Black holes and gravitational effects in two-dimensional dilaton gravity

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(Received 5 July 1995)

We consider the model of two-dimensional dilaton gravity inspired by string theory modified by the inclusion of an additional term. We solve the model exactly in a Schwarzschild-like gauge to obtain a black hole solution. We also examine the post-Newtonian and the weak-field approximations as well as stellar structure in the model.

PACS number(s): 04.70.Dy, 04.60.Kz

I. INTRODUCTION

In recent years there has been much study of relativistic theories of gravitation in two spacetime dimensions. Because of their relative tractability, it is hoped that such investigations might shed light on the (3+1)-dimensional theory. In particular two such theories, the so-called "$R=T$" theory of Refs. [1,2] and the string-inspired dilaton gravity theory of Refs. [3,4], have attracted a lot of attention. This is mainly due to the fact that their field equations admit black hole solutions. In Ref. [5] Callan, Giddings, Harvey, and Strominger (CGHS) included matter fields so that the classical action for dilaton gravity reads

$$S=\frac{1}{2\Pi}\int d^2x\sqrt{-g}\left\{e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2]-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right\}, \quad (1)$$

where g, ϕ, and f_i are the metric, dilation, and matter fields, respectively, and λ^2 is a cosmological constant. Furthermore, CGHS added to the classical action a Liouville term which accounts for the one-loop corrections due to the N matter fields. The action (1) gives rise to singular classical solutions that describe the formation of a black hole by incoming matter and the conformal anomaly of the matter fields is used to demonstrate Hawking emission [6] from this background geometry. Moreover, CGHS, by introducing anomaly-induced terms into the equations of motion, proposed a semiclassical description of the back reaction of the Hawking radiation on the geometry.

A solution in closed form for the quantum-corrected equations had not been found. This led Russo, Susskind, and Thorlacius (RST) [7], following previous work by Bilal and Callan [8] and de Alwis [9], to propose a modification of the CGHS model by adding a local covariant term to the action. The one-loop effective action of the so-called RST model is given by [7]

$$S=\frac{1}{\Pi}\int d^2x\left[e^{-2\phi}(2\partial_+\partial_-\rho-4\partial_+\phi\partial_-\phi+\lambda^2e^{2\rho})+\frac{1}{2}\sum_{i=1}^{N}\partial_+f_i\partial_-f_i-\kappa(\partial_+\rho\partial_-\rho+\phi\partial_+\partial_-\rho)\right]. \quad (2)$$

The above action is written in the conformal gauge $g_{++}=g_{--}=0$, $g_{+-}=-\frac{1}{2}e^{2\rho}$, where it assumes a local form. The constant κ in front of the one-loop quantum correction term has the value $\kappa=N/12$. The first quantum correction term in (2) arises from the one-loop conformal anomaly, and the second term is the covariant local counterterm that defines the RST model. By performing suitable field redefinitions RST were able to solve the field equations following from the model exactly. Subsequently the RST model has been investigated extensively by a number of authors [10–14].

In this work we study the model formed by adding the ϕR type of term considered by RST to the classical dilaton gravity action. In fact, as is well known, actions based on this type of term were proposed some time ago by Jackiw and, independently, by Teitelboim [15] and have since been investigated by several authors [16]. In the model we consider, the dilaton-Ricci scalar term is treated as part of the classical action and allowed to enter with an arbitrary coupling constant κ that is not restricted to have the value $N/12$ of the RST model. We demonstrate that this model can be solved exactly in the Schwarzschild-like gauge. The solution describes a black hole the mass of which is found to depend linearly on κ. We also study some classical aspects of the model. In particular we study the post-Newtonian and weak-field approximations for the model. We also set up the equation for stellar equilibrium in the model and solve it for a given equation of state and for large $|\kappa|$.

This paper is organized as follows. In Sec. II we set up the model, derive the field equations, and solve them in the Schwarzschild-like gauge. Properties of the black hole solution are described. In Sec. III we describe the post-Newtonian approximation while Sec. IV is devoted to the study of the weak-field approximation. In Sec. V we consider stellar structure and in Sec. VI we offer some concluding remarks.

II. DILATON GRAVITY MODEL

The classical action for our two-dimensional dilaton gravity model is

$$S=\frac{1}{2\Pi}\int d^2x\sqrt{-g}\left\{e^{-2\phi}[R+4(\nabla\phi)^2+J]-\frac{\kappa}{2}\phi R+L_M\right\}, \quad (3)$$

where J is a source for the dilaton field ϕ and L_M is a matter Lagrangian. The constant κ that appears before the Jackiw-Teitelboim term [15] is taken to be arbitrary. From Eq. (3) follows the dilaton equation of motion

$$\left(1+\frac{\kappa}{4}e^{2\phi}\right)R-4(\nabla\phi)^2+4\nabla^2\phi+J=0 \tag{4}$$

and the metric equations of motion

$$\left(e^{-2\phi}+\frac{\kappa}{4}\right)R_{\mu\nu}+2\left(e^{-2\phi}+\frac{\kappa}{4}\right)\nabla_\mu\nabla_\nu\phi-\frac{\kappa}{2}g_{\mu\nu}\nabla^2\phi$$
$$=8\Pi G T_{\mu\nu}. \tag{5}$$

In Eq. (5) the matter energy-momentum tensor $T_{\mu\nu}$ is given by

$$-\frac{2\Pi}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\equiv 8\Pi G T_{\mu\nu}, \tag{6}$$

where S_M is the matter action and we have made explicit Newton's constant G. In a local neighborhood that excludes the critical points of the dilaton, one can choose a gauge in which the dilaton is proportional to one of the coordinates [3]. Thus

$$\phi=\frac{1}{2}Qx. \tag{7}$$

Then choosing the time coordinate orthogonal to ϕ the metric becomes

$$g_{\alpha\beta}=\mathrm{diag}[g_{tt},g_{xx}]. \tag{8}$$

Now consider Eq. (5) in the absence of matter fields: $T_{\mu\nu}=0$. Upon using the fact that $R_{\alpha\beta}=\frac{1}{2}g_{\alpha\beta}R$ identically in two dimensions, taking the trace of Eq. (5), and feeding back into the equation we obtain

$$\nabla_\mu\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\nabla^2\phi=0. \tag{9}$$

Using Eq. (7) in Eq. (9) and by repeating the analysis of Ref. [3] one can show that the metric can be written as

$$g_{\mu\nu}=\mathrm{diag}[-g(x),g^{-1}(x)]. \tag{10}$$

Next by taking the tt or xx component of Eq. (5) we readily derive that

$$\left(e^{-Qx}+\frac{\kappa}{4}\right)g''-Q\left(e^{-Qx}-\frac{\kappa}{4}\right)g'=0, \tag{11}$$

where the prime indicates differentiation with respect to x. Turning to the dilaton Eq. (4) and taking $J=c$, a constant, we obtain a second equation for the metric function g:

$$\left(1+\frac{\kappa}{4}e^{Qx}\right)g''+Q^2g-2Qg'-c=0. \tag{12}$$

Integrating Eq. (11) yields

$$g=\frac{Ae^{Qx}}{Q\left(1+\frac{\kappa}{4}e^{Qx}\right)}+B, \tag{13}$$

where A,B are constants. Substituting Eq. (13) into Eq. (12) we determine B as

$$B=\frac{c}{Q^2}. \tag{14}$$

The constant B can be set equal to unity by a suitable rescaling of the coordinates. We then have $Q^2=c$ and writing $A/Q=-a$ we obtain

$$g=1-\frac{ae^{Qx}}{1+\frac{\kappa}{4}e^{Qx}}. \tag{15}$$

The scalar curvature corresponding to the metric (15) is given by

$$R=-\frac{aQ^2e^{Qx}\left(1-\frac{\kappa}{4}e^{Qx}\right)}{\left(1+\frac{\kappa}{4}e^{Qx}\right)^3}; \tag{16}$$

where $\kappa\to 0$, the solution given in Eq. (15) reduces to that obtained in Refs. [3] and [4] when the action does not include the Jackiw-Teitelboim term. For the sake of brevity and when comparing our results to those of Refs. [3] and [4] we shall refer to the case they considered as the $\kappa=0$ case. Below we shall list some properties of the solution given in Eq. (15).

First we note that the equation $Q^2=c$ admits of the two roots $Q=\pm\sqrt{c}$ (note that $c=4\lambda^2$ in the notation of CGHS), and hence there are two solutions that are related by a parity transformation. Next we observe that for positive $a-\kappa/4, g(x)$ has a zero at

$$\left(a-\frac{\kappa}{4}\right)e^{Qx}-1=0. \tag{17}$$

There is thus an event horizon at

$$x_0=-\frac{1}{Q}\ln\left(a-\frac{\kappa}{4}\right). \tag{18}$$

The parameters a and κ are therefore related to the mass of the black hole. The event horizon is thus shifted in position in comparison with the $\kappa=0$ case. Moreover, for $\kappa<0$, $g(x)$ has a singularity when

$$1+\frac{\kappa}{4}e^{Qx}=0. \tag{19}$$

We note that the Ricci scalar R is finite at the position of the event horizon x_0 but becomes infinite for

$$x_0'=\frac{1}{Q}\ln\frac{4}{|\kappa|}$$

when Eq. (19) holds. In the $\kappa=0$ case the solution has a curvature singularity with $|R|$ becoming infinite at $x=+\infty$ for $Q>0$ and at $x=-\infty$ for $Q<0$. In contrast we see from Eq. (16) that in our case

$$\lim_{|x|\to\infty} R=0 \tag{20}$$

for both signs of Q. The inclusion of the $\kappa\phi R$ term in the action has thus removed the curvature singularity at infinity, but as noted above, for negative κ a new curvature singularity arises at finite x.

Next we turn our attention to a determination of the mass of the black hole. For this purpose we use a formula for the mass function given by Mann [17]. Specifically for a general two-dimensional dilaton gravity action expressed as

$$S'=\int d^2x\sqrt{-g}[H(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + D(\phi)R+V(\phi;\psi_M)], \tag{21}$$

where H and D are arbitrary functions of the dilaton field ϕ and V is a matter Lagrangian depending both on ϕ and the matter field ψ_M. The action S' generalizes one considered by Banks and O'Loughlin [18] in which $H=\frac{1}{2}$ and there are no matter fields. For S' Mann obtains the following formula for the mass function:

$$M=\frac{F_0}{2}\left[\int^{\phi} ds D'V\exp\left(-\int^{s}dt\,\frac{H(t)}{D'(t)}\right) -(\nabla D)^2\exp\left(-\int^{\phi}dt\,\frac{H(t)}{D'(t)}\right)\right]. \tag{22}$$

In Eq. (22) the prime indicates differentiation with respect to the functional argument: $D'(\phi)=dD/d\phi$. F_0 is a constant defined through

$$F=F_0\int^{\phi}ds\ D'\exp\left(-\int^{s}dt\frac{H(t)}{D'(t)}\right) \tag{23}$$

and may be obtained by the requirement that $dF/dx\to 1$ for large $|x|$. The quantity M is a generalization of a mass function considered in Ref. [19]. The notes that M is constant when the equation of motion for the metric [i.e., $T_{\mu\nu}=0$] is satisfied. One also notes that Eq. (22) gives the expression for the Arnowitt-Deser-Misner (ADM) mass for the black hole obtained in Ref. [4]. Using Eqs. (22) and (23) we derive the following formula for the mass of the black hole in our model:

$$M=-\frac{Q}{2}\left(a-\frac{\kappa}{4}\right). \tag{24}$$

Requiring M to be positive we see that $a-\kappa/4$ must be positive for negative Q and vice versa.

We now turn to a closer examination of the singularity structure of the metric. For definiteness we shall take $a-\kappa/4$ to be positive and Q negative. The opposite case would be similarly handled. The singularity at $x=x_0$, Eq. (18), is a coordinate one that can be removed by a coordinate transformation. This is done in the usual manner by examining the null geodesics. For $x>x_0$ one arrives at the Kruskal coordinates

$$\bar{u}=-\left[1-\left(a-\frac{\kappa}{4}\right)e^{Qx}\right]^{1/2}\exp\left(\frac{Q\left(a-\frac{\kappa}{4}\right)(t-x)}{2a}\right),$$
$$\bar{v}=-\left[1-\left(a-\frac{\kappa}{4}\right)e^{Qx}\right]^{1/2}\exp\left(\frac{-Q\left(a-\frac{\kappa}{4}\right)(t+x)}{2a}\right), \tag{25}$$

and the metric reads

$$ds^2=-\left[\frac{2a}{Q\left(a-\frac{\kappa}{4}\right)}\right]^2\frac{\exp\frac{Q\left(a-\frac{\kappa}{4}\right)x}{a}}{1+\frac{\kappa}{4}e^{Qx}}\,d\bar{u}d\bar{v}. \tag{26}$$

Alternatively one can use the combinations

$$u=\frac{1}{2}(\bar{v}-\bar{u}),\quad v=\frac{1}{2}(\bar{v}+\bar{u}) \tag{27}$$

to obtain the more familiar form

$$u=\left[1-\left(a-\frac{\kappa}{4}\right)e^{Qx}\right]^{1/2}\exp\left(\frac{-Q\left(a-\frac{\kappa}{4}\right)}{2a}x\right)\times\cosh\left[\frac{Q\left(a-\frac{\kappa}{4}\right)}{2a}t\right],$$
$$v=\left[1-\left(a-\frac{\kappa}{4}\right)e^{Qx}\right]^{1/2}\exp\left(\frac{-Q\left(a-\frac{\kappa}{4}\right)}{2a}x\right)\times\sinh\left[\frac{-Q\left(a-\frac{\kappa}{4}\right)}{2a}t\right]. \tag{28}$$

For $x<x_0$ we have

$$u=\left[\left(a-\frac{\kappa}{4}\right)e^{Qx}-1\right]^{1/2}\exp\left(\frac{-Q\left(a-\frac{\kappa}{4}\right)}{2a}x\right)\times\sinh\left[\frac{-Q\left(a-\frac{\kappa}{4}\right)}{2a}t\right], \tag{29}$$

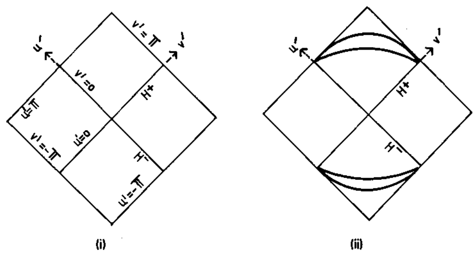

FIG. 1. (i) Penrose diagram for the case $\kappa \geqslant 0$ displaying the event horizons $H^{\pm}$. (ii) Penrose diagram for the case $\kappa < 0$. A thick curve is that of the singularity. A curve is determined by the value of the parameter $C = ae^{Q\kappa' x_0/4a}$. Curves shown correspond to two different values of $C > 1$. In the upper (lower) part of the diagram the higher (lower) curve corresponds to a large value of C. Increasing C pushes the curve towards the boundary of the diagram. For $C = 1$ the curve becomes a straight line. For the $0 < C < 1$ the curves in the upper part of the diagram arch downwards with the lowest corresponding to the smallest value of C. The opposite behavior occurs in the lower part of the diagram.

$$v = \left[\left(a - \frac{\kappa}{4}\right) e^{Qx} - 1\right]^{1/2} \exp\left(\frac{-Q\left(a - \frac{\kappa}{4}\right)}{2a} x\right) \times \cosh\left[\frac{Q\left(a - \frac{\kappa}{4}\right)}{2a} t\right].$$

Thus for $\kappa \geqslant 0$ the metric given in Eq. (26) is regular. One then carries out the transformation

$$u' = 2\arctan\bar{u}, \quad v' = 2\arctan\bar{v} \tag{30}$$

to arrive at the Penrose diagram shown in Fig. 1(i) where the line H^+ and H^-, represent the event horizons.

Next we turn to the $\kappa < 0$ case. As mentioned before, and can be seen in Eq. (26), the metric has a singularity at

$$x_0' = \frac{1}{Q} \ln\left(\frac{4}{|\kappa|}\right). \tag{31}$$

This is a true singularity as evident from the fact that the curvature invariants diverge at x_0'. Thus, for example,

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{a^2 Q^4 e^{2Qx}\left(1 - \frac{\kappa}{4} e^{Qx}\right)^2}{\left(1 + \frac{\kappa}{4} e^{Qx}\right)^6}. \tag{32}$$

In fact, in two dimensions, both the Riemann and Ricci tensors are uniquely determinable in terms of the Ricci scalar and this is singular at x_0' as can be seen from Eq. (16). We recall that for definiteness we have chosen $Q < 0$. Then for $\kappa < 0$ we can write the position of the event horizon x_0 and the singularity x_0', respectively, as

$$x_0 = \frac{1}{|Q|} \ln\left(a + \frac{|\kappa|}{4}\right),$$

$$x_0' = \frac{1}{|Q|} \ln\left(\frac{|\kappa|}{4}\right). \tag{33}$$

Clearly for $a > 0$ we have $x_0 > x_0'$ and the singularity lies to the left of the event horizon. If $a < 0$ and $a + |\kappa|/4 > 0$, then we have $x_0 < x_0'$ and the singularity lies to the right of the event horizon. Thus for both cases the solution has two asymptotic regions: one with a naked singularity and the other with the singularity shielded by an event horizon. For the values of Q and $a + |\kappa|/4$ under consideration, the mass of the black hole, as given by Eq. (24), is positive. For a black hole of negative mass, and with Q being negative, Eq. (24) implies that $a - \kappa/4 < 0$. We then note that for a negative value of $a - \kappa/4$, Eq. (17) has no solution for real x. Thus for a black hole of negative mass there is no associated event horizon and only a naked singularity can arise for $\kappa < 0$. If $\kappa > 0$, then the solution for a blackhole of negative mass is regular everywhere.

The singularity curve in the Penrose diagram is described by the equation

$$u' = 2\arctan\left(\frac{a e^{\frac{Q\kappa x_0'}{4a}}}{\tan\frac{1}{2} v'}\right). \tag{34}$$

It intersects the u' and v' axes at $\pm\Pi$ and is shown schematically in Fig. 1(ii). Its precise shape, however, depends on the values assigned to the parameters Q, κ, a.

Finally we note that for $a = \kappa/4$ there is no event horizon, the metric being regular for $\kappa \geqslant 0$ while for $\kappa < 0$ there arises the singularity x_0' described above.

III. POST-NEWTONIAN CALCULATIONS

In this section we describe the post-Newtonian approximation [20] for our model. The aim of this approximation is to supply higher order terms in the expansion of physical quantities, the small quantity being the square of the speed $\bar{v}^2$. Such calculations have been carried out previously for the $R = T$ model by Sikkema and Mann [2] and for the dilaton gravity model without the Jackiw-Teitelboim term by Mann and Ross [21]. We intend to examine in particular the effects arising from the presence of this term.

Following Weinberg [20] we write the metric as

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \cdots,$$

$$g_{ij} = \delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \cdots,$$

$$g_{i0} = \overset{3}{g}_{i0} + \overset{5}{g}_{i0} + \cdots, \tag{35}$$

where $\overset{N}{g}_{\mu\nu}$ indicates the terms in $g_{\mu\nu}$ of order $\bar{v}^N$. In the harmonic gauge defined by

$$g^{\mu\nu} \Gamma^{\lambda}_{\mu\nu} = 0, \tag{36}$$

the components of the Ricci tensor are given by

$$\overset{2}{R}_{00}=-\frac{1}{2}\partial_1^2\overset{2}{g}_{00},$$
$$\overset{4}{R}_{00}=-\frac{1}{2}\partial_1^2\overset{4}{g}_{00}+\frac{1}{2}\partial_0^2\overset{2}{g}_{00}+\frac{1}{2}\overset{2}{g}_{11}\partial_1^2\overset{2}{g}_{00}-\frac{1}{2}(\partial_1\overset{2}{g}_{00})^2,$$
$$\overset{3}{R}_{01}=0,$$
$$\overset{2}{R}_{11}=-\frac{1}{2}\partial_1^2\overset{2}{g}_{11}. \tag{37}$$

In (37) ∂_1 denotes $\partial/\partial x$. We also write

$$T_{00}=\overset{0}{T}_{00}+\overset{2}{T}_{00}+\cdots,$$
$$T_{01}=\overset{1}{T}_{01}+\overset{3}{T}_{01}+\cdots,$$
$$T_{11}=\overset{2}{T}_{11}+\overset{4}{T}_{11}+\cdots, \tag{38}$$

for the energy density, the momentum density, and momentum flux, respectively. We also have

$$\phi=\overset{2}{\phi}+\overset{4}{\phi}+\cdots,$$
$$J=c+\overset{2}{J}+\overset{4}{J}+\cdots. \tag{39}$$

Next we expand the field equations in powers of $\overline{v^2}$. From the 00 component of Eq. (5) we then readily obtain that

$$\partial_1^2\left[-\left(1+\frac{\kappa}{4}\right)\overset{2}{g}_{00}+\kappa\overset{2}{\phi}\right]=16\Pi G\overset{0}{T}_{00}. \tag{40}$$

Integrating (40) we obtain

$$-\left(1+\frac{\kappa}{4}\right)\overset{2}{g}_{00}+\kappa\overset{2}{\phi}=4\xi, \tag{41}$$

where

$$\xi(x,t)=2\pi G\int dx'|x-x'|\overset{0}{T}_{00}(x',t) \tag{42}$$

is the Newtonian potential. Similarly from the 11 component of Eq. (5) we derive that

$$\overset{2}{g}_{11}=4\left(1+\frac{\kappa}{4}\right)^{-1}\overset{2}{\phi}. \tag{43}$$

An examination of Eq. (4) to order 2 gives

$$\overset{2}{g}_{11}=4\left(1+\frac{\kappa}{2}\right)^{-1}\beta, \tag{44}$$

where the field β is defined by

$$\beta(x,t)=\int dx'|x-x'|\left(\frac{1}{2}\partial_1'^2\xi-\frac{1}{4}\overset{2}{J}\right)$$
$$=\xi-\frac{1}{4}\int dx'|x-x'|\overset{2}{J}. \tag{45}$$

From (43) and (44) we then have that

$$\overset{2}{\phi}=\left(1+\frac{\kappa}{2}\right)^{-1}\beta. \tag{46}$$

From (41) and (46) we obtain

$$\overset{2}{g}_{00}=\kappa\left(1+\frac{\kappa}{4}\right)^{-2}\beta-4\left(1+\frac{\kappa}{4}\right)^{-1}\xi. \tag{47}$$

Next from the 01 component of Eq. (5) we find that

$$\partial_0\partial_1\beta=4\Pi G\overset{1}{T}_{01}. \tag{48}$$

In particular $\overset{3}{g}_{01}$ is not constrained by this equation and in fact from the harmonic gauge condition (36) we have that

$$\partial_1^2\overset{3}{g}_{01}=0. \tag{49}$$

The solution of (49) that vanishes at infinity is

$$\overset{3}{g}_{01}=0. \tag{50}$$

Finally we compute $\overset{4}{g}_{00}$ by considering the 00 component of Eq. (5) to order 4. After some algebra we find that

$$\overset{4}{g}_{00}=\left(1+\frac{\kappa}{4}\right)^{-1}\int dx'|x-x'|\Bigg\{-8\Pi G\overset{2}{T}_{00}+2\partial_0^2\beta-2\partial_0^2\xi-\frac{\kappa^2}{2}\left(1+\frac{\kappa}{4}\right)^{-3}(\partial_1'\beta)^2$$
$$-8\left(1+\frac{\kappa}{4}\right)^{-1}(\partial_1'\xi)^2+4(\kappa-1)\left(1+\frac{\kappa}{4}\right)^{-2}\partial_1'\beta\partial_1'\xi-\frac{1}{2}\left(1+\frac{\kappa}{4}\right)^{-3}(\kappa^2+4\kappa+24)\beta\partial_1'^2\xi-\frac{\kappa}{2}\left(1-\frac{\kappa}{2}\right)\left(1+\frac{\kappa}{4}\right)^{-3}\beta\overset{2}{J}$$
$$+2\kappa\left(1+\frac{\kappa}{4}\right)^{-2}\xi\left(\partial_1'^2\xi-\frac{1}{2}\overset{2}{J}\right)\Bigg\}. \tag{51}$$

From the equations for the metric tensor components one can proceed and compute the Christoffel symbols and the Ricci tensor.

Our results show some differences from those of Ref. [21] which correspond to the limit $\kappa\to 0$ in our equations. The most notable difference concerns $\overset{3}{R}_{01}$ which these authors claim to be different from zero and hence use it to generate a nonzero result for $\overset{0}{g}_{31}$. In fact one easily finds from the results of Ref. [20] that in two spacetime dimensions $\overset{3}{R}_{01}$ vanishes identically even before the selection of the harmonic gauge.

IV. WEAK-FIELD APPROXIMATION

We now discuss the weak-field approximation in our model. In this section we denote the dilaton field by Φ. The metric is considered to be a perturbation on a Minkowski background and the dilaton field to be a perturbation about a solution Φ of the vacuum equations

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}, \quad \Phi=\phi+\varphi. \tag{52}$$

We also take the source J to be

$$J=c\epsilon+\mathscr{J}. \tag{53}$$

Two possible choices for the vacuum exist, namely, $T_{\mu\nu}=0$ and $J=0$ or $T_{\mu\nu}=0$ and $J=c$. These correspond, respectively, to $\epsilon=0$ or $\epsilon=1$.

We first treat the $\epsilon=0$ case. We take the trace of Eq. (5) and in the resulting equation, and in Eq. (4) as well, we substitute $\Phi=\phi+\varphi$ and $J=\mathscr{J}$. We then deduce from the zeroth order equations that $\phi=0$. For the linear order Ricci scalar $R^{(1)}$ we obtain the equation

$$\left(1+\frac{\kappa}{4}\right)^2 R^{(1)}=16\Pi GT+\left(1-\frac{\kappa}{4}\right)\mathscr{J}. \tag{54}$$

Now in two dimensions, as a consequence of the identity $R_{\mu\nu}=\frac{1}{2}g_{\mu\nu}R$, we have the relationship

$$\partial^2 h_{\mu\nu}=\frac{1}{2}\,\eta_{\mu\nu}\partial^2 h. \tag{55}$$

Hence we may choose coordinates so that $h_{\mu\nu}=\frac{1}{2}\eta_{\mu\nu}h$ and (54) becomes

$$\left(1+\frac{\kappa}{4}\right)^2 \partial^2 h_{\mu\nu}=-32\Pi GT-\left(2-\frac{\kappa}{2}\right)\mathscr{J}. \tag{56}$$

The solution of this wave equation is

$$h(x,t)=\pm 16\Pi G\left(1+\frac{\kappa}{4}\right)^{-2}\int dx' \times\int^t dt'\,\mathscr{H}(x',t',\mp|x-x'1|), \tag{57}$$

where

$$\mathscr{H}(x,t)=T(x,t)+\frac{\left(1-\frac{\kappa}{4}\right)}{16\Pi G}\mathscr{J}(x,t). \tag{58}$$

To the right-hand side (RHS) of (57) we may add any solution of the homogeneous equation $\partial^2 h=0$. Note that h decreases as κ increases and thus the spacetime becomes increasingly flat with increasing κ.

Consider now a system of oscillating matter such that T and $\mathscr{J}$ can be expressed as a sum over frequencies or as a Fourier integral [20]. A single Fourier component is described as

$$\mathscr{H}(x,t)=H(x,w)e^{-iwt}+\text{c.c.} \tag{59}$$

The retarded potential solution is then

$$h(x,t)=\pm 16\pi G\left(1+\frac{\kappa}{4}\right)^{-2}\int dx' \times\int^t dt'\,H(x',w)e^{-iw(t'-|x-x'1|)}+\text{c.c.} \tag{60}$$

For a point in space located in the wave zone outside the source such that $r=|x|\gg R$, where R is the maximum extension of a finite source, we then have that

$$h(x,t)=16i\Pi\left(1+\frac{\kappa}{4}\right)^{-2}w^{-1}e^{-iwt} \times\int dx'\,H(x',w)e^{-iw(r--x'\hat{x})}+\text{c.c.}, \tag{61}$$

where $\hat{x}=x/r$. As wr is assumed large this looks just like a plane wave with

$$h(x,t)=e(x,w)e^{ik_\mu x^\mu}+\text{c.c.} \tag{62}$$

and where

$$e(x,w)=16i\Pi G\left(1+\frac{\kappa}{4}\right)^{-2}w^{-1}\int dx'\,H(x',w), \qquad k_0=w, \quad k_1=w\hat{x}. \tag{63}$$

Next we turn to the case of the dilaton vacuum where $\epsilon=1$. The full system of equations has the solution given in Eqs. (7) and (15). It is convenient when discussing the weak-field expansion to redefine variables. Thus we perform the conformal transformation

$$g_{\mu\nu}=e^{2\Phi}\hat{g}_{\mu\nu} \tag{64}$$

in Eqs. (5) and (4). This leads to the equations

$$-\left(1+\frac{\kappa}{4}e^{-2\Phi}\right)\hat{\nabla}_\mu\hat{\nabla}_\nu e^{-2\Phi}+\hat{g}_{\mu\nu}\left\{\hat{\nabla}^2 e^{-2\Phi}+\frac{\kappa}{2}[(\hat{\nabla}\Phi)^2-\hat{\nabla}^2\Phi]\right\}=8\Pi GT_{\mu\nu}+\frac{1}{2}J\hat{g}_{\mu\nu}, \tag{65}$$

$$\left(e^{-2\Phi}+\frac{\kappa}{4}\right)\hat{R}-\kappa\hat{\nabla}^2\Phi=8\Pi G\hat{g}^{\mu\nu}T_{\mu\nu}. \tag{66}$$

With $\sigma\equiv e^{-2\Phi}$ the weak-field approximation is now defined by [21]

$$\hat{g}_{\mu\nu}=\eta_{\mu\nu}+\hat{h}_{\mu\nu}, \quad \sigma=\sigma_0+\varphi. \tag{67}$$

From Eq. (65) we then derive at the zeroth level that

$$\left(\sigma_0^2+\frac{\kappa}{4}\sigma_0\right)\partial^2\sigma_0-\frac{\kappa}{4}(\partial\sigma_0)^2=c\sigma_0^2, \tag{68}$$

where $\partial^2=\eta^{\mu\nu}\partial_\mu\partial_\nu$. We look for solutions of (68) such that

$$\sigma_0=\sigma_0(\tau), \quad \tau=(x-x_0)^2. \tag{69}$$

Using (69) in (68) yields

$$4\left(\sigma_0^2+\frac{\kappa}{4}\sigma_0\right)(\tau\sigma_0''+\sigma_0')-\kappa\tau\sigma_0'^2=c\sigma_0^2, \tag{70}$$

where the prime denotes differentiation with respect to τ. It is difficult to solve Eq. (70) in a closed form. We can easily obtain a series solution

$$\sigma_0=\sum_{j=0}^{\infty} a_j\tau^j, \tag{71}$$

where the first few terms are given by (assuming $a_0\neq 0$)

$$a_1=\frac{ca_0}{4a_0+\kappa},$$

$$a_2=\frac{ca_0}{2(4a_0+\kappa)^2}\left(1-\frac{4a_0c}{4a_0+\kappa}\right), \tag{72}$$

$$a_3=\frac{1}{9a_0(4a_0+\kappa)}[c(a_1^2+2a_0a_2)-40a_0a_1a_2-4a_1^3 - \kappa a_1a_2], \text{ etc.}$$

The series solution is, however, only useful for small τ and to gain insight into the nature of the closed form solutions of Eq. (70) we shall investigate two limits, namely, those of small and large values of $|\kappa|$, respectively.

For small $|\kappa|$ we seek solutions of (70) in the form

$$\sigma_0=\bar{\sigma}_0+\kappa\psi, \tag{73}$$

where $\bar{\sigma}_0$ is a solution of the $\kappa=0$ equation [21]:

$$\bar{\sigma}_0=\frac{c}{4}\tau-M, \tag{74}$$

with M being a constant. Using (73) in (70) yields an equation for ψ which is easily solved to give

$$\psi=-\frac{1}{4}\int\frac{\ln\left(\tau-\frac{4M}{c}\right)}{\tau}d\tau+A\ln\tau+B, \tag{75}$$

where A,B are constants. The metric $g_{\mu\nu}$ is thus determined to be

$$g_{\mu\nu}=\frac{\eta_{\mu\nu}}{\frac{c}{4}\tau-M_0}\left\{1+\frac{\kappa}{c\tau-4M_0}\times\left[\int\frac{\ln\left(\tau-\frac{4M}{c}\right)}{\tau}d\tau-4A\ln\tau\right]\right\}, \tag{76}$$

where $M_0=M-\kappa B$. The first term on the RHS of (76) is equivalent to the black hole solution of Refs. [3,4] under a change of coordinates [21]. The κ term then represents a first order perturbation about the black hole solution.

Next we consider the large $|\kappa|$ limit and approximate Eq. (70) to read

$$\kappa\sigma_0(\tau\sigma_0''+\sigma_0')-\kappa\tau\sigma_0'^2=c\sigma_0^2. \tag{77}$$

Note that we have kept the term on the RHS of (60) but dropped the term in the coefficient of $\tau\sigma_0''+\sigma_0'$ that does not involve κ. By making the transformation $\sigma_0=e^{\zeta}$ it is possible to solve (77) and thereby obtain that

$$\sigma_0=\frac{\beta}{\tau^{\alpha}}e^{\tau c/\kappa}, \tag{78}$$

thereby leading to

$$g_{\mu\nu}=\frac{\tau^{\alpha}}{\beta}e^{-\tau c/\kappa}\eta_{\mu\nu}, \tag{79}$$

where α,β are constants.

We now go back to Eq. (65) and seek an equation for the dilaton perturbation using (67). We derive that

$$\left(1+\frac{\kappa}{4\sigma_0}\right)\partial^2\varphi+\frac{2\varphi}{\sigma_0}\left(1+\frac{\kappa}{8\sigma_0}\right)\partial^2\sigma_0-\frac{2c\varphi}{\sigma_0} -\frac{\kappa}{2\sigma_0^2}\eta^{\alpha\beta}\partial_\alpha\sigma_0\partial_\beta\varphi=8\Pi GT+\frac{1}{2}c\hat{h}+\mathscr{J}. \tag{80}$$

This equation is difficult so solve in a closed form and so as an illustration we consider the large $|\kappa|$ limit in which it simplifies to become

$$\sigma_0\partial^2\varphi+\varphi\partial^2\sigma_0-2\eta^{\alpha\beta}\partial_\alpha\sigma_0\partial_\beta\varphi\simeq 0. \tag{81}$$

To arrive at (81) we have retained only terms that are proportional to κ in (80). We now take $\varphi=\varphi(\tau)$ and substitute for σ_0 from (78) to obtain

$$\tau^2\varphi''+(1+2\alpha)\tau\varphi'-\alpha\varphi\simeq 0. \tag{82}$$

Using the substitution $\tau=e^u$ we can solve (82) to get

$$\varphi\simeq B_1\tau^{m_1}+B_2\tau^{m_2}, \tag{83}$$

where

$$m_{1,2}=-\alpha\pm(\alpha^2+\alpha)^{1/2}. \tag{84}$$

To determine $\hat{h}$ we consider Eq. (66) to first order and thereby obtain

$$\hat{h}=\pm\frac{1}{2}\int dx'\int^t dt' N(x',t'\mp|x-x'|), \tag{85}$$

where

$$N\equiv\left(e^{-2\Phi}+\frac{\kappa}{4}\right)^{-1}(16\Pi GT+2\kappa\hat{\nabla}^2\Phi). \tag{86}$$

In the large $|\kappa|$ limit (85) simplifies to

$$\hat{h}=\pm 4\int dx'\int^{t}dt'\hat{\nabla}^2\Phi(x',t'\mp|x-x'|). \qquad (87)$$

Finally we note that including the first order perturbations gives the metric as

$$g_{\mu\nu}=\sigma_0^{-1}\left(1+\frac{1}{2}\hat{h}-\frac{\varphi}{\sigma_0}\right)\eta_{\mu\nu}. \qquad (88)$$

We were able to obtain an explicit expression for $g_{\mu\nu}$ only in the large $|\kappa|$ limit. This is accomplished through the results (78) and (73) which also determine $\hat{h}$ via (87).

V. STELLAR STRUCTURE

In this section we study the equations that govern the existence of "stars" in this two-dimensional universe. To do that we consider a static metric of the form

$$ds^2=-B^2(x)dt^2+dx^2. \qquad (89)$$

Using $R_{\mu\nu}=\frac{1}{2}g_{\mu\nu}R$ in Eq. (5) and substitution for R from the trace of Eq. (5) gives

$$2\left(e^{-2\phi}+\frac{\kappa}{4}\right)\nabla_\mu\nabla_\nu\phi-g_{\mu\nu}e^{-2\phi}\nabla^2\phi-\frac{\kappa}{4}g_{\mu\nu}\nabla^2\phi$$
$$=8\Pi G\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right). \qquad (90)$$

We employ the energy-momentum tensor for a perfect fluid,

$$T_{\alpha\beta}=\rho g_{\alpha\beta}+(p+\rho)U_\alpha U_\beta. \qquad (91)$$

Using (89) and (91) in (90) then yields

$$\left(e^{-2\Phi}+\frac{\kappa}{4}\right)\left(\phi''-\frac{B'}{B}\phi'\right)=4\Pi G(p+\rho). \qquad (92)$$

Next we take the trace of Eq. (5) and substitute for $\nabla^2\phi$ from Eq. (4) into the resulting equation. This leads to

$$\phi'^2=4\Pi G(p-\rho)e^{2\phi}\left(1-\frac{\kappa}{4}e^{2\phi}\right)^{-1}$$
$$+\left(1+\frac{\kappa}{4}e^{2\phi}\right)^2\left(1-\frac{\kappa}{4}e^{2\phi}\right)^{-1}\frac{B''}{B}+\frac{1}{4}J. \qquad (93)$$

Now we substitute from (92) for ϕ'' in the formula for $\nabla^2\phi$,

$$\nabla^2\phi=\phi''+\frac{B'}{B}\phi', \qquad (94)$$

and employ the resulting expression in the trace of Eq. (5). This procedure gives

$$\frac{B'}{B}\phi'=\delta\frac{B''}{B}+\epsilon, \qquad (95)$$

where

$$\delta=\frac{1}{2}\left(1+\frac{\kappa}{4}e^{2\phi}\right)\left(1-\frac{\kappa}{4}e^{2\phi}\right)^{-1},$$

$$\epsilon=-\frac{4\Pi Ge^{2\phi}\left(\rho-\frac{\kappa}{4}pe^{2\phi}\right)}{\left(1+\frac{\kappa}{4}e^{2\phi}\right)\left(1-\frac{\kappa}{4}e^{2\phi}\right)}. \qquad (96)$$

The equation of hydrostatic equilibrium [20] reads in our case

$$\frac{B'}{B}=-\frac{p'}{p+\rho}. \qquad (97)$$

Using (97) in (95), squaring, and equating to (93) gives a quadratic equation for B''/B which can be solved to give

$$\frac{B''}{B}=8\Pi Ge^{2\phi}\left(1+\frac{\kappa}{4}e^{2\phi}\right)^{-2}\left(\rho-\frac{\kappa}{4}pe^{2\phi}\right)+\left(1-\frac{\kappa}{4}e^{2\phi}\right)\left(\frac{p'}{p+\rho}\right)^2$$
$$\pm\left(1+\frac{\kappa}{4}e^{2\phi}\right)\left(1-\frac{\kappa}{4}e^{2\phi}\right)\frac{p'}{p+\rho}\left[16\Pi Ge^{2\phi}p+\left(1+\frac{\kappa}{4}e^{2\phi}\right)^2\left(\frac{p'}{p+\rho}\right)^2+J\right]^{1/2}. \qquad (98)$$

Now differentiating (97) gives

$$p''=-(p+\rho)\frac{B''}{B}+\frac{p'(2p'+\rho')}{p+\rho}. \qquad (99)$$

With B''/B given by (98) we see that given an equation of state $p=p(\rho)$, the solution of (98) will give ρ and hence the metric. The resulting equation is, however, quite complicated and difficult to solve in general. Once again we study the large $|\kappa|$ limit in which (98) becomes

$$\frac{B''}{B}\simeq-\frac{\kappa e^{2\phi}}{2}\left(\frac{p'}{p+\rho}\right)^2 \text{ or } 0. \qquad (100)$$

Concentrating on the first case in (100) and for a general equation of state $p=(\gamma-1)\rho$, Eq. (99) then gives

$$\rho\rho''+\beta\rho'^2=0, \qquad (101)$$

where

$$\beta=\frac{-\frac{\kappa}{2}(\gamma-1)e^{2\phi}+2\gamma-1}{\gamma}. \tag{102}$$

For constant ϕ and by multiplying (101) by $\rho^{\beta-1}$ we can solve and obtain

$$\rho=(\beta+1)^{1/(\beta+1)}(Fx+G)^{1/(\beta+1)}, \tag{103}$$

where F,G are constants. With this expression for ρ Eq. (95) then gives

$$B''+\frac{\xi}{(Fx+G)^2}B=0, \tag{104}$$

where

$$\xi=\frac{\kappa e^{2\phi}(\gamma-1)}{2\gamma^2(\beta+1)^2}. \tag{105}$$

We solve (104) and finally obtain, for the metric,

$$B=C_1(Fx+G)^{\nu_1}+C_2(Fx+G)^{\nu_2}, \tag{106}$$

where

$$\nu_{1,2}=\frac{1}{2}\pm\frac{1}{2}\left(1-\frac{4\xi}{F^2}\right)^{1/2}. \tag{107}$$

We observe that for $|\kappa|\to\infty$, $\nu_{1,2}\to 1{,}0$ and both the metric and curvature remain finite.

Finally we note that in the Newtonian limit $p\to 0$, $\phi\to 0$, $J\to 0$, (98) reduces to

$$\frac{B''}{B}=\frac{8\Pi G\rho}{1+\frac{\kappa}{4}}. \tag{108}$$

Comparing this to the result of Ref. [21] we see that it differs by a factor $(1+\kappa/4)^{-1}$ multiplying G. Now the Newtonian equation for stellar structure is

$$-p''=4\Pi G\rho^2-\frac{p'\rho'}{\rho}. \tag{109}$$

Thus Eq. (99), in the Newtonian limit with B''/B given by (108), will differ from Eq. (109) in that G appears multiplied by $2(1+\kappa/4)^{-1}$.

VI. CONCLUSIONS

In this work we considered the model of two-dimensional dilaton gravity inspired by string theory modified by the inclusion of Jackiw-Teitelboim gravity with an arbitrary coupling κ. We solved the model exactly in the Schwarzschild-like gauge, obtaining a black hole solution. This solution is a generalization of the one found previously in dilaton gravity and reduces to it when the coupling of the additional term is made to vanish. Including an energy-momentum tensor for matter we also examined the post-Newtonian and weak-field approximation in the model as well as stellar structure. The post-Newtonian approximation is fairly simple and similar to that of general relativity. In the weak-field approximation the situation was rather complicated and we could obtain some explicit results only in limiting cases of small or large $|\kappa|$. Similar remarks apply to stellar structure where we could compute the metric for the situation of large $|\kappa|$ only.

Our treatment in this work has been completely classical. Now the great attention devoted in recent years to two-dimensional gravity theories stems from the fact that they provide an arena for the investigation of the dynamical processes of black hole formation and evaporation and the attendant issues of information loss. Most treatments of such issues so far have been semiclassical. Attempts to go beyond the semiclassical approximation and quantize two-dimensional dilaton gravity have been made by several authors and we cite in particular the work of Hirano *et al.* [22] and that of Louis-Martinez *et al.* [23]. It would be interesting to investigate the quantization of the model discussed in this work. Moveover, one should examine how the spacetime geometry can be extracted from the quantitized model [24]. We hope to return to these and other issues in the near future.

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