Quantum formation of black holes and wormholes in the gravitational collapse of a dust shell

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A quantum-mechanical model of a self-gravitating dust shell is considered. To clarify the relation between classical and quantum spacetime which the shell collapse forms, we consider various time slicings on which quantum mechanics is developed. By considering the static time slicing which corresponds to an observer at a constant circumference radius, we obtain the wave functions of the shell motion and the discrete mass spectra which specify the global structures of spherically symmetric spacetime formed by the shell collapse. It is found that wormhole states are forbidden when the rest mass is comparable to the Planck mass scale due to the zero-point quantum fluctuations.

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I. INTRODUCTION

Historically, many works have been devoted to constructing quantum theories of gravitational collapse. One of the motivations is to resolve the problems concerning the final fates of black hole evaporation due to Hawking radiation [1], which may lead to the paradox of information loss [2]. However, quantum gravity itself includes not only technical difficulties but also conceptual ones such as the interpretation of wave function, the nature of time, and the definition of observables. Since any fully consistent theory is not yet established, the present step would be to develop various useful toy models to shed a new light on some features of quantum effects of gravity. In particular, quantum gravitational collapse of a spherically symmetric shell, on which we focus our attention in this paper, has been studied as one of such toy models [3–7].

In this model, one considers the spherically symmetric spacetime which is in vacuum except the δ-function distribution of incoherent dust shell at a finite circumference radius. By the virtue of the spherical symmetry, the inner side of the shell may be a Minkowski spacetime and the outer one is the Schwarzschild spacetime which has a finite gravitational mass parameter M. The vanishing divergence of the energy-momentum tensor of the dust shell tells us only the fact that the matter energy μ of the shell is conserved and μ can be regarded as the rest mass of the shell. The two vacuum regions should be joined at the shell according to the Einstein equations. This program can be accomplished by using Israel's junction condition [8,9], which works as the equation of motion of the dust shell, and the dynamical variable is limited to the circumference radius R of the shell and the global structure of spacetime can be classified by the value of $E \equiv M/\mu$.

The original idea to quantize this dynamical system was proposed by Berezin *et al.* [3,4]. If any quantum effect of incoherent dust shell is ignored, this matter shell can be characterized by μ. Both the vacuum regions in the spacetime are also classically treated. Then, only the equation of motion of the shell which can be regarded as the energy equation is quantized by constructing the Hamiltonian operator. Although the Hamiltonian operator to be constructed is not unique, Berezin *et al.* investigated the Hamiltonian operator

$$H_B = \mu \cosh\left(-\frac{i m_P}{m} \frac{\partial}{\partial R} \right) - \frac{\mu^2}{2R}, \tag{1.1}$$

where $m = \mu/m_P$ and m_P is the Planck mass (throughout this paper we denote the Planck constant by $\hbar = m_P^2$ and use units such that $G = c = 1$) and the time coordinate describing the shell dynamics is chosen to be the proper time of a comoving observer. Then, they obtained the spectrum of the eigenvalues E of H_B,

$$E = 1 - \frac{m^4}{8(n+1)^2}, \tag{1.2}$$

by using WKB approximation where n is a non-negative integer. From their analysis, it is unclear that the spectrum (1.2) is also valid when $m^2 >> n$. However, Hájíček *et al.* [7] gave a definite formula of the spectrum E adapting a simpler equation for the wave function. They introduce the super-Hamiltonian on an extended minisuperspace which leads to the "Wheeler-DeWitt" equation

$$\left(-i\hbar \frac{\partial}{\partial T} - \frac{\mu^2}{2R} \right)^2 \Psi + \frac{\partial^2}{\partial R^2} \Psi - \mu^2 \Psi = 0, \tag{1.3}$$

where T is the Minkowskian time of the inner side of the shell. Furthermore, they introduced "energy" and "charge" currents and by imposing the boundary condition for Ψ which means that "energy" and "charge" currents must vanish at $R = 0$ and infinity, they obtained the formula of E for the bound states defined by $-1 < E < 1$:

$$E = \frac{2(\kappa + n)}{\sqrt{m^4 + 4(\kappa + n)^2}}, \quad \kappa = \frac{1}{2} + \frac{1}{2}\sqrt{1 - m^4}. \tag{1.4}$$

Nevertheless the two quantum treatments arrive at the same conclusion when $m^2 << n$, (1.4) becomes meaningless when $\mu > m_P$ which means classical limit cannot be obtained from

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this eigenvalue. Furthermore, it is shown that in the bound states only the black hole formation corresponding to the range $1/2<E<1$ is allowed, while the wormhole formation corresponding to $0<E<1/2$ is possible as classical solutions.

Here, we must mention about the quantum effects in this system. Only quantum effect of gravity which can be considered by this model is that of the shell motion. There are many gravitational effects in general relativity, and this shell motion is also one of them. In this model, the values of E determine not only the shell motion but also global structure of the spacetime (the black hole formation or wormhole formation). If the shell motion changes, this change affects to the global structure of the spacetime. Then, nevertheless both vacuum regions of spacetime are treated classically, to consider the quantization of the shell motion also means the quantization of the spacetime through the quantized values of E. In this sense, the results obtained by Hájíček *et al.* means that only black hole formation is possible as a result of the quantum collapse of the dust shell, while both black hole and wormhole formation is possible as a classical solution. Then, it is not clear whether one can consider the classical limit of this quantum collapse of the shell.

Our purpose in this paper is to clarify the relation between quantum and classical spacetime of this system. For this purpose, we pay attention to the time slicing on which the quantum mechanics is developed. Note that the previous treatments are essentially based on the comoving time slicing. It is also possible to construct quantum mechanics based on the other time slicing, in which the dust shell is really moving. Since the canonical formalism is based on the decomposition of spacetime into space and time, a special attention must be paid to the problem of time slicing which determines the foliation of spacetime. Furthermore, it is well known in classical relativity how a description of black hole spacetime depends on the choice of time slicing: In the usual static chart the Schwarzschild horizon plays a role of the infinite redshift surface, while in the synchronous chart corresponding to a freely falling observer it is not any special surface [10]. This means that the horizon can be regarded as a sort of boundary of the foliated spacetime only for a static observer. So the foliation of spacetime or the choice of observers in the spacetime is more important when one consider a black hole spacetime.

To study the quantum mechanics of dust shell collapse, we use various time slicings in hope that the physical essence is independent of the time slicing. The black hole horizon is not a special surface for an observer who uses the proper time along the shell history or the Minkowskian time. However, a static observer outside the horizon will require a different boundary condition for the wave function due to the existence of the horizon for him. By developing quantum mechanics for a static observer, we can obtain the mass spectrum which corresponds to (1.2) and (1.4). Furthermore, we will show that wormhole states are also possible for a static observer who stays inside the wormhole but when μ is same order of m_P no wormhole state is allowed owing to the zero-point fluctuation of the shell motion.

This paper is organized as follows. In Sec. II, we derive the classical equation of motion of the shell in terms of various observers who define time slicings on the shell trajectory in spacetime. Although infinitely many observers or time slicings can be assumed in general, we restrict our consideration to a one-parameter family which contains two typical time slicings corresponding to the Gaussian normal coordinates of a comoving observer and the static Schwarzschild coordinates of an observer who stays at a finite circumference radius, and derive the possible spacetimes as the solutions of the Einstein equations. In Sec. III, we introduce the Hamiltonian constraint which corresponds to the time-time component of the Einstein equations and gives the equation of motion in a similar manner to the Wheeler-DeWitt procedure. Following the earlier works, we solve classically the Einstein equations except the Hamiltonian constraint for the shell motion. This means that the momentum constraint is classically treated just like many minisuperspace models.

In Sec. IV, we consider the quantum mechanics of collapsing shell, by using the one-parameter family of time slicings introduced in the previous sections and we also discuss the global structure of spacetime in which the dust shell collapse forms. Our main result will be obtained under the static time slicing. Furthermore, our consideration is restricted to the so-called bound states $-1<E<1$, which have the discrete mass eigenvalues, and we must consider the cases $E>1/2$ and $E<1/2$ separately in the static time slicing, these cases correspond to the black hole states and wormhole states respectively. Then, we discuss the relation between quantum and classical solutions of this system and show that there is no quantum states when $E<1/2$ and $\mu\sim m_P$. We also consider the quantum mechanics on nonstatic time slicing to confirm that our arguments are natural extension of the result obtained in the comoving frame. It can be shown by the fact that the quantum version of the Hamiltonian constraint becomes identical with the radial equation of (1.3) in the comoving limit. Finally, our consideration is summarized and we make some discussions in Sec. V. Although our simple model is a preliminary approach to quantum gravity, it would give a useful clue when one investigates the problem of time slicing in a more complete theory.

II. CLASSICAL EQUATION OF MOTION

In this section, we derive the equation of motion for a collapsing dust shell, which is described by various time slicing. Since the spacetime is spherically symmetric, one can choose the metric of the form

$$ds^2=g_{ab}dx^a dx^b+R^2 d\Omega_2^2 \quad (a,b=0,1), \tag{2.1}$$

where R is the circumference radius and $d\Omega_2^2$ is the metric of unit two-sphere. For this metric, matter fields depend on x^0 and x^1 which are time and radial coordinates. Furthermore, the spacetime is assumed to be in vacuum except the δ-function matter distribution at a finite circumference radius. We denote the world volume of this spherically symmetric shell as Σ which is a $(1+2)$-dimensional hypersurface. In a neighborhood of Σ, one can choose the Gaussian normal coordinates [9]

$$ds^2=-d\tau^2+d\eta^2+R^2 d\Omega_2^2, \tag{2.2}$$

where τ is the proper time which would be measured by an observer comoving with the shell. The coordinate η is the

proper distance from Σ along the geodesics which are orthogonal to Σ, and Σ is assigned to the world volume of $\eta=0$. Imposing the spacetime to be orientable on Σ, we may regard one side of Σ as being the ''outer side'' ($\eta>0$) and the other side as being ''inner side'' ($\eta<0$).

The whole spacetime is constructed by connecting the Schwarzschild spacetimes of the outer side and the Minkowski spacetime of the inner side. The Einstein equations give the ''junction condition'' for the neighborhood of Σ, which is well known as Israel's formula [8,9]. In the case of the Minkowski-Schwarzschild junction, the nontrivial conditions are given by [9]

$$[\partial_\eta R]=-\frac{\mu}{R},\quad [\partial_\tau A]=0, \tag{2.3}$$

where ∂ denotes the partial derivative with respect to its subscripted coordinate, and A represents all the metric functions on the spacetime. The square brackets $[A]$ mean the difference of A between the outer and inner sides:

$$[A]=\lim_{\eta\to+0} A-\lim_{\eta\to-0} A. \tag{2.4}$$

The first equation of (2.3) contains the total energy of dust shell defined by $\mu=\sigma R^2$ where σ is the surface energy density of the dust shell. From the vanishing divergence of the energy-momentum tensor of dust shell, one can easily see that μ is a constant of motion.

For our convention of calculation, let us introduce quasilocal mass defined by

$$M=\frac{R}{2}(1-g^{\mu\nu}\partial_\mu R\partial_\nu R), \tag{2.5}$$

which is conserved in each vacuum region of the spherically symmetric spacetime. Of course, $M=0$ in the Minkowski side, and $M(\neq 0)$ in the Schwarzschild side represents the gravitational mass of the shell. The formula (2.5) is useful to estimate the derivatives of R in both sides, which are involved in (2.3).

Now we discuss the time slicings to describe the shell motion. ''Time slicings'' usually mean foliations of a whole spacetime. Foliations are spaces in a spacetime in which one can set observers. In our model, however, the equation of motion for the collapsing shell can be reduced to a local equation on Σ. Hence, the necessary procedure is to set a radial direction of coordinate system near Σ, which is called ''time slicing'' in this paper. Let us denote the radial coordinate by x and rewrite the metric into the form

$$ds^2=-N^2dt^2+U^2dx^2+R^2d\Omega_2^2. \tag{2.6}$$

We can refer to a local observer near Σ whose world line is along a constant x. There are two typical observers. One is a comoving observer corresponding to the time slicing $x=\eta$. The world line of this observer is embedded in Σ. Another is a static observer who stays at a constant circumference radius $x=R$ [$dx=dR$ in (2.6)]. We call the above time slicings ''comoving slicing'' and ''static slicing,'' respectively. Our idea is to give a more general form of the time slicing as

$$dx=\xi dR+\zeta d\eta, \tag{2.7}$$

where ξ and ζ are constant parameters at the outside the shell. This form is useful because one can recover the static and comoving time slicings by choosing the parameters to be $\zeta=0$, and $\xi=0$, respectively. Except the special cases the time slicing corresponds to an observer who is not comoving with the shell but infalling toward the origin $R=0$. In the following we will study the parameter dependence of the equation of the shell motion in both classical and quantum levels.

The first task is to rewrite the junction condition (2.3) using the coordinates x and t. The general coordinate transformations from (2.2) into (2.6) should be

$$d\tau=N\cosh\phi dt+U\sinh\phi dx, \tag{2.8}$$

$$d\eta=N\sinh\phi dt+U\cosh\phi dx. \tag{2.9}$$

The boost angle ϕ is related to the velocity of the shell in the (t,x) frame as

$$\tanh\phi=-\frac{U}{N}\left(\frac{dx}{dt}\right)_\eta, \tag{2.10}$$

where the subscript η means the derivative along a line of constant η. In this frame the junction conditions (2.3) are given by

$$[(\partial_x R)_t]=-\frac{\mu}{R}U\cosh\phi,\quad [(\partial_t R)_x]=-\frac{\mu N}{R}\sinh\phi, \tag{2.11}$$

where t and x are treated as independent variables in the calculation of the partial derivatives.

On the other hand, because the quasilocal mass M defined by (2.5) invariant under the coordinate transformation (2.8) and (2.9), it must satisfy with the conditions

$$\frac{2M}{R}-1=\frac{(\partial_t R)_{x+}^2}{N^2}-\frac{(\partial_x R)_{t+}^2}{U^2}, \tag{2.12}$$

in the Schwarzschild side, and

$$-1=\frac{(\partial_t R)_{x-}^2}{N^2}-\frac{(\partial_x R)_{t-}^2}{U^2}, \tag{2.13}$$

in the Minkowski side. By the virtue of the time slicing (2.7) and the coordinate transformations (2.8) and (2.9), we obtain the relations

$$\left(\frac{dx}{dt}\right)_{\eta^\pm}=\xi\left(\frac{dR}{dt}\right)_{\eta^\pm}, \tag{2.14}$$

$$\left(\frac{dR}{dt}\right)_{x^\pm}=-\frac{\zeta}{\xi}\left(\frac{d\eta}{dt}\right)_{x^\pm}=-\frac{\zeta}{\xi}N\sinh\phi, \tag{2.15}$$

$$\left(\frac{dR}{dx}\right)_{t^\pm}=\frac{1}{\xi}-\frac{\zeta}{\xi}\left(\frac{d\eta}{dx}\right)_{t^\pm}=\frac{1}{\xi}-\frac{\zeta}{\xi}U\cosh\phi. \tag{2.16}$$

Here we choose the metric components N^2 and U^2 in the coordinate system (t,x) are assumed to change continuously at the shell. This continuous change of N^2 and U^2 is possible like a Gaussian normal coordinate and the radial coordinate x can match continuously in the sense C^{1-} class at the shell by this choice. Then the substitution of (2.15) and (2.16) into (2.11) leads to

$$[\zeta]=\frac{\mu\xi}{R}, \quad [\xi]=0. \tag{2.17}$$

Note that the parameter $\zeta=\zeta_+$ at the outer side must be different from $\zeta=\zeta_-$ at the inner side. Since we are interested in an observer located at the outer side of the shell, only ζ_+ is treated as a parameter for specifying the time slicing. Then ζ_- becomes a function depending on R. Furthermore, (2.12) and (2.13) for the quasilocal mass in both sides are rewritten into the forms

$$\left(\frac{2M}{R}-1\right)\xi^2=\frac{2\zeta_+\cosh\phi}{U}-\zeta_+^2-\frac{1}{U^2}, \tag{2.18}$$

and

$$-\xi^2=\frac{2\zeta_-\cosh\phi}{U}-\zeta_-^2-\frac{1}{U^2}, \tag{2.19}$$

respectively. If ζ_- and ϕ are eliminated from Eqs. (2.17), (2.18), and (2.19), we obtain the metric component

$$\frac{1}{U^2}=\zeta_+^2+\left(1-\frac{2M}{R}\right)\xi^2+\zeta_+\xi\left(\frac{2M}{\mu}-\frac{\mu}{R}\right), \tag{2.20}$$

which shows that $g_{xx}=1$ in the comoving time slicing $(\xi=0,\zeta_+=1)$, while $g_{xx}=1/(1-2M/R)$ in the static time slicing $(\xi=1,\zeta_+=0)$. The velocity of the shell motion measured by the time coordinate t should be defined by $(dR/dt)_\eta$. Then, by using (2.10), (2.14), (2.18) and (2.20), we arrive at the final form of the equation of motion

$$\frac{\mu}{2N^2}\left(\frac{dR}{dt}\right)_\eta^2+\frac{1}{2\mu}V(M,\mu,\lambda,R)=0, \tag{2.21}$$

where the potential $V=V(M,\mu,\lambda,R)$ is given by

$$V=V_\infty\left(1+\frac{V_\infty}{\mu^2\left[\lambda+\frac{1}{2}\left(2E-\frac{\mu}{R}\right)\right]^2}\right), \tag{2.22}$$

$$V_\infty=1-\frac{1}{4}\left(2E+\frac{\mu}{R}\right)^2, \tag{2.23}$$

and λ and E are constants defined by $\lambda=\zeta_+/\xi$ and $E=M/\mu$, respectively. In the limit $\lambda\to\infty$, the potential V coincide with V_∞ in the comoving system. This represents the motion of the shell described by an observer corresponding to the time slicing parameter "λ" who is accelerated against gravity produced by the shell.

It is worth noting that the metric component $N^2=g_{tt}$ cannot be determined by the junction conditions. The reason can be seen in the proof of Birkhoff's theorem [11]. When one chooses R to be the radial coordinate, the vacuum Einstein equations for the spherically symmetric vacuum spacetime give

$$N^2=\left(1-\frac{2M}{R}\right)f(t)^2, \tag{2.24}$$

where $f(t)$ is an arbitrary function of t. The factor $f(t)$ may be eliminated by using the coordinate transformation $dT=f(t)dt$. However, one cannot determine this function by the Einstein equations. Since the junction conditions are local conditions, all the metric components on the shell depend only on the time coordinate t, and the lapse function $N(t)$ is treated as an arbitrary function by the virtue of the gauge freedom $f(t)$ of a choice of t by a local observer. In the next section, we develop the canonical quantum theory using this arbitrariness of N.

Before discussing the quantization procedure, it is better to give the brief derivation of the classical solution of this system, since the vacuum spacetimes in the outer and inner regions of the shell are classically treated. In the comoving frame, the difference of the quasilocal mass between the outer and inner sides of Σ leads to the formula

$$M=\frac{\mu}{2}(R'_-+R'_+), \tag{2.25}$$

where prime denotes the derivative with respect to η. Then the junction conditions (2.3) allow us to write explicitly R'_- and R'_+ as

$$R'_-=\frac{1}{2}\left(2E+\frac{\mu}{R}\right), \quad R'_+=\frac{1}{2}\left(2E-\frac{\mu}{R}\right). \tag{2.26}$$

Because the classical motion for bound states is limited in the range

$$0\leqslant R\leqslant\frac{\mu}{2(1-E)}, \tag{2.27}$$

the derivatives R'_+ and R'_- must satisfy the condition

$$R'_+\leqslant 2E-1, \quad R'_-\geqslant 1. \tag{2.28}$$

The equalities hold just at the turning point $R=\mu/[2(1-E)]$ of the shell motion. Recall that $R'_\pm$ are proportional to the components of the extrinsic curvatures $K^\pm_{\theta\theta}$ of Σ of the outer and inner sides. Hence $K^-_{\theta\theta}$ is always positive, while $K^+_{\theta\theta}$ becomes negative when $E<1/2$. The signs of $K^\pm_{\theta\theta}$ are essential to the global structure of spacetime constructed by the junction of the outer and inner spacetimes. In the range of E given by $0<E<1/2$ and $1/2<E<1$, the junction clearly shows a wormhole formation and a black hole formation, respectively (see Fig. 1) [9]. The gravitational mass M can be also negative, even if the local energy condition $\mu=\sigma R^2>0$ is imposed, and it does not contradict to the positive mass theorem [12] because it has no asymptotically

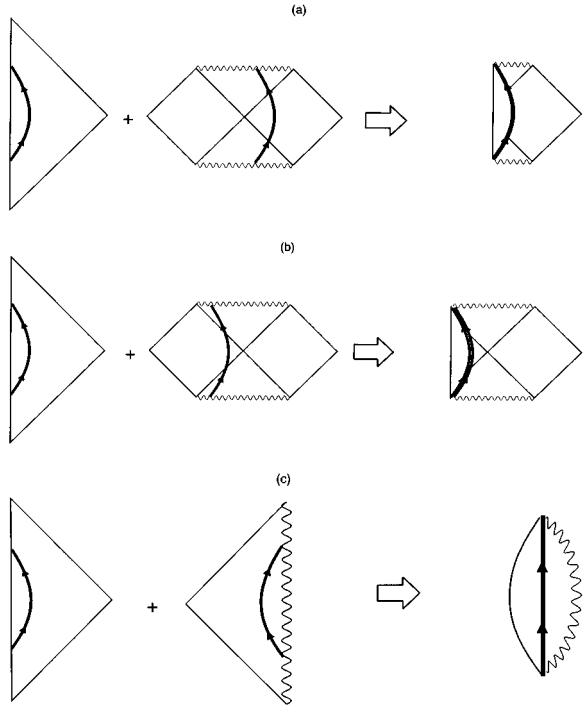

FIG. 1. Penrose diagrams of possible spacetimes constructed by the Minkowski-Schwarzschild junction with $E<1$. The trajectories of the dust shell in the Minkowski and Schwarzschild spacetimes are drawn. The two spacetimes are glued at the shell boundary. These figures show (a) a black hole spacetime for $1/2<E<1$ (in which the junction condition requires that both R'_+ and R'_- are positive), (b) a wormhole spacetime for $0<E<1/2$ (in which $R'_+<0$ and $R'_->0$), and (c) a negative mass spacetime (in which $R'_+<0$ and R'_-) without any asymptotic flat region.

flat region and has a timelike singularity at the Schwarzschild side. Thus the relation between the value of E and the global structure of spacetime is very clear, and we use this relation also in quantum mechanics of the shell, in which the motion is specified by a discrete eigenvalue of E.

III. CANONICAL QUANTIZATION

Now we give the Hamiltonian which generates the equation of motion given by (2.21). The procedure is to use the gauge freedom previously mentioned. Although the Hamiltonian is not uniquely determined, we seek the simplest one here. For this purpose, we consider the Lagrangian for (2.21)

$$L=\frac{\mu}{2N}\left(\frac{dR}{dt}\right)_\eta^2-\frac{N}{2\mu}V. \tag{3.1}$$

Because (3.1) does not include $\dot{N}$, there is a primary constraint that is the canonical momentum conjugate to N must weakly vanish. Furthermore, there is a secondary constraint which can be obtained by the variation of (3.1) with respect to N. It is easy to confirm that the secondary constraint coincides with (2.21). The Euler-Lagrange equation which can be derived by the variation with respect to R is the first derivative of (2.21). Because we have the Lagrangian (3.1), the usual step of the canonical formalism leads to the canonical momentum conjugate to R,

$$P=\frac{\mu}{N}\left(\frac{dR}{dt}\right)_\eta, \tag{3.2}$$

and the Hamiltonian

$$H=\frac{N}{2\mu}(P^2+V). \tag{3.3}$$

We wish to emphasize that (2.21) is a result of the Hamiltonian constraint $H=0$ which is generated by the variation with respect to N. Our model can keep the property of a constrained system in a similar manner to the full canonical theory of general relativity.

The next step is to quantize the Hamiltonian constraint given by

$$P^2+V=0. \tag{3.4}$$

If we use the usual commutation relation $[R,P]=i\hbar$ and the simplest factor ordering, (3.4) is reduced to the Schrödinger equation

$$-\hbar^2\frac{d^2}{dR^2}\Psi+V\Psi=0, \tag{3.5}$$

where potential V is given by (2.22) and has the form

$$V=\frac{\mu^2[\lambda^2+1+2\lambda E](1-E^2)}{(\lambda+E)^2}\times\frac{(R-R_0)(R-R_1)(R+R_2)}{R(R-R_3)^2}, \tag{3.6}$$

and the constants R_0, R_1, R_2, and R_3 are given by

$$R_0=\frac{m(2E+\lambda)\hbar^{1/2}}{\lambda^2+1+2\lambda E},\quad R_1=\frac{m\hbar^{1/2}}{2(1-E)}, \tag{3.7}$$

$$R_2=\frac{m\hbar^{1/2}}{2(1+E)},\quad R_3=\frac{m\hbar^{1/2}}{2(\lambda+E)}, \tag{3.8}$$

where $m=\mu/\hbar^{1/2}=\mu/m_P$. Let us explain some implications of these radii. Because of the condition $|E|<1$ of the bound state, the classical motion of the shell has the maximum radius R_1 which corresponds to a turning point. In the time slicing parameterized by λ, the classical motion has also the minimum radius $R=R_0$ where the infinite redshift occurs for the corresponding observer. Note that R_0 is equal to $2M$ in the static limit $\lambda=0$, which coincides with the true horizon radius. From the potential (3.6), $R=R_0$ is a turning point of the shell. Since the WKB feature, in general, breaks down at turning points, this means that the semiclassical description become meaningless at $R=R_0$ [13]. The potential V diverges at $R=R_3$ where we obtain the regular singular point of the differential equation (3.5). This radius R_3 becomes smaller than R_0, if the inequality $E>1/2-\lambda/2$ holds. Then, we can consider the region $R_3<R<R_0$ in (3.5), which is classically

forbidden and disappears only in the comoving limit $\lambda\to\infty$ (i.e., $R_0\to 0$). Only when $\lambda<3$ there exist the additional bound states with E in the range $-1<E<1/2-\lambda/2$, for which we obtain the classically allowed region $R_0<R<R_1$ and classically forbidden regions $0<R<R_0$ and $R_1<R<R_3$.

The Schrödinger equation (3.5) means that we have the Hilbert spaces $\mathscr{H}_\lambda$ of Ψ which depends on the time slicing parameter λ, in particular, as a consequence of the existence of the classically forbidden region $R_3<R<R_0$ (or $R_0<R<R_3$ when $-1<E<1/2-\lambda/2$ and $\lambda<3$). Hence, for each λ, we can give a discrete set of the eigenvalues of E, which is the unique observable in this quantum system. Because the vacuum spacetimes outside and inside the shell are classically treated, the global structure of the whole spacetime is specified only by E. Then the Hilbert space can be regarded as a set of spherically symmetric spacetimes (such as black holes, wormholes) which the collapsing shell forms.

IV. MASS EIGENVALUES

In this section, we study the quantum mechanics on various time slicings using the Schrödinger equation (3.5). We mainly consider the static time slicing $\lambda=0$ and then the other time slicing is considered to confirm that our argument is the natural extension of that in [7].

A. Static time slicing

First, we consider the typical time slicing which corresponds to a static observer. When $\lambda=0$, the Schrödinger equation (3.5) can be written in the form

$$-\hbar^2\frac{d^2}{dR^2}\Psi+\frac{\mu^2(1-E^2)}{E^2}\frac{(R-R_0)(R-R_1)}{(R-R_3)^2}\frac{R+R_2}{R}\Psi=0, \tag{4.1}$$

where

$$R_0=2M=2mE\hbar^{1/2},\quad R_1=\frac{m\hbar^{1/2}}{2(1-E)},$$

$$R_2=\frac{m\hbar^{1/2}}{2(1+E)},\quad R_3=\frac{m\hbar^{1/2}}{2E}. \tag{4.2}$$

As previously mentioned, for the bound states in the range $1/2<E<1$, only the bounded region $R_0\leqslant R\leqslant R_1$ is classically allowed. Though a quantum penetration of the wave function is possible in the region $R<R_0$, it must stop at $R=R_3$ owing to the infinite potential barrier. Therefore the boundary conditions which we adopt here is that the wave function vanishes at $R=R_3$ and $R\to\infty$. Although it is difficult to solve (4.1) exactly, an approximate calculation of the eigenvalue E is possible. Notice that the factor $1+R_2/R$ satisfies the inequality

$$1<1-\frac{R_2}{R}<1-\frac{R_2}{R_3}=1+\frac{E}{1+E}, \tag{4.3}$$

since the wave function is defined in the region $R_3<R<\infty$. As will be seen later, the eigenvalue E is in the range $1/2<E<1$. Then, the factor

$$1<\alpha\equiv 1+\frac{R_2}{R}<\frac{3}{2}, \tag{4.4}$$

remains nearly constant in (4.1). Then, introducing the nondimensional variable z,

$$z=\frac{2m\sqrt{\alpha(1-E^2)}}{E}\frac{R-R_3}{\hbar^{1/2}}, \tag{4.5}$$

the approximate form of (4.1) can be written as

$$\frac{d^2}{dz^2}\Psi+\left(-\frac{1}{4}+\frac{k}{z}-\frac{p^2-1/4}{z^2}\right)\Psi=0, \tag{4.6}$$

where α is treated as a constant, and

$$p^2=\frac{\alpha m^4(1-2E)^2(1+2E)(1+E)}{4E^4}+\frac{1}{4}, \tag{4.7}$$

$$k=\frac{m^2(2E-1)(2+E-2E^3)}{4E^2}\sqrt{\frac{(1+E)\alpha}{1-E}}. \tag{4.8}$$

The general solution is a superposition of the two Wittaker's function $M_{k,p}(z)$ and $M_{k,-p}(z)$, one of which is defined by

$$M_{k,P}(z)=z^{p+(1/2)}e^{-z}$$

$$\times\sum_{n=0}^{\infty}\frac{\Gamma(2p+1)\Gamma[p-k+n+(1/2)]}{\Gamma(2p+n+1)\Gamma[p-k+(1/2)]}\frac{z^n}{n!}. \tag{4.9}$$

The boundary condition at $z\to\infty$ selects the unique solution $W_{k,p}(z)$ which exponentially decrease as z increases and is written by the superposition

$$W_{k,p}(z)=\frac{\Gamma(-2p)}{\Gamma(1/2-p-k)}M_{k,p}(z)+\frac{\Gamma(2p)}{\Gamma(1/2+p-k)}M_{k,-p}(z). \tag{4.10}$$

Now another boundary condition at the regular singular point $z=0$ $(R=R_3)$ determines the eigenvalue E. The behavior $M_{k,P}(z)\to z^{p+1/2}$ in the limit $z\to 0$ means that Ψ can satisfy the boundary condition only when

$$\Gamma(1/2+p-k)=\pm\infty, \tag{4.11}$$

because the Γ function does not vanish on the real axis. Since the Γ function has poles at nonpositive integers, this condition is reduced to

$$n=k-p-1/2, \tag{4.12}$$

where n is a nonnegative integer. From (4.12) we can give the limiting behavior of the mass eigenvalue: In the limit $n\gg m^2$,

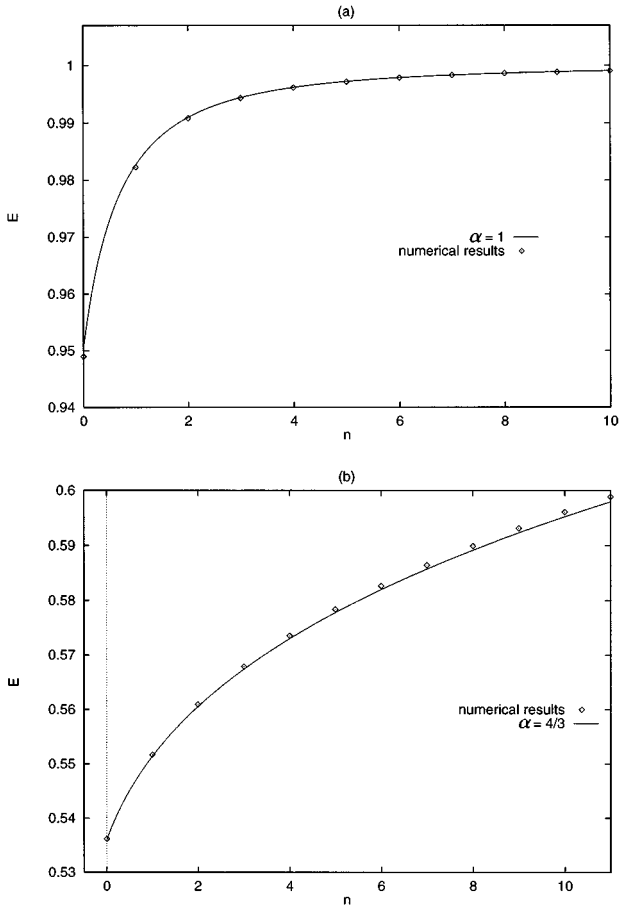

FIG. 2. The eigenvalues of E for the wave functions of black hole formation in a static time slicing. The validity of the approximate formula (4.12) checked by numerical calculations. The value of m is chosen to be (a) $m=1.0000$ and (b) $m=100.00$. The spectrum given by (4.12) is drawn by solid lines which corresponds to (a) $\alpha=1$ and (b) $\alpha=4/3$, respectively.

$$E\sim 1-\frac{\alpha m^4}{8(n+1)^2}. \tag{4.13}$$

If $\alpha=1$, (4.13) corresponds to the spectrum (1.2) obtained by Berezin [4] and (1.4) Hájíček *et al.* [7] in the limit $n>>1$ of highly excited states. On the other hand, in the opposite limit $n<<m^2$, the quantum effect becomes more important, and we have

$$E\sim\frac{1}{2}+\left(\frac{2n+1}{\sqrt{3\alpha}m^2}\right)^{\frac{1}{3}}. \tag{4.14}$$

Although the spectrum of E obtained here is consistent with the assumption $1/2<E<1$, the validity can be also checked by numerical calculation of (4.1), which is based on the standard shooting method of solving two-point boundary-value problems [14]. In Fig. 2 the eigenvalues of E are plotted for $m=1$ and $m=100$, which corresponds to the quantum numbers of n which runs from 0 to 10. We note that the approximate spectrum (4.12) coincides with the numerical results within the accuracy of our numerical code if choosing the constant $\alpha=1$ in (4.13) and $\alpha=4/3$ in the case of (4.14).[1] Hence we can claim that (4.12) is useful to discuss the qualitative behavior of the spectrum $E=E(n)$.

The behaviors of the spectrum of E written by (4.13) and (4.14) are plausible as a quantum version of gravitational collapse. Recall that n is the quantum number of the shell motion, and $1-E$ can be regarded as the gravitational binding energy per unit rest mass energy. We can find that the gravitational binding energy remains small when the kinetic energy of the shell motion dominates the inertia of the shell ($n\gg m^2$), while the binding energy becomes large when the inertia dominates the kinetic energy. For the ground state ($n=0$) we obtain $E\to 1/2$ in the limit $m\gg 1$, which corresponds to the classical limit of black hole formation. The quantum zero-point fluctuations generate the term $(2/m^2)^{1/3}$ in (4.14), where α is taken to be equal to 4/3. The contribution of these zero-point fluctuations can seriously affect the motion of the shell, if m is not so large.

Note that the restriction $m<1$ which arise from the inner boundary condition by Hájíček *et al.* [7] in the comoving time slicing disappears in this static time slicing. Hájíček *et al.* construct the quantum theory of this system by imposing that ''energy'' and ''charge'' currents must vanish at the central singularity, while in this static time slicing, the singularity is hidden by the infinite redshift surface, and the inner boundary condition is set up at a finite R. Hence we can construct quantum mechanics of the collapsing dust shell including the case $m>1$. This might means that their ''energy'' or ''charge'' current can not vanish at the central singularity when $m>1$.

The eigenvalues (4.12) [or (4.13), (4.14)] shows that the global structure of spacetime which corresponds to these eigenvalues is limited to only a black hole formation, i.e., the wave functions which have a support only in $R_3<R<\infty$ corresponds to the black hole states [Fig. 1(a)]. This does not mean that there is no wormhole state. When $E<1/2$, we can also consider the wave function whose support exists only in $0<R<R_3$. In this case, the classically allowed region also exists in $R_0<R<R_1$ where the inequalities $0<R_0<R_1<R_3$ holds due to the condition $E<1/2$. In this case, one cannot use the approximation that α defined by (4.4) is nearly constant. So, we must numerically solve the Schrödinger equation (4.1) and the result is shown in Fig. 3: For fixed m, the eigenvalue of E monotonically decreases in the range $0<E<1/2$ as n increases. These eigenvalues plotted in Fig. 3 corresponds to the global structure of a wormhole formation, i.e., the wave functions whose support exist only in $0<R<R_3$ correspond to the wormhole states [Fig. 1(b)]. When $m>>1$, there are many bound states whose eigenvalues of E satisfy the inequality $E<1/2$. Together with the case $E>1/2$, we can take the limit to the positive mass classical solutions of bound state in which E takes an arbitrary value in $0<E<1$. Note that there is no bound state in $\mu<m^*\sim 2.4m_P$. This means wormhole spacetimes are not

[1] $\alpha=4/3$ can be easily obtained by the estimation as follows. When $E\to 1/2$, the classically allowed region becomes narrow due to the $R_1-R_0\to 0$. Since the eigenvalues of E is determined by the behavior of the wave function in the classically allowed region, $\alpha\sim 1+R_2/R_0\sim 1+R_2/R_1\sim 4/3$ when $E\to 1/2$.

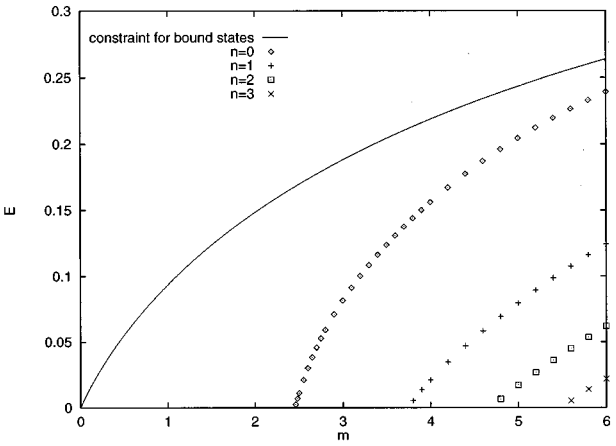

FIG. 3. The eigenvalues of E for the wave functions of wormhole formation in a static time slicing. The m dependence of E is shown for the quantum numbers $n=0$, 1, 2, and 3. The solid line given by $R_3 > \pi/\sqrt{-V_{\min}}$ means a rough upper boundary of the allowed range of E.

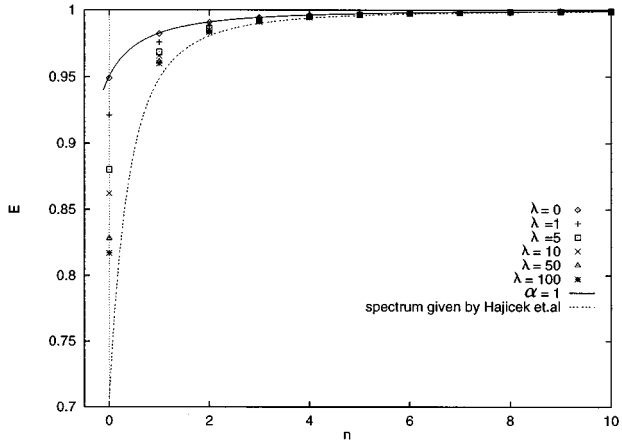

FIG. 4. The λ dependence of the spectrum $E(n)$ for $m=1.0000$. The time slicing parameter is chosen to be $\lambda=0.0000$, 1.0000, 5.0000, 10.000, 50.000, and 100.00. As λ increases, the spectrum of E approaches to (1.4) drawn by the dashed line.

permissible in the case $\mu < m^*$, and this might be due to the zero point fluctuations of the shell motion.

We must also note that the state of $E=1/2$ is forbidden like the case $E<1/2$. It can be also seen from the necessary condition for the existence of bound states. Let us denote the minimum value of the potential V by $V_{\min}(E)<0$ which depends on E. Since we impose the boundary condition $\Psi|_{R=0}=\Psi|_{R=R_3}=0$, all eigenvalues E of bound states must satisfy the condition

$$R_3(E) > \frac{\pi}{\sqrt{-V_{\min}(E)}}. \tag{4.15}$$

When $E=1/2-\delta$ ($\delta<<1$), we obtain $V_{\min}(E) \sim -m^2\delta^2(1+\delta)$. Then the inequality is reduced to $m^2 > \pi/\delta$, which cannot be satisfied when $\delta \to 0$ even if $\mu \gg m^*$. Furthermore we can give a physical interpretation of $E \neq 1/2$. The classically allowed region is $R_0 < R < R_1$, and $R_1 - R_0 \to 0$ as $E \to 1/2$, so the shell is confined in this narrow region. However, it is impossible due to the uncertainty relation $\Delta P \cdot \Delta R \sim \hbar$, thus $E=1/2$ is forbidden due to the quantum effect of the shell motion. This interpretation is also valid when $E>1/2$. Since (4.14) tells us $E \neq 1/2$ when $1/2<E<1$ (even if $\mu \gg m_P$ due to the zero point fluctuations as mentioned before, $E \neq 1/2$ is valid in the static time slice.

B. Nonstatic time slicing

Based on the result obtained in the static time slicing, we consider the Schrödinger equation (3.5) in the case $\lambda \neq 0$ except that there is no bound state in the region $0<R<R_3$ when $\lambda>3$. Any essential property of (3.5) is not so much different from the case $\lambda=0$, and we also impose the boundary condition; the wave function must vanish at the regular singular points $R=m\hbar^{1/2}/[2(\lambda+E)]$ and $R=\infty$. Then, we can derive the eigenvalues E in a similar manner. The approximation that $\alpha=1-R_2/R$ is constant in (3.5) leads to

$$E \sim 1 - \frac{\alpha m^4 (1+\lambda)^2}{8(n+1)^2} \tag{4.16}$$

in the limit $n>>m^2$. This shows that the λ dependence of E is not so sensitive in the limit $n>>m^2$. However, the approximated form (4.16) will not remain valid as λ become infinitely large. To clarify the behavior of E for the ground states in the limit $\lambda>>1$, we must solve numerically the Schrödinger equation (3.5). By varying the parameter λ, we can consider the extrapolation from the static time slicing to the comoving one. In particular, for $m \leqslant 1$, we can compare the spectrum of E with (1.4) in the comoving limit. The numerical results for $m=1$ are plotted in Fig. 4, which confirm that the mass spectrum converges to (1.4) as λ increases and E remains larger than 1/2. On the other hand, as an example of the spectrum for $m>1$, the eigenvalues E for $m=10$ are plotted in Fig. 5. We find the common tendency

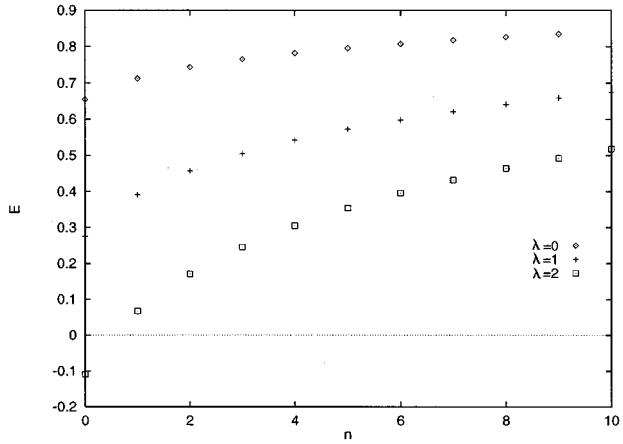

FIG. 5. The λ dependence of the spectrum $E(n)$ for $m=10.000$. The time slicing parameter is chosen to be $\lambda=0.0000$, 1.0000, and 2.0000. It is shown in this figure that the spectrum does not remain in the range $1>E>1/2$ and even the negative mass states are allowed.

that the energy level at the ground state decreases as λ increases. The remarkable point for $m>1$ (i.e., $\mu>m_P$) is that the mass spectrum does not keep the condition of $E>1/2$ nevertheless the wave function has its support only in $R_3<R<\infty$. As was suggested in [7], even the states of $E<0$ are allowed, if λ is sufficiently large. [This might means that H_B in (1.1) cannot be positive self-adjoint operator in general.] Then observable states exist in the range $-1<E<1$ when μ is sufficiently larger than m_P, and these correspond to the global structure of black hole and wormhole formations (see Fig. 1).

Finally, we consider the wave function given by the Schrödinger equation (3.5) in the comoving frame ($\lambda\to\infty$), which has the form

$$-\hbar^2\frac{d^2}{dR^2}\Psi+\mu^2\left[1-\frac{1}{4}\left(2E+\frac{\mu}{R}\right)^2\right]\Psi=0. \quad (4.17)$$

Note that this Schrödinger equation corresponds to the Wheeler-DeWitt equation (1.3), and our Schrödinger equation may be regarded as a natural extension of (1.3). In terms of the requirement of unitarity and of positivity of energy, Hájíček *et al.* [7] gave the boundary condition that for the bound state $E<1$ the wave function must vanish at the origin and at infinity. Then the discrete spectrum of E was shown to be (1.4) which becomes meaningless when $m>1$. As discussed by Hájíček *et al.*, this means that when $m>1$ one cannot impose the positivity of E or the regularity of the wave function at $R=0$. In our approach the regularity of wave function at $R=R_3$ are set up in any time slicing parameter λ, even if $m>1$. Therefore, we can discuss the comoving limit through the extrapolation of the results to the range $\lambda\gg1$. In this sense, as previously mentioned, observable states exist in the range $-1<E<1$ when μ is sufficiently larger than m_P, there is no essential difference in a set of observable states. On the other hand, there are no wormhole states $0<E<1/2$ when $\mu<m_P$. These states are suppressed by the zero point fluctuations of the shell motion.

V. SUMMARY AND DISCUSSION

In summary, by studying this model of dust shell collapse, we obtained the wave functions for the bound states with positive eigenvalues of E under the static time slicing. When $\mu>>m_P$, these can recover the classical solutions which describe wormhole or black hole formation. This also means that ''energy'' or ''charge'' current which discussed by Hájíček *et al.* must not vanish when $\mu>>m_P$. In the static time slicing, wave functions which has its support only in $0<R<R_3$ correspond to the wormhole states, and wave functions which has its support only in $R_3<R<\infty$ correspond to the black hole states.

This situation is analogous to the quantum field theory in Rindler space [15]. In this theory, nevertheless the set of mode functions in the left- and right-handed wedge of the Rindler space separately is not complete on the whole Minkowski space, both sets together are so complete. Let us suppose that the regions $0<R<R_3$ and $R_3<R<\infty$ corresponds to the left- and right-handed wedge of Rindler spacetime, respectively. And suppose that one may regard that the set of states with positive eigenvalues of E in our model, black hole and wormhole states correspond to the complete set of mode functions in whole Minkowski spacetime, the mode functions in left- and right-handed wedge, respectively. Then, the complete set of classical solutions can be constructed by the both sets of the wave functions which have its support in left- and right-handed wedge together.

Furthermore, we showed that the wormhole states cannot exist when $\mu<m^*\sim2.4m_P$. Although we couldn't discuss about the factor m^*/m_P in detail, the necessary condition (4.15) for the existence of bound states tells us the tendency of wormhole suppression and this condition is closely related to the zero point fluctuations of the shell motion. So this may be due to the result of this zero point fluctuations. For a comoving time slicing also, we can see the same quantum effect. Moreover, for noncomoving and nonstatic time slicings which are charactorized by finite nonvanishing λ, one can also see the same quantum effect at least $m<1$. Although it is unclear whether $E<0$ is permissible or not, observable states exist in the range $-1<E<1$ when $\mu>m_P$, while only black hole states are possible when $\mu<m_P$. Thus, our conclusion is that the wormhole formation with small mass is efficiently suppressed in the quantum collapse of dust shell due to the quantum fluctuations of the dust shell motion.

Although our model is concerned about the local geometries of the dust shell and both vacuum regions of spacetime are treated classically, one might be able to consider the quantization of both vacuum region. Recently, Kuchař propose the definitive treatment of the spherically-symmetric quantum problem [16]. When one considers the quantization of the vacuum region, one might use Kuchař's treatment. In this case, the world volume of the dust shell will be treated as a boundary of a spacetime manifold and one must consider the boundary term of the Einstein Hilbert action of spherically symmetric spacetime at the world volume of the shell. Since, in his treatment, the boundary term of the action and the foliations of spacetime near the boundary play essential roles, one must consider the foliation of spacetime near the world volume of the shell. Various foliations of spacetime near the shell are possible and our foliations which characterized by the parameter λ are examples of that. Then one might be able to consider the foliation of the vacuum regions of spacetime from near the shell to asymptotic flat region using our foliation parameter λ. The parameter λ will become more important in this case. It might be interesting to clarify that the mass eigenvalues are also obtained or not when one consider the wave functional of the vacuum region and together with the wave function of the dust shell motion. If these considerations were done, the difference of the wormhole states and black hole states become more meaningful. Because ''wormhole'' in our dust shell collapse means that there is a wormhole in the space which is a foliation of the Schwarzschild spacetime beyond the black hole horizon and connect the untrapped Minkowski region to untrapped region of Schwarzshild space.

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