String tension and Chern-Simons fluctuations in the vortex vacuum of $d=3$ gauge theory

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Based on a model of the $d=3$ SU(2) pure gauge theory vacuum as a monopole-vortex condensate, we give a quantitative (if model-dependent) estimate of the relation between the string tension and a gauge-invariant measure of the Chern-Simons susceptibility, due to vortex linkages, in the absence of a Chern-Simons term in the action. We also give relations among these quantities and the vacuum energy and gauge-boson mass. Both the susceptibility and the string tension come from the same physics: the topology of linking, twisting, and writhing of closed vortex strings. The closed vortex string is described via a complex scalar field theory whose action has a precisely specified functional form, inferred from previous work giving the exact form of a gauge-theory effective potential at low momentum. Applications to high-T phenomena, including $B+L$ anomalous violation, are mentioned.

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I. INTRODUCTION

In this paper we consider a $d=3$ parity-conserving [no Chern-Simons (CS) term] pure-gauge (no fermions or scalars) non-Abelian gauge theory, or just $d=3$ gauge theory for short. Our specific considerations will be for the gauge group SU(2), but the generalization to other gauge groups is straightforward.

We supply here quantitative details of a scenario proposed earlier [1] for $d=3$ gauge theory, which is strongly coupled in the infrared (IR) because the gauge bosons are perturbatively massless. In this scenario, the gauge-theory vacuum is dominated by an entropy-driven condensate of closed strings[1] of thickness M^{-1}, where $M\sim g_3^2$ is the perturbative gauge-boson mass, proportional to the $d=3$ coupling constant g_3^2 (for high$=T$, $d=4$ gauge theory, $g_3^2=g^2T$, where g^2 is the $d=4$ coupling at temperature T). Each string has a magnetic flux such that a large Wilson loop in the fundamental representation

$$W(\Gamma)=P\exp(\oint_\Gamma d\vec{x}\cdot\vec{A}) \tag{1}$$

takes on values in the center of the gauge group if the Wilson-loop Γ is topologically linked with the closed string, and is unity otherwise. (A large Wilson loop is one whose length scales and distance to the string are $\gg M^{-1}$.) For SU(2), the only nontrivial value of W is -1. The functional average $\langle W(\Gamma)\rangle$ over all $\vec{A}$ then gives an area law (described in terms of a string tension), just as for $d=4$ gauge theory [2], where the condensate is formed from closed two-surfaces.[2] Note that in (1) we have absorbed a factor of i [see Eq. (4) below] as well as the coupling constant g_3 into the definition of the gauge potential. So $\vec{A}$ has dimensions of mass, while the canonically defined vector potential, which is found by dividing $\vec{A}$ by g_3, has units of the square root of mass.

There is another quantity of interest which also depends on linking and related topological quantities: the vortex part of the Chern-Simons susceptibility, which we label $\chi_{\rm CS}$. We will see that this can be defined gauge invariantly and without uninteresting perturbative contributions. Begin by defining [5] the Chern-Simons (CS) number as usual

$$W_{\rm CS}=\int d^3x\Omega(\vec{x}), \tag{2}$$

$$\Omega=\frac{1}{4\pi^2}\epsilon_{ijk}{\rm Tr}\left(\frac{1}{2}A_i\partial_jA_k-\frac{1}{3}A_iA_jA_k\right), \tag{3}$$

where the SU(2) matrices A_i are defined as

$$A_i=\frac{1}{2i}A_i^a\sigma_a. \tag{4}$$

Now, $W_{\rm CS}$ is not gauge invariant; under a gauge transformation with winding number N,

$$W_{\rm CS}\to W_{\rm CS}+N. \tag{5}$$

But it is possible to define a gauge-invariant susceptibility as that part of the zero-momentum correlator,

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[1]The strings we discuss are not the same as electroweak, or Z, strings [3]. Z strings occur for $T<T_c$ and owe their existence to a Higgs vacuum expectation valve (VEV); they are entropy disfavored because they occur in the weak-coupling regime of electroweak theory.

[2]This is the mechanism of confinement in non-Abelian lattice gauge theory; see, e.g., Ref. [4] and references therein. In SU(2) lattice gauge theory, one introduces monopoles as the site of a junction between a spread-out (continuum) vortex tube and an infinitesimally thin Z_2 string; the latter is suppressed in the continuum limit, leaving closed vortices without monopoles as the means of confinement.

$$\int d^3x\langle\Omega(\vec{x})\Omega(0)\rangle, \tag{6}$$

which is independent of the space volume $V\equiv\int d^3x$ as $V\to\infty$ and thus can have no contributions from disconnected parts with $\langle W_{\rm CS}\rangle=0$, which is a kind of topological neutrality condition for the string condensate. The last step in defining $\chi_{\rm CS}$ is simply to restrict the gauge potentials A_i of (3) to those describing closed strings. In view of the essentially Abelian nature of these strings [their holonomy group (1) is Abelian], only the Abelian (first) term of (3) contributes, and when a string configuration is used to evaluate (6), one finds [1,6] that it is a sum of Gauss linking numbers for mutually linked different strings and twist plus writhe, or self-linking number [7], for a single string. This is strictly true if the string centers are large in the sense previously described for the Wilson loop; at short distances, there are corrections coming from the gauge-boson mass M which automatically regulate the usual divergences associated with self-intersections. Finally, we can define

$$\chi_{\rm CS}=\int d^3x\langle\Omega(\vec{x})\Omega(0)\rangle_{\rm conn}, \tag{7}$$

where only vortex configurations are to be included. Note that $\chi_{\rm CS}$ scales like M^3, or $(g^2T)^3$ in a high-T gauge theory. Similarly, the string tension K_F scales like M^2, so one can write $\chi_{\rm CS}=\text{const}\times(K_F)^{3/2}$. Our purpose here is to calculate quantitatively this constant and the ratios $\chi_{\rm CS}/M^3$ and K_F/M^2, within the string condensate model, as well as similar dimensionless relations involving the vacuum energy and the gauge-boson mass. If we could calculate any one of these dimensionful quantities in appropriate units of g^2T, all the others would be determined.

The relation (7) can be written in an interesting way, which has its counterpart in $d=4$ gauge theory. We define the operator $\Theta(\vec{x})$ as the square of the usual (without the coupling constant factor) field strength:

$$\Theta(\vec{x})=\frac{1}{4g_3^2}\Sigma(G_{ij}^a)^2(\vec{x}); \tag{8}$$

note that Θ is proportional to the trace of the stress-energy tensor. We can write (7) as

$$\chi_{\rm CS}=\int d^3x\langle\Omega(\vec{x})\Omega(0)\rangle_{\rm conn}=\frac{\xi}{(4\pi)^4}\langle\Theta\rangle, \tag{9}$$

where the problem is to calculate the dimensionless quantity ξ. One may check that some such factor as $(4\pi)^4$ is natural, so that ξ is nominally of order 1. Equation (8) is the $d=3$ version of a sum rule conjectured, on various grounds [8], for $d=4$ gauge theory, where Ω is replaced by the topological charge density and Θ is the (anomalous) trace of the $d=4$ stress-energy tensor. We can, in fact, estimate ξ because our work yields an estimate for $\langle\Theta\rangle$ in terms of M^3. We have already said that the Wilson-loop area law stems from linking of condensate strings to a fixed test string, the Wilson loop itself. There is, then, a natural relation between the string tension and the CS susceptibility, which is also expressed in terms of linkages.

Before turning to the string-condensate model, which involves construction of a complex scalar field theory, let us mention the natural field of application of these results. Perhaps the most interesting use of $d=3$ gauge theory is in electroweak (or other gauge-theory) processes in the high-T early Universe with $T>T_c$, where T_c is the phase-transition temperature and there is no Higgs boson mass-generation mechanism. A finite-T gauge theory splits into sectors labeled by the Matsubara frequencies $\omega_N=2\pi NT$; all of the sectors with $N\neq 0$ are well behaved in the IR and can be treated more or less perturbatively [9], if the coupling is small enough. Even the $N=0$ electric sector (gauge-potential A_0) generates a perturbative Debye mass and causes no special difficulties of principle. But the $N=0$ magnetic sector, equivalent to $d=3$ gauge theory, has IR divergences which must be cured nonperturbatively.

Perturbation theory fails here because no IR-regulating mass can be generated in perturbation theory; gauge invariance prevents it. With no magnetic mass, the would-be expansion parameter of high-T theory is g^2T/k, where k is the spatial momentum of the process considered. This has two consequences: (1) At scales k set by nonzero Matsubara frequencies, or by the Debye mass $\sim gT$, a genuine perturbation expansion is possible for small g; (2) No such expansion is possible at $k\leqslant g^2T$, and the theory is strongly coupled. Of course, these IR effects go away below the phase transition where the gauge bosons get a mass from the Higgs mechanism.

The essence of all nonperturbative effects is the generation of a magnetic mass M of order g^2T [10–13]. This mass is, in principle, to be found by solving some sort of nonlinear gauge-invariant Schwinger-Dyson equation, but analytic progress in this direction has been slow [13–15]. There is a recent lattice determination of this mass [16], but the technique used is not quite gauge invariant. We will not attempt to determine M, but given its existence, we will analyze further nonperturbative effects which can be expressed in terms of it [see the discussion below Eq. (6)].

These effects include high-T sphalerons [17,18] and vortex strings which can either close or terminate on magnetic monopoles; we need only consider closed strings. These are like Nielsen-Olesen strings except that there is no Higgs effect and no symmetry breaking. Because $d=3$ gauge theory is strongly coupled, these strings form an entropy-driven condensate. The existence of a condensate follows directly from certain exact results [19] for $d=3$ gauge theory, which include among others $\langle\Theta\rangle>0, \epsilon_{\rm vac}=-(1/3)\langle\Theta\rangle<0$ [20] where Θ is the squared field strength [see (8)] and $\int d^3x\,\epsilon_{\rm vac}$ is the vacuum energy (or equivalently the $N=0$ contribution to βF in the thermal case, with F the Helmholtz free energy).[3] Of course, the negative sign of F reflects entropy domination.

The fluctuations in CS number measured by the susceptibility (7) play an important role in determining the rate of anomalous $B+L$ violation [21–23] from high-T electroweak processes in the early universe. However, it is not possible to

[3]Shaposhnikov, in Ref. [20], also attempts to estimate the free energy by semiclassical means; we will compare his value to ours in the conclusions.

determine this $B+L$ rate/unit volume Γ_{BL} solely from the essentially static fluctuations of $W_{\rm CS}$ in the $d=3$ gauge-theory sector; one really needs to understand fluctuations in the rate of change $\dot{W}_{\rm CS}$ of CS number, which are directly related to fluctuations of topological charge (and $B+L$, by the anomaly equations). To calculate Γ_{BL} requires much more than $d=3$ gauge theory, and we will not attempt such a calculation here. The general wisdom [21,22] is that for $T>T_c$, Γ_{BL} scales like $(\alpha_W T)^4$, that is, like M^4, where M is the magnetic mass. This is just dimensional analysis, if M is the only relevant scale; the hard part is to calculate the overall coefficient. Usually, one argues [21,22] that there is no sphaleron barrier for $T>T_c$ so the coefficient is of order 1, but this is not really clear. Given a magnetic mass M, there should (Cornwall, Ref. [17]; Philipsen [18]) be a sphaleron of mass $M_{\rm Sp}\sim M/\alpha_W\sim T$, and the thermal activation factor $\exp(-M_{\rm Sp}/T)$ is the exponent of a negative pure number, which is not necessarily small in absolute value. We will not consider sphalerons further here, but it will certainly be important to estimate their contributions to $\chi_{\rm CS}$, etc.[4]

In our vortex model, Γ_{BL} depends on the rate of topological reconnection [24] of strings in the condensate. This dynamical process involves electric gauge fields, which lie outside the $d=3$ gauge-theory sector we study here, and we will not discuss it in detail. However, we will make a few remarks on how the $d=3$ strings are promoted to closed two-surfaces in $d=4$ and on how an intersection-number [25] topological invariant of these surfaces is related to the usual topological charge.

Now, let us return to pure $d=3$ gauge theory and our method of approach to its nonperturbative effects. It has been known for decades [26] that a condensate of strings can be described by a (possibly complex) scalar field theory, with a wrong-sign mass term representing entropy effects and an interaction term like $\lambda\phi^4$ representing the repulsion, or increase in free energy, of two strings trying to occupy the same region of space. The underlying connection between a string condensate and a scalar field theory is that a field-theoretic propagator can be represented as a sum over paths, or strings, and, in particular, the field-theory vacuum functional becomes a partition function of closed strings.

Such a connection between a $d=3$ string condensate and a field theory would not have been useful in the past, because there would have been more parameters for the scalar-field theory than for the underlying gauge theory. However, we will exploit here the exactly known [19] form of the zero-momentum effective action for the operator Θ of Eq. (8). Given this effective action, we show that the corresponding classical action for the scalar-field theory is uniquely[5] of the form

$$S=\int d^3x\left(|\vec{\nabla}\psi|^2+\frac{\lambda|\psi|^4}{4}\right), \tag{10}$$

where the coupling constant λ depends on M. From this classical action we will derive loop terms, automatically consistent with the gauge-theory effective action, which are essential to describe the condensate entropy. These loop terms simply modify the classical action of (10) to an effective action

$$\Gamma=\int d^3x\left(|\vec{\nabla}\psi|^2+\frac{\lambda|\psi^4|}{4}-\frac{\lambda v|\psi|^3}{3}\right), \tag{11}$$

where v is the expectation value of $|\psi|$, also expressible in terms of M. Like all effective actions, (11) should be treated classically. Although we can and do derive the form (11) from loop corrections to the classical action (10), the functional form of (11) is completely dictated by the gauge-theory effective action. Note that (11) has a phase with $\langle|\psi|\rangle=v$ and short-range fluctuations, as required to describe the gauge-string condensate.

Given this complex scalar field theory, how does one describe the topological effects involved in the string tension and the CS susceptibility? In both cases, one uses auxiliary Abelian gauge potentials, but in different ways. For the string tension, the auxiliary gauge potential is itself a fixed vortex, described in terms of the Wilson-loop contour; this fixed potential is coupled to the scalar-field theory in the usual way (which requires a complex field). At large distances, all that is relevant for the string tension, this Wilson-loop potential becomes a pure gauge of the type $\vec{\nabla}\Lambda$, but Λ is singular and $\exp(i\Lambda)$ is not single valued. [This is because we seek the fundamental-representation string tension; for the adjoint representation, $\exp(i\Lambda)$ is single valued and the string tension vanishes.] The multivaluedness is in conflict with the necessary single valuedness of the complex scalar field, or equivalently, the underlying $d=3$ gauge potentials, and when one minimizes the scalar-field effective action in the presence of the auxiliary gauge potential one finds that this conflict is resolved by a surface of zeros of the scalar field, coinciding with the minimal spanning surface of the Wilson loop. As a result, a string tension $\sim M^2$ is generated; the numerical coefficient is an elliptic integral.

To calculate the CS susceptibility, we use a technique given long ago by Edwards [27] to study topological entanglement of polymers. A new Abelian gauge potential $\vec{V}(\vec{x})$ is introduced, with a standard coupling to the scalar field and a self-action which is a pure CS term (except for very important short-distance modifications, coming from the gauge mass M, which make everything finite). Functional integration over $\vec{V}$ yields a power series in the V-field coupling constant, whose coefficients are related to expectation values of Gauss linking integrals. In particular, we can read off the CS susceptibility from this series. To construct the series to the requisite (fourth) order, we need to calculate a large number of three-loop vacuum graphs, all of which are finite and scale like M^3. We have calculated the sum of graphs mostly numerically, after reducing the graphs to Feynman-parameter form.

[4]Actually, sphalerons are closely related to the vortices we use [J. M. Cornwall and G. Tiktopoulos, Phys. Lett. B **181**, 353 (1986); M. Hindmarsh and T. W. B. Kibble, Phys. Rev. Lett. **55**, 2398 (1985)]. A sphaleron interpolates between regions of oppositely directed magnetic flux on a vortex, and can be pulled apart into two monopoles of the type that can terminate a thick vortex string [4].

[5]We will see later why a complex rather than a real field is chosen.

II. SCALAR-FIELD THEORY DESCRIPTION OF THE VORTEX CONDENSATE

The usual discussion [26] of mapping a string condensate onto a field theory begins with the observation that the number of configurations of a closed string on a lattice, of length l and step length L, is roughly (ignoring unimportant effects) given by the path integral

$$\oint (d\vec{x})\exp\left[-\int_0^l\left(\frac{\dot{x}^2}{2}-\ln 2d\right)dl'/L\right] \tag{12}$$

up to an irrelevant normalization. Here, d is the number of dimensions in which the string lives, and the entropy factor $(2d)^{l/L}$ is roughly the number of ways the string can turn at its L/l vertices, without consideration of self-avoidance effects. If the strings are oriented, as they are for us, one multiplies this result by $\int dl/l\exp(-Ul)$ and exponentiates to get the string partition function. Here, U is the internal energy/unit length of the string. (The l in the denominator of the integral adjusts for overcounting the point on the string arbitrarily chosen as a starting point.) The final result is recognized as the functional integral of a free two-component real (or one complex) scalar field, where the logarithm of the propagator of the field has been expressed as a proper-time integral. The field has a mass term which may be normal or tachyonic, depending on whether the internal energy dominates the entropy or otherwise. A string condensate will form only if the entropy dominates, as it does for $d=3$ gauge theory.

Next, one must add self-avoidance effects, which amounts to adding interaction terms for the scalar field inside the functional integral. It is, of course, natural to have a four-field interaction to represent the simplest kind of intersection of two strings, but the most general renormalizable field theory in $d=3$ has other types of interactions, ranging up to sixth-order terms. The question is what terms and what coupling constants are needed.

We will address this problem in the reverse order of the remarks above, where the possible tachyonic mass term (signaling a string condensate) is conceptually introduced before self-avoidance effects. Based on earlier exact results for the gauge-theory effective action [19], we will construct a classical scalar action containing a fourth-order interaction but no mass terms at all. We will then show that one-loop effects contain a negative contribution to the effective action, equivalent to a tachyonic mass term. The resulting effective action has a minimum corresponding to a string condensate.

The idea behind the gauge-theory results is very simple: There is only one coupling constant, g_3^2, and it has dimensions of mass. The action of the gauge theory can be written as

$$S_g=\int d^3x\Theta=\frac{1}{4g_3^2}\int d^3x\Sigma(G_{ij}^a)^2 \tag{13}$$

[where Θ is the squared field strength of Eq. (8)]. The partition function Z, which is the functional integral of $\exp(-S_g)$, gives $\exp(-\int d^3x\epsilon_{\rm vac})$. By naive dimensionality, $\epsilon_{\rm vac}\sim g_3^6$, and repeated differentiation of Z and of $\epsilon_{\rm vac}$ with respect to g_3^2, yields an infinite set of sum rules for the zero-momentum vacuum matrix elements of Θ. The functional differentiation of Z acts only on the explicit g_3^{-2} in the action (13), and the specific form of the sum rules, or effective action which generates them, depends only on the specific powers of g_3 in Z and $\epsilon_{\rm vac}$. The effective action can be found by introducing a constant source J into the functional integral, defining a vacuum functional $\mathbf{W}$ as

$$Z=\exp[-\mathbf{W}(J)]=\int (dA)\exp[-S_g(1-J)]. \tag{14}$$

Because the only dependence on g_3^2 is an overall inverse factor in the action, adding the source is the same as changing g_3^2 to $g_3^2(1-J)^{-1}$. Then

$$\mathbf{W}(J)=\epsilon_{\rm vac}(1-J)^{-3}. \tag{15}$$

By the usual Legendre transform, one finds the effective action for the zero-momentum matrix elements of Θ:

$$\Gamma(\Theta)=\int d^3x\left(\Theta-\frac{4}{3}\Theta^{3/4}\langle\Theta\rangle^{1/4}\right). \tag{16}$$

One sees that Γ has a minimum at $\Theta=\langle\Theta\rangle$ of value $-\int d^3x(1/3)\langle\Theta\rangle=\int d^3x\epsilon_{\rm vac}$.

What are the consequences for the scalar theory? A little thought shows that this theory must have only one coupling constant λ, and this coupling can be chosen to have dimensions of mass. Then, in order to reproduce the form of the effective action as in (16), $\epsilon_{\rm vac}$ must scale like λ^3, corresponding to the scaling of the gauge action as g_3^6, and it must be possible to rescale ψ so that the only appearance of λ in the action is an overall factor of λ^{-1}. So the scalar action (10),

$$S=\int d^3x\left[|\vec{\nabla}\psi|^2+\frac{\lambda|\psi|^4}{4}\right], \tag{17}$$

is acceptable. Of course, other interaction terms could be included, such as $\lambda^{3/2}|\psi^3|$, but they will not do. The reason is that with these other interaction terms, one cannot rescale ψ in such a way that the action has an overall factor of λ^{-1}. But such a rescaling is essential to the derivation of the gauge-theory effective action.

The upshot is that only the classical action (17) can yield loop corrections consistent with the gauge-theory effective action (16). There is one subtlety of the action (17): it generates quadratic and sextic terms from loops, which appear to be divergent and require bare terms in the action to accommodate the needed renormalizations. However, demanding consistency with the gauge-theory effective action, fixes these renormalizations. A simple way to do this is to introduce a free energy $\mathbf{W}_s(J)$ for the scalar theory, analogous to $\mathbf{W}(J)$ for the gauge theory introduced above. Here, J is a constant source for the scalar action S. Then, we will require

$$\mathbf{W}_s(J)=(1-J)^{-3}\mathbf{W}_s(0),\quad \mathbf{W}_s(0)=\epsilon_{\rm vac}, \tag{18}$$

just as for the gauge theory as shown in Eq. (15).

Next, we turn to the calculation of one-loop corrections to the classical action (17), which express the entropy of strings. Rather than calculate $\mathbf{W}(J)$ as outlined above, we

will directly calculate the effective action (essentially the free energy) using the Cornwall-Jackiw-Tomboulis (CJT) [28] formalism. Following the one-loop calculation, we will estimate two-loop contributions, which are appreciable, and presumably suggest the accuracy of the calculations (perhaps $\pm 30\%$). Please note that the CJT approach does not deal with bare loops, that is, perturbative effects, but instead with dressed loops constructed from the full propagator. To start, write

$$Z=\exp(-\beta F)=\int (D\psi D\bar{\psi})\exp(-S) \tag{19}$$

$$=\exp\left\{\mathrm{tr}\ \ln\frac{G}{G_0}-\mathrm{tr}(GG_\psi^{-1}-1)-\int\frac{\lambda}{4}|\psi^4|+2\mathrm{PI\ graphs}\right\}. \tag{20}$$

Here, G is the full propagator of the theory, G_ψ is the classical propagator in the field ψ, and 2PI graphs are connected graphs with more than one loop that are two-particle irreducible. The notation tr indicates an integral over all space, or a sum over all momenta. The physical free energy is found by searching for the extrema of βF in the functional variable G as well as in the field ψ. To this end, we introduce a variational parameter m in G, writing in momentum space

$$G^{-1}=k^2+m^2. \tag{21}$$

We are only interested in small-momentum phenomena, so ψ can be treated as a constant, and one readily finds the classical propagator

$$G_\psi^{-1}=k^2+\lambda|\psi|^2. \tag{22}$$

All one-loop graphs, with G as the propagator, are found by dropping the 2PI graphs in (19). The next step is to insert G and G_ψ into the first two terms of (19) and do the integrals. Divergences appear, which can be removed in either of two equivalent ways: dimensional regularization, or by the substitution

$$\int d^3kF(k^2)\rightarrow -2\int d^3k\left(1+k^2\frac{\partial}{\partial k^2}\right)F. \tag{23}$$

It is easily checked that this substitution leaves convergent integrals unchanged, and eliminates divergences.

Before removing the divergences, we should comment on their physical meaning in the case of interest, finite-T gauge theory. The underlying $d=3$ gauge theory has no divergences, except those which can be interpreted as coming from the other sectors which go to make up the full finite-T theory. For example, any divergence in the coupling constant would have to come from an underlying $d=4$ divergence. In general, ultraviolet divergences are associated with scales such as T, rather than g^2T which is appropriate to the $d=3$ gauge theory only. We are justified in regulating the scalar theory in a way which reproduces known results of the $d=3$ gauge theory, such as the effective action form (16).

It is a remarkable property of three dimensions that integrals whose integrands depend on m^2, yield odd powers of m. In fact, these odd powers must be understood as absolute values, in view of the symmetry $m\rightarrow -m$ of the integrand. Ultimately, these powers of m will yield odd powers of $|\psi|$, as in (11). Moreover, the one-loop integrals after regularization yield negative contributions to the effective potential. For further discussion and more references, see Ref. [29].

Now, do the integrals in (19), following the above prescriptions, to find

$$\beta F=\int d^3x\left[\frac{\lambda|\psi|^4}{4}-\frac{|m|^3}{6\pi}-\frac{|m|}{4\pi}(\lambda|\psi|^2-|m|^2)\right]. \tag{24}$$

Upon varying m, one finds[6] $m^2=\lambda|\psi|^2$, and substituting yields

$$\beta F=\int d^3x\left[\frac{\lambda|\psi|^4}{4}-\frac{\lambda^{3/2}|\psi|^3}{6\pi}\right]. \tag{25}$$

Except for the gradient terms, which will be considered later, this is of the form (11) and consonant with the gauge-theory effective action (16), provided that one identifies

$$\Theta=\frac{\lambda|\psi|^4}{4},\ \ \langle\Theta\rangle=\frac{\lambda^3}{4(2\pi)^4}. \tag{26}$$

Further, minimization on ψ yields

$$\langle|\psi|\rangle\equiv v=\left[\frac{m}{2\pi}\right]^{1/2},\ \ m=\frac{\lambda}{2\pi},\ \ \epsilon_{\mathrm{vac}}=V^{-1}\beta F_{\mathrm{min}}=-\frac{m^3}{24\pi}. \tag{27}$$

Here, V is the volume of all space. It is important to note that the variational parameter m is not the physical scalar-mass value, which is found from the second derivative of the effective action at the minimum. We require that the scalar mass, or correlation length, must be the same as for the $d=3$ gauge theory, and we will use the same notation M for both of these quantities. It is easy to find from (25) that

$$M^2=\frac{1}{2}\lambda v^2=\frac{1}{2}m^2. \tag{28}$$

Next, we ask what the effect of two-loop terms is. There is only one 2PI two-loop graph, the double bubble, and its value, to be added to the integrand of the one-loop value (24), is $+\lambda m^2/32\pi^2$. As before, vary to find m, with the result

$$m=-\frac{\lambda}{8\pi}+\left[\lambda\psi^2+\left(\frac{\lambda}{8\pi}\right)^2\right]^{1/2}. \tag{29}$$

When this result is inserted in the two-loop effective action, new terms involving ψ appear which are not of the desired form (11). This is no problem of principle; once one goes beyond one-loop graphs, one really cannot recover the desired form (11) without considering loops of all orders.

[6]This value of m means that $G=G_\psi$, as required by the variational equation for G which comes from Eq. (20) when the 2PI graphs are omitted.

The reason is that a generic multiloop graph corresponds to an expansion in powers of λ/ψ^2, as one readily checks, and such terms are already found in the expansion of m in (29). A simple way to include the needed parts of other graphs to correct the explicit two-loop result has already been given in (18): one calculates the true vacuum energy $\mathbf{W}_s(\mathbf{0})$ with zero source J at two loops, and then writes the vacuum functional in the presence of the source as in (18) by multiplying by $(1-J)^{-3}$. After some uninteresting algebra, one finds the two-loop parameters

$$\epsilon_{\rm vac}=-\frac{m^3}{24\pi},\quad v=\left(\frac{m}{2\pi}\right)^{1/2},\quad m=M. \tag{30}$$

Note that, in spite of a very different algebraic structure, the expression of $\beta F_{\rm min}$ and v in terms of m are the same for the one- and two-loop effective actions; the only difference is the relation between the physical mass M and the variational parameter m, which differ in the two cases by a factor of $\sqrt{2}$. This can be taken as a rough measure of the (not very high) accuracy of the first few terms of the CJT loop expansions, although the error encountered in any specific quantity may be more or less than this. Perhaps this is an acceptable discrepancy in analytic calculations of a strongly coupled gauge theory, where there is no obvious small parameter in the kind of dressed-loop expansion we are using.

The point here is not to show that the effective scalar action must have the form (11), which is guaranteed by the underlying gauge theory, but to estimate the parameters λ, v which occur in it in terms of the physical mass M. This we have done in Eqs. (27), (28), and (30). The next step is to calculate the string tension K_F in terms of the same parameters.

III. THE STRING TENSION

To calculate the string tension via the expectation value of the Wilson loop

$$\langle W(\Gamma)\rangle=\langle \mathrm{Tr}P\ \exp(\oint_\Gamma d\vec{x}\cdot\vec{A})\rangle, \tag{31}$$

we write for $\vec{A}$ a representative vortex configuration and sum over the collective coordinates of the vortices. This will be recognized as the coupling of a string condensate to a fixed Abelian vortex gauge potential generated by the Wilson-loop contour itself. The usual rules [26] for converting the string expectation value to the scalar-field theory generate a standard minimal gauge coupling of the Wilson-loop vortex to the scalar field. Classical minimization of the resulting effective action leads to differential equations whose solution gives rise to the string tension.

As has been discussed elsewhere [2,1], the gauge-field vortex is a classical solution to an effective gauge action[7] which includes a gauge-invariant mass term [30,12]. This action is

$$S_v=S_g-\int d^3x\frac{M^2}{g_3^2}\mathrm{Tr}(V^{-1}\vec{D}V)^2, \tag{32}$$

where S_g is the usual massless gauge action [see (13)], V is an auxiliary SU(2) matrix, and $\vec{D}$ is the covariant derivative $\vec{\nabla}+\vec{A}$. The gauge transformation laws,

$$\vec{A}\to U\vec{A}U^{-1}+U\vec{\nabla}U^{-1}, \tag{33}$$

$$V\to VU, \tag{34}$$

show that the mass term in (32) is locally gauge invariant (it is just a gauged nonlinear sigma model). The V field can be functionally integrated out at the classical level, just by solving the V-field equations of motion. These turn out to be the same as the vanishing of the covariant derivative of the $\vec{A}$ equations of motion.

Up to a local gauge transformation, the vortex solution can be written

$$\vec{A}(\vec{x})=2\pi Q\vec{\nabla}\times\oint d\vec{z}[\Delta_M(\vec{x}-\vec{z})-\Delta_0(\vec{x}-\vec{z})], \tag{35}$$

where Q is an SU(2) generator such as $\sigma_3/2i$ with the property that $\exp(2\pi Q)$ is in the center of SU(2). Here, Δ_M is the scalar propagator for mass M

$$\Delta_M(\vec{x})=\frac{1}{(2\pi)^3}\int d^3k\frac{e^{i\vec{k}\cdot\vec{x}}}{k^2+M^2} \tag{36}$$

and Δ_0 is the corresponding massless propagator. The loop integral is over a closed string which describes the center of the vortex, whose field strength extends a distance $\sim M^{-1}$ from this center. However, the potential itself has a long-range pure-gauge part associated with Δ_0; this part comes from the V field in the effective action (32). The long-range pure-gauge part is, as we will see [2], responsible for confinement and the string tension. The pure-gauge term has its role at short distances too, where the leading singularities at $\vec{x}\sim\vec{z}$ cancel between the two propagators in (35). This will be crucial in deriving finite results for the CS susceptibility, but is not important for the string tension; the only singularities cured by a mass term are in perimeter-law pieces of the Wilson loop.

The Wilson-loop expectation now becomes

[7]This effective action summarizes some of the quantum effects which generate the mass, but to the extent that it generates new short-distance (scales $\ll M^{-1}$) singularities, it is not suitable for use at such scales. To the extent that the mass is important in shielding short-distance singularities, as we find in the present work, there is no reason to doubt the correctness of the effective action. Consistency requires not only that the mass M vanish at short distances, but also that there are other group-singlet scalar excitations, or glueballs, which do not need discussion in this paper.

$$\langle W\rangle = \langle\langle \exp(\pm i\pi \oint_\Gamma d\vec{x}\cdot\vec{\nabla}\times\oint_s d\vec{z}[\Delta_M-\Delta_0])\rangle\rangle \quad (37)$$

where the double brackets indicate an expectation value over the string partition function. The contour Γ is, as before, the Wilson-loop contour, and s labels a set of string contours.

At this point, we can appreciate the connection between confinement and linking of the Wilson loop with a closed string. The concept of string tension is appropriate for a large Wilson loop, with scales $\gg M^{-1}$; this means that we can drop the Δ_M term in (37). The remaining term in the exponent immediately becomes $\pm i\pi L$, where L is the Gauss linking integral:

$$L = \oint_\Gamma \oint_s d\vec{x}\times d\vec{z}\cdot\frac{(\vec{x}-\vec{z})}{4\pi|\vec{x}-\vec{z}|^3}. \quad (38)$$

The quantity L measures (with sign) the number of times the contours s and Γ are linked. Clearly, an even (odd) number of linkages contribute a term $\exp(i\pi L)=+1(-1)$ to the Wilson loop. The Wilson loop is large, so L is a sum of a large number N of random positive or negative integers, and L has a Poisson distribution. Moreover, N is proportional to the (minimal) area A of the Wilson loop, since all unlinked strings give no contribution to the string tension: $N=\rho A$, where ρ is a two-dimensional density of strings in the vacuum. It is then an elementary exercise to calculate the expectation of $\exp(i\pi L)$ in the Poisson distribution (which we approximate by a Gaussian), with an area law as the result:

$$\langle\langle \exp(i\pi L)\rangle\rangle = \exp(-\pi^2 N/2) = \exp(-\pi^2\rho A/2) \equiv \exp(-K_F A), \quad (39)$$

where K_F is the string tension. We see that the string tension measures in some sense the fluctuations of linkages of the string condensate with the fixed string of the Wilson loop, so it is natural to expect a close connection between the string tension and another measure of linkage fluctuation given by the CS susceptibility.

Note, by the way, that had we desired $\langle W\rangle$ in the adjoint representation, π in (39) would be replaced by 2π and there would be no string tension.

It is difficult to estimate the string density ρ in a straightforward elementary way, so instead we turn to the description of the string condensate by the scalar field theory. Observe that the Wilson-loop expectation value (37) has a standard gauge action S_W coupling the strings to the fixed Abelian gauge potential $\vec{W}$:

$$S_W = i\oint d\vec{z}\cdot\vec{W}, \quad (40)$$

$$\vec{W}(\vec{z}) = \pi\vec{\nabla}\times\oint_\Gamma d\vec{y}[\Delta_M(\vec{z}-\vec{y})-\Delta_0(\vec{z}-\vec{y})]. \quad (41)$$

This is of the form of the original gauge vortex in (35), with the Wilson-loop contour Γ as the center of the vortex. Such a coupling, expressed in scalar-field language through the proper-time formalism, amounts to writing the scalar-effective action given in (11) with a gauge-covariant derivative:

$$S_W = \int d^3x\left[|\vec{D}\psi|^2 + \frac{\lambda|\psi^4|}{4} - \frac{\lambda v|\psi|^3}{3}\right], \quad (42)$$

where the covariant derivative is

$$\vec{D} = \vec{\nabla} - i\vec{W}. \quad (43)$$

So, the Wilson-loop value is

$$\langle W\rangle = Z^{-1}\int (D\psi D\bar{\psi})\exp(-S_W), \quad (44)$$

where Z is the scalar-field partition function in the absence of the Wilson-loop gauge potential. At the classical level, Z is just

$$Z = \int (D\psi D\bar{\psi})\exp\left(+\int d^3x\frac{\lambda v^4}{12}\right), \quad (45)$$

that is, a functional integral over the action evaluated at the $\vec{W}=0$ saddle point $S=-\int d^3x\lambda v^4/12$. Then, we can write the logarithm of the Wilson loop, or $K_F A$, as

$$K_F A = -\ln\langle W\rangle = \int d^3x\left[|\vec{D}\psi|^2 + \frac{\lambda|\psi^4|}{4} - \frac{\lambda v|\psi|^3}{3} + \frac{\lambda v^4}{12}\right]. \quad (46)$$

The idea is to find a classical solution (i.e., minimum of the action) and to read off the string tension from the above integral.

As mentioned above, the Wilson loop is large and effects associated with the Δ_M term in $\vec{W}$ of (41) do not contribute to the area-law part of the action, only to curing short-range divergences in perimeter-law contributions. Let us see what happens when we drop this term. The remaining Δ_0 term of $\vec{W}$ is pure gauge, as can be shown directly with Stokes' theorem; it corresponds to the V term in (32). A pure-gauge term can normally be compensated by choosing the phase of the field to which the gauge potential is coupled, so it is tempting to say that, if

$$\vec{W} = \vec{\nabla}\Lambda, \quad (47)$$

then a solution to the classical equations is just

$$\psi = v e^{i\Lambda}, \quad (48)$$

for which the action integral (46) vanishes. This argument is globally quite correct for the string tension in the adjoint representation, for which Λ is twice as big as in the fundamental representation, but not for the fundamental representation. The reason is that in the fundamental representation Λ of (47), which is multivalued, is such that $\exp(i\Lambda)$ is two valued when either (but not both) of the points $x=\pm a$ are

encircled in the xy plane. Multivaluedness of ψ is not allowed; like the gauge potentials themselves, this field must be single valued.

We now show that there is another solution for the fundamental representation, which has a finite string tension. The first step is to specify the Wilson loop, which is taken to be two infinite straight lines parallel to the z axis and passing through the points $x=\pm a, y=0$. We require $Ma \gg 1$, so that the Wilson loop is large, justifying the dropping of Δ_M in $\vec{W}$. One readily calculates from (41), the gauge function Λ of (47):

$$\Lambda(\vec{x})=\frac{1}{2}[\phi_a-\phi_{-a}], \tag{49}$$

where

$$\phi_{\pm a}=\arctan\frac{y}{x\mp a}. \tag{50}$$

It is, of course, the 1/2 in (49) that makes $e^{i\Lambda}$ multivalued, and which would be missing in the adjoint representation. We can make $e^{i\Lambda}$ single valued in a cut plane, by giving $\phi_{\pm a}$ each a cut along the positive x axis, starting at $x=\pm a$. Then, the gauge function $e^{i\Lambda}$ has no cut (is single valued) for $|x|>a$, but its phase has a jump of π across the portion of the x axis joining the two sides of the Wilson loop. We can explicitly exhibit the structure of $e^{i\Lambda}$ on this cut, and it is convenient to do so at $x=0$:

$$e^{i\Lambda}=\epsilon(y)\frac{y+ia}{\sqrt{y^2+a^2}} \tag{51}$$

which changes sign from i to $-i$ upon crossing the x axis, and approaches unity at $y=\pm\infty$. This change of sign holds not only at $x=0$ but all along the x axis.

Now, separate ψ into a real part and a phase:

$$\psi=Re^{iQ}. \tag{52}$$

Here, R is real, but can be positive or negative. We can choose $Q=\Lambda$ outside a circle in the xy plane of radius $r\gg a$, where there is no cut, but we cannot do so for $|x|<a$ because of the jump discontinuity. However, it is possible to find another solution for $|x|<a$ to the equation

$$\vec{\nabla}Q=\vec{\nabla}\Lambda, \tag{53}$$

which is valid for $|x|<a$ except on the cut at $y=0$. This solution, unlike $e^{i\Lambda}$ of (51), has no jump. It is sufficient for the gradients to agree in order to compensate for $\vec{W}$ in the covariant derivative. At $x=0$, we can again exhibit this solution explicitly:

$$e^{iQ}=\frac{y+ia}{\sqrt{y^2+a^2}}. \tag{54}$$

There is no cut, and e^{iQ} approaches unity at $y=+\infty$. But for $y=-\infty$, this phase factor approaches -1.

Consider the behavior of ψ as $r\to\infty$. In this domain, we can take

$$\psi=ve^{i\Lambda} \tag{55}$$

as the classical solution, up to terms exponentially small in Mr. This is just what we have already done for $|x|>a$, but not for $|x|<a$. In the latter case, the phase factor e^{iQ} has opposite signs in the upper and lower half planes, which is inconsistent with choosing the real factor $R=v$. Clearly, ψ cannot change sign for sufficiently large r without leading to infinite action (46). The simple solution is to allow R to change sign too, canceling the sign change in e^{iQ}. This sign change is brought about by requiring ψ to vanish along the cut $|x|<a$, $y=0$, which we do by requiring R to be odd in y for $|x|<a$. This is consistent with the field equations. Given that (53) holds, the gauge potential drops out of the equations of motion. In these equations, the x derivatives are only important in a region of $O(M^{-1})$ around the Wilson loop, so we keep only the y derivatives. Then, the equation of motion is

$$-R''+\frac{\lambda}{2}R^2[R-\epsilon(R)v]=0. \tag{56}$$

Here, the primes indicate y derivatives and the ϵ factor comes from differentiating $|\psi|^3$. We choose R to have the same sign as y, and find the solution for $y>0$, $x=0$ which vanishes at $y=0$:

$$y=\int_0^R dx\left[2\left(\frac{\lambda}{4}x^4-\frac{\lambda v}{3}x^3+\frac{\lambda}{12}v^4\right)\right]^{1/2}. \tag{57}$$

Expansion of R in (56) around $y=0$, plus $R(-y)=-R(y)$, shows that R has continuous derivatives through third order at the origin.

So far we have constructed some useful pieces of a full solution to the classical equations of motion. To find an exact solution would require some smoothing and matching along the lines $x=\pm a$, but we will not do that here because it contributes nothing to the string tension. For $|x|>a$, the approximate solution is $\psi=ve^{i\Lambda}$, while for $|x|<a$, the solution is Re^{iQ} as shown in (54) and (56). These must be patched together in a neighborhood of size $\sim M^{-1}$ around $|x|=a, y=0$, where the two solutions differ, and this would require keeping the Δ_M term in $\vec{W}$.

The string tension-area product is given in the action integral (46). The area factor arises as a product of $\int dz$ times the integral over x from $-a$ to a, and the string tension K_F is the remaining factor, an integral over y. Some standard manipulations which convert the integral over y to an integral over R, plus some rescaling, put the string tension in the form

$$K_F=4Mv^2\int_0^1 du\left[\frac{1}{6}(3u^4-4u^3+1)\right]^{1/2}. \tag{58}$$

This elliptic integral yields

$$K_F=1.17Mv^2. \tag{59}$$

Using the one- and two-loop results of Eqs. (27), (28), and (30), we can convert this into $K_F=\text{const}\times M^2$; numerical

results will be discussed in the concluding section, after we find the CS susceptibility in the next section.

IV. THE CHERN-SIMONS SUSCEPTIBILITY

First, we observe that the CS number $W_{\rm CS}$ [see (2),(3)], associated with strings or vortices, can also be expressed as a modified Gauss linking number, just as the string tension can. The modifications are due to the finite mass M and come from the Δ_M term of the vortex (35); they vanish for linkages of large well-separated strings, but are vital to regulate string intersections, including self-intersections. We insert the vortex form (35) into the CS number [(2),(3)], to find for the CS number coming from two distinct vortices (labeled 1 and 2):

$$W_{\rm CS}(1,2)=-\frac{1}{8\pi^2}\int d^3x\ {\rm Tr}[\vec{A}(1)\cdot(\vec{\nabla}\times\vec{A}(2))+(1\leftrightarrow 2)]. \tag{60}$$

Insert the vortex expression (35) to come to

$$W_{\rm CS}(1,2)=\frac{\pm 1}{2}\oint_1 d\vec{z}_1\times\oint_2 d\vec{z}_2\cdot\frac{\vec{R}}{4\pi R^3}F(MR). \tag{61}$$

In this equation,

$$\vec{R}=\vec{z}_1-\vec{z}_2 \tag{62}$$

and the function F comes from the massive and massless propagators in the vortex expression (35):

$$F(R)=\frac{1}{2}\int_0^{MR}du u^2 e^{-u}. \tag{63}$$

For large well-separated strings, $MR\gg 1$ and $F\to 1$, so

$$W_{\rm CS}=\pm\frac{1}{2}L(1,2), \tag{64}$$

where $L(1,2)$ is the Gauss linking number of the two strings. But as $MR\to 0, F\sim(MR)^3$ which completely removes the singularity in the Gauss integral when $\vec{R}=0$.

The CS number of (64) is fractional, but one should expect for the gauge group SU(N) to encounter CS numbers which are multiples of $1/N$; this is equivalent to periodicity $2\pi N$ in the dependence on the vacuum angle. Such dependence is not in contradiction to the requirement that the CS number be an integer, which is a global requirement; the linking of any two particular strings is local.

Self-linking of a single string also contributes to the CS number an amount $(1/2)W_{\rm CS}(1,1)$ [see (61)]. For a large string with no self-intersections and minimum chord length $\gg M^{-1}$, one again recovers a Gauss linking integral which, as is well known [7], expresses the self-linkage as the sum of two integers, twist and writhe (which separately have no invariant topological meaning). For a string with self-intersections or near misses, the function $F(MR)$ regulates possible divergences, somewhat in the spirit of the usual ribbon framing procedure. However, the strings are fat, and the self-linking CS number can take any value. Again, this does not interfere with the global requirement that the total CS number be an integer.

We wish to calculate the fluctuation in linkages of the string condensate. The first step, following Edwards [27], is to write the sum over strings of (61) in propagator form:

$$W_{\rm CS}=\frac{1}{4}\int d^3x\int d^3y J_i(\vec{x})\Delta_{ij}(\vec{x}-\vec{y})J_j(\vec{y}), \tag{65}$$

where

$$\Delta_{ij}(\vec{x})=-i\int d^3k e^{i\vec{k}\cdot\vec{x}}\epsilon_{ijl}k_l\frac{M^4}{k^2(k^2+M^2)^2} \tag{66}$$

and the current is the usual one

$$\vec{J}(\vec{x})=\sum_{\rm strings}\oint d\vec{z}\,\delta(\vec{x}-\vec{z}). \tag{67}$$

Now, introduce an Abelian gauge field $\vec{V}(\vec{x})$ for which Δ is the propagator, and a functional integral over $\vec{V}$ which generates expectation values of powers of $W_{\rm CS}$:

$$Z_V(\zeta)=\int(DV)\exp\left\{i\int[\tfrac{1}{2}V_i\Delta_{ij}^{-1}V_j+\zeta\vec{J}\cdot\vec{V}]\right\}$$
$$\equiv\exp[-\omega(\zeta)]. \tag{68}$$

[Strictly speaking, Δ has no inverse, and a gauge-fixing term should be added to (68) in order that the inverse exist. This term contributes nothing and will be omitted from the explicit discussion.] The action for $\vec{V}$ is of CS type, but with a propagator modified at short distances according to (66); at long distances, this propagator is just the usual CS one. The coupling constant ζ is introduced to allow construction of matrix elements of $W_{\rm CS}$ by differentiation, after doing the functional integral over $\vec{V}$. This integral is, up to an irrelevant normalization,

$$Z_\zeta=\left\langle\left\langle\exp\left(\frac{-i\zeta^2}{2}\int\int J\cdot\Delta\cdot J\right)\right\rangle\right\rangle, \tag{69}$$

where, as in (37), the double brackets indicate a string expectation value. Then, from (65) and (69) the connected CS susceptibility $\xi_{\rm CS}$ [see (7)] is

$$\frac{1}{4V}\left(\frac{\partial}{\partial\zeta^2}\right)^2\omega(\zeta)|_{\zeta=0}, \tag{70}$$

where V is the volume of all space.

As before, we calculate $\omega(\zeta)$ in the scalar-field form, which amounts to changing the scalar action (17) to

$$S_\zeta=\int d^3x\left[|\vec{D}_\zeta\psi|^2+\frac{\lambda|\psi|^4}{4}\right], \tag{71}$$

where the new covariant derivative is

$$\vec{D}_\zeta=\vec{\nabla}+i\zeta\vec{V}. \tag{72}$$

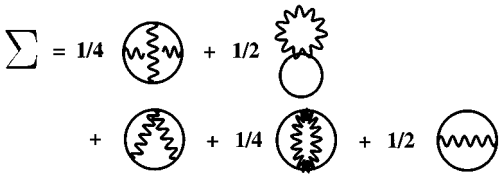

FIG. 1. Dressed-propagator graphs for the 2PI graphs Σ (defined in the text). The wavy lines are $\vec{V}$ propagators, and the solid lines are dressed ψ propagators, including dressing with $\vec{V}$ lines as shown in Fig. 2.

Then, Z_ζ is found from a functional integral over ψ and $\vec{V}$. While the effect of the $\vec{V}$ field can be calculated perturbatively, we already know that nonperturbative effects of the ψ field are crucial. Once again, we use the CJT formalism to express these effects, including graphs with two $\vec{V}$ lines [thus of $O(\zeta^4)$] in the 2PI sum [see (20)]. The calculation will be done on shell (that is, $\psi \to v$, the physical expectation value). As before, we express the free energy as the sum of one-particle terms and 2PI graphs:

$$-\omega(\zeta)=\mathrm{tr}\left\{\ln\frac{G}{G_0}-\frac{G}{G_0}+1+\frac{1}{2}\ln\frac{\Delta}{\Delta_0}-\frac{1}{2}(\Delta\Delta_0^{-1}-1)\right\}+\Sigma \tag{73}$$

and use a simple massive form for the dressed propagator G, as in Eq. (21), with variational parameter m:

$$G(k)=(k^2+m^2)^{-1}. \tag{74}$$

The term Σ represents the graphs shown in Fig. 1, where the solid lines represent the propagator G. This propagator differs from our previous solution by powers of ζ, and is to be determined from its variational equation. The solution is a nonperturbative one as far as ψ goes (which we have worked out at one- and two-loop levels earlier, see Sec. II), modified by $\vec{V}$ corrections. The result is the set of graphs shown in Fig. 2, where the solid lines now represent a massive propagator with the *physical* mass M. A couple of comments are in order: (1) The weights given in the figures include the factor of 1/2 required for closed loops consisting of one or two $\vec{V}$ lines; (2) All the *bare* graphs of $O(\zeta^2)$ vanish, because they contain a single ϵ symbol. This means that the expectation value of the CS number is zero, as one expects for a parity-conserving theory. However, the corresponding $O(\zeta^4)$ *dressed* graphs do not vanish, since they have two ϵ symbols.

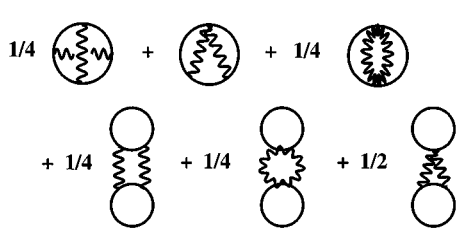

FIG. 2. Graphs of Fig. 1 expanded to show $\vec{V}$ lines occurring in the propagator G of that figure. The solid line now means the physical ψ propagator in the absence of $\vec{V}$.

It remains to evaluate these graphs, which have conventional Feynman rules except for the $\vec{V}$ propagator, which is given by (66). Note that this falls off, like k^{-6} in momentum space, so rapidly that all the graphs we need to calculate have no divergences from $\vec{V}$ lines. [We regulate any ψ divergences as before, using (23).] The evaluation is straightforward but lengthy, and cannot be done completely analytically. Our approach is to introduce Feynman parameters and to do the momentum-space integrals analytically, then to do the remaining Feyman-parameter integrals numerically. There are integrable singularities in the Feynman-parameter integrals, but these cause no difficulties. In certain cases, some of the Feynman-parameter integrals can be done analytically, but the generic case is a seven-fold integral [there are usually eight parameters, constrained by a delta function, because of the nonstandard form of the $\vec{V}$ propagator in (66)]. Another approach to this propagator is to write it as

$$\frac{1}{k^2}\left(\frac{M^2}{k^2+M^2}\right)^2=\int_0^{M^2}d\sigma\frac{1}{(\sigma+M^2)^2}-\frac{M^2}{(k^2+M^2)^2}. \tag{75}$$

This is not necessarily an easier way to do things, but it affords a check on our results since it provides an alternative Feynman-parameter form for numerical integration. Both ways give the same answer.

The final result for the CS susceptibility, expressed in terms of M, is [using (70)]

$$\chi_{\mathrm{CS}}=0.010\frac{M^3}{(4\pi)^4}. \tag{76}$$

In the concluding section below, we will evaluate this and other results in terms of the string tension and the vacuum energy.

V. NUMERICAL RESULTS AND CONCLUSIONS

First, we discuss the numerical results of our calculations, and then briefly point out how reconnection of strings leads to topological charge change.

For purposes of comparison with lattice and other calculations, it is useful to express all quantities in terms of the string tension K_F, whose value is fairly accurately known from lattice work [31,32]. This value is

$$K_F=0.11-0.13(g^2T)^2 \tag{77}$$

(here and in what follows, we always write g^2T for g_3^2, in view of applications to high-T gauge theory). Below, we show values of ϵ_{vac}, M, and χ_{CS} in appropriate units of K_F and g^2T, at one and two loops, using a nominal value $K_F^{1/2}=0.36g^2T$. These results are based on Eqs. (27), (28), (30), (59), and (76). We use superscripts (1),(2) to indicate one- and two-loop values:

$$\epsilon_{\mathrm{vac}}^{(1)}=-0.28K_F^{3/2}=-0.013(g^2T)^3, \tag{78}$$

$$\epsilon_{\mathrm{vac}}^{(2)}=-0.17K_F^{3/2}=-8\times10^{-3}(g^2T)^3, \tag{79}$$

$$M^{(1)}=1.95K_F^{1/2}=0.70g^2T, \tag{80}$$

$$M^{(2)}=2.32K_F^{1/2}=0.83g^2T, \tag{81}$$

$$\chi_{\rm CS}^{(1)}=0.074\frac{K_F^{3/2}}{(4\pi)^4}=\frac{3.5\times10^{-3}}{4\pi}(\alpha_W T)^3, \tag{82}$$

$$\chi_{\rm CS}^{(2)}=0.12\frac{K_F^{3/2}}{(4\pi)^4}=\frac{5.6\times10^{-3}}{4\pi}(\alpha_W T)^3. \tag{83}$$

One can get a rough idea of the errors involved in truncating the loop expansion from these numbers; they depend on the dimensionality of the quantity involved. For M, with dimensions of mass, the error is about 30%, with errors in other quantities growing as the dimension grows.

There are only a few other calculations one might compare to these numbers. Concerning the nonperturbative vacuum energy, Refs. [32,33] give values for $-\epsilon_{\rm vac}$ in the range $0.016-0.027(g^2T)^3$, while Shaposhnikov [20] gives $0.033(g^2T)^3$. The first two references cited [32,33] above are lattice works for the full electroweak theory at finite T (near the transition temperature) and include Higgs and $N\neq0$ modes. However, these should not contribute substantially to a nonperturbative quantity like $\epsilon_{\rm vac}$. Reference [20] is based on a dilute-monopole gas approximation, and contains some factors arbitrarily taken to be unity. One sees here a wide spread in these other calculated values, which are not too far from the results we calculate.

For the gluon mass M, a recent lattice determination [16] gives $M=0.46g^2T$, somewhat smaller than we give. This is not a pole mass determination, which would be gauge invariant, but we do not expect there to be a serious gauge dependence in this number. References [14,15] give a small value $M=g^2T/3\pi=0.11g^2T$ from a continuum one-loop gap equation, but again this value is not gauge invariant and one has no reason to believe that a one-loop gap equation with no vertex corrections is at all reliable. Reference [13] gives a lower limit on M of about $0.58g^2T$, which is consistent with our values. Reference [13] uses a nonlinear *gauge-invariant* [12] one-dressed-loop gap equation with vertex corrections and the sea gull graph included. The reason [12] that only a lower limit can be given is that at two-loop order, there is a logarithmic divergence in the sea gull graph even in perturbation theory, and this must (because there is no mass counterterm) be canceled by other two-loop contributions which nobody has studied yet. In other words, without imposing gauge invariance on a two-loop or higher gap equation, one has no control over perturbative effects which lead to (ultimately canceling) ultraviolet divergences. So the authors of Ref. [13] showed that their nonlinear equation had no solution at all unless M were a certain minimum value, no matter what happened to the sea gull graph.

There are no other computations known to us of $\chi_{\rm CS}$ with which our result can be compared. Various authors have estimated a related quantity, which is Γ_{BL}, the rate at which $B+L$ is violated, or equivalently (up to a factor of N_F, the number of flavors), the rate at which the CS number diffuses. Given the static mean-square fluctuation $X_{\rm CS}$, one can estimate Γ_{BL} by multiplying it by a rate. If we take this rate to be M and ignore factors which we hope are of order 1, this leads to

$$\Gamma_{BL}/N_F\approx M\chi_{\rm CS}=7-8\times10^{-3}(\alpha_W T)^4. \tag{84}$$

Philipsen [18] has done a calculation of Γ_{BL} in high-T electroweak theory, based on the proposed existence (Cornwall, Ref. [17]) of sphalerons above the phase transition temperature, where the Higgs VEV vanishes. His result maximizes at about $0.01(\alpha_W T)^4$ at $M=0.1g^2T$, and falls off rapidly on either side. The dependence of Philipsen's result on $x\equiv M_{\rm Sp}/T$ is complicated, but somewhat similar to the usual semiclassical sphaleron rate below the transition temperature [22]:

$$\Gamma_{BL}\sim(\alpha_W T)^4x^7e^{-x}. \tag{85}$$

If, as given in Refs. [34,1], $M_{\rm Sp}\approx5.3M/\alpha_W$, one finds a maximum rate at $M\approx0.1g^2T$ $(x=7)$, a mass value proposed before [14,15]. But, if the values of M in Eqs. (80), and (81) are used, the quantity x is very large, about 45–50, and Philipsen's rate would be very small. Given such sensitivity of sphaleron rates to $M_{\rm Sp}$, it is simply not clear yet whether high-T sphalerons are or are not important, but our numbers taken at face value would not leave much room for sphalerons to dominate $B+L$ decay above the phase transition temperature.

We close by showing how the conventional $d=4$ topological charge is related to a change in vortex linkage. For $d=4$, the form of the vortex corresponding to the $d=3$ form (35) has long been known (see, e.g., Ref. [2]): it is

$$A_\mu(x)=2\pi Q\epsilon_{\mu\nu\alpha\beta}\partial_\nu\oint d\sigma d\tau\frac{1}{2}[\dot z_\alpha z'_\beta-(\alpha\leftrightarrow\beta)]\times[\Delta_M(z-x)-\Delta_0(z-x)], \tag{86}$$

where the Δ's are $d=4$ propagators, Q is a group generator as in (35), and $z(\sigma,\tau)$ describes a closed two-surface. This potential is a solution to the $d=4$ action analogous to the $d=3$ action in (32), and in the static limit $(z=(\vec z(\sigma),\tau))$ reduces to (35). When two strings cross each other in such a way as to change their linkage (Gauss integral), the topological charge q expresses this change as a $d=4$ integral giving the intersections of the two closed surfaces. We give the result only for two large surfaces, where the Δ_M term can be neglected:

$$q\equiv-\frac{1}{16\pi^2}\int d^4x\ \ {\rm tr}G_{\mu\nu}\tilde G_{\mu\nu}=\frac{1}{2}\int d\sigma_1 d\sigma_2 d\tau_1 d\tau_2\epsilon_{\mu\nu\alpha\beta}\dot z_{1\mu}z'_{1\nu}\dot z_{2\alpha}z'_{2\beta}\delta(z_1-z_2). \tag{87}$$

Here, $z_1(\sigma_1,\tau_1)$, $z_2(\sigma_2,\tau_2)$ are the equations of the two surfaces. This integral clearly measures the (generically pointlike) intersections of the two surfaces. A similar integral can be written for self-intersections (see also Ref. [25]), and the finite-M corrections worked out. The topological charge for SU(2) is half-integral, like the CS number, but it is a local contribution to the global charge, which must be integral, and is no obstruction to an integral global charge.[8] The

[8]For a construction of localized instantons of half-integral charge, tied together by a sphaleron world line, see Cornwall and Tiktopoulos as cited in footnote 4.

reader can easily construct kinematic configurations for the surfaces which correspond to the time evolution of a change of linkage, and see how the topological charge is generated thereby. Of course, such time evolution necessarily involves electric fields and goes beyond the $d=3$ gauge theory we have considered here.

There still remains much to be done on $d=3$ gauge theory before we can have any confidence in attacking such difficult dynamical problems as $B+L$ washout at high T. The next significant step will be to construct a Schwinger-Dyson equation for the gauge-boson mass M which, like the work reported in Cornwall *et al.* [13], is gauge invariant and includes vertex parts consistent with the Ward identities, but which goes to at least two dressed loops in order to deal with incipient perturbative two-loop divergences in the mass. Then, one must face up to the problem inherent in all analytic treatments of strongly interacting gauge theories, with no obvious expansion parameter: How accurate are the results? The only self-contained approach is to keep more 2PI graphs in the CJT effective potential. Aside from that, we can only compare with lattice computations. It would be valuable to have a lattice calculation of the CS susceptibility for this purpose.

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