

## Three-dimensional BTZ black hole as a cylindrical system in four-dimensional general relativity

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(Received 2 October 1995)

It is shown how to transform the three-dimensional BTZ black hole into a four-dimensional cylindrical black hole (i.e., black string) in general relativity. This process is identical to the transformation of a point particle in three dimensions into a straight cosmic string in four dimensions.

PACS number(s): 04.40.Nr, 04.20.Cv

### I. INTRODUCTION

Black holes were predicted within four-dimensional general relativity as objects emerging from complete gravitational collapse of massive objects, such as stars [1]. They have also appeared as exact solutions of several gravity theories in two, three, four, and higher dimensions. String theory has provided a great variety of black hole solutions as well as extended black objects such as black strings and black membranes [2]. The study of black holes in dimensions lower than four has proved fruitful for a better understanding of some physical features in a black hole geometry.

Recently, it has been shown [3] that three-dimensional (3D) general relativity with a negative cosmological constant admits a black hole solution with constant curvature, the Bañados-Teitelboim-Zanelli (BTZ) black hole. This black hole has mass and angular momentum. It has many features similar to the Kerr black holes in four-dimensional (4D) general relativity. It is interesting, as well as important, to formulate other theories for the same solution. For instance, the BTZ black hole is also a solution to string theory in three dimensions [4]. Our aim in this Brief Report is to show that the BTZ black hole is also a black hole solution of cylindrical general relativity.

### II. FROM 3D TO 4D GENERAL RELATIVITY

In order to set the nomenclature we write the action in  $D$  dimensions as

$$S^{(D)} = S_g^{(D)} + S_{\text{matter}}^{(D)}, \quad (1)$$

where  $D = 3, 4$ .  $S_g^{(D)}$  is the gravitational Einstein-Hilbert action ( $G = c = 1$ ),

$$S_g^{(D)} = \frac{1}{16\pi} \int d^D x \sqrt{-g^{(D)}} (R^{(D)} - 2\Lambda), \quad (2)$$

where  $g^{(D)}$  is the determinant of the metric,  $R^{(D)}$  is the

Ricci scalar, and  $\Lambda$  is the cosmological constant. The matter action can be written as

$$S_{\text{matter}}^{(D)} = \int d^D x \sqrt{-g^{(D)}} \mathcal{L}_{\text{matter}}^{(D)}, \quad (3)$$

where  $\mathcal{L}_{\text{matter}}^{(D)}$  is the Lagrangian for the matter. Variation of (1) with respect to  $g^{(D)}_{ab}$  yields the equations of motion

$$G^{(D)}_{ab} + \Lambda g^{(D)}_{ab} = 8\pi T^{(D)}_{ab}, \quad (4)$$

where  $G^{(D)}_{ab}$  is the Einstein tensor and  $T^{(D)}_{ab}$  is the energy-momentum tensor, defined by  $T^{(D)ab} = \frac{2}{\sqrt{-g^{(D)}}} \frac{\delta(\sqrt{-g^{(D)}} \mathcal{L}_{\text{matter}}^{(D)})}{\delta g^{(D)}_{ab}}$ .

Now, in 3D we write the metric in the form

$$ds^{(3)2} = g^{(3)}_{ab} dx^a dx^b. \quad (5)$$

In addition, in 3D the Riemann tensor can be written as

$$R^{(3)ab}_{cd} = \epsilon^{abe} \epsilon_{cdf} G^{(3)f}_e. \quad (6)$$

Thus, from Eq. (4), for a source free region ( $T^{(3)}_{ab} = 0$ ), one has a space of constant curvature. If one further sets  $\Lambda = 0$ , one has flat spacetime.

Now, we want to relate general relativity in 3D with general relativity in 4D, in such a way that a solution in 3D is also a solution in 4D. This can be achieved by setting the 4D metric as

$$ds^{(3)2} = g^{(3)}_{ab} dx^a dx^b + dz^2, \quad (7)$$

where  $z$  is a Killing direction in the 4D spacetime. Then, the determinants of the metric are the same  $g^{(4)} = g^{(3)}$ , the Ricci scalars are also the same,  $R^{(4)} = R^{(3)}$ , and through dimensional reduction one obtains 3D from 4D general relativity. Thus, having found a 3D solution one can now find the corresponding 4D solution by choosing an appropriate energy-momentum tensor in 4D. In the next section we will do this for the well-known example

of 3D point particles and 4D cosmic strings. Afterwards, we will find the 4D counterpart of the 3D BTZ black hole.

### III. THE EXAMPLE OF 3D POINT PARTICLES AND 4D COSMIC STRINGS

For a point particle with mass  $m$  and spin  $s$  in 3D [with Cartesian coordinates  $(t, x, y)$ ], the energy-momentum tensor can be written as

$$T^{(3)0}_0 = m\delta(x)\delta(y), \quad (8)$$

$$T^{(3)i}_0 = s\epsilon^{ij}\partial_j\delta(x)\delta(y), \quad (9)$$

$$T^{(3)i}_j = 0, \quad (10)$$

where  $i, j = x, y$ . Then, by integrating Einstein's equations (with  $\Lambda = 0$ ), one finds the metric [5]

$$ds^{(3)2} = -dt^2 - 8sdt d\varphi + dr^2 + r^2(1 - 4m)^2 d\varphi^2, \\ -\infty < t < \infty, \quad 0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi, \quad (11)$$

where  $(r, \varphi)$  are the polar coordinates associated to  $(x, y)$ . There are two Killing vectors,  $\frac{\partial}{\partial t}$  and  $\frac{\partial}{\partial \varphi}$ , and the full symmetry group of this spacetime is  $R \times \text{SO}(2)$  [6]. By the coordinate transformation  $\bar{t} = t - 4s\varphi$  and  $\bar{\varphi} = (1 - 4m)\varphi$  one obtains a flat spacetime,

$$ds^{(3)2} = -d\bar{t}^2 + dr^2 + r^2 d\bar{\varphi}^2, \quad (12)$$

but now with the obligatory identifications  $(\bar{t}, r, \bar{\varphi}) = (\bar{t} + 8\pi s, r, \bar{\varphi} + 2\pi(1 - 4m))$ , which mean space is conical and time is helical.

Now, from Eqs. (5) and (7), we can put metric (11) in the 4D form

$$ds^{(4)2} = -dt^2 - 8sdt d\varphi + dr^2 + r^2(1 - 4m)^2 d\varphi^2 + dz^2, \quad (13)$$

with  $-\infty < z < \infty$ . In 4D, Einstein's equations for metric (13) can be split into

$$G^{(4)}_{ab} = 8\pi T^{(4)}_{ab}, \quad (14)$$

$$G^{(4)}_{zz} = 8\pi T^{(4)}_{zz}, \quad (15)$$

and the cross equation  $G^{(4)}_{az}$  is identically satisfied. Here,  $a, b = 0, 1, 2$ . Then, from Eq. (4), we can deduce

$$G^{(4)} = -R^{(4)} = -R^{(3)} = 8\pi T^{(4)}, \quad (16)$$

where  $G^{(4)}$  is the trace of the Einstein tensor in 4D, etc. From (4) and (8)–(10), we also deduce

$$R^{(3)} = -16\pi T^{(3)} = -16\pi T^{(3)0}_0. \quad (17)$$

But  $T^{(4)} = T^{(3)} + T^{(4)z}_z = T^{(3)0}_0 + T^{(4)z}_z$ . Then, from (16) and (17) and the fact that  $T^{(4)0}_0 = T^{(3)0}_0$ , we obtain

$$T^{(4)z}_z = T^{(4)0}_0. \quad (18)$$

Therefore, metric (13) is a solution of 4D general relativity if the energy-momentum tensor is given by

$$T^{(4)0}_0 = m\delta(x)\delta(y), \quad (19)$$

$$T^{(4)i}_0 = s\epsilon^{ij}\partial_j\delta(x)\delta(y), \quad (20)$$

$$T^{(4)z}_z = m\delta(x)\delta(y), \quad (21)$$

with  $i, j = x, y$  and the other components are zero. This is the energy-momentum tensor of a straight cosmic string [7]. But now  $m$  and  $s$  are the mass per unit length and the angular momentum per unit length of the string, respectively. There are three Killing vectors  $\frac{\partial}{\partial t}$ ,  $\frac{\partial}{\partial \varphi}$  and  $\frac{\partial}{\partial z}$  and the full symmetry group of this spacetime is  $R^2 \times \text{SO}(2)$ .

### IV. THE 3D BTZ BLACK HOLE AND THE BLACK STRING IN 4D

The 3D BTZ black hole appears as a solution of Einstein's gravity with a negative cosmological term,  $\alpha^2 \equiv -\Lambda > 0$ , and  $T^{(3)}_{ab} = 0$ . The metric is given by [3, 8],

$$ds^{(3)2} = -(\alpha^2 r^2 - 8M)dt^2 - 8Jdt d\varphi \\ + \frac{dr^2}{\alpha^2 r^2 - 8M + \frac{16J^2}{r^2}} + r^2 d\varphi^2, \quad (22)$$

where  $M$  and  $J$  are the mass and angular momentum of the black hole. (Note that the normalizations for  $M$  and  $J$  are different from those in [3]. We are using  $G = 1$ , instead of  $G = \frac{1}{8}$ .) This solution has constant curvature and appears through identification of points in the anti-de Sitter spacetime. The asymptotic symmetry group is the conformal group in two dimensions which has the anti-de Sitter group  $\text{SO}(2,2)$  as a subgroup [6, 8].

We note that the cosmological term in Einstein's equations [i.e.,  $\Lambda g^{(3)}_{ab}$  in Eq. (4)] can be regarded, if one wishes, as an energy-momentum tensor for a perfect fluid [9]. Indeed, if we write

$$T^{(3)}_{ab} = (\rho + p)u_a u_b + p g_{ab}, \quad (23)$$

where  $\rho$  is the energy density and  $p$  is the pressure, and set

$$8\pi\rho = -8\pi p = -\alpha^2, \quad (24)$$

we obtain the cosmological term. In this description, the cosmological term does not appear explicitly in Einstein's equations (4), which are then given by,  $G^{(3)}_{ab} = 8\pi T^{(3)}_{ab}$ , with  $T^{(3)}_{ab}$  given by Eqs. (23) and (24). This fluid obeys the strong but not the weak energy condition since its energy density is negative [10].

Now we want to translate the above black hole solution (22) into a solution of 4D general relativity. Then, using (7), we have

$$ds^{(4)2} = -(\alpha^2 r^2 - 8M)dt^2 - 8Jdt d\varphi \\ + \frac{dr^2}{\alpha^2 r^2 - 8M + \frac{16J^2}{r^2}} + r^2 d\varphi^2 + dz^2. \quad (25)$$

Using the formalism of Brown and York [11], we find that  $M$  and  $J$  are now the mass per unit length and the angular momentum per unit length, respectively, of the black string solution, given in (25) (for an analysis of a more general case see [12]).

To know what is the stress-energy tensor for this solution, we take the trace of (4) to find

$$-R^{(4)} + 4\Lambda = 8\pi T^{(4)}. \quad (26)$$

But

$$R^{(4)} = R^{(3)} = 6\Lambda = -6\alpha^2. \quad (27)$$

Then from (26) and (27), we obtain

$$8\pi T^{(4)} = -2\Lambda = 2\alpha^2. \quad (28)$$

Since here  $T^{(3)}_{ab} = 0$ , we are left with

$$8\pi T^{(4)z}_z = 2\alpha^2, \quad (29)$$

and the other components are equal to zero. One can see this fluid as one having pressure in the  $z$  direction alone, and no energy density. Such a fluid also obeys the strong but not the weak energy condition.

However, in view of the fact that the cosmological term can be regarded as a perfect fluid, we can also switch the 4D description into a fluid with the components

$$8\pi T^{(4)0}_0 = -8\pi\rho = \alpha^2, \quad (30)$$

$$8\pi T^{(4)r}_r = 8\pi p = \alpha^2, \quad (31)$$

$$8\pi T^{(4)\varphi}_\varphi = 8\pi p = \alpha^2, \quad (32)$$

$$8\pi T^{(4)z}_z = 8\pi p_{zz} = 3\alpha^2. \quad (33)$$

As in the 3D case, in this description, the cosmological term in Einstein's equations (4) does not appear explicitly. Metric (25) is thus a solution of Einstein's field equations,  $G^{(4)}_{ab} = 8\pi T^{(4)}_{ab}$ , with  $T^{(4)}_{ab}$  given in equations (30)–(33). Again, this fluid obeys the strong but not the weak energy condition. It is a fluid with anisotropic pressures.

Thus, the BTZ black hole can be translated into a black string (or cylindrical black hole) in 4D general relativity. All the features appearing in 3D, such as 3D collapse [13] and 3D stars [14] can be translated into 4D cylindrical general relativity. The asymptotic symmetry group is  $R$  times the conformal group in two dimensions of which  $R \times \text{SO}(2, 2)$  is a subgroup.

## V. CONCLUSIONS

It is remarkable that one can connect solutions in theories with different dimensions. Here, we have connected 3D and 4D general relativities by an appropriate choice of metric and energy-momentum tensor in 4D. Such a type of connection can also be found from other 3D theories to 4D general relativities [15, 12], as well as from 2D theories to 4D general relativity [16, 17].

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- [1] J.R. Oppenheimer and H. Snyder, *Phys. Rev. D* **56**, 455 (1939); J. A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964), p. 317; R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965).
- [2] G. T. Horowitz, in *String Theory and Quantum Gravity 1992*, edited by J. Harvey *et al.* (World Scientific, Singapore, 1993).
- [3] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [4] G. T. Horowitz and D. L. Welch, *Phys. Rev. Lett.* **71**, 328 (1993); N. Kaloper, *Phys. Rev. D* **48**, 2598 (1993). The BTZ black hole is also a solution of 4D effective string theory which solution can be interpreted as a rotating black cosmic string; see N. Kaloper, *ibid.* **48**, 4658 (1993).
- [5] R. Jackiw, in *Proceedings of the 7th Latin American Symposium on Relativity and Gravitation, SILARG VII*, edited by J. C. Olivo, E. Nahmad-Achar, M. Rosenbaum, M. P. Ryan, Jr., L. F. Urrutia, and F. Zertuche (World Scientific, Singapore, 1991), p. 74.
- [6] J. D. Brown, *Lower Dimensional Gravity* (World Scientific, Singapore, 1988).
- [7] A. Vilenkin, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), p. 499.
- [8] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993).
- [9] W. H. McCrea, *Proc. R. Soc. London* **A206**, 562 (1951).
- [10] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
- [11] J.D. Brown and J.W. York, Jr., *Phys. Rev. D* **47**, 1407 (1993).
- [12] J. P. S. Lemos and V. T. Zanchin, "Rotating Charged Black String and Three Dimensional Black Holes," Observatorio Nacional (Rio de Janeiro) Report No. hep-th/9511188 (unpublished).
- [13] R. B. Mann and S. F. Ross, *Phys. Rev. D* **47**, 3319 (1993).
- [14] N. Cruz and J. Zanelli, *Class. Quantum Grav.* **12**, 975 (1995).
- [15] J. P. S. Lemos, *Phys. Lett. B* **352**, 46 (1995); "Cylindrical Black Hole in General Relativity," Report No. gr-qc/9404041 (unpublished); P. M. Sá, A. Kleber, and J. P. S. Lemos, "Black Holes in Three Dimensional Dilation Gravity Theories," Report No. DF/IST-2.95, UATP-95/01, hep-th/9503089 [*Class. Quantum Grav.* (to be published)].
- [16] C. Callan, D. Friedan, E. Martinec, and M. Perry, *Nucl. Phys.* **B262**, 593 (1985).
- [17] J. P. S. Lemos, *Class. Quantum Grav.* **12**, 1081 (1995); J. P. S. Lemos and P. M. Sá, *Phys. Rev. D* **49**, 2897 (1994); *Mod. Phys. Lett. A* **9**, 771 (1994); *Class. Quantum Grav.* **11**, L11 (1994).