparison between chiral perturbation calculations and the unitarization program of current algebra.

Exactly as in the previous analysis, of pion-pion scattering, we would like to emphasize that, despite the error of not including all t-channel contributions, the basic structure of the amplitudes presented in (2.3) and in (2.4), concerning the dependence on the variables s, t, and u, is the same. On the other hand, only ChPT contains meson mass-dependent, nonanalytic terms. We remark also that the methods lead to amplitudes with different numbers of free parameters, by a particular choice made in the Ref. [6].

Concerning phase-shifts definition, we considered that, as current algebra gives real amplitudes, any method intending to go beyond threshold for meson processes must explore the imaginary part it implies.

A good feature of the current algebra unitarization program is the possibility of constructing next-order approximation in a very simple way, as explained in Ref. [11].

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