Trace anomaly and the Hawking effect in generic two-dimensional dilaton gravity theories

Mariano Cadoni*
Dipartimento di Scienze Fisiche, Università di Cagliari, Via Ospedale 72, I-09100 Cagliari, Italy
and Istituto Nazionale di Fisica Nucleare, Sezione di Cagliari, I-09100 Cagliari, Italy
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Black hole solutions in the context of a generic matter-coupled two-dimensional dilaton gravity theory are discussed both at the classical and semiclassical level. Starting from general assumptions, a criterion for the existence of black holes is given. The relationship between the conformal anomaly and Hawking radiation is extended to a broad class of two-dimensional dilaton gravity models. A general and simple formula relating the magnitude of the Hawking effect to the dilaton potential evaluated on the horizon is derived.

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I. INTRODUCTION

Two-dimensional (2D) dilaton gravity [1–10] is a subject that has been intensively investigated in the last years not only because of its intrinsic mathematical interest but also because it provides a simple toy model for studying the Hawking radiation of black holes. Moreover the relation with string theory in noncritical dimensions and the fact that gravity in two dimensions is renormalizable could make these models very useful in a full quantum description of physical black holes. Several models of 2D dilaton gravity have been analyzed in the literature [5–10]. Most of them are more or less motivated either by their relation with string theory or by the connection with four-dimensional (4D) black-hole physics. We have, for example, the Callan-Giddings-Harvey-Strominger (CGHS) model [5], which has been used to describe back reaction effects in the black hole evaporation process, and the Jackiw-Teitelboim (JT) theory [6,7], which is historically the first 2D dilaton gravity theory. Other models of current interest are spherically symmetric gravity (SSG) that is obtained by retaining only the radial modes of 4D Einstein gravity and string-inspired models that admit black-hole solutions in 2D anti-de Sitter spacetime [8,9].

Even though the spectrum of models for 2D dilaton gravity is large and composite, a unified and complete description of the theory still exists [1–4]. It turns out that dilaton reparametrizations and Weyl rescalings of the metric relate different models in such a way that the most general action for 2D dilaton gravity depends only on a function of the dilaton field (the potential). The resulting theory in its general form is simple and, in the absence of matter, is exactly solvable both at the classical and quantum-mechanical level. Though the classical structure of the theory, including the solutions and the classical observables for black holes, is well understood, there are still two main unsolved problems. First, in the generic model one can find, using arguments related to the local structure of the spacetime, strong evidence for the existence of black holes [4]. A rigorous proof of the existence of black holes implies a detailed knowledge of the global structure of the spacetime. For the particular models mentioned above one can explicitly show that black holes do exist but analogous statements for the generic model are still lacking. In principle, the potential being the only free input in the action, one should be able to impose conditions on the functional form of the potential, which would assure that black holes really exist. Second, assuming that black holes exist, one should be able to describe them semiclassically. In particular one would like to use here the well-known relationship between trace anomaly and the Hawking effect [11]. Again, this has been done for some special cases [5,7,8,12] but a treatment for the general model is still lacking.

In this paper we will focus on these two problems and we will find that they are strongly related. We will set general conditions on the functional form of the potential, which will be enough to assure that black holes exist. On the other hand we will find, considering a matter-coupled dilaton gravity theory, that the same conditions are crucial for having a consistent semiclassical description of black holes. We will show that the relation between the conformal anomaly and Hawking radiation can be extended to generic 2D dilaton gravity introducing local, dilaton-dependent, counterterms in the semiclassical action. We will also derive a simple expression relating the magnitude of the Hawking effect to the potential evaluated on the event horizon.

The paper is organized in the following way. In Sec. II we briefly review the features of generic 2D dilaton gravity that are relevant to our investigation. In Sec. III we analyze the global structure of the solution and we show how the functional form of the potential can be constrained so that black holes exist. In Sec. IV we study the theory coupled to scalar matter fields. In Sec. V we discuss the relationship between the conformal anomaly and Hawking radiation in the context of generic 2D dilaton gravity. In Sec. VI we discuss some relevant special cases. Finally, in Sec. VII we draw our conclusions.

II. CLASSICAL 2D DILATON GRAVITY

Our starting point is the most general two-dimensional action depending on the metric $\hat{g}_{\mu\nu}$ and the dilaton ϕ, which is invariant under coordinate transformations and contains at

*Electronic address: CADONI@CA.INFN.IT

most two derivatives of the fields. This action takes the form [1,2]

$$S[\hat{g}_{\mu\nu},\phi]=\frac{1}{2\pi}\int d^2x\sqrt{-\hat{g}}\left[D(\phi)\hat{R}+\frac{1}{2}(\hat{\nabla}\phi)^2+\lambda^2W(\phi)\right], \tag{2.1}$$

where $\hat{R}$ is the Ricci scalar and D and W are arbitrary functions of the dilaton ϕ. The model represents a generalization of well-known 2D gravity theories such as the CGHS model [5] and the JT model [6]. In its general form, given by (2.1), it has already been analyzed in the literature [1–4]. In the following we will briefly review some basic features of the model that are relevant to our further investigation. The model defined by (2.1) actually depends only on the dilaton potential $W(\phi)$, since the Weyl rescaling

$$g_{\mu\nu}=\exp\left(\frac{1}{2}\int\frac{d\phi}{dD/d\phi}\right)\hat{g}_{\mu\nu} \tag{2.2}$$

and the reparametrization of ϕ, $\Phi=D(\phi)$, bring the action (2.1) into the form

$$S[g_{\mu\nu},\Phi]=\frac{1}{2\pi}\int d^2x\sqrt{-g}[\Phi R+\lambda^2V(\Phi)], \tag{2.3}$$

where $V(\Phi)$ is an arbitrary function of Φ. The field equations derived from the action (2.3) have the simple form

$$R=-\lambda^2\frac{dV}{d\Phi}, \tag{2.4}$$

$$\nabla_\mu\nabla_\nu\Phi-\frac{\lambda^2}{2}g_{\mu\nu}V=0. \tag{2.5}$$

One can show that a generalized Birkhoff's theorem holds for the theory: for each choice of the potential V and modulo spacetime diffeomorphisms, the general static solutions of the theory form a one-parameter family of solutions. In the Schwarzschild gauge the solutions can be written as

$$ds^2=-\left(J(\lambda r)-\frac{2M}{\lambda}\right)dt^2+\left(J(\lambda r)-\frac{2M}{\lambda}\right)^{-1}dr^2, \tag{2.6}$$

$$\Phi=\lambda r, \tag{2.7}$$

where $J(\Phi)=\int^\Phi d\tau V(\tau)$. The parameter M labeling the solutions is a constant of motion, which can be interpreted as the mass of the solution and can be expressed in the coordinate invariant form

$$M=-\frac{\lambda}{2}\left[\frac{1}{\lambda^2}(\nabla\Phi)^2-J(\Phi)\right]. \tag{2.8}$$

The solutions admit a Killing vector k_μ, whose magnitude is

$$k_\mu k^\mu=\frac{2M}{\lambda}-J(\Phi). \tag{2.9}$$

If the equation

$$J(\Phi)=\frac{2M}{\lambda} \tag{2.10}$$

admits at least one solution Φ_0, such that in a neighborhood of Φ_0 the function $J(\Phi)$ is monotonic, the Killing vector (2.9) becomes spacelike at Φ_0, signalizing the presence of an event horizon. One is then led to interpret the solution as a black hole. But the existence of event horizons cannot be inferred by studying only local properties of the solution. In the next section we will discuss the global structure of the spacetime and we will show that for a broad class of models the interpretation of (2.6) as a black hole is possible.

In the next sections we will need the solutions (2.6), (2.7) written in the conformal gauge, $ds^2=-\exp(2\rho)dx^+dx^-$. Fixing the residual gauge freedom relative to the conformal subgroup of diffeomorphisms, the solutions can be written as

$$e^{2\rho}=\left(J-\frac{2M}{\lambda}\right), \tag{2.11}$$

$$\int^\Phi\frac{d\tau}{J(\tau)-\frac{2M}{\lambda}}=\frac{\lambda}{2}(x^+-x^-). \tag{2.12}$$

Assuming that the solutions represent black holes one can associate to them thermodynamical parameters. For the temperature of a generic black hole one has

$$T=\frac{\lambda}{4\pi}V(\Phi_0). \tag{2.13}$$

III. GLOBAL STRUCTURE OF THE SPACETIME

As stated in the previous section a rigorous proof of the existence of black hole involves a detailed analysis of the global structure of the spacetime. The potential V being the only free input in the action (2.3), it is evident that the information about this global structure is encoded in the particular form of the function $V(\Phi)$. In the following, starting from general conditions that assure the existence of a black-hole solution, we will single out a broad class of models for which the interpretation of (2.6) as a black hole can be well established. These conditions will be translated in some constraints about the functional form of the potential V. We will consider for simplicity only black holes with a single event horizon. Our discussion can be easily generalized to the case of multiple horizons.

A crucial role in our analysis is played by the field Φ. Because of its scalar character, Φ gives a coordinate-independent notion of location and can therefore be used to define the asymptotic region, the singularities, and the event horizon of our 2D spacetime. We will consider only the spacetime region for which $0\leqslant\Phi<\infty$. This restriction is justified by the fact that the natural coupling constant of the theory is $g=(\Phi)^{-1/2}$; the spacetime can therefore be divided into a strong-coupling region $(\Phi=0)$ and a weak-coupling asymptotic region $(\Phi=\infty)$. Φ is the correspondent of the radial coordinate r in 4D spherically symmetric solutions (this analogy is particularly evident for SSG, where the area of the transverse two-sphere is proportional to Φ).

The next step in our analysis is to write down a set of conditions that, if satisfied make the interpretation of (2.6) as a black hole meaningful. Let us assume that (a) the equation (2.10) has only one solution for $\Phi=\Phi_0>0$ with $V(\Phi_0)\neq 0$ and $V(\Phi)>0$ for $\Phi>\Phi_0$, (b) for $M>0$ naked singularities are not present (the states with $M<0$ describe naked singularities), (c) in the $\Phi=\infty$ asymptotic region the Killing vector (2.9) is timelike for every finite value of the mass M, and (d) in the $\Phi=\infty$ asymptotic region the curvature R is finite. Condition (a) is necessary for the presence of an event horizon. It implies that the Killing vector (2.9) is timelike for $\Phi>\Phi_0$ and becomes spacelike for $\Phi<\Phi_0$. Condition (b) is necessary for the existence of black holes for every positive value of the mass and to assure that the vacuum $M=0$ has no event horizons. It implies that for every curvature singularity Φ_1 we have $\Phi_1<\Phi_0$ and that the function $J(\Phi)$ has no zeros. Both conditions translate into very weak constraints on the functional form of V. Conditions (c) and (d) strongly constrain the asymptotic behavior of V. In fact (c) implies that $J(\Phi)\to\infty$ as $\Phi\to\infty$. This can be easily demonstrated, assuming that $J(\Phi=\infty)=l$ with l finite. If this is the case for $M>\lambda l/2$ the Killing vector (2.9) becomes spacelike. Furthermore condition (d) determines the degree of divergence of J. By looking at equation (2.4) one easily realizes that if the curvature must stay finite as $\Phi\to\infty$ the function J must diverge lesser than or equal to Φ^2. In conclusion black holes do exist if the function V behaves asymptotically as

$$V\sim\Phi^a, \quad -1<a\leqslant 1. \tag{3.1}$$

We have correspondingly two classes of black holes: for $-1<a<1$, $R\to 0$ as $\Phi\to\infty$; for $a=1$, $R\to\text{const}$ as $\Phi\to\infty$. The asymptotic behavior $R\to 0$ is not enough to assure that the spacetime is asymptotically flat. Asymptotic flatness requires that asymptotically the metric can be put in a Minkowski form. This issue will be settled at the end of this section after the analysis of the global structure of the spacetime.

At this point the alert reader could object that the interpretation of (2.6) as a black hole fails, even though conditions (a)–(d) are satisfied, if the spacetime has no curvature singularities and can be maximally extended to describe a regular spacetime. This happens, for example, for $V(\Phi)=1$ [12] and for the JT theory [$V(\Phi)=2\Phi$] [7]. In these models the black-hole spacetime can be maximally extended to become the whole of the Minkowski and anti–de Sitter spacetime, respectively, for the two cases. As pointed out [7,12] if one cuts the spacetime at the line $\Phi=0$ (as we do here) this extension is not possible and the spacetime does indeed represent a black hole. For the general model, in absence of curvature singularities, we will therefore consider the line $\Phi=0$ as the boundary of the spacetime.

Two-dimensional dilaton gravity models, in which the potential behaves like (3.1) with arbitrary a, have been discussed in Ref. [13]. In that paper the author points out that the solutions with $a<1$ and $a>1$ present essentially the same physical behavior if one interchanges $\Phi=0$ with $\Phi=\infty$. Here, we will not consider this possibility because we want to maintain the identification of the asymptotic region with the weak-coupling region $\Phi=\infty$.

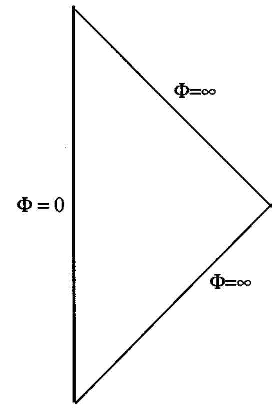

FIG. 1. Penrose diagram of the spacetime (3.2) with $-1<a\leqslant 0$.

The previous discussion enables us to single out those models in (2.3) for which the solutions (2.6) can be consistently interpreted as black holes. A detailed description of the causal structure of the spacetime depends of course on the specific form of the potential V. The form of the Penrose diagram will depend on the presence of single or multiple horizons and on the nature of the singularities. Yet, the knowledge of the asymptotic behavior of the spacetime enables us to infer some general conclusions about the causal structure of the spacetime and to draw a qualitative Penrose diagram. For our class of models the metric behaves asymptotically as

$$ds^2=-(\lambda r)^{a+1}dt^2+(\lambda r)^{-(a+1)}dr^2,$$
$$-1<a\leqslant 1, \quad 0\leqslant r<\infty, \tag{3.2}$$

where we make use of Eq. (2.7). Performing the coordinate transformations $|a|\lambda y=(\lambda r)^{-a}$, $x^+=t+y$, and $x^-=t-y$, the metric becomes

$$ds^2=-\left[\frac{|a|\lambda}{2}(x^+-x^-)\right]^{-(a+1)/a}dx^+dx^-.$$

From this expression of the metric one can read off the Penrose diagram, which turns out to depend on the value of a. For $-1<a\leqslant 0$ the metric can be put asymptotically in a Minkowski form. The spacetime is asymptotically flat, the line $r=\infty$ ($\Phi=\infty$) is lightlike whereas $r=0$ ($\Phi=0$) is timelike. This Penrose diagram is shown in Fig. 1. For $0<a<1$, $R\to 0$ as $r\to\infty$, but the metric singularity at $r=\infty$ ($x^+=x^-$) cannot be removed by any coordinate transformation. We have a strange situation where even though the curvature is asymptotically zero, the metric cannot be put asymptotically in a Minkowski form. The line $r=\infty$ is timelike whereas $r=0$ is lightlike. This Penrose diagram is depicted in Fig. 2. The case $a=1$ is analogous to the previous one. The only difference is that now $R\to-\lambda^2$ as $r\to\infty$ so that the spacetime is asymptotically anti–de Sitter. Typical Penrose diagrams for black holes with a single event horizon and with the asymptotic behavior (3.1) are depicted in Figs. 3 and 4.

We end this section with a brief discussion of the ground state of the theory. Generally we regard the solutions with

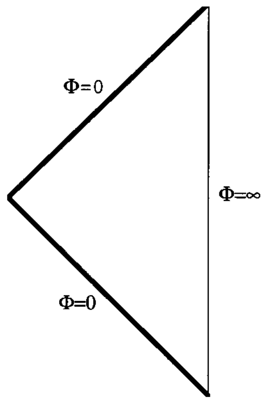

FIG. 2. Penrose diagram of the spacetime (3.2) with $0<a\leqslant 1$.

$M=0$ as the ground states of the theory. Condition (b) assures that this state does not describe a black hole, but it does not guarantee that it describes a regular spacetime. For example, for $V(\Phi)=1$ it has been shown that the vacuum is not a regular spacetime but rather a Minkowski space endowed with a null boundary [12]. At the semiclassical level this fact poses nontrivial questions about the stability of the ground state [12]. Here we will follow the same approach as in [12], i.e., we will use a *cosmic censorship* conjecture to rule out the states of negative mass from the physical spectrum.

IV. COUPLING TO (CONFORMAL) MATTER

Two-dimensional dilaton gravity has no propagating degrees of freedom. If one wants to describe a dynamical situation in which a black hole forms and then (at the semiclassical level) eventually evaporates, one has to couple the gravity-dilaton sector to matter fields. We will consider here the simplest case of N, conformally coupled, scalar matter fields f. The classical action is

$$S[g_{\mu\nu},\Phi,f]=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(\Phi R+\lambda^2 V(\Phi)-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right). \tag{4.1}$$

In the conformal gauge the equation of motion and the constraints that follow from this action are

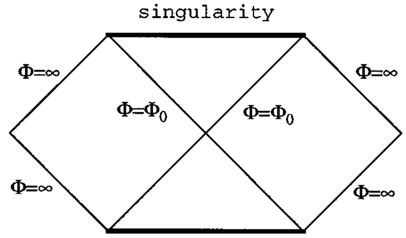

FIG. 3. Typical Penrose diagram of a black hole with a single event horizon at $\Phi=\Phi_0$ and with an asymptotic behavior characterized by $-1<a\leqslant 0$.

$$\partial_+\partial_-\rho=-\frac{\lambda^2}{8}e^{2\rho}\frac{dV}{d\Phi},\quad \partial_+\partial_-\Phi=-\frac{\lambda^2}{4}e^{2\rho}V,$$

$$\partial_+\partial_- f_i=0,\quad \partial_\pm^2\Phi-2\partial_\pm\rho\partial_\pm\Phi=-T^f_{\pm\pm}, \tag{4.2}$$

where $T^f_{\pm\pm}=\frac{1}{2}\Sigma_{i=1}^N(\partial_\pm f_i)^2$ is the classical energy-momentum tensor for the matter fields. For generic V and $T_{\pm\pm}$ this system of differential equations is very hard to solve. Still a solution can be found, maintaining a generic V, when we have only incoming matter in the form of a shock wave of magnitude M at $x^+=x_0^+$, described by

$$T^f_{++}=M\,\delta(x^+-x_0^+),\quad T^f_{--}=0.$$

Thanks to Birkhoff's theorem we can find the solution simply by patching together a vacuum solution [(2.11) and (2.12) with $M=0$] and a black hole solution along the trajectory of the shock wave. We have

$$e^{2\rho}=J,$$

$$\int^{\Phi}\frac{d\tau}{J(\tau)}=\frac{\lambda}{2}(x^+-x^-) \tag{4.3}$$

for $x^+\leqslant x_0^+$ and

$$e^{2\rho}=\left(J-\frac{2M}{\lambda}\right)F'(x^-),$$

$$\int^{\Phi}\frac{d\tau}{J(\tau)-\frac{2M}{\lambda}}=\frac{\lambda}{2}[x^+-x_0^+-F(x^-)], \tag{4.4}$$

for $x^+\geqslant x_0^+$, where

$$F'(x^-)=\frac{dF}{dx^-}=\left(\frac{J}{J-\frac{2M}{\lambda}}\right)_{x^+=x_0^+}. \tag{4.5}$$

Notice that the form of the function F is such that both ρ and Φ are continuous along the line $x^+=x_0^+$.

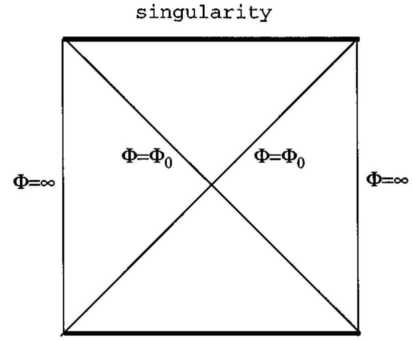

FIG. 4. Typical Penrose diagram of a black hole with a single event horizon at $\Phi=\Phi_0$ and with an asymptotic behavior characterized by $0<a\leqslant 1$.

V. TRACE ANOMALY AND HAWKING RADIATION

So far our discussion has been purely classical. In a first approximation one can describe quantum effects at the semiclassical level, by quantizing the matter fields in the fixed classical background of a black hole formed by collapsing matter. In two dimensions this semiclassical description is greatly simplified by the relation between the conformal anomaly and Hawking radiation discovered in Ref. [11]. In the context of 2D dilaton gravity this relation has already been used to study the evaporation of the CGHS black holes [5] and of black holes in anti–de Sitter spacetime [8]. The generalization to a generic theory of 2D dilaton gravity is still not trivial. In fact using the one-loop conformal anomaly contribution stemming from the usual Liouville-Polyakov term one obtains an expression for the quantum corrected energy-momentum tensor $\langle T^f_{\mu\nu}\rangle$ that is different from zero when evaluated on the ground state of the theory. Besides, a simple calculation shows that for the class of models discussed in Sec. III, $\langle T^f_{\mu\nu}\rangle_{gs}\sim\Phi^{2a}$ asymptotically. For $a>0$ the semiclassical energy-momentum tensor diverges in the $\Phi\to\infty$ asymptotic region. At first glance this behavior seems to make it impossible to use conformal anomaly arguments for the study of the Hawking effect. This conclusion is strictly true only if one assumes that the contributions to the trace anomaly come entirely from the usual nonlocal Liouville-Polyakov-action. In general we have the freedom to add local, convariant, dilaton-dependent counterterms to the semiclassical action. The presence of dilaton-dependent contribution to the trace anomaly is natural if one treats the metric and the dilaton on the same footing, and is crucial if one wants to relate the trace anomaly to the Hawking effect. The inclusion of such terms in the semiclassical action has already been discussed in the literature. In particular Strominger [14] has observed that the form of the Polyakov anomaly action depends on the metric used to define the path-integral measure. The form of the counterterms can be chosen in order to satisfy some physical conditions, for instance that they respect all the symmetries of the classical theory [15], that the theory becomes a conformal field theory [16], or that the reparametrization ghosts decouple from the outgoing energy flux [14]. Here we will follow an approach similar to the one used in Ref. [12], where the form of the dilaton-dependent counterterms was determined by imposing physical conditions on the semiclassical energy-momentum tensor evaluated on the classical vacuum of the theory.

Let us now consider the general form of the semiclassical action:

$$S=S_{\rm cl}-\frac{N}{96\pi}\int d^2x\sqrt{-g}\left[R\frac{1}{\nabla^2}R-4H(\Phi)R+4G(\Phi)(\nabla\Phi)^2\right], \tag{5.1}$$

where $S_{\rm cl}$ is the classical action (4.1) and $H(\Phi),G(\Phi)$ are two arbitrary functions. The first term above is the usual nonlocal Liouville-Polyakov term (we set to zero the contribution of the ghosts) whereas the other two represent the most general local covariant, with no second derivative counterterms one can add to the semiclassical action [we do not consider modifications of the cosmological constant action, i.e., terms of the type $\lambda^2N(\Phi)$, because they are irrelevant for our considerations].

The Liouville-Polyakov term in the action (5.1) becomes local in the conformal gauge, so that in this gauge one can easily derive the quantum contributions of the matter fields to the energy-momentum tensor

$$\langle T^f_{+-}\rangle=-\frac{N}{12}(\partial_+\partial_-\rho+\partial_+\partial_-H), \tag{5.2}$$

$$\langle T^f_{\pm\pm}\rangle=-\frac{N}{12}[\partial_\pm\rho\partial_\pm\rho-\partial^2_\pm\rho+2\partial_\pm\rho\partial_\pm H-\partial^2_\pm H-G\partial_\pm\Phi\partial_\pm\Phi+t_\pm(x^\pm)]. \tag{5.3}$$

The functions $t_\pm(x^\pm)$ reflect the nonlocal nature of the anomaly and are determined by boundary conditions. Next, we have to find the form of the functions H and G. They can be fixed by the condition that the energy-momentum tensor vanishes identically when evaluated on the classical ground state of the theory:

$$\langle T^f_{\mu\nu}\rangle_{gs}=0. \tag{5.4}$$

Assuming that the functions $t_\pm$ vanish for the ground state and using Eqs. (5.2), (5.3), and (4.3), from (5.4) we get

$$H(\Phi)=-\frac{1}{2}\ln J(\Phi)+c\int^\Phi\frac{d\tau}{J(\tau)},\qquad G(\Phi)=-\frac{1}{4}\frac{1}{J^2(\Phi)}\left(\frac{dJ}{d\Phi}\right)^2+\frac{c}{J^2(\Phi)}\frac{dJ}{d\Phi}. \tag{5.5}$$

The constant c appearing in the previous equations is arbitrary. It turns out that the Hawking radiation effect does not depend on c so that we can take, without loss of generality, $c=0$.

The expressions (5.2) and (5.3) for the energy-momentum tensor are now unambiguously determined and we can turn to the calculation of the Hawking radiation from a black hole formed by collapse of a f shock wave as in (4.3), (4.4). For $x^+\leqslant x_0^+$ the solution is given by the vacuum (4.3) so that we have $\langle T^f_{\mu\nu}\rangle=0$. For $x^+\geqslant x_0^+$ the solution is given by the black hole solution (4.4) so that using (5.5) the expressions (5.2) and (5.3) become, respectively,

$$\langle T^f_{++}\rangle=-\frac{N\lambda M}{48}\left[\frac{1}{J}\left(1-\frac{M}{\lambda}\frac{1}{J}\right)\left(\frac{dJ}{d\Phi}\right)^2-\left(1-\frac{2M}{\lambda}\frac{1}{J}\right)\frac{d^2J}{d\Phi^2}\right], \tag{5.6}$$

$$\langle T^f_{--}\rangle=(F')^2\langle T^f_{++}\rangle+\frac{N}{24}\{F,x^-\}, \tag{5.7}$$

$$\langle T^f_{+-}\rangle = -\frac{N\lambda M}{24}\left[\frac{1}{J}\left(1-\frac{M}{\lambda}\frac{1}{J}\right)\left(\frac{dJ}{d\Phi}\right)^2 - \frac{1}{2}\left(1-\frac{2M}{\lambda}\frac{1}{J}\right)\frac{d^2J}{d\Phi^2} - \frac{1}{2J}\frac{dJ}{d\Phi}\right], \qquad (5.8)$$

where F is given by (4.5) and $\{F,x^-\}$ denotes the Schwarzian derivative of the function $F(x^-)$. Using the expression (3.1) for the asymptotic behavior of the function $V(\Phi)$ one can now read off the values of the energy momentum tensor in the asymptotic region by taking the limit $\Phi\to\infty$. For asymptotically flat spaces $(-1<a\leqslant 0)$ this limit can be taken in two different ways, either $x^+\to\infty$ or $x^-\to-\infty$. Because we are interested in the value taken by the energy-momentum tensor in the future null infinity region we will let $x^+\to\infty$ as $\Phi\to\infty$ at fixed x^-. For $0<a\leqslant 1$ the line $\Phi=\infty$ is timelike and can be reached by letting $x^+\to x^-$. In both cases the result of the limit will be a function of the retarded coordinate x^- and will depend on the value of the parameter a that characterizes the asymptotic behavior of the spacetime. For $-1<a<1$ we have

$$\langle T^f_{++}\rangle\to 0, \quad \langle T^f_{+-}\rangle\to 0, \qquad (5.9)$$

$$\langle T^f_{--}\rangle\to\frac{N}{24}\{F,x^-\}. \qquad (5.10)$$

For $a=1$ the spacetime is asymptotically anti–de Sitter and $\langle T^f_{++}\rangle\to A$, with A constant. The constant term in the asymptotic expression for $\langle T^f_{++}\rangle$ and $\langle T^f_{--}\rangle$ can be eliminated with an appropriate choice of the functions $t_\pm$ appearing in (5.3). Setting $t_+(x^+)=(12/N)A\,\theta(x^+-x_0^+)$ and $t_-(x^-)=(12/N)A\,\theta(x^--x_0^+)$ we obtain also for $a=1$ the same results (5.9), (5.10). On the other hand we still have $\langle T^f_{+-}\rangle\to$const. This constant term can be interpreted as the quantum correction to the vacuum energy of the anti–de Sitter background. It is important to note that the asymptotic behavior (3.1) of the potential V is crucial for having a well behaved expression for $\langle T^f_{\mu\nu}\rangle$. For example if V behaves asymptotically as in (3.1) but with $a>1$, $\langle T^f_{\mu\nu}\rangle$ will diverge as $\Phi\to\infty$. Thus Eq. (3.1) not only assures the existence of black holes, but also assures that a semiclassical description of them is possible.

The limiting value $\langle T^f_{--}\rangle_{\rm as}$ in (5.10) is the flux of f particle energy across future infinity. However insertion of F given by (4.5) in (5.10) shows that $\langle T^f_{--}\rangle_{as}$ diverges as the horizon is approached. This is due to the bad behavior of our coordinate system on the horizon. The divergence can be easily eliminated by defining the new light-cone coordinate $\hat{x}^-=F(x^-)$, with F given by Eq. (4.5). Because in (5.3) we have set $t_-=0$, $\langle T^f_{--}\rangle$ transforms anomalously under conformal coordinate transformations. Using the anomalous transformation law of $\langle T^f_{--}\rangle$ (see for example [8]), one finds

$$\langle \hat{T}^f_{--}\rangle_{\rm as}=\frac{N}{24}\frac{1}{(F')^2}\{F,x^-\}. \qquad (5.11)$$

This expression is well behaved on the horizon. Inserting Eq. (4.5) into (5.11) we find that as the horizon $\Phi=\Phi_0$ is approached the Hawking flux reaches the constant (thermal) value

$$\langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N}{12}\frac{\lambda^2}{16}[V(\Phi_0)]^2. \qquad (5.12)$$

This is the main result of this paper and is consistent with naive thermodynamical arguments based on the formula (2.13) for the temperature of the black hole. In fact, using (2.13) one can express the magnitude of the Hawking effect (5.12) as a function of the temperature,

$$\langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N\pi^2}{12}T^2.$$

VI. SPECIAL CASES

The general model (2.3) with the potential V satisfying the conditions discussed in Sec. III contains, as particular cases, models that have already been investigated in the literature both classically and semiclassically. In this section we will show how previous results on the Hawking effect can be obtained as particular cases of Eq. (5.12). Also we will work out an example of a model that admits black-hole solutions with multiple horizons.

A. String inspired dilaton gravity

This is the most popular 2D dilaton gravity model. In its original derivation [5], due to CGHS, the action has the form (2.1). The Weyl-rescaled model is of the form (2.3) with $V(\Phi)=1$. The model admits asymptotically flat black-hole solutions [12]. Using Eqs. (2.13) and (5.12) we find for the temperature and magnitude of the Hawking effect

$$T=\frac{1}{4\pi}\lambda, \quad \langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N}{12}\frac{\lambda^2}{16}. \qquad (6.1)$$

This result coincides, after the redefinition $\lambda\to 2\lambda$ needed to match the conventions of Refs. [5,12], both with the CGHS results [5] and with the result of Ref. [12] for the Weyl-rescaled model.

B. Spherically symmetric gravity

This model is obtained by retaining only the radial modes of 4D Einstein gravity. It is characterized by $V(\Phi)=1/\sqrt{2\Phi}$ [4]. According to our classification of Sec. III we have $a=-1/2$; therefore the model admits asymptotically flat black-hole solutions with an event horizon at $\Phi_0=2M^2/\lambda^2$. The corresponding Penrose diagram is that shown in Fig. 3. Using Eqs. (2.13) and (5.12) we obtain

$$T=\frac{1}{8\pi}\frac{\lambda^2}{M}, \quad \langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N}{12}\frac{\lambda^4}{64M^2}. \qquad (6.2)$$

The black holes of this model have negative specific heat.

C. The Jackiw-Teitelboim theory

The JT theory is obtained from the action (2.3) by taking $V(\Phi)=2\Phi$ (we use the conventions of Ref. [7]). Being characterized by $a=1$ the model admits black holes with anti–de Sitter behavior. More precisely, as shown in [7], the black-hole spacetime is obtained from a particular parametrization of 2D anti–de Sitter spacetime endowed with a boundary. The black-hole horizon is at $\Phi_0=\sqrt{2M/\lambda}$ and Eqs. (2.13) and (5.12) give now

$$T=\frac{1}{2\pi}\sqrt{2M\lambda},\quad \langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N}{24}M\lambda. \tag{6.3}$$

The same result for the Hawking radiation rate has been obtained in Ref. [7] performing the canonical quantization of the scalar fields f in the anti–de Sitter background geometry. Note that the black holes have positive specific heat, indicating the emergence of a stable state as the mass of the hole goes to zero.

D. 2D black holes in anti–de Sitter spacetime

The models discussed in Refs. [8,9], characterized by the action

$$S[\hat{g}_{\mu\nu},\phi]=\frac{1}{2\pi}\int d^2x\sqrt{-\hat{g}}e^{-2\phi}\left[\hat{R}+\frac{8k}{k-1}(\hat{\nabla}\phi)^2+\lambda^2\right], \tag{6.4}$$

with $-1<k\leqslant 0$, admit black-hole solutions in anti–de Sitter spacetime. The CGHS and the JT models appear as limiting cases of this general class of dilaton gravity models for $k=-1,0$ respectively. The action (6.4) can be mapped by a Weyl rescaling of the metric of the form (2.2) into action (2.3) with

$$V(\Phi)=\Phi^{(k+1)/(1-k)}. \tag{6.5}$$

For $-1<k\leqslant 0$ the parameter a characterizing the asymptotic behavior of the black-hole solutions verifies $0<a\leqslant 1$. The black-hole solutions of these models give evidence of the peculiar asymptotic behavior in Sec. III. The Penrose diagram relative to them is represented in Fig. 4. The event horizon of the black hole is at $\Phi=\Phi_0=[4M/((1-k)\lambda)]^{(1-k)/2}$. The temperature and the flux of Hawking radiation are

$$T=\frac{1}{2\pi}\left(\frac{2M}{1-k}\right)^{(k+1)/2}\left(\frac{\lambda}{2}\right)^{(1-k)/2},$$

$$\langle \hat{T}^f_{--}\rangle^h_{\rm as}=\frac{N}{48}\left(\frac{2M}{1-k}\right)^{(k+1)}\left(\frac{\lambda}{2}\right)^{(1-k)}. \tag{6.6}$$

The result (6.6) coincides with that found in Ref. [8] for the model (6.4), after the redefinition $\lambda\to\sqrt{2/(1-k)}\lambda$, needed to match the conventions of [8].

E. 2D black holes with multiple horizons

We conclude this section with a model that admits a black-hole solution with two event horizons. Let us consider the action (2.3) with

$$V(\Phi)=1-(C/\Phi)^2, \tag{6.7}$$

where C is an arbitrary positive constant. According to our general classification of Sec. III we will have asymptotically flat black-hole solutions. The solutions (2.6), (2.7) become now

$$ds^2=-\left(\lambda r+\frac{C^2}{\lambda r}-\frac{2M}{\lambda}\right)dt^2+\left(\lambda r+\frac{C^2}{\lambda r}-\frac{2M}{\lambda}\right)^{-1}dr^2,$$

$$\Phi=\lambda r. \tag{6.8}$$

For $M>\lambda C$ the solution (6.8) describes black holes with a singularity at $r=0$ and two horizons at $r=r_\pm=\lambda^{-2}[M\pm\sqrt{M^2-(C\lambda)^2}]$. The black hole becomes extremal for $M=\lambda C$. The potential V evaluated on the outer horizon is

$$V(\Phi_0^{\rm out})=1-\left(\frac{C\lambda}{M+\sqrt{M^2-(C\lambda)^2}}\right)^2. \tag{6.9}$$

Both the temperature and the Hawking radiation rate decrease with the mass of the hole and become zero in the extremal case.

VII. CONCLUSIONS

We have been able to give a unified description at both the classical and semiclassical level of the black-hole solutions of a general 2D dilaton gravity theory. A criterion for the existence of black holes has been formulated and the relationship between the conformal anomaly and Hawking radiation has been extended to a broad class of 2D dilaton gravity models. In particular we could write down a very simple and general formula relating the magnitude of the Hawking effect to the dilaton potential evaluated on the horizon. The price that we had to pay for achieving this general description is a strong constraint on the functional form of the potential, in particular on its asymptotic behavior. The conditions discussed in Sec. III rely very heavily on the form of the action (2.3), and they are sensitive to the Weyl rescaling (2.2) that brings the action into the form (2.1). Though the global structure of the solutions does not change under this transformation, local quantities such as the Ricci curvature do change so that some conditions of Sec. III should be reformulated when one considers the Weyl-rescaled model. It may therefore be possible that a more general description exists that takes full account of the equivalence of models under Weyl rescalings of the metric. It may also be possible that in such general framework the existence of black-hole solutions could be derived imposing weaker restrictions to the form of the potential. At the semiclassical level a description that takes full account of the equivalence of models under Weyl rescalings would be even more opportune. In Sec. VI we saw that the Hawking radiation rate for string-inspired gravity and for the models (6.4) does not change under a Weyl rescaling of the form (2.2). It would be very interesting to see if this fact is a peculiarity of these models or a general feature of 2D dilaton gravity. On the other hand our discussion omitted the backreaction of the geometry on the radiation, and quantum gravity effects. Both are expected to be crucial in order to understand the end point of the evaporation process. The inclusion of the backreaction makes the theory very hard to solve at least in its general form. Up to now an exact solution could be found only for a modified version of the CGHS model (the Russo-Susskind-Thorlacius model [15]).

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