Tree-level string cosmology

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In this paper we examine the classical evolution of a cosmological model derived from the low-energy tree-level limit of a generic string theory. The action contains the metric, dilaton, central charge and an antisymmetric tensor field. We show that with a homogeneous and isotropic metric, allowing spatial curvature, there is a formal equivalence between this system and a scalar field minimally coupled to Einstein gravity in a spatially flat metric. We refer to this system as the shifted frame and using it we describe the full range of cosmological evolution that this model can exhibit. We show that generic solutions begin (or end) with a singularity. As the system approaches a singularity the dilaton becomes large and loop corrections will become important.

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I. INTRODUCTION

String theory is the most promising candidate for the unification of gravity with the other fundamental forces of nature. However, string theory is most likely to cause significant modifications to classical general relativity near the Planck scale, which is far beyond the range of direct terrestrial experimentation. Since these energy scales are typically associated with the big bang, it is natural to view cosmology as a laboratory for testing string theoretic modifications to gravity. Conversely, cosmologists can hope that if string theory does provide a deeper understanding of gravitational physics than general relativity, then some of the ''standard'' problems of conventional cosmology will be resolved by string theory. Consequently, the cosmological dynamics of superstring theories are the subject of intense scrutiny. Typically, one proceeds by perturbatively expanding the full superstring theory and extracting a ''low-energy'' Lagrangian that contains Einstein gravity and the lowest-order corrections from string theory. For this approach to be valid we need to restrict our attention to sub-Planckian scales where quantum effects can be ignored and the higher-order terms in the perturbative expansion do not contribute significantly.

We consider the tree-level action [1–3] which contains contributions from the metric, the dilaton, the central charge, and an antisymmetric tensor field. We assume that the background spacetime is four dimensional and that it and the fields defined upon it are homogeneous and isotropic. Further, we assume that degrees of freedom associated with any compactified metric can be ignored. Such cosmologies have received considerable study [4–21], with particular attention being paid to the effect of the dilaton field and its self-interaction potential. However, the roles of the antisymmetric tensor field and the spatial curvature of the metric have often been neglected. Copeland, Lahiri, and Wands [22] give general analytic solutions to the equations of motion for the antisymmetric tensor field in a background with nonzero spatial curvature, for the case when the central charge is zero. In this paper we consider this system with a nonzero central charge. Previously, Tseytlin [23] has given several exact solutions. Also Goldwirth and Perry [24] use a phase-plane analysis to describe the cosmological properties of solutions to the equations of motion of models with a nonzero central charge in a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) background.

Many previous authors have exploited the equivalence of the action written in terms of conformally related metrics. In this paper we show that there is an additional, formal, equivalence between the equations of motion for our homogeneous fields in *spatially curved* FLRW metrics and those of a scalar field minimally coupled to Einstein gravity in a *spatially flat* FLRW metric. We refer to this system as the shifted frame, and use it to succinctly describe the full range of cosmological evolution that is possible within this model. As the name suggests, it is based on the shifted dilaton field [12,13]. While we cannot give an analytic solution to the equations of motion for the general case when the central charge is nonzero, we do find an exact result for a particular choice of parameters.

II. TREE-LEVEL STRING EFFECTIVE ACTION

We take as our starting point the tree-level action [1–3]

$$S=\frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g_D}e^{-\phi}\left[R_D+(\nabla\phi)^2-\Lambda-\frac{1}{12}H^2\right], \qquad (1)$$

where g_{ab} is the graviton, ϕ is the dilaton, and $H_{abc}=\partial_{[a}B_{bc]}$ is the antisymmetric tensor field where lowercase roman indices run from 0 to $D-1$ and the central charge deficit is denoted by $-\Lambda$. Henceforth we will assume that this D-dimensional theory has undergone compactification,

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leaving only four macroscopic dimensions, and that the terms corresponding to degrees of freedom on the compact metric are held fixed. The real world may be more complicated, but, in the absence of compelling reasons for choosing any particular compactification scheme, this simplification will allow us to examine the dynamics due solely to the degrees of freedom associated with the macroscopic dimensions. This permits us to reduce the action to

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-\phi}\left[R+(\nabla\phi)^2-\Lambda-\frac{1}{12}H^2\right], \quad (2)$$

where the spacetime manifold is now four dimensional.

We want to consider a homogeneous and isotropic spacetime, and so the curvature scalar is that of an FLRW universe. The string frame line element is, thus,

$$ds^2=s(\eta)^2\left(-n(\eta)^2d\eta^2+\frac{1}{1-kr^2}dr^2 + r^2[d\theta^2+\sin^2(\theta)d\phi^2]\right), \quad (3)$$

where η is the conformal time if we set the arbitrary lapse function $n=1$. Open, flat, or closed spatial hypersurfaces correspond to $k=-1$, 0, or $+1$, respectively. The antisymmetric tensor H has only one degree of freedom, and can be written

$$H^{\mu\nu\lambda}=e^{\phi}\epsilon^{\mu\nu\lambda\kappa}\partial_\kappa\Theta, \quad (4)$$

where Θ is a pseudoscalar field. Assuming that the dilaton and antisymmetric tensor field are homogeneous like our metric, $H^2=6e^{2\phi}\Theta'^2$, where the prime denotes differentiation with respect to η. Furthermore, since the Lagrangian does not depend on Θ, the corresponding momentum

$$p_\Theta=\frac{\partial\mathscr{L}}{\partial\Theta'}=-\frac{e^{\phi}s^2\Theta'}{n}=q \quad (5)$$

is conserved and q is a constant. Finally, by adding the total derivative

$$-6\frac{d}{d\eta}\left(\frac{e^{-\phi}s's}{n}\right) \quad (6)$$

to the integrand, we obtain

$$S=\frac{1}{2\kappa^2}\int d\eta\left[ne^{-\phi}\left(-6\frac{s'^2}{n^2}-\frac{\phi'^2}{n^2}s^2+6\frac{\phi's's}{n^2}+6ks^2 - \Lambda s^4-\frac{q^2}{2s^2}\right)\right]. \quad (7)$$

The action now has a simpler form, but notice that it includes the crossed kinetic term $\phi's'$. This term vanishes in the Einstein frame, which is related to the string frame by the conformal transformation

$$s(\eta)=e^{\phi/2}a(\eta), \quad (8)$$

where a is the Einstein frame scale factor. In the Einstein frame the action takes the form

$$S=\frac{1}{2\kappa^2}\int dte^{3\alpha}\left[-6\dot{\alpha}^2+\frac{\dot{\phi}^2}{2}-U(\alpha,\phi)\right], \quad (9)$$

where for convenience we let $a=e^{\alpha}$, and choose $n=1/a$ so that η coincides with the Einstein frame proper time, t.

The potential is a function of α and ϕ:

$$U=\frac{q^2}{2}e^{-2\phi-6\alpha}+\Lambda e^{\phi}-6ke^{-2\alpha}. \quad (10)$$

However, this potential is simpler when written in terms of the original string frame scale factor, as it then is a separable function of s and ϕ:

$$U=e^{\phi}\left(\Lambda+\frac{q^2}{s^6}-\frac{6k}{s^2}\right). \quad (11)$$

In fact, the Einstein frame represents only one of infinitely many different choices of variables which diagonalize the kinetic terms in the Lagrangian for homogeneous fields. As we shall now see, it is possible to further simplify the system by choosing alternative variables which have orthogonal kinetic terms and respect the symmetry of the potential.

III. THE SHIFTED FRAME

While we can simplify the kinetic terms by converting to the Einstein frame, we do so at the cost of introducing a more complicated potential. We now introduce a choice of variables that combines the advantages both of the string frame (separable potential) and the Einstein frame (no crossed kinetic terms). Transform α and ϕ into a new pair of variables r and ψ,

$$\begin{pmatrix}\phi\\ \alpha\end{pmatrix}=\begin{pmatrix}3 & -3\\ -\frac{1}{2} & \frac{3}{2}\end{pmatrix}\begin{pmatrix}\psi\\ r\end{pmatrix}, \quad (12)$$

and choose the lapse function to be

$$n=\frac{3}{2}e^{-\alpha-(\phi/2)}=\frac{3}{2}e^{-\psi}. \quad (13)$$

In terms of the transformed variables the action and potential are

$$S=\frac{1}{2\kappa^2}\int dte^{3r}[-6\dot{r}^2+2\dot{\psi}^2-V(\psi)], \quad (14)$$

$$V(\psi)=\frac{3}{4}q^2e^{-6\psi}+\frac{3}{2}\Lambda-9ke^{-2\psi}. \quad (15)$$

The potential $V(\psi)$ is plotted for a variety of different parameter values in Fig. 1.

Up to a rescaling of the field, this action is identical to that of a scalar field ψ with potential V, minimally coupled to Einstein gravity in a spatially flat universe, with scale factor $R=e^r$. The equations of motion are

$$-6\left(\frac{dr}{dT}\right)^2+2\left(\frac{d\psi}{dT}\right)^2+V(\psi)=0, \quad (16)$$

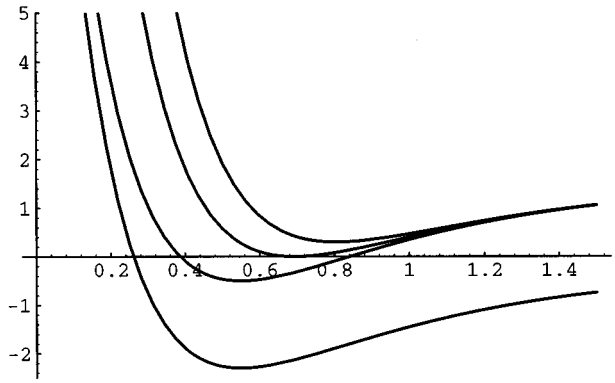

FIG. 1. The potential $V(\psi)$ is shown for a variety of parameter values. In ascending order, the plots correspond to $k=1$, $\Lambda=-0.2$, and $q=6$, and $\Lambda=1$, $k=1$ with $q=6$, 8, and 10.

$$\frac{d^2r}{dT^2}=-\left(\frac{d\psi}{dT}\right)^2, \tag{17}$$

$$\frac{d^2\psi}{dT^2}+3\frac{dr}{dT}\frac{d\psi}{dT}+\frac{1}{4}\frac{d}{d\psi}V(\psi)=0. \tag{18}$$

The time T corresponding to our choice of lapse is proportional to the proper time in the original string frame. It is related to the proper time in the Einstein frame, t, by

$$\frac{dT}{dt}=\frac{2}{3}e^{\phi/2}$$

$$\Rightarrow t=\frac{3}{2}\int e^{-\phi/2}dT. \tag{19}$$

We will refer to this choice of variables as the shifted frame because, apart from a numerical factor, r coincides with the shifted dilaton previously used [12,23,17] to simplify the equations of motion. Our scalar field is actually the logarithm of scale factor in the string frame, $\psi\equiv\ln s$, and the shifted scale factor $R=e^r$ represents the variation of the fields in the orthogonal direction. The shifted dilaton, or equivalently r, reflects the symmetries of the underlying string theory better than the ''renormalized'' field ϕ [12]. For instance, it remains invariant under the scale-factor duality transformation $s\rightarrow s^{-1}$. However, our action is only invariant under this transformation if $dV/d\psi=0$, which requires both the spatial curvature and antisymmetric tensor field to vanish.

This change of variables is not a conformal transformation, which is an identity between two actions for all field configurations, as the shifted frame only exists for homogeneous fields. Also there is no equivalent choice of variables if we include loop corrections to the dilaton coupling. However, if we are going to restrict our attention to the tree-level action which contains only the dilaton, antisymmetric tensor, and central charge terms with a homogeneous and isotropic string metric, then working in the shifted frame allows us to write this system in a particularly simple form.

The practical advantage of working in this shifted frame is that the spatial curvature of the string metric, k/s^2, appears as a term in the potential $V(\psi)$, and the shifted FLRW metric is always spatially flat. The dynamical system in the shifted frame is thus the basis of almost all inflationary models and its properties are familiar and well understood. Moreover, we will see in the next section that it is particularly simple to identify the $(+)$ or $(-)$ branches of pre-big-bang string cosmology [19,21] with the contracting or expanding scale factor in the shifted frame.

The qualitative dynamics of our system are strongly influenced by whether or not the values of q, Λ, and k permit the inequality $V<0$ to be satisfied. If $\Lambda<0$, then $V<0$ for sufficiently large values of ψ, but when $\Lambda>0$, a negative potential region can only exist if $k>0$. Then (for $k=1$) $V(\psi)$ has its minimum value when $\psi=\frac{1}{2}\ln(|q|/2)$, which is negative if $\Lambda|q|<8$ (see Fig. 1). Thus the negative potential region exists when one of the following is true:

$$\Lambda|q|<8 \text{ and } k=1,$$
$$\Lambda<0. \tag{20}$$

In the next section we use the formalism of the shifted frame to discuss exact solutions to the equations of motion. Analytic solutions are known to exist when any one of the three terms in $V(\psi)$ is nonzero, and we show that these can be simply expressed in the shifted frame. Second, we employ techniques developed for obtaining exact scalar field cosmologies to derive a new particular solution where all three terms in the potential are nonzero. In Sec. V we then utilize these exact solutions as limiting cases of the solutions to the equations of motion to catalog all the different possible types of cosmological behavior this model can produce.

IV. EXACT SOLUTIONS TO THE EQUATIONS OF MOTION

Exact solutions to the system of equations (16)–(18) for specific potentials have been analyzed by a number of authors. Clearly, the only solutions of relevant to the system considered here are those where the potential takes the form of Eq. (15). It is often useful to parametrize the motion by the field ψ [25–28,30]. Setting $dr/dT=H$ (the ''Hubble parameter'' in the shifted metric), Eq. (17) gives $dH/d\psi=-d\psi/dT$, where the prime in Eqs. (21)–(23) denotes differentiation with respect to ψ. We derive the following expressions for the potential, scale factor r, and time T:

$$V(\psi)=6H(\psi)^2-2H(\psi)'^2, \tag{21}$$

$$r(\psi)-r(\psi_0)=-\int_{\psi_0}^{\psi}\frac{H}{H'(\psi)}d\psi, \tag{22}$$

$$T(\psi)-T(\psi_0)=-\int_{\psi_0}^{\psi}\frac{1}{H'(\psi)}d\psi. \tag{23}$$

It is convenient to make the extra substitution [31,32,25]

$$x=\sqrt{3}\psi, \tag{24}$$

$$f(x)=\sqrt{\frac{|V|}{6}}, \tag{25}$$

$$y^2(x)=1-\frac{V}{6H^2}. \tag{26}$$

There is a sign ambiguity of H corresponding to a time reversal, which transforms an expanding solution in the shifted frame into a contracting one. In terms of y, Eq. (21) becomes

$$yy_x=(1-y^2)\left(y-\frac{f_x}{f}\right), \tag{27}$$

where y_x denotes y differentiated with respect to x. When f_x/f is constant (or zero) this equation can be integrated immediately. Fortunately these special cases correspond to a potential that is either a constant or a single exponential term, which is precisely what we need to discuss the asymptotic solutions in Sec. V.

A. Pure dilaton cosmology

First, note that in the absence of a central charge, antisymmetric tensor field, and spatial curvature in the string frame, then $V=0$ and we have the standard result for a massless scalar field in the shifted frame:

$$R=R_0\left|\frac{T}{T_0}\right|^{1/3}, \tag{28}$$

$$\psi-\psi_0=\pm\frac{1}{\sqrt{3}}\ln\left|\frac{T}{T_0}\right|. \tag{29}$$

In terms of the string frame scale factor and dilaton this is the usual pure dilaton cosmology:

$$s=s_0\left|\frac{T}{T_0}\right|^{\pm 1/\sqrt{3}}, \tag{30}$$

$$\phi-\phi_0=(\pm\sqrt{3}-1)\ln\left|\frac{T}{T_0}\right|. \tag{31}$$

The choice of signs in the above equations corresponds to an increasing or decreasing dilaton. In addition there are two branches corresponding to T less than or greater than zero, denoted as the $(+)$ and $(-)$ branches by Brustein and Veneziano [19]. We see that in the shifted frame these two branches correspond simply to a contracting or expanding scale factor R, respectively.[1] Here the $(+)$ branch approaches a singularity at $T=0$, while the $(-)$ branch starts from the singularity at $T=0$.

B. Dilaton cosmology with a central charge

When $\psi\gg 0$, $V(\psi)$ is dominated by the central charge term Λ and we approximate the potential by $V=3\Lambda/2$. For this case the solution is straightforward, since $f_x=0$ and

$$y(x)=\begin{cases}\tanh(x-x_0) & \text{for } \Lambda>0,\\ \coth(x-x_0) & \text{for } \Lambda<0.\end{cases} \tag{32}$$

When Λ is positive, $f=\sqrt{\Lambda}/2$ and

$$R(\psi)=R_0\{\sinh[\sqrt{3}(\psi-\psi_0)]\}^{-1/3}, \tag{33}$$

$$T(\psi)-T_0=\mp\frac{2}{3\sqrt{\Lambda}}\ln\left\{\tanh\left[\frac{\sqrt{3}}{2}(\psi-\psi_0)\right]\right\}. \tag{34}$$

This is the general solution for a massless field plus cosmological constant in a flat FLRW metric [29]. The choice of upper or lower sign reflects whether we choose the $(+)$ or $(-)$ branch corresponding to the contracting or expanding solutions, respectively, in the shifted frame. The string frame scale factor and dilaton are [21]

$$s=s_0\left[\tanh\left(\frac{3\sqrt{\Lambda}}{2}|T-T_0|\right)\right]^{\pm 1/\sqrt{3}}, \tag{35}$$

$$e^{\phi-\phi_0}=\frac{\left[\tanh\left(\frac{3\sqrt{\Lambda}}{4}|T-T_0|\right)\right]^{\pm\sqrt{3}}}{\sinh\left(\frac{3\sqrt{\Lambda}}{2}|T-T_0|\right)}. \tag{36}$$

The corresponding solution with $\Lambda<0$ has $f=\sqrt{|\Lambda|}/2$ and

$$R(\psi)=R_0\{\cosh[\sqrt{3}(\psi-\psi_0)]\}^{-1/3}, \tag{37}$$

$$|T(\psi)-T_0|=\frac{4}{3\sqrt{\Lambda}}\tan^{-1}\{\exp[\sqrt{3}(\psi-\psi_0)]\}. \tag{38}$$

There is no choice of $(+)$ or $(-)$ branches, as every solution for $R(\psi)$ starts expanding [$(-)$ branch], turns around when $\psi=\psi_0$, and recollapses [$(+)$ branch]. Written in terms of the string frame variables we have

$$s=s_0\left[\tan\left(\frac{3\sqrt{|\Lambda|}}{4}(T-T_0)\right)\right]^{\pm 1/\sqrt{3}}, \tag{39}$$

$$e^{\phi-\phi_0}=\frac{\left[\tan\left(\frac{3\sqrt{|\Lambda|}}{4}(T-T_0)\right)\right]^{\pm\sqrt{3}}}{\cos\left(\frac{3\sqrt{|\Lambda|}}{2}(T-T_0)\right)}. \tag{40}$$

Note that the $\Lambda<0$ solution has a finite lifetime, as $|T-T_0|$ is bounded between 0 and $2\pi/3\sqrt{|\Lambda|}$. The string scale factor is monotonic, and increases if we take $T>T_0$ and decreases otherwise.

C. Dilaton and antisymmetric tensor field

When $\psi\ll 0$, the potential has the form $V(\psi)=3q^2e^{-6\psi}/4$, and so $f_x/f=-\sqrt{3}$. For this case we give y parametrically as

$$\psi(y)-\psi_0=-\frac{1}{4}\ln\left[\left(\frac{1-y}{1+y}\right)^{1/\sqrt{3}}\frac{(\sqrt{3}+y)^2}{1-y^2}\right], \tag{41}$$

$$R(y)=R_0\left[\left(\frac{1-y}{1+y}\right)^{\sqrt{3}}\frac{(\sqrt{3}+y)^2}{1-y^2}\right]^{1/12}. \tag{42}$$

[1]This can seen from Eq. (4) of [19] where the choice of $\pm$ sign coincides with the sign of $-\dot{r}$.

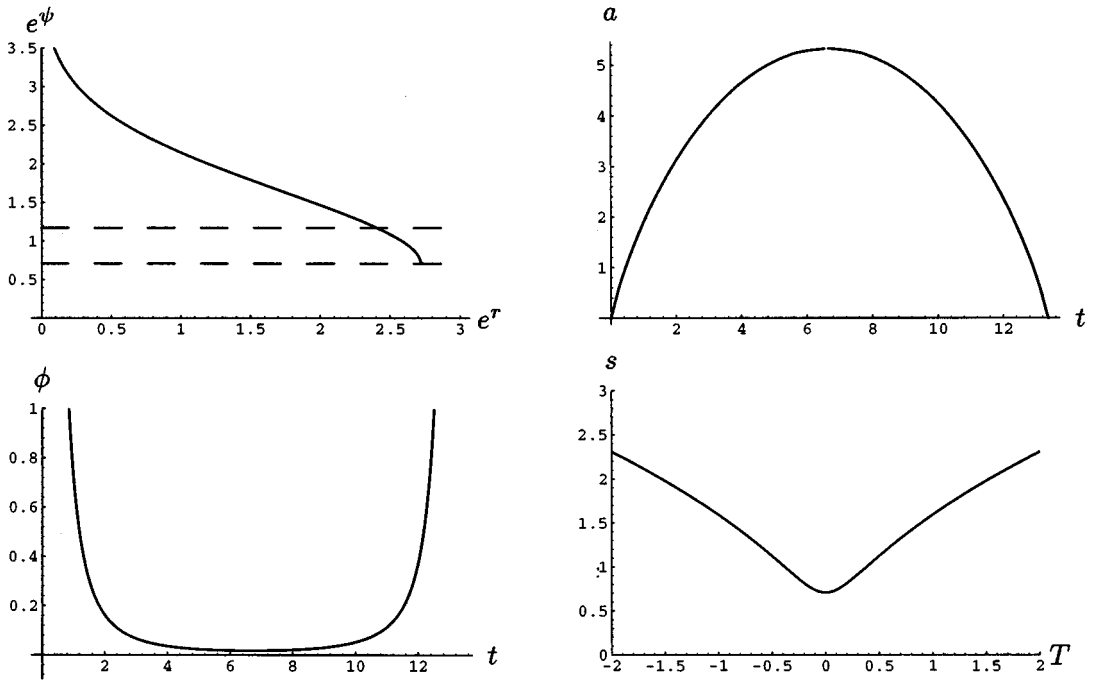

FIG. 2. This figure displays the exact solution to the string equations of motion given by Eqs. (47)–(50) for the parameter values $A=1$ ($\Lambda=4$ and $q^2=2$) and $r_0=1$. In the top left plot the motion in the shifted frame is shown. The path is reflected at the boundary of the negative potential region, which lies between the dashed horizontal lines. The evolution of the dilaton ϕ is plotted at bottom left and the Einstein frame scale factor is plotted at top right. The coordinate time t in the Einstein frame runs over a finite interval, but in the string frame T runs from $-\infty$ to $+\infty$. At the bottom right, we plot the string frame scale factor s when it is near its minimum value.

A scalar field with an exponential potential is the basis of power-law inflation [33,34] and is known to have an exact solution [26]. The situation in the shifted frame is not analogous to power-law inflation, due to the steepness of the potential. In particular, we find that the value of ψ is always bounded below, whereas the exponential potentials which drive power-law inflation admit solutions where the field is a monotonic function of the time. For $|f_x/f|>1$ the value of ψ cannot decrease indefinitely, irrespective of the initial conditions. The minimum value occurs when $y=0$, while at early or late times, as $y\to\pm1$, we have $\psi\to+\infty$. Thus the string frame scale factor $s=e^{\psi}$ always has a minimum value. This behavior is also seen in the exact solution given, in rather different form, by Copeland, Lahiri, and Wands [22] for this system with $\Lambda=0$.

D. Special case: Particular solution with $k=+1$

We have found a new exact solution to the equations of motion, for a case where all the terms in the potential (15) are nonzero. This is generated by choosing

$$H(\psi)=\pm(A-Be^{-2\psi})^{3/2}, \tag{43}$$

where A and B are both positive. The plus and minus signs correspond to increasing and decreasing r, respectively, with $H=0$ when $\psi=\psi_0\equiv(1/2)\ln(B/A)$. It is straightforward to write down the potential

$$V(\psi)=6A^3-18A^2Be^{-2\psi}+12B^3e^{-6\psi}. \tag{44}$$

If either A or B vanishes, then the potential reduces to one of the special cases we have already considered, and so we assume that they are both nonzero. By comparison with Eq. (15), we see that this provides us with a nontrivial solution to the equations of motion when $\Lambda>0$ and

$$A=\left(\frac{\Lambda}{4}\right)^{1/3}, \quad B=\frac{1}{2A^2} \tag{45}$$

$$q^2\Lambda^2=32. \tag{46}$$

Performing the integrals in Eqs. (22) and (23) yields

$$r(\psi)-r_0=\tfrac{1}{3}(\psi-\psi_0)-\tfrac{1}{6}(e^{2(\psi-\psi_0)}-1), \tag{47}$$

$$T(\psi)-T_0=\mp\frac{1}{3\sqrt{\Lambda}}[e^{\psi-\psi_0}\sqrt{e^{2(\psi-\psi_0)}-1} + \ln(e^{\psi-\psi_0}+\sqrt{e^{2(\psi-\psi_0)}-1})]. \tag{48}$$

This solution is displayed in Fig. 2. Notice that ψ_0 is the minimum value attained by ψ, and quantities with the subscript 0 refer to their value at $\psi=\psi_0$. The maximum value of r is r_0. The upper sign in the expression for T corresponds to the expanding phase ($T<T_0$), and the lower sign to the contracting phase ($T>T_0$). This solution thus interpolates between the $(-)$ branch and the $(+)$ branch.

The ambition of many studies in string cosmology has been to show whether a nonsingular universe can be found as a solution to the equations of motion [19–21,35]. Because the lifetime in the shifted frame, and thus in the string frame, for our solution is infinite, it might appear to be just such a nonsingular cosmology. However, the dilaton

$$\phi-\phi_0=2(\psi-\psi_0)+\tfrac{1}{2}(e^{2(\psi-\psi_0)}-1) \tag{49}$$

becomes arbitrarily large when $|T|\to\infty$. This is the strong coupling limit of the string theory, and so the tree-level action from which our solution is derived becomes unreliable. This is a consequence of the change at T_0 from the $(-)$ to $(+)$ branches, rather than changing from $(+)$ to $(-)$, as envisaged in the pre-big-bang scenario [19].

The Einstein frame scale factor $a=e^{\alpha}$ is given by

$$\alpha-\alpha_0=-\tfrac{1}{4}(e^{2(\psi-\psi_0)}-1). \tag{50}$$

The time in this frame is given by Eq. (19), and so

$$t(\psi)-t_0=\pm\frac{e^{\alpha_0+1/4}}{2\sqrt{2}}\int_z^1\frac{e^{-1/z'}}{z'^{3/2}\sqrt{1-z'}}dz', \tag{51}$$

where we have made the additional substitution $z=e^{-2(\psi-\psi_0)}$. Evaluating this integral with the lower limit $z=0$ shows that the time between the ''big bang,'' when $a=0$, and the time t_0 when the universe attains its maximum size is finite in the Einstein frame.

Finally, we remark that this particular solution is unstable to small perturbations. In the shifted frame, our solution corresponds to the critical case where the ψ field reaches an infinite value in an infinite time with vanishing velocity. If it rolled more slowly, it would eventually be reflected back towards the minimum of the potential, whereas a faster evolution would see it become infinite in a finite time.

E. Special case: Static solution with $k=+1$

Finally, there is a particular solution when the ψ sits in the minimum of its potential, if this minimum value is nonnegative. We can therefore find the result, with $\dot{\psi}=0$, first given by Tseytlin [23]:

$$\psi=\frac{1}{2}\ln\frac{|q|}{2}, \tag{52}$$

$$r(T)-r(0)=\pm\sqrt{\frac{2}{|q|}-\frac{\Lambda}{4}}T, \tag{53}$$

when $k=+1$ and $\Lambda|q|\geqslant 8$. Since $s=e^{\psi}$, the string frame scale factor s is a constant. The dilaton

$$\phi-\phi_0=\mp 3\sqrt{\frac{2}{|q|}-\frac{\Lambda}{4}}T \tag{54}$$

is linear with respect to the string frame time. The Einstein frame scale factor a is monotonic and proportional to the Einstein frame time t. Like our previous particular solution, the lifetime in the string frame is infinite, but the dilaton becomes large at either early or late times, here depending on whether we are on the $(+)$ or $(-)$ branch, rendering the tree-level action invalid. There is a corresponding singularity in the Einstein frame when the scale factor becomes zero.

This $(-)$ branch solution is stable at late times, as perturbations of ψ about the minimum are damped by the expansion of the shifted frame. Conversely, the $(+)$ branch solution is unstable at late times, but is the general solution at early times.

V. COSMOLOGICAL BEHAVIOR FOR THE GENERAL CASE

The exact solutions examined in the previous section only apply to a small portion of the full parameter space. However, using the shifted frame we can give a qualitative account of the properties of the general solution to the equations of motion.

For large negative and decreasing ψ with $q\neq 0$, the system must eventually evolve into a region where Eqs. (41) and (42) accurately describe the motion. This shows that ψ cannot decrease to arbitrarily large negative values. This immediately establishes that the string frame scale factor $s=e^{\psi}$ always has a nonzero lower bound in the presence of an antisymmetric tensor field. Conversely, we will show that ψ always reaches arbitrarily large values at early and/or late times, except in the particular static solution of Eq. (52). The evolution of the string frame scale factor can be quite complicated but in the shifted frame the evolution is straightforward.

If the scale factor in the shifted frame is growing, the energy density must decrease and the field will eventually evolve towards the minimum of its potential. This naturally splits the analysis into two subcases, depending on whether or not the values of q, Λ, and k admit a negative potential region, and we treat them separately.

A. Motion with a negative potential region

We showed in Sec. III that a negative potential region exists whenever $\Lambda<0$ or $k>0$ and $\Lambda|q|<8$. By examining the constraint, Eq. (16), we can catalog the possible extrema of $r(T)$ and $\psi(T)$.

$\dot{r}=0$, $\dot{\psi}\neq 0$. We see that for $\dot{r}=0$ we must have $V(\psi)<0$, and so branch changing between contracting and expanding solutions can only occur in the negative potential region. In addition, Eq. (17) implies that all turning points of $r(T)$ are maxima, and so all branch changes must be from the $(-)$ to the $(+)$ branch. Therefore r has at most only one turning point.

$\dot{\psi}=0$, $\dot{r}\neq 0$. Extrema of $\psi(T)$ can only occur outside the negative potential region and reflect ψ back towards the minimum of the potential.

$\dot{r}=\dot{\psi}=0$. This special case can only occur on the boundary of the negative potential region. Again, the value of $r(T)$ is a maximum, and the trajectory is reflected back towards $V<0$. Our exact solution, Eqs. (47) and (48), exhibits this type of extremum.

When $\dot{r}>0$, even if the field ψ is evolving away from the negative potential region, the frictional damping will force ψ towards the minimum of the potential. Eventually the energy density in the shifted frame (kinetic plus potential energy of ψ) reaches zero, leading to a turning point for $r(T)$. In the contracting phase the energy density increases. The presence of the antisymmetric tensor field will ensure that ψ is always reflected back from large negative values towards the minimum of the potential. However, at large ψ

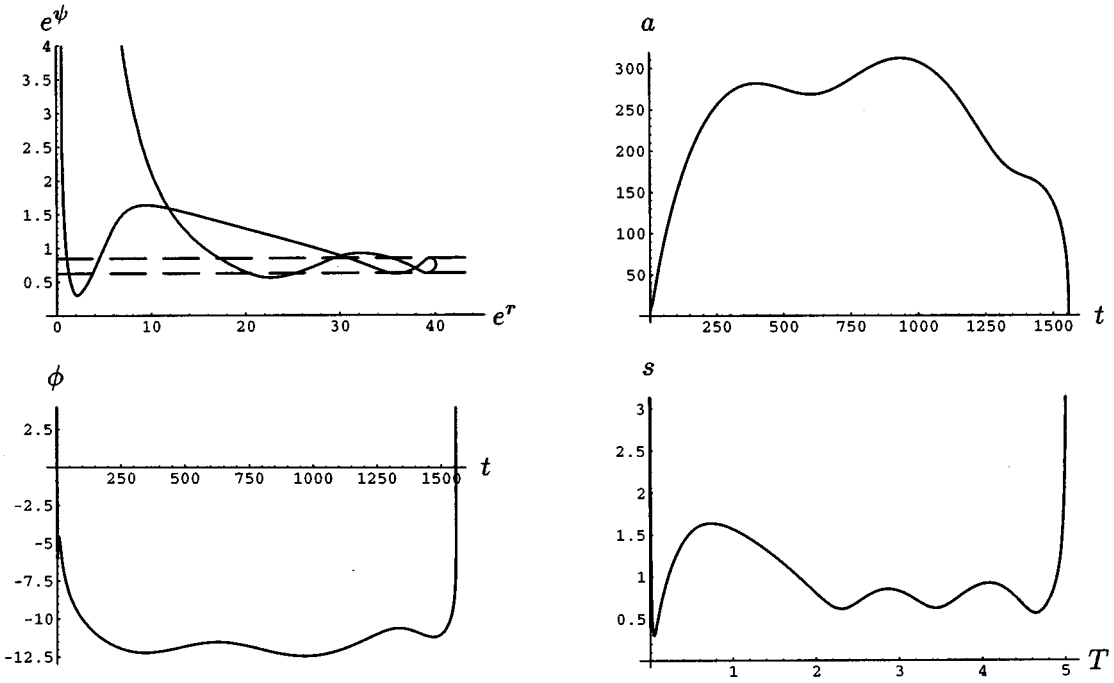

FIG. 3. Numerical solution of the equations of motion with $q=1$, $\Lambda=7$, and $k=1$ is plotted here for the initial data $r=1$, $\psi=-1$, $\dot{r}=10$ with $\dot{\psi}$ chosen to satisfy the constraint. At the top left, the motion in the shifted frame is shown. The boundary of the negative potential region has also been plotted (the two horizontal lines) and the oscillations around it can be clearly seen. The evolution of the dilaton is shown in the bottom left, while the right-hand plots depict the Einstein and string frame scale factors. Note that while this model is singular (when $a=0$), the Einstein frame scale factor can have several phases of expansion and contraction.

the potential energy tends towards a finite value $3\Lambda/2$. Once the total energy density exceeds this value, it must continue to increase if $\dot{r}<0$ and ψ will escape to infinity. Similarly, extrapolating back in time, we find that ψ must always originate at infinity.

For $\Lambda>0$ (and thus $k=+1$) the field ψ may oscillate about the minimum of the potential many times both during the expanding and contracting phases, as shown in Fig. 3. If $\Lambda\leqslant 0$, the energy density always exceeds $3\Lambda/2$ and thus the field escapes to infinity without passing through a local maximum.

Our particular solution, Eqs. (47) and (48), corresponds to the critical case where the asymptotic energy density is exactly $3\Lambda/2$ and ψ reaches infinity with zero kinetic energy as $|T|\to\infty$. Such a late (or early) time solution exists for any choice of parameters (when $\Lambda>0$), but our exact solution with $q^2\Lambda^2=32$ is the special case where the turning points for r and ψ coincide and the evolution is symmetrical about T_0.

For $k=+1$ the Einstein frame scale factor may possess both local maxima and local minima, which implies that string matter, after transformation to the Einstein frame, does not satisfy the strong energy condition. This behavior can be seen both by numerically integrating the full equations of motion or by considering the equation of motion for $\ddot{\alpha}$. However, if a negative potential region exists, there is always an upper bound on the Einstein frame scale factor. Conversely, as we shall see in the next section, if the potential is everywhere nonnegative, then the Einstein frame scale factor is monotonic.

B. Motion without a negative potential region

In this case $V\geqslant 0$ at all points. From the constraint equation, Eq. (16), $\dot{r}=0$ requires both V and $\dot{\psi}$ to be zero, which is a special case of the static solution given in Eqs. (52) and (53). Otherwise, without a negative potential region r is monotonic, and therefore there are no branch changing solutions.

Turning points in ψ will still occur. However, for $k\leqslant 0$ the potential is a decreasing, monotonic function of ψ and any extremum will be a global minimum as there can be no further turning points. In this case ψ becomes infinitely large at both early and late times, as illustrated by Fig. 4.

For $k=+1$, the potential has a minimum value at $\psi=1/2\ \ln|q|/2$. When r is increasing [$(-)$ branch], the field oscillates with decreasing amplitude about this minimum, and the static solution given in Eqs. (52) and (53) is a stable attractor at late times. Note that closed models therefore can escape recollapse in the Einstein frame if $\Lambda|q|\geqslant 8$. This type of motion is shown in Fig. 5. The $(+)$ branch is simply the time-reversed solution, and so is unstable at late times.

When $k=-1$ the behavior of this system at late times can be probed using the slow rolling approximation. The criteria for the validity of this approximation are that the potential be much larger than its first and second derivatives, which holds well when $V\approx 3\Lambda/2$ and $V'(\psi)\approx 18ke^{-2\psi}$. Dropping the appropriate terms from Eqs. (16) and (18), we integrate the approximate system to obtain the following asymptotic solution for large ψ:

$$r\to\frac{\sqrt{\Lambda}}{2}T, \tag{55}$$

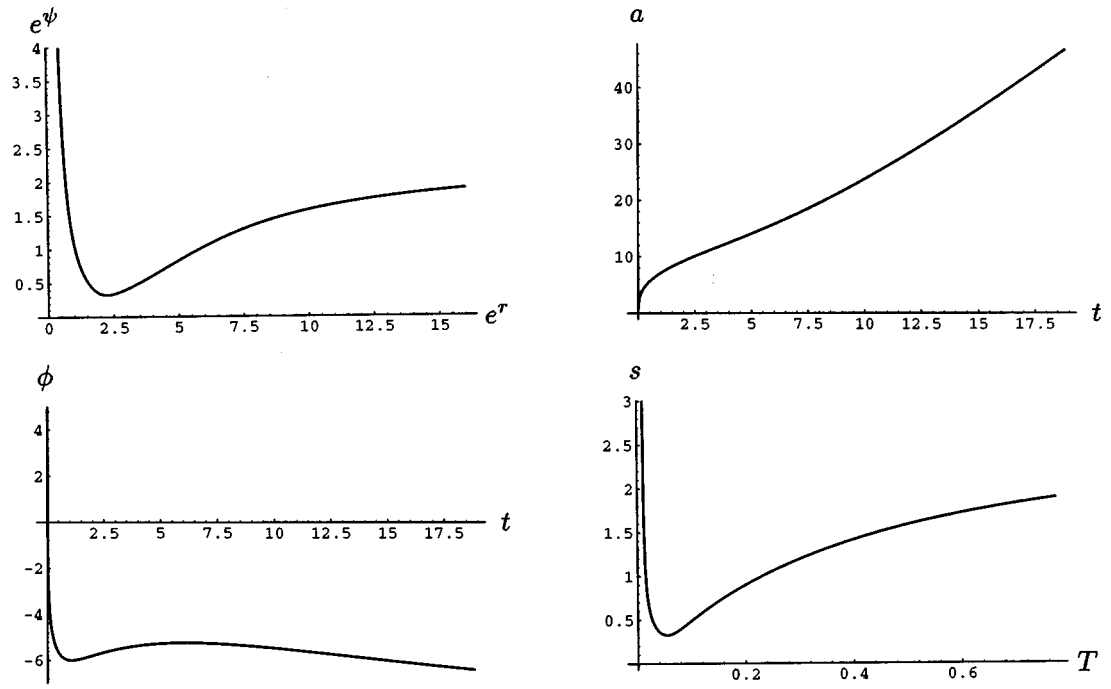

FIG. 4. The plot shows the solution with $\Lambda=9$ and $k=-1$, and the other parameters are the same as for the plot in Fig. 3. Since $k=-1$, the field ψ does not oscillate. At late times $r\to\infty$ and $\psi\to-\infty$. The dilaton (bottom left) decreases indefinitely, while the Einstein frame scale factor (top right) expands without limit. The string frame scale factor (bottom right) is initially infinite, and is proportional to T at late times.

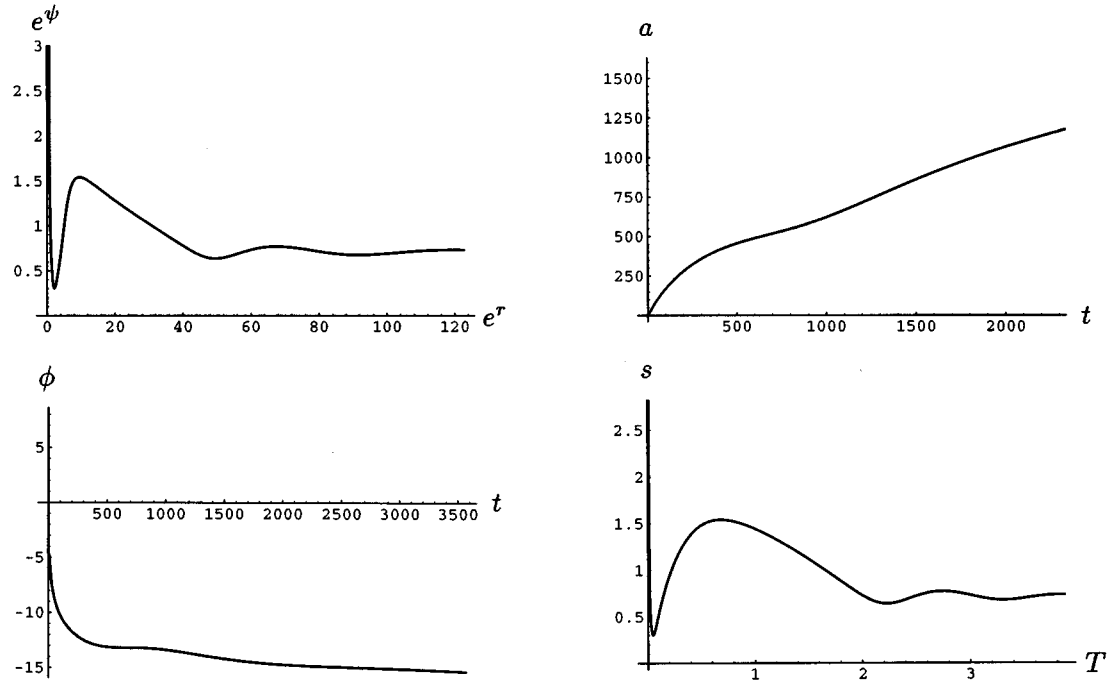

FIG. 5. This figure shows the solution to the equations of motion when the parameter values are the same as those in Fig. 3, except for setting $\Lambda=9$. In this case there is no negative potential region as $\Lambda|q|>8$. Since $k=1$, the field ψ oscillates about the value $e^{-2\psi}=2/|q|$ and at late times the solution tends towards that given by Eqs. (52) and (53). This can be observed in the plots of the dilaton (bottom left) and the Einstein frame scale factor (top right), as well as the string frame scale factor (bottom right) which is asymptotically constant.

$$\psi \rightarrow \frac{1}{2}\ln\left(\frac{6}{\sqrt{\Lambda}}T\right). \tag{56}$$

At late times, therefore, $r \gg \psi$, and thus, from Eq. (12), $\phi \rightarrow -\infty$ and $a \rightarrow \infty$. A similar analysis will show that the late time behavior for $\Lambda, |q| > 0$ and $k = 0$ is similar to that for $k = -1$.

VI. CONCLUSIONS

We have succeeded in describing the full range of cosmological evolution that can be found for the string motivated action, Eq.(2), containing the dilaton, central charge, and antisymmetric tensor field with a homogeneous and isotropic but spatially curved metric. We have done this by showing that this system is formally equivalent to a self-interacting scalar field ψ, minimally coupled to Einstein gravity in a spatially flat FLRW metric, and by using this shifted frame to understand the cosmological evolution. The parameters of the theory determine the form of the self-interaction potential $V(\psi)$.

If the potential for the scalar field in the shifted frame is positive definite, then the generic evolution of the shifted frame scale factor is monotonic and with a semi-infinite lifetime. Solutions either start or end at a singularity where the scale factor vanishes. Monotonically contracting or expanding solutions correspond to the $(+)$ or $(-)$ branches, respectively, of the pre-big-bang scenario [18–21]. If $V < 0$, then a turning point is possible, but this is always a maximum corresponding to a change from the $(-)$ to $(+)$ branch. These general conclusions will remain valid for any potential $V(\psi)$. Generic solutions to this system are singular. Only exceptional cases, for which we have analytic solutions, have an infinite lifetime in the string frame. However, the dilaton always diverges at early and/or late times, taking the solution into the strong coupling regime.

In the string or Einstein frames the solutions exhibit a diverse range of behavior, depending on both the curvature of the spatial hypersurfaces of the background spacetime (described by k) and the other parameter values Λ and q. As long as the antisymmetric tensor field is nonzero, the string frame scale factor is always bounded from below [22].

For all values of k, including the case with positive curvature when $\Lambda > 8/|q|$, there are choices of the parameter values for which the Einstein frame scale factor expands indefinitely from an initial singularity. If $k = +1$ and $\Lambda < 8/|q|$, the Einstein frame scale factor can pass through several local maxima and minima, but the lifetime of the universe is finite. If $\Lambda < 0$, then the Einstein frame scale factor always has a finite maximum value, irrespective of the values of k and q.

Each term in the action we have considered turns out to play an important role at different stages in the cosmological evolution. Consequently, we have found new types of behavior not seen in previous studies which omit one or more of the terms. By the same token, our own conclusions may be sensitive to the inclusion of further terms in the action. Nonetheless, the absence of solutions which interpolate between weak coupling regimes rules out the possibility of successfully implementing the pre-big-bang scenario [18] in our system. This complements the work of Kaloper, Madden, and Olive [21] who reach similar conclusions considering the effect of an explicit potential for the dilaton and loop corrections to the dilaton coupling, but without an antisymmetric tensor field or spatial curvature. All possible solutions to our equations of motion contain phases where the coupling becomes strong. Therefore this tree-level limit of the full string theory predicts its own downfall, where higher-order corrections cannot be neglected.

Note added in proof. In a recent report Kaloper, Madden, and Olive [36] have extended their earlier no-go theorem to include the antisymmetric tensor field and an axion-dilaton potential in a spatially flat FLRW universe.

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