Equivalence principle and the Moon

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The perturbation of the lunar motion caused by a hypothetical violation of the equivalence principle is analytically worked out in terms of power series in the manner of Hill and Brown. The interaction with the quadrupolar tide is found to amplify the leading order term in the synodic range oscillation by more than 62%. Confirming a recent finding of Nordtvedt, we show that this amplification has a pole singularity for an orbit beyond the lunar orbit. This singularity is shown to correspond to the critical prograde orbit beyond which, as found by Hénon, Hill's periodic orbit becomes exponentially unstable. It is suggested that ranging between prograde and retrograde orbits around outer planets might provide future high precision orbital tests of the equivalence principle. It is argued that, within the context of string-derived non-Einsteinian theories, the theoretical significance of orbital tests of the universality of free fall is to measure the basic coupling strength of some scalar field primarily through composition-dependent effects. Present Lunar Laser Ranging data yield, within such models, the value $\bar{\gamma}=(-0.9\pm 1.3)\times 10^{-7}$ for the effective Eddington parameter $\bar{\gamma}\equiv\gamma-1$ measuring this coupling strength.

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I. INTRODUCTION

Gravity seems to enjoy a remarkable universality property: all bodies are experimentally found to fall with the same acceleration in an external gravitational field, independently of their mass and composition. Although Galileo [1] was the first [2] to suggest in a clear and general way that this property of universality of free fall might hold true, it was left to Newton [3] to realize the remarkable conceptual status of this universality: exact proportionality between a particular force (the weight) and the general dynamical measure of inertia (the mass). Newton went further in performing the first precise laboratory tests of the universality of free fall (pendulum experiments; precision $\sim 10^{-3}$). It is less well known that Newton went even further and suggested to test the universality of free fall of celestial bodies by looking for a possible miscentering of the orbits of satellites around Jupiter, Saturn, and the Earth [4]. More precisely, Newton considers a possible violation of the ratio weight (w) over mass (m), i.e.,

$$\bar{\delta}_{12}\equiv\frac{(w/m)_1}{(w/m)_2}-1\neq 0\,, \tag{1.1}$$

where 1 labels a satellite and 2 a planet, the weights w_1, w_2 being the gravitational forces exerted by the Sun (label 3). He says, without giving any details, that he has found "by some computations" that the center $\mathbf{x}_c$ of the orbit of the satellite 1 around the planet 2 will be displaced (in the Sun-planet direction and away from the Sun if $\bar{\delta}_{12}>0$) by the amount

$$\delta|\mathbf{x}_c-\mathbf{x}_3|=+\frac{1}{2}\bar{\delta}_{12}a'\,, \tag{1.2}$$

where a' denotes the radius of the orbit of the planet 2 around the Sun. In modern phraseology, one can say that Newton predicted a "polarization" of the satellite's orbit in the Sun-planet direction (away from the Sun if $\bar{\delta}_{12}>0$). Then Newton used his theoretical estimate (1.2) to conclude from the observed good centering of the orbits of the satellites of Jupiter that $|\bar{\delta}_{12}|<10^{-3}$, a number comparable to the result of his pendulum experiments. Actually, this upper limit obtained from Jovian satellites is wrong, as Newton's theoretical estimate (1.2) is incorrect both in magnitude (being a gross overestimate in general) and in sign (see below). We could not find any information about Newton's original calculations in his published papers. It is surprising that Newton did not remark that, as a consequence of his estimate (1.2), a value $|\bar{\delta}_{12}|=10^{-3}$ would also entail an unacceptably large polarization (one-fifth) of the Moon's orbit.

As far as we are aware, Laplace was the first to realize that the best celestial system to test a possible violation of the universality of free fall (1.1) is the Earth-Moon system (1=Moon, 2=Earth). In [5] he derived a rough estimate of the main observable effect of $\bar{\delta}_{12}$ on the angular motion of the Moon. Then, he noticed that even a very small $\bar{\delta}_{12}\neq 0$ would spoil the agreement linking his theoretical derivation of the solar perturbation term called "parallactic inequality" of the Moon, the set of observations of the lunar motion, and

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the direct measurements of the solar parallax. He concluded that an upper bound to the fractional difference in acceleration of the Moon and the Earth toward the Sun is

$$|\bar{\delta}_{12}| < \frac{1}{3\,410\,000} \simeq 2.9\times 10^{-7}. \tag{1.3}$$

It is remarkable that the limit (1.3) is much better than the limit ($|\bar{\delta}_{AB}| < 2\times 10^{-5}$) obtained some years later by Bessel through improved pendulum experiments [6], and has been superseded (though not by much) only by the work of Eötvös in 1890 ($|\bar{\delta}_{AB}| < 5\times 10^{-8}$) [7]. (The later results of Eötvös, Pekár, and Fekete improved the bound to 3×10^{-9} [8].) As we discuss in Appendix D, in spite of some obscurities in his reasonings and the lack of a fully accurate calculation of the effect of $\bar{\delta}_{12}$ in longitude, Laplace's final bound (1.3) turns out to be a conservative upper limit, given the information he had.

In 1907, Einstein [9] deepened the conceptual implications of the property of universality of free fall by raising it to the level of a "hypothesis of complete physical equivalence" between a gravitational field and an accelerated system of reference. This heuristic hypothesis was used very successfully by Einstein in his construction of the theory of general relativity, and later became enshrined in the name "principle of equivalence."

Within the context of relativistic gravitational theories, the use of the Moon as a sensitive probe of a possible violation of the equivalence principle for massive bodies has been rediscovered by Nordtvedt in 1968 [10]. His idea was that self-gravitational energies might couple nonuniversally to an external gravitational field in theories having a different structure than that in general relativity [11,12]. (Let us note that though Dicke had mentioned this possibility earlier [13–15], he had not explored its consequences in detail.) Anyway, Nordtvedt, unaware both of the old ideas of Newton and Laplace, and of the more recent ones of Dicke, realized that the planned Lunar Laser Ranging (LLR) experiment was providing an exquisitely sensitive tool for testing the universality of free fall of massive bodies [10]. Performing a *first-order* perturbation analysis of the lunar orbit (assumed circular and planar) in presence of a violation of the equivalence principle, $\bar{\delta}_{12} \neq 0$ [see Eq. (1.1) with the labels 1 and 2 denoting the Moon and the Earth, respectively], he provided the first analytical estimate of the corresponding range oscillation:

$$(\delta r)^{(1)} = C^{(1)} \bar{\delta}_{12} a' \cos[(n-n')t + \tau_0], \tag{1.4}$$

with

$$C^{(1)} = \frac{1+2n/(n-n')}{n^2-(n-n')^2} n'^2. \tag{1.5}$$

Here, n denotes the (mean) sidereal angular velocity of the Moon around the Earth, n' the (mean) sidereal angular velocity of the Earth around the Sun, and a' denotes the radius of the orbit of the Earth around the Sun (assumed circular). The angle $\tau \equiv (n-n')t + \tau_0$ is equal to the difference between the mean longitude of the Moon and the mean longitude of the Sun (as seen from the Earth). For completeness we derive Eqs. (1.4) and (1.5) in Appendix A (see also Ref. [16] which gives an alternative derivation).

Let us note in passing that the $\cos\tau$ dependence of the range oscillation (1.4) is equivalent (when disregarding the perturbations of the motion in longitude) to displacing the center $\mathbf{x}_c$ of a circular lunar orbit in the Earth-Sun direction (toward the Sun if $\bar{\delta}_{12} > 0$) by the amount

$$\delta|\mathbf{x}_c - \mathbf{x}_3| = -C^{(1)} \bar{\delta}_{12} a'. \tag{1.6}$$

The result (1.2) of Newton can therefore be viewed as implying a range oscillation of the type (1.4) with $C^{\rm Newton} = -1/2$, independently of n and n'. By contrast, the first order estimate (1.5) contains the small dimensionless parameter[1]

$$m = \frac{n'}{n-n'}, \tag{1.7}$$

which is $m \simeq 1/12.3687$ for the Moon and much smaller for the (Galilean) satellites of Jupiter (e.g., $m \simeq 3.86\times 10^{-3}$ for Jupiter IV). More precisely, Eq. (1.5) can be rewritten as

$$C^{(1)} = \frac{3}{2} m \frac{1+\frac{2}{3}m}{1+\frac{1}{2}m} = \frac{3}{2} m \left[1 + \frac{1}{6}m - \frac{1}{12}m^2 + \cdots\right]. \tag{1.8}$$

In 1973, Nordtvedt [17] suggested that a more accurate value of the coefficient C in the $\cos\tau$ (or "synodic") range oscillation (1.4) would be obtained by replacing, in the denominator of $C^{(1)}$, Eq. (1.5), the first term n^2 by $n_{\rm rad}^2$ where $n_{\rm rad}$ denotes the frequency of radial perturbations: $n_{\rm rad} = \dot{l} = \overline{\rm c} n$. The perturbation series giving $\overline{\rm c}$ reads (see, e.g., [18,19])

$$\overline{\rm c} = 1 - \frac{\dot{\bar{\omega}}}{n} = 1 - \frac{3}{4}m^2 - \frac{177}{32}m^3 - \cdots. \tag{1.9}$$

[In the case of the Moon, the series (1.9) is very slowly convergent. The full value of $1-\overline{\rm c} \simeq 0.008\,572\,573$ [18] is more than twice the lowest-order correction $\frac{3}{4}m^2$.] The correction of Ref. [17] amounts numerically to increasing the first-order result (1.5) by about 13%.

In 1981, Will [20] tried, more systematically, to estimate the higher-order corrections in the coefficient C due to the mixing between the perturbation (1.4) at frequency $n-n'$ and the tidal perturbations at frequencies 0 and $2(n-n')$. He suggested that the first-order result should be multiplied by a factor $1+2n'/n = 1+2m+O(m^2)$, i.e., amplified by about 15%. As a result of these (coincidentally equivalent) prescriptions, the literature on the "Nordtvedt effect" [21,22,20,23,24] has, for many years, used as a standard estimate for the range oscillation $\delta r = C\bar{\delta}_{12} a' \cos\tau$ a value $C \simeq 1.14 C^{(1)}$ (corresponding to about $9.3\eta\ \cos\tau$ meters in metrically-coupled theories; see below).

[1]This is the version of the small parameter which is appropriate to our Hill-Brown treatment. Beware of the fact that the more traditional perturbation approaches denoted by the letter m the quantity $\bar{m} \equiv n'/n$.

Actually, as recently found by Nordtvedt [25] and studied in fuller detail in the present paper, both modifications (suggested in [17] and [20]) of the first-order result fall short of giving an accurate estimate of the effects due to higher powers of m. In fact, they do not even give correctly the second order in m. For completeness, we compute in Appendix A, by the standard perturbation theory of de Pontécoulant [26], the contribution at order $O(m^2)$ and find that it amounts to multiplying the first-order result by $1+\frac{9}{2}m+O(m^2)$, i.e.,[2]

$$C^{(1)}+C^{(2)}=\left[1+\frac{9}{2}m+O(m^2)\right]C^{(1)} =\frac{3}{2}m\left[1+\frac{14}{3}m+O(m^2)\right]. \tag{1.10}$$

In his recent work [25] Nordtvedt showed, by studying what is essentially a truncated version of Hill's perturbation equation (discussed in Appendix C below), that the interaction with the orbit's tidal deformation causes a rather large numerical amplification of the synodic oscillation (1.4). The numerical result he got for the synodic oscillation sensitivity of the lunar orbit is $\delta r \simeq 2.9\times 10^{12}\bar{\delta}_{12}\cos\tau$ cm and agrees well (within the quoted precision) with our result, Eq. (1.11) below. However, his treatment gives only an incomplete theoretical analysis of this amplification. The only explicit literal result he quotes [his Eq. (2.33)] matches the second-order result (1.10) and captures the important feature of the existence of a simple pole in m, but does not accurately determine the location of the pole.

The aim of the present paper is to provide, for the first time, a full-fledged Hill-Brown analytical treatment of the orbital perturbations caused by a violation of the equivalence principle. Our results will notably allow us to give a precise numerical value for the full range oscillation in the case of the actual Moon.[3] Namely, we obtain below

$$\delta r = 2.9427\times 10^{12}\bar{\delta}_{12}\cos\tau \ \text{cm}, \tag{1.11}$$

corresponding to a full coefficient $C=\frac{3}{2}m\times 1.622\,01$ which is larger than the first-order value (1.8) by more than 60%. More generally, we shall be able to discuss in detail the dependence on m of the range oscillation: see Eqs. (2.60)–(2.62), Eq. (3.2), and Appendix B. These results are summarized in Fig. 1 below. Our results confirm the prediction of Ref. [25] that when m increases (corresponding to prograde orbits beyond the actual lunar orbit) the $\cos\tau$ range oscillation eventually becomes resonant and is (formally) infinitely amplified. We have some doubts, however, about the practical utility of such a resonant orbit, notably because we show that it occurs precisely at the value

$$m=m_{\rm cr}=0.195\,103\,996\,6\ldots \tag{1.12}$$

[corresponding to a sidereal period $T_{\rm cr}=m_{\rm cr}/(1+m_{\rm cr})$ yr $=1.959\,03$ month] where the orbit becomes exponentially unstable. Armed with our theoretical understanding of the m dependence of the $\cos\tau$ oscillation, we suggest other orbits that might be practically interesting (retrograde orbits and orbits around outer planets). Finally, we emphasize that within the context of modern unified theories, the most probable theoretical significance of *orbital* tests of the universality of free fall is the same as that of *laboratory* tests, namely, to measure, through composition-dependent effects, the strength of the coupling to matter of some long range scalar field(s). The basic measure of this coupling strength is embodied in an effective Eddington parameter $\bar{\gamma}\equiv\gamma-1$ which governs both the standard post-Newtonian effects (including the violation of the strong equivalence principle $\propto\eta\equiv 4\beta-\bar{\gamma}$) and the composition-dependent couplings (violation of the weak equivalence principle). Actually, string theory suggests that the former contribution (proportional to the gravitational binding energy) is, in the Earth-Moon case (but not necessarily in other cases), negligible compared to the one due to a violation of the weak equivalence principle. Interpreting the latest LLR observational results [27,16] within a recently studied class of string-derived theoretical models, we conclude that present orbital tests give the excellent constraint $\bar{\gamma}=(-0.9\pm 1.3)\times 10^{-7}$. [This limit is comparable to the (similarly interpreted) constraint coming from laboratory tests [28] of the weak equivalence principle: $\bar{\gamma}=(-0.8\pm 1.0)\times 10^{-7}$.]

The plan of this paper is as follows. Section II presents our Hill-Brown approach and gives the analytical results obtained with it. Section III discusses the physical consequences of our results. Many technical details are relegated to appendices: Appendix A presents the standard de Pontécoulant treatment of lunar theory and uses it to derive the second-order result (1.10), Appendix B gives some details of our Hill-Brown treatment, Appendix C treats the link between certain commensurabilities of frequencies, linear instability, and the presence of pole singularities in perturbed motions, and finally Appendix D discusses Laplace's derivation of the remarkably good limit (1.3) on $\bar{\delta}_{12}$.

II. HILL-BROWN TREATMENT OF EQUIVALENCE-PRINCIPLE-VIOLATION EFFECTS

A. Introduction

Relativistic effects in the lunar motion have been investigated by many authors. The pioneers in this field are de Sitter [29] (who computed the general relativistic contributions to the secular motions of the lunar perigee and node as observed in a global, barycentric frame) and Brumberg [30] (who gave a comprehensive Hill-Brown treatment of the post-Newtonian three-body problem). Later works studied non-Einsteinian effects, notably those associated with the Eddington post-Newtonian parameters β and γ. The most comprehensive and accurate analytical study of post-Newtonian effects in the lunar motion (described in a bary-

[2]By contrast, [17] and [20] give 11/12 and 13/6, respectively, for the coefficient of m in the correcting factor within square brackets in the second equation (1.10). Note that $\frac{14}{3}m=37.7\%$ for the Moon.

[3]Note, however, that we consider only the Main Lunar Problem, i.e., that we neglect the terms proportional to the squares of the lunar and solar eccentricities, and to the square of the lunar inclination, which are expected to modify our numerical estimates by $\lesssim 1\%$.

centric frame) is due to Brumberg and Ivanova [19]. For general accounts and more references see the books [31,32]. Let us also mention the semianalytical treatment of the general relativistic perturbations of the Moon by Lestrade and Chapront-Touzé [33].

However, apart from the work of Nordtvedt [10,17,25], the studies of non-Einsteinian effects in the lunar motion have not considered the effect of a violation of the equivalence principle. The results of the present paper can therefore be considered as a completion of Ref. [19] which gave an accurate Hill-Brown theory of all the other Einsteinian and non-Einsteinian effects. In fact, as pointed out long ago by Nordtvedt, the effects of a violation of the universality of free fall are the most prominent non-Einsteinian effects in the lunar orbit, and therefore deserve an accurate study. Indeed, most of the non-Einsteinian effects are *non-null* effects, i.e., correspond to modifications proportional to $\bar{\beta}\equiv\beta-1$ or $\bar{\gamma}\equiv\gamma-1$ of observable relativistic effects (as seen in a local, geocentric frame) predicted by Einstein's theory. As the latter are at the few cm level [34,31,35,25], which is the precision of the LLR data, they can be of no use for measuring $\bar{\beta}$ or $\bar{\gamma}$ at an interesting level (say $<10^{-2}$). An exception must be made for secular effects and for the parameters describing the temporal and spatial transformation linking a local, geocentric frame to a global, barycentric one, e.g., the parameters entering the de Sitter-Fokker ("geodetic") precession. (Recent work [16] concludes that geodetic precession alone constraints $\bar{\gamma}$ at the 1% level.)

In addition to the "Nordtvedt effect" proper (i.e., the effect of $\bar{\delta}_{12}\neq 0$), that we discuss here, there are some other *null* effects which are more sensitive to $\bar{\beta}$ and $\bar{\gamma}$ than the non-Einsteinian modifications of non-null general relativistic effects. A subdominant null effect comes from the violation of the equivalence principle associated with the gravitational binding energy of the Earth-Moon system. In lowest approximation (linear in m), it is equivalent [see, e.g., Eq. (3.14b) of Ref. [36]] to replacing $\bar{\delta}_{12}$ by

$$\tilde{\bar{\delta}}_{12}=\bar{\delta}_{12}-\frac{1}{3}\,\eta\left(\frac{na}{c}\right)^2, \tag{2.1}$$

where a denotes the semi-major axis of the lunar orbit and where η denotes, as usual, the combination

$$\eta\equiv 4\bar{\beta}-\bar{\gamma}=4\beta-\gamma-3. \tag{2.2}$$

In general, $\bar{\delta}_{12}$ is the sum of two physically independent contributions

$$\bar{\delta}_{12}=(\hat{\delta}_1-\hat{\delta}_2)+\eta\left(\frac{E_1^{\rm grav}}{m_1c^2}-\frac{E_2^{\rm grav}}{m_2c^2}\right). \tag{2.3}$$

The first contribution $\hat{\delta}_{12}\equiv\hat{\delta}_1-\hat{\delta}_2$ is generically expected to be present because the best motivated modified theories of gravity violate the "weak equivalence principle," i.e., contain, in addition to Einstein's universal tensor interaction, some composition-dependent couplings that make laboratory bodies fall in a nonuniversal way (see, e.g., [37,38]). The second contribution on the right-hand side of Eq. (2.3) [proportional to η such as the correction in Eq. (2.1)] contains the gravitational self-energy of the bodies ($A=1,2$),

$$E_A^{\rm grav}=-(G/2)\int_A\int_A d^3x\,d^3x'\,\rho(\mathbf{x})\rho(\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|, \tag{2.4}$$

and is the one first pointed out by Nordtvedt [11,12]. As indicated by Dicke [13,14], it is present in all gravity theories where the effective, locally measured gravitational "constant" may vary from place to place (see, e.g., Sec. V B of [37]). We shall take as nominal values for the gravitational self-energies of the Moon (label 1) and the Earth (label 2) the values adopted by Williams, Newhall, and Dickey [16], namely, $E_1^{\rm grav}/m_1c^2=-0.19\times10^{-10}$, $E_2^{\rm grav}/m_2c^2=-4.64\times10^{-10}$, so that

$$\frac{E_1^{\rm grav}}{m_1c^2}-\frac{E_2^{\rm grav}}{m_2c^2}=4.45\times10^{-10}. \tag{2.5}$$

We then find numerically that the modification due to the gravitational binding energy of the Earth-Moon system, $-\frac{1}{3}\eta n^2a^2/c^2=-\frac{1}{3}\eta G(m_1+m_2)/ac^2$ in Eq. (2.1), is (to first-order) equivalent to decreasing the nominal value (2.5) by -0.039×10^{-10}. This represents a fractional change of (2.5) by -0.87% which is probably smaller than the uncertainty in the estimate (2.5) associated with our imperfect knowledge of the internal structures of the Earth and the Moon. These orders of magnitude illustrate the fact that the overwhelmingly dominant sensitivity of the lunar motion to non-Einsteinian effects comes from the terms proportional to $\bar{\delta}_{12}$ that we concentrate upon in the following.

B. Three-body Lagrangian

The Lagrangian describing the N-body problem in the currently best-motivated relativistic theories of gravity, i.e., those where gravity is mediated both by a tensor field and a scalar field with, generically, composition-dependent couplings (see, e.g., [37,38]), can be written as

$$L_{\bar{\beta},\bar{\gamma},\bar{\delta}}=L_{\rm GR}+L_{\bar{\beta}}+L_{\bar{\gamma}}+L_{\bar{\delta}}, \tag{2.6}$$

where $L_{\rm GR}$ denotes the general relativistic contribution (in which one should use an effective value of the gravitational coupling constant G which incorporates the composition-independent part of the interaction mediated by the scalar field),

$$L_{\bar{\gamma}}=\frac{1}{2c^2}\,\bar{\gamma}\sum_{A\neq B}\frac{Gm_Am_B}{r_{AB}}(\mathbf{v}_A-\mathbf{v}_B)^2 \tag{2.7}$$

denotes the (nontensor-like) velocity-dependent part of the two-body scalar interaction (one-scalaron exchange level),

$$L_{\bar{\beta}}=-\frac{1}{c^2}\,\bar{\beta}\sum_{B\neq A\neq C}\frac{G^2m_Am_Bm_C}{r_{AB}r_{AC}} \tag{2.8}$$

denotes the modification of the nonlinear, three-body general relativistic interaction due to the scalar interaction, and

$$L_{\bar{\delta}}=\frac{1}{2}\sum_{A\neq B}(\bar{\delta}_A+\bar{\delta}_B)\frac{Gm_Am_B}{r_{AB}}, \tag{2.9}$$

with

$$\bar{\delta}_A = \hat{\delta}_A + \eta \frac{E_A^{\rm grav}}{m_A c^2}, \tag{2.10}$$

represents (to lowest order) the combined effect of the composition-dependent couplings ($\hat{\delta}_A \neq 0$; violation of the "weak equivalence principle") and of the Dicke-Nordtvedt contribution due to the spatial variability of the effective gravitational coupling constant ($\eta = 4\bar{\beta} - \bar{\gamma} \neq 0$; violation of the "strong equivalence principle"). For a direct, field-theory derivation of $L_{\bar{\gamma}}$ and $L_{\bar{\beta}}$ and the expression of the phenomenological Eddington parameters $\bar{\gamma} = \gamma - 1$ and $\bar{\beta} = \beta - 1$ in terms of the basic coupling parameters of the scalar field (as well as the generalization of these results to the case of strongly self-gravitating bodies) see Ref. [39].

We assume here that all the general relativistic contributions to the lunar motion (and to its observation through laser ranging) are separately worked out with sufficient accuracy, using, for instance, the new, complete framework for relativistic celestial mechanics of Ref. [40] (which provides the first consistent relativistic description of the multipole moments of extended bodies). Following the discussion above, we henceforth discard the subdominant contributions coming from $L_{\bar{\beta}}$ and $L_{\bar{\gamma}}$ to concentrate upon the effects due to $L_{\bar{\delta}}$. (The barycentric frame contributions of $L_{\bar{\beta}}$ and $L_{\bar{\gamma}}$ have been accurately computed by Brumberg and Ivanova [19] and can be linearly superposed with the ones of $L_{\bar{\delta}}$.) Finally, it is enough to consider the sum of the lowest-order approximation to $L_{\rm GR}$ and of $L_{\bar{\delta}}$, namely,

$$L(\mathbf{x}_A, \mathbf{v}_A) = \sum_A \frac{1}{2} m_A \mathbf{v}_A^2 + \frac{1}{2} \sum_{A \neq B} \frac{G_{AB} m_A m_B}{r_{AB}}, \tag{2.11}$$

where

$$G_{AB} = G[1 + \bar{\delta}_A + \bar{\delta}_B], \tag{2.12}$$

with $\bar{\delta}_A$ of the form given in (2.10), represents the effective, composition-dependent gravitational coupling between the massive bodies A and B. In Eq. (2.11) $\mathbf{v}_A = d\mathbf{x}_A/dt$ denotes the (barycentric) velocity of body A, m_A the (inertial) mass of A, and $r_{AB} = |\mathbf{x}_A - \mathbf{x}_B|$ the distance between A and B. We consider a three-body problem, and, more particularly, the Moon-Earth-Sun system (1 = Moon, 2 = Earth, 3 = Sun). Evidently, all our analytical results will apply if 2 denotes another planet, and 1 one of its natural or artificial satellites. (Note, however, that the relative orders of magnitude of the non-Einsteinian effects is different for low-orbit artificial Earth satellites. See [36] for a recent discussion.)

Starting from Eq. (2.11), we first separate the variables describing the motion of the center of mass of the Earth-Moon system,

$$m_0 \mathbf{x}_0 \equiv m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2, \tag{2.13}$$

$$m_0 \equiv m_1 + m_2, \tag{2.14}$$

and $\mathbf{v}_0 \equiv d\mathbf{x}_0/dt$, from those describing the relative lunar motion,

$$\mathbf{x}_{12} \equiv \mathbf{x}_1 - \mathbf{x}_2, \tag{2.15}$$

$$\mu_{12} \equiv \frac{m_1 m_2}{m_1 + m_2}, \tag{2.16}$$

and $\mathbf{v}_{12} \equiv d\mathbf{x}_{12}/dt$. This yields

$$L(\mathbf{x}_0, \mathbf{x}_{12}, \mathbf{x}_3; \mathbf{v}_0, \mathbf{v}_{12}, \mathbf{v}_3) = \frac{1}{2} m_0 \mathbf{v}_0^2 + \frac{1}{2} \mu_{12} \mathbf{v}_{12}^2 + \frac{1}{2} m_3 \mathbf{v}_3^2 + G_{12} \frac{m_1 m_2}{r_{12}} + G_{13} \frac{m_1 m_3}{r_{13}} + G_{23} \frac{m_2 m_3}{r_{23}}, \tag{2.17}$$

where

$$r_{13} = |\mathbf{x}_3 - \mathbf{x}_0 - X_2 \mathbf{x}_{12}|, \tag{2.18a}$$

$$r_{23} = |\mathbf{x}_3 - \mathbf{x}_0 + X_1 \mathbf{x}_{12}|, \tag{2.18b}$$

$$X_1 \equiv m_1/m_0, \quad X_2 \equiv m_2/m_0 = 1 - X_1. \tag{2.18c}$$

Expanding r_{13}^{-1} and r_{23}^{-1} in powers of r_{12}/r_{30} ($r_{30} = |\mathbf{x}_{30}|$ with $\mathbf{x}_{30} = \mathbf{x}_3 - \mathbf{x}_0$) leads to

$$L = L_{03}(\mathbf{x}_0, \mathbf{x}_3, \mathbf{v}_0, \mathbf{v}_3) + \mu_{12} \hat{L}_{12}(\mathbf{x}_{12}, \mathbf{v}_{12}, \mathbf{x}_{30}), \tag{2.19}$$

$$L_{03} = \frac{1}{2} m_0 \mathbf{v}_0^2 + \frac{1}{2} m_3 \mathbf{v}_3^2 + (X_1 G_{13} + X_2 G_{23}) \frac{m_0 m_3}{r_{03}}, \tag{2.20}$$

$$\hat{L}_{12} = \frac{1}{2} \mathbf{v}_{12}^2 + G_{12} \frac{m_0}{r_{12}} + R_1 + R_2 + R_3 + \cdots, \tag{2.21}$$

and

$$R_1 = m_3(-G_{13} + G_{23}) x_{12}^i \partial_i^{(3)} \frac{1}{r_{30}}, \tag{2.22a}$$

$$R_2 = \frac{1}{2!} m_3 (G_{13} X_2 + G_{23} X_1) x_{12}^i x_{12}^j \partial_{ij}^{(3)} \frac{1}{r_{30}}, \tag{2.22b}$$

$$R_3 = \frac{1}{3!} m_3 (-G_{13} X_2^2 + G_{23} X_1^2) x_{12}^i x_{12}^j x_{12}^k \partial_{ijk}^{(3)} \frac{1}{r_{30}}, \tag{2.22c}$$

where $\partial_i^{(3)} \equiv \partial/\partial x_3^i$, $\partial_{ij}^{(3)} \equiv \partial^2/\partial x_3^i \partial x_3^j, \ldots$. Note that the suffices 1,2,3, ... in R_n's have nothing to do with the body labels $A, B = 1,2,3$, but keep track of the successive powers of $\mathbf{x}_{12}$.

To a very good approximation we can consider that, in the (normalized) Earth-Moon Lagrangian (2.21), the motion of the Sun with respect to the Earth-Moon barycenter, $\mathbf{x}_{30}(t)$, is obtained by solving the two-body Lagrangian (2.20). After separating in Eq. (2.20) the motion of the center of mass of the Earth-Moon-Sun system, the reduced Lagrangian describing the dynamics of the relative motion $\mathbf{x}_{30}$ is

$$\hat{L}_{03}^{\rm relative} = \frac{1}{2} \mathbf{v}_{03}^2 + G_{03} \frac{m_0 + m_3}{r_{03}}, \tag{2.23}$$

where we introduced

$$G_{03} \equiv X_1 G_{13} + X_2 G_{23}. \tag{2.24}$$

Therefore, seen from the Earth-Moon barycenter, the Sun undergoes a Keplerian, elliptic motion corresponding to a total effective gravitational mass $G_{03}(m_0+m_3)=G_{03}(m_1+m_2+m_3)$. If we denote (as is traditional in lunar theory) the mean angular velocity and the semimajor axis of this elliptic motion as n' and a', respectively, we can write

$$n'^2a'^3=G_{03}(m_0+m_3)=G_{03}(m_1+m_2+m_3). \quad (2.25)$$

For simplicity's sake, we shall, in fact, consider the "Main Problem" of lunar theory in which the Sun is considered as moving in a circle of radius a' with the constant angular velocity n'.

Evaluating the derivatives with respect to $\mathbf{x}_3$ in Eqs. (2.22), and using Eq. (2.12), yields

$$R_1=m_3^{\rm grav}\bar{\delta}_{12}\frac{\mathbf{N}\cdot\mathbf{r}}{r'^2}, \quad (2.26a)$$

$$R_2=m_3^{\rm grav}[1+(X_2-X_1)\bar{\delta}_{12}]\frac{3(\mathbf{N}\cdot\mathbf{r})^2-\mathbf{r}^2}{2r'^3}, \quad (2.26b)$$

$$R_3\simeq m_3^{\rm grav}(X_2-X_1)\frac{5(\mathbf{N}\cdot\mathbf{r})^3-3(\mathbf{N}\cdot\mathbf{r})\mathbf{r}^2}{2r'^4}. \quad (2.26c)$$

To simplify the notation, we have written $\mathbf{r}\equiv\mathbf{x}_{12}=\mathbf{x}_1-\mathbf{x}_2$, $\mathbf{N}\equiv\mathbf{x}_{30}/r_{30}$ (directed toward the Sun), $r'\equiv r_{30}$, $m_3^{\rm grav}\equiv G_{03}m_3$, where we recall that G_{03} is the weighted average (2.24), and

$$\bar{\delta}_{12}\equiv\bar{\delta}_1-\bar{\delta}_2, \quad (2.27)$$

which agrees with our previous expressions (1.1) or (2.3). In the (much smaller) term R_3 we have neglected the δ-modifications.

C. Hill's equations of motion

Following Euler's second lunar theory and Hill [41,42], it is convenient to refer the motion of the Moon to axes rotating with the mean angular velocity of the Sun. For simplicity, we shall consider the main lunar problem in which (i) the Sun is considered as moving in a circle of radius a' with the uniform angular velocity n', (ii) the Moon moves in the same plane as the Sun, and (iii) one looks for a periodic motion of the Moon in the frame rotating with the angular velocity n'. Taking into account the lunar and solar eccentricities e and e', and the lunar inclination I, is expected to modify the results for the terms discussed in this paper by contributions of order $O(e^2,e'^2,\sin^2 I)\lesssim 1\%$.

With respect to the rotating frame $(\mathbf{e}_X(t),\mathbf{e}_Y(t))$, with $\mathbf{e}_X=\mathbf{N}$ directed toward the Sun, the position and velocity vectors of the Moon read $\mathbf{r}=\mathbf{x}_{12}=X\mathbf{e}_X+Y\mathbf{e}_Y$, $\mathbf{v}=\mathbf{v}_{12}=(\dot{X}-n'Y)\mathbf{e}_X+(\dot{Y}+n'X)\mathbf{e}_Y$ (the overdot denoting d/dt). When expanding the kinetic terms in the reduced, relative Lagrangian (2.21), i.e., $\frac{1}{2}\mathbf{v}^2=\frac{1}{2}(\dot{X}-n'Y)^2+\frac{1}{2}(\dot{Y}+n'X)^2$, one recognizes the usual Coriolis (terms linear in $\dot{X}$ and $\dot{Y}$) and centrifugal (terms quadratic in X and Y) effects. The centrifugal terms can be gathered with the contribution R_2, Eq. (2.26b), which is also quadratic in X and Y. The resulting Lagrangian describing the dynamics in the rotating frame reads

$$\hat{L}_{12}(X,Y,\dot{X},\dot{Y})=\frac{1}{2}(\dot{X}^2+\dot{Y}^2)+n'(X\dot{Y}-Y\dot{X})+F(X,Y). \quad (2.28)$$

Here, $F(X,Y)=G_{12}m_0/r+R_1+[R_2+1/2n'^2(X^2+Y^2)]+R_3+\cdots=F_0+F_1+F_2+F_3+\cdots$ is a *time-independent* potential with

$$F_0=\frac{G_{12}m_0}{\sqrt{X^2+Y^2}}, \quad (2.29a)$$

$$F_1=\frac{n'^2a'}{1+m_0/m_3}\bar{\delta}_{12}X, \quad (2.29b)$$

$$F_2=\frac{1}{2}\frac{n'^2}{1+m_0/m_3}\left\{[1+(X_2-X_1)\bar{\delta}_{12}](3X^2)+\left[\frac{m_0}{m_3}-(X_2-X_1)\bar{\delta}_{12}\right](X^2+Y^2)\right\}, \quad (2.29c)$$

$$F_3\simeq\frac{1}{2}\frac{n'^2}{a'}(X_2-X_1)[5X^3-3X(X^2+Y^2)]. \quad (2.29d)$$

In writing out Eqs. (2.29) we have replaced $m_3^{\rm grav}=G_{03}m_3$ by the expression $m_3^{\rm grav}=n'^2a'^3/(1+m_0/m_3)$ obtained from Eq. (2.25). The equations of motion corresponding to the Lagrangian (2.28) read

$$\frac{d^2X}{dt^2}-2n'\frac{dY}{dt}=\frac{\partial F}{\partial X}, \quad (2.30a)$$

$$\frac{d^2Y}{dt^2}+2n'\frac{dX}{dt}=\frac{\partial F}{\partial Y}. \quad (2.30b)$$

We see from Eqs. (2.29) that a violation of the equivalence principle has several consequences in the lunar theory: (i) the effective gravitational constant appearing in front of the Earth-Moon total mass $m_0=m_1+m_2$, namely $G_{12}=G(1+\bar{\delta}_1+\bar{\delta}_2)$, differs from the one appearing in the theory of the Earth orbital motion, G_{03}, defined in Eq. (2.24), (ii) there is a new term, linear in X, in the Lagrangian, F_1, Eq. (2.29b), (iii) the usual tidal plus centrifugal potential F_2 (as well as the higher-order tidal potentials) is fractionally modified by $\bar{\delta}_{12}\neq 0$. The effect (i) has practically no observational consequences as, for instance, the "GM" of the Earth is measured much more accurately from Earth satellites (artificial or natural) than from the correction it brings in the Earth-Sun interaction. The effect (ii) is the one discussed by Newton, Laplace, and Nordtvedt, that we shall study in detail below. As for the effects (iii) it will be clear from the following that they are numerically negligible compared to the effects of F_1 because the corresponding source terms in the equations of motion are smaller by a factor $r/a'\simeq 1/400$, and, moreover, the corresponding solution is not amplified (as the F_1 effects) by a small divisor $1/m\simeq 12$ because they correspond to the driving frequency $2(n-n')$ (instead of $n-n'$ for F_1). Finally, as we can also

neglect the fractional correction $(1+m_0/m_3)^{-1}$ to $\bar{\delta}_{12}$ in (2.29b) ($m_0/m_3 \simeq 1/328900$), we shall keep Eqs. (2.29a) and (2.29d) and replace (2.29b) and (2.29c), respectively, by

$$F_1 \simeq n'^2 a' \bar{\delta}_{12} X, \tag{2.31a}$$

$$F_2 \simeq \frac{3}{2} n'^2 X^2. \tag{2.31b}$$

The contribution F_3 to the potential F (octupolar tide) generates the so-called "parallactic" terms in the lunar motion. Compared to the usual (quadrupolar) tidal contribution, they contain the small parameter $r/a' \simeq 1/400$. Hill's approach consists of solving first exactly (in the sense of infinite power series) the dynamics defined by the truncated potential $F_{\rm Hill} = F_0 + F_2 = G_{12} m_0/r + \frac{3}{2} n'^2 X^2$. (The parallactic terms are obtained later by perturbing Hill's main problem.) In presence of a violation of the equivalence problem we have to add the term F_1, Eq. (2.31a) ("dipolar tide"), to $F_{\rm Hill}$. The resulting equations of motion (2.30) read explicitly

$$\frac{d^2X}{dt^2} - 2n' \frac{dY}{dt} = -\frac{G_{12} m_0}{r^3} X + 3n'^2 X + n'^2 a' \bar{\delta}_{12}, \tag{2.32a}$$

$$\frac{d^2Y}{dt^2} + 2n' \frac{dX}{dt} = -\frac{G_{12} m_0}{r^3} Y. \tag{2.32b}$$

They admit (in the general case of a time-independent potential F) the Jacobi energy integral

$$\frac{1}{2}(\dot{X}^2 + \dot{Y}^2) - F(X,Y) = C = \text{ const}. \tag{2.33}$$

D. Iterative solution of Hill's equations

In spite of the apparent simplicity of Eqs. (2.32) and of the existence of the first integral (2.33), the corresponding dynamics contains all the richness and complexity of the circular, planar, restricted, three-body problem. Hill's idea was first to find an exact *periodic* solution of Eqs. (2.32) (with $\bar{\delta}_{12} = 0$). The existence (in a mathematical sense) of Hill's periodic solution, and the convergence of the power series [in the parameter $m \equiv n'/(n-n')$] giving its explicit form, have been proven by Poincaré [43] using an analytic-continuation argument, and by Wintner [44] using majorant series (see [45] for more references). The existence of such a one-parameter family of coplanar, periodic solutions, and the convergence of the associated perturbation series in m, are stable under the addition of the full series of higher-order tidal terms [46]. Poincaré's analytic-continuation method shows that it will still exist when one adds the "dipole tidal" term F_1, if m and $\bar{\delta}_{12}$ are small enough.

To construct explicitly the perturbation series in m giving the periodic solutions of Eqs. (2.32), it is convenient to rewrite them in terms of new variables. Following a standard notation [18] (except that we do not introduce a separate letter for the complex conjugate of u), we define

$$u = X + iY, \tag{2.34a}$$

$$\bar{u} = X - iY, \tag{2.34b}$$

$$\tau = (n-n')t + \tau_0, \tag{2.34c}$$

$$\zeta = e^{i\tau}, \tag{2.34d}$$

$$D = \frac{1}{i}\frac{d}{d\tau} = \zeta \frac{d}{d\zeta}. \tag{2.34e}$$

Here, n denotes the mean sidereal orbital velocity of the periodic solution one is looking for (in other words, the rotating frame quantities $X, Y, u, \bar{u}$ are supposed to be periodic functions of τ with period 2π). The parameter $m = n'/(n-n')$ is the remaining free parameter of the problem. It takes positive values for prograde orbits (going in the same sense as the Sun[4]: $0 < n' < n$), and negative values for retrograde orbits $n < 0$. With this notation the general equations of motion (2.30) read

$$D^2 u + 2mDu = -2\frac{\partial \hat{F}}{\partial \bar{u}} \tag{2.35}$$

(and its complex conjugate: $u \to \bar{u}$, $D \to \bar{D} = -D$), where $\hat{F} \equiv (n-n')^{-2} F = m^2 n'^{-2} F$. For small values of m (i.e., orbits of small radius around the Earth) the periodic solution of Eq. (2.35) is of the form $u \simeq (\text{const})\zeta$, $\bar{u} \simeq (\text{const})\bar{\zeta} = (\text{const})\zeta^{-1}$. It is convenient to replace u by a variable w which tends to zero with m. Following Liapunov [47] and Brumberg and Ivanova [19], one defines first a fiducial lunar semimajor axis $\tilde{a}$ by writing

$$\frac{G_{12} m_0}{(n-n')^2 \tilde{a}^3} = \kappa(m), \tag{2.36}$$

where

$$\kappa(m) \equiv 1 + 2m + \frac{3}{2} m^2. \tag{2.37}$$

With this definition of $\tilde{a}$ one introduces the variable w by

$$u = \tilde{a}\zeta(1+w), \tag{2.38a}$$

$$\bar{u} = \tilde{a}\zeta^{-1}(1+\bar{w}). \tag{2.38b}$$

The Lagrangian $\mathscr{L} = -2\tilde{a}^{-2}(n-n')^{-2}\hat{L}_{12}$ can be written as

$$\mathscr{L}(w,\bar{w},Dw,D\bar{w}) = Dw D\bar{w} + 2(m+1) w D\bar{w} - G(w,\bar{w}) + (\text{total derivative}), \tag{2.39}$$

where $G \equiv 2\tilde{a}^{-2}\hat{F} + (1+2m)(1+w)(1+\bar{w})$, and the associated equations of motion read

[4] We shall not consider the prograde orbits with $0 < n < n'$ which are highly unstable; see below.

$$D^2w+2(m+1)Dw=-\frac{\partial G}{\partial\bar{w}}. \tag{2.40}$$

The explicit form of the potential $G(w,\bar{w})$ in our problem [i.e., when $(n-n')^2\hat{F}\equiv F=F_0+F_1+F_2$] reads

$$\begin{aligned}G(w,\bar{w})=&\kappa(m)[2(1+w)^{-1/2}(1+\bar{w})^{-1/2}+(1+w)(1+\bar{w})]\\&+\hat{\lambda}[\zeta(1+w)+\zeta^{-1}(1+\bar{w})]\\&+\frac{3}{4}m^2[\zeta^2(1+w)^2+\zeta^{-2}(1+\bar{w})^2],\end{aligned} \tag{2.41}$$

where we have introduced, instead of $\bar{\delta}_{12}$, the small dimensionless parameter

$$\hat{\lambda}\equiv m^2\bar{\delta}_{12}\frac{a'}{\tilde{a}}. \tag{2.42}$$

The corresponding equations of motion read

$$\begin{aligned}&D^2w+2(m+1)Dw+\kappa[1+w-(1+w)^{-1/2}(1+\bar{w})^{-3/2}]\\&+\hat{\lambda}\zeta^{-1}+\frac{3}{2}m^2\zeta^{-2}(1+\bar{w})=0.\end{aligned} \tag{2.43}$$

A last transformation [47,19] consists of separating off the square bracket multiplied by κ in (2.43), its nonlinear piece, namely,

$$\begin{aligned}Q(w,\bar{w})\equiv&(1+w)^{-1/2}(1+\bar{w})^{-3/2}-1+\frac{1}{2}w+\frac{3}{2}\bar{w}\\=&\frac{3}{8}w^2+\frac{15}{8}\bar{w}^2+\frac{3}{4}w\bar{w}+O(w^3,w^2\bar{w},w\bar{w}^2,\bar{w}^3).\end{aligned} \tag{2.44}$$

We can now define a *linear* operator acting on $(w,\bar{w})$,

$$L(w,\bar{w})\equiv D^2w+2(m+1)Dw+\frac{3}{2}\kappa(m)(w+\bar{w}), \tag{2.45}$$

and an effective source term (containing source terms and nonlinearities)

$$W(w,\bar{w})\equiv-\hat{\lambda}\zeta^{-1}-\frac{3}{2}m^2\zeta^{-2}(1+\bar{w})+\kappa(m)Q(w,\bar{w}). \tag{2.46}$$

In terms of these definitions, the equations of motion read

$$L(w,\bar{w})=W(w,\bar{w}), \tag{2.47}$$

and its complex conjugate equation. This is the form used by Brumberg and Ivanova [19] in their study of relativistic effects in the lunar motion.

Note that the source of all equivalence-principle-violation effects is the contribution $-\hat{\lambda}\zeta^{-1}$ on the right-hand side of Eq. (2.46), with $\hat{\lambda}$ defined by Eq. (2.42). Even when restricting oneself (as we shall) to the effects linear in $\hat{\lambda}$, the corresponding contributions in the solution are quite complicated because of the interplay with the quadrupole tidal effects (from F_2), i.e., mathematically, because of the mixed term $-\frac{3}{2}m^2\zeta^{-2}\bar{w}$ and the nonlinear term $\propto Q(w,\bar{w})$ in $W(w,\bar{w})$.

When putting back the octupole tidal effects [F_3, Eq. (2.29d)], they add to the effective source term (2.46) the contribution

$$\begin{aligned}W_3(w,\bar{w})=&-\frac{3}{8}\hat{\pi}[\zeta(1+w)^2+2\zeta^{-1}(1+w)(1+\bar{w})\\&+5\zeta^{-3}(1+\bar{w})^2],\end{aligned} \tag{2.48}$$

where

$$\hat{\pi}\equiv m^2(X_2-X_1)\frac{\tilde{a}}{a'}=m^2\frac{m_2-m_1}{m_2+m_1}\frac{\tilde{a}}{a'}. \tag{2.49}$$

The Eq. (2.47) can be solved by iteration: first, one keeps only the linear source terms which exist when $w=0$, namely, $W^{(1)}(\zeta,\zeta^{-1})=W(0,0)=-\hat{\lambda}\zeta^{-1}-\frac{3}{2}m^2\zeta^{-2}$ [with the addition of $W_3(0,0)=-\frac{3}{8}\hat{\pi}(\zeta+2\zeta^{-1}+5\zeta^{-3})$ when including parallactic terms]. Second, one solves the linear equations $L(w,\bar{w})=W^{(1)}(\zeta,\zeta^{-1})$ to get the corresponding first-order solution: $w^{(1)}=\hat{\lambda}w^{(1)}_{\lambda}+m^2w^{(1)}_{m^2}+\hat{\pi}w^{(1)}_{\pi}$, which is valid up to terms of higher order in the formal expansion parameters $\hat{\lambda}$, m^2, and $\hat{\pi}$. The next step is to insert the first-order solution $w^{(1)}$ in the full source term $W(w,\hat{w})$ and to collect the second-order source term $W^{(2)}(\zeta,\zeta^{-1})$ of formal order $(\hat{\lambda}+m^2+\hat{\pi})^2$. The corresponding second-order solution $w^{(2)}$ is obtained by solving the linear equation $L(w^{(2)},\bar{w}^{(2)})=W^{(2)}(\zeta,\zeta^{-1})$, etc. At each stage of the iteration, one deals with a source term which is a linear combination (with real coefficients[5]) of a finite number of integer powers of ζ and ζ^{-1}, say

$$\begin{aligned}W_\star=&W_0+W_1\zeta+W_{-1}\zeta^{-1}+W_2\zeta^2+W_{-2}\zeta^{-2}+\cdots+W_k\zeta^k\\&+W_{-k}\zeta^{-k}.\end{aligned} \tag{2.50}$$

It is easy to check that there is a unique solution of the linear system $L(w,\bar{w})=W_\star$, $\bar{L}(w,\bar{w})=\bar{W}_\star$ and that it is given by

$$\begin{aligned}w_\star=&w_0+w_1\zeta+w_{-1}\zeta^{-1}+w_2\zeta^2+w_{-2}\zeta^{-2}+\cdots+w_k\zeta^k\\&+w_{-k}\zeta^{-k},\end{aligned} \tag{2.51}$$

with [19]

$$w_0=\frac{1}{3\kappa(m)}W_0, \tag{2.52a}$$

$$w_k=\frac{1}{\Delta_k}\left[\left(k^2-2(m+1)k+\frac{3}{2}\kappa\right)W_k-\frac{3}{2}\kappa W_{-k}\right], \tag{2.52b}$$

$$w_{-k}=\frac{1}{\Delta_k}\left[\left(k^2+2(m+1)k+\frac{3}{2}\kappa\right)W_{-k}-\frac{3}{2}\kappa W_k\right], \tag{2.52c}$$

[5]The reality of all coefficients W_k in Eq. (2.50) and w_k in Eq. (2.51) is easily proven by induction, given the reality of the coefficients in the exact definition of $W(w,\bar{w})$ and in the iterative solution equation (2.52).

Here, $\Delta_k(=\Delta_{-k})$ denotes the determinant of the 2×2 system of equations satisfied by w_k and w_{-k} (when $k\neq 0$). Its value is

$$\Delta_k = k^2[k^2+3\kappa-4(m+1)^2] = k^2[k^2-1-2m+\tfrac{1}{2}m^2]. \tag{2.53}$$

This determinant never vanishes ($k\neq 0$), but it takes a small value of order m when $k=\pm 1$. This small divisor is one of the origins of the peculiar amplification which affects both equivalence-principle-violation effects [$W^{(1)}(F_1)=-\hat{\lambda}\zeta^{-1}$] and a part of the octupolar-tide effects [$W^{(1)}(F_3)=-\frac{3}{8}\hat{\pi}(\zeta+2\zeta^{-1}+\cdots)$]. A consequence of the small divisor $\Delta_1=-2m+\frac{1}{2}m^2$ is that, when collecting from the iterative solution the contributions proportional to $\hat{\lambda}\zeta^{\pm 1}$ and $\hat{\pi}\zeta^{\pm 1}$, they are found to proceed according to the powers of $m^2/\Delta_1=O(m)$ instead of the powers of m^2 as formally expected from the structure (2.46).[6]

The first steps of the iteration can be done by hand. From Eqs. (2.52) the linearized solution (without parallactic terms) has the form

$$w^{(1)} = w_1^{(1)}\zeta + w_{-1}^{(1)}\zeta^{-1} + w_2^{(1)}\zeta^2 + w_{-2}^{(1)}\zeta^{-2}, \tag{2.54}$$

with

$$w_1^{(1)} = +\frac{3}{2}\frac{\kappa}{\Delta_1}\hat{\lambda}, \tag{2.55a}$$

$$w_{-1}^{(1)} = -\frac{1+2(m+1)+\frac{3}{2}\kappa}{\Delta_1}\hat{\lambda}, \tag{2.55b}$$

$$w_2^{(1)} = +\frac{9}{4}\frac{\kappa}{\Delta_2}m^2, \tag{2.55c}$$

$$w_{-2}^{(1)} = -\frac{3}{2}\frac{4+4(m+1)+\frac{3}{2}\kappa}{\Delta_2}m^2. \tag{2.55d}$$

Here, as defined above, $\kappa\equiv 1+2m+\frac{3}{2}m^2$, $\Delta_1=-2m+\frac{1}{2}m^2$, and $\Delta_2=4(3-2m+\frac{1}{2}m^2)$. The insertion of the linearized solution $w^{(1)}$, Eqs. (2.55), into $W(w,\bar{w})$, Eq. (2.46), generates a second-order source term with the following structure: $W^{(2)}\sim\hat{\lambda}m^2(\zeta^{\pm 1}+\zeta^{\pm 3})+m^4(\zeta^0+\zeta^{\pm 2}+\zeta^{\pm 4})+O(\hat{\lambda}^2)$. Let us focus on the terms in the solution which are linear in $\hat{\lambda}$ and contain the "resonant" frequencies $\zeta^{\pm 1}$. Their source terms are found to be $W_1^{(2)}=\frac{99}{64}m\hat{\lambda}[1+O(m)]$ and $W_{-1}^{(2)}=-\frac{207}{64}m\hat{\lambda}[1+O(m)]$. From Eqs. (2.52) the corresponding solutions can be written as $w_1^{(2)}=-\frac{45}{16}\hat{\lambda}[1+O(m)]$, $w_{-1}^{(2)}=-3w_1^{(2)}[1+O(m)]$. At this approximation we have, when expanding $w_{\pm 1}^{(1)}$ in powers of m,

$$w_1^{(1)}+w_1^{(2)} = \left[1+\frac{15}{4}m+O(m^2)\right]w_1^{(1)} = -\frac{3}{4}\frac{\hat{\lambda}}{m}[1+6m+O(m^2)], \tag{2.56a}$$

$$w_{-1}^{(1)}+w_{-1}^{(2)} = \left[1+\frac{15}{4}m+O(m^2)\right]w_{-1}^{(1)} = \frac{9}{4}\frac{\hat{\lambda}}{m}\left[1+\frac{46}{9}m+O(m^2)\right]. \tag{2.56b}$$

E. Radial and angular perturbations due to a violation of the equivalence principle

Let us relate the results (2.56) to the radial and angular perturbations of the lunar motion associated with the parameter $\hat{\lambda}\propto\bar{\delta}_{12}$. The radius vector $r=(X^2+Y^2)^{1/2}$ and longitude θ of the Moon (with respect to the rotating vector $\mathbf{e}_X=\mathbf{N}$, i.e., with respect to the Sun) are such that $u=re^{i\theta}=\tilde{a}\zeta(1+w)$, where we recall that $\zeta=e^{i\tau}$. Hence,

$$r^2 = u\bar{u} = \tilde{a}^2(1+w)(1+\bar{w}), \tag{2.57a}$$

$$e^{2i\theta} = \frac{u}{\bar{u}} = e^{2i\tau}\frac{1+w}{1+\bar{w}}. \tag{2.57b}$$

Working linearly in $\hat{\lambda}$, we get the following radial and longitudinal equivalence-principle-violation perturbations

$$\frac{\delta_\lambda r}{\tilde{a}} = \mathrm{Re}\left[\left(\frac{1+\bar{w}}{1+w}\right)^{1/2}\delta_\lambda w\right], \tag{2.58a}$$

$$\delta_\lambda\theta = \mathrm{Im}\left[\frac{\delta_\lambda w}{1+w}\right]. \tag{2.58b}$$

At the approximation (2.56) we can write $\delta_\lambda r/\tilde{a}\simeq(w_1+w_{-1})\cos\tau+(w_3+w_{-3})\cos 3\tau$, and $\delta_\lambda\theta\simeq(w_1-w_{-1})\times\sin\tau+(w_3-w_{-3})\sin 3\tau$. In the approximation, the observable synodic effects are entirely described by $w_1\pm w_{-1}$. However, in higher approximations, $w_{\pm 3}, w_{\pm 5}$, etc., feed down to the synodic effects in r and θ. [Let us note in passing that, when averaging over time, the mean shift of the Cartesian components $u=X+iY=\tilde{a}\zeta(1+w)$ is given, to all orders, by w_{-1} alone: $\langle X\rangle=\tilde{a}w_{-1}$, $\langle Y\rangle=0$.]

Focusing on the contributions at the synodic frequency $n-n'$, we get, at this stage (in agreement with Appendix A),

$$\left(\frac{\delta_\lambda r}{\tilde{a}}\right)_{\text{synodic}} = \frac{3}{2}\frac{\hat{\lambda}}{m}\left[1+\frac{14}{3}m+O(m^2)\right]\cos\tau, \tag{2.59a}$$

$$(\delta_\lambda\theta)_{\text{synodic}} = -3\frac{\hat{\lambda}}{m}\left[1+\frac{16}{3}m+O(m^2)\right]\sin\tau. \tag{2.59b}$$

A straightforward, though slightly more involved, calculation allowed us to compute by hand the $O(m^2)$ contributions to the square brackets on the right-hand sides of Eqs. (2.59). In particular, we found that the square bracket in the range perturbation, Eq. (2.59a), reads $1+\frac{14}{3}m+\frac{1061}{48}m^2$

[6] Following Poincaré [48] (see also Ref. [18]), one can clarify the iterative process by giving a new name to the parameter m appearing in the second term of W, Eq. (2.46), leaving unchanged the other occurrences of m in L and $\kappa(m)$.

$+O(m^3)$. In view of the large coefficients appearing in this expansion, which create large corrections to the lowest-order effect (for the Moon, $\frac{14}{3}m=0.3773$ and $\frac{1061}{48}m^2=0.1445$), we have decided to take the bull by the horns and to solve iteratively the equations of motion (2.47) to a very high order by using the dedicated computer manipulation program MINIMS written by M. Moons (see [49]). Some details on the application of this program to our problem are given in Appendix B. Let us quote here the form of the results. Replacing $\hat{\lambda}$ by its definition (2.42), we see that $\tilde{a}$ drops out when writing the range perturbation $\delta_\lambda r$. Finally, we can write

$$(\delta_\lambda r)_{\rm synodic}=C(m)\bar{\delta}_{12}a'\cos\tau, \tag{2.60a}$$

$$(\delta_\lambda \theta)_{\rm synodic}=-C'(m)\bar{\delta}_{12}\frac{a'}{\tilde{a}}\sin\tau, \tag{2.60b}$$

where

$$C(m)=\frac{3}{2}m\left(1+\sum_{k\geqslant 1}c_k m^k\right)\equiv\frac{3}{2}mS(m), \tag{2.61a}$$

$$C'(m)=3m\left(1+\sum_{k\geqslant 1}c'_k m^k\right)\equiv 3mS'(m). \tag{2.61b}$$

The beginning of the power series $S(m)$ entering the synodic range perturbation is

$$S(m)=1+\frac{14}{3}m+\frac{1061}{48}m^2+\frac{2665}{24}m^3+\frac{145\,683}{256}m^4 +\frac{6\,729\,119}{2304}m^5+\frac{1\,656\,286\,531}{110\,592}m^6+\cdots. \tag{2.62}$$

The coefficients of the series $S(m)$ and $S'(m)$ are given in Appendix B up to the power m^{17} included. They are found to grow fast. The ratio between two successive coefficients c_k/c_{k-1} or c'_k/c'_{k-1} is found, numerically, to converge rapidly to the value $5.125\,471\,7\ldots$, thereby mimicking a geometric series in $m/m_{\rm cr}$ with $m_{\rm cr}=(5.125\,471\,7\ldots)^{-1}\simeq 0.195\,103\,996\ldots$. In the case of the Moon, with $m=0.080\,848\,937\,5\ldots$ [50], this means that the series $S(m)$ and $S'(m)$ converge rather slowly, as geometric series of ratio $m/m_{\rm cr}\simeq 0.4144$. The truncation to order m^{17} is just enough to estimate the values of the series to the 10^{-5} accuracy. As discussed below, the method of Padé approximants allows us to improve this precision. We find, for the Moon,

$$S=1.622\,01\ldots, \quad S'=1.723\,48\ldots, \tag{2.63}$$

so that the full coefficients appearing in the synodic effects (2.60) are, respectively, $C=0.196\,707\ldots$, $C'=0.418\,025\ldots$. Finally, using the recommended value of the semimajor axis of the Earth orbit, $a'=a_{\rm Earth}\,A=1.495\,980\,221\times 10^{13}$cm [51] (where A denotes the astronomical unit), the amplitude of the range oscillation of the Moon, due to an equivalence principle violation, is numerically found to be

$$Ca'\bar{\delta}_{12}=2.9427\times 10^{12}\bar{\delta}_{12}\quad \text{cm}. \tag{2.64}$$

In the case where one assumes the absence of violation of the weak equivalence principle, i.e., $\hat{\delta}_A\equiv 0$ in Eq. (2.10), the result (2.64) gives

$$(\delta_\lambda r)_{\rm synodic}=13.10\,\eta\,\cos\tau\ \ \text{m}, \tag{2.65}$$

if we use Eq. (2.5) [16] as nominal value for the difference of gravitational binding energies. Our final result is approximately 60% larger than the lowest-order estimate first derived by Nordtvedt in 1968 [10] and recalled in Eqs. (1.4), (1.5), and (1.8) above. On the other hand, it confirms the recent finding of Nordtvedt [25][7] that the interaction with the orbit's tidal deformation significantly amplifies the synodic range oscillation and substantiates it by providing a more complete analytical treatment of this effect.

III. PHYSICAL DISCUSSION

A. Resonances and instability

We have seen in the previous section that the series in powers of m giving the amplitudes of synodic perturbations (2.60) appear to be close to geometric series in $m/m_{\rm cr}$ with $m_{\rm cr}\simeq 0.195\,104$. This suggests the existence of pole singularities $\propto(m_{\rm cr}-m)^{-1}$ at $m=m_{\rm cr}\simeq 0.195\,104$ in those series. Nordtvedt [25] predicted the presence of such a pole singularity at $m\simeq 0.2$ (i.e., for a sidereal period of about 2 months) on the basis that for such a high orbit a determinant appearing in his study as denominator of the synodic perturbation vanishes. He also mentioned that for this orbit the driving frequency (n' in a nonrotating frame) becomes equal, in his truncated model, to the rate of perigee advance ($d\varpi/dt$). We have substantiated these predictions, as well as obtained by several independent approaches a much more precise value for $m_{\rm cr}$, namely,

$$m_{\rm cr}=0.195\,103\,996\,6\ldots \tag{3.1}$$

by making use, notably, of the work of Hénon [52] on the three-body problem. To relieve the tedium, the details of our arguments are relegated to Appendix C. Let us summarize our approach and our results.

Our approach consists of putting together the (numerical) results of Hénon [52] on the stability of the periodic orbits in Hill's problem, with some knowledge of the general structure of Hamiltonian perturbations, and a more specific use of the analytical structure of the solutions of Hill's variational equations in presence of ''forcing'' terms, such as the ones coming from the potentials F_1, Eq. (2.29b), and F_3, Eq. (2.29d), which are neglected in Hill's main problem. Our conclusions are that when m increases up to $m_{\rm cr}$, Eq. (3.1), there is a confluence of correlated singularities: on the one hand, as found by Hénon, the free perturbations of Hill's orbit (those not driven by any additional force) lose their stability, and on the other hand, all perturbations driven by perturbing potentials of any odd frequency in the rotating

[7]This reference quotes 2.9×10^{12} for the numerical coefficient in Eq. (2.64).

frame [i.e., containing terms $\propto \exp[\pm(2k+1)i\tau]$] develop pole singularities $\propto (m_{\rm cr}-m)^{-1}$. Note that, reasoning conversely, it is the existence of such a pole singularity, located at a rather small value of $m_{\rm cr}$, which "explains" the slow convergence of the perturbation series in m giving the amplitudes of the various synodic effects. [See also the footnote 19 of Appendix C for the link with the bad convergence of the perturbation series giving the perigee precession.]

As indicated by Nordtvedt, the value $m=m_{\rm cr}$ does correspond to a simple 1:1 commensurability $d\varpi/dt=n'$ or (in terms of rotating-frame variables) $c\equiv dl/d\tau=1$ (where $l=nt+\epsilon-\varpi$ is the mean anomaly). Note, however, that, contrary to what happens in the familiar case of a harmonic oscillator, the basic frequency of the driving force does not need to have a 1:1 resonance with the natural frequency of the orbit (perigee precession) to generate poles $\propto (m_{\rm cr}-m)^{-1}$; the odd commensurabilities 3:1, 5:1, etc., generate similar poles. Therefore, both the (hypothetical) equivalence-principle-violation effects (potential F_1), and many of the (really existing) parallactic effects (potentials $F_3, F_5, \ldots$) will have pole singularities $\propto (m_{\rm cr}-m)^{-1}$. Moreover, these poles are present not only in the synodic terms (that we concentrate upon here) but in the terms at frequencies $3(n-n')$, $5(n-n')$, ... The consequences of this situation are explored in the following subsections.

B. Padé approximant of equivalence-principle-violation effects

The analysis of Appendix C shows that the amplitudes of the synodic perturbations (2.60) considered as functions of m have a simple pole (but no branch point) on the positive real axis at $m=m_{\rm cr}$, Eq. (3.1), and have no singularities on the negative real axis down to $m=-1$ (because of the stability of the retrograde orbits [52]).[8] This simple analytical behavior suggests that the numerical validity of the power series representation (2.61) can be efficiently extended by using Padé approximants, i.e., by rewriting the power series $S(m)$, $S'(m)$ truncated at order m^{17} as quotients $N(m)/D(m)$, $N'(m)/D'(m)$ of two power series truncated at order m^8. The explicit coefficients of the Padé approximants[9] are given in Appendix B. We have done several checks of the conjecture that these Padé approximants provide a numerically accurate representation of the exact solution $S(m)$ on the entire interval $(-1, m_{\rm cr})$. First, the real zeros of smallest absolute value of the denominators $D(m)$ and $D'(m)$ are, respectively, found to be 0.195 103 996 68... and 0.195 103 996 60 ... in excellent agreement with Hénon's value (3.1). Second, we found that the Padé approximants truncated to order m^7 numerically agree all over the interval $(-1, m_{\rm cr})$ with those at order m^8 within better than 1%. [Actually, the difference is much smaller than 10^{-3} except very near $m=-1$.] We plot in Fig. 1 the Padé approximant of the coefficient $C(m)$ of the radial synodic effect,

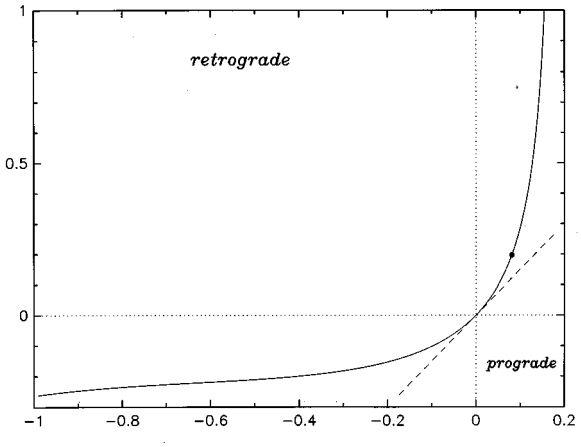

FIG. 1. Coefficient $C(m)$ of the synodic range oscillation [defined in Eq. (2.60a)] as a function of $m=n'/(n-n')$. The solid line represents the Padé approximant of $C(m)$, while the dashed line represents the linearized approximation $\frac{3}{2}m$. The dot indicates the actual lunar value.

$$C_{\rm Padé}(m)=\frac{3}{2}m\frac{N_8(m)}{D_8(m)}, \tag{3.2}$$

over the interval $(-1, m_{\rm cr})$. Let us note the two numerical values (using $m_{\rm Moon}=0.080\,848\,937\,5\ldots$; [50])

$$C_{\rm Padé}(m_{\rm Moon})=0.196\,707, \tag{3.3a}$$

$$C_{\rm Padé}(-1)=-0.267\,706. \tag{3.3b}$$

In Fig. 1 the lunar value (3.3a) is indicated by a dot. The difference between the linearized approximation $\frac{3}{2}m$ (dashed line in Fig. 1) and the exact value of $C(m)$ (solid line) illustrates the importance of nonlinear effects in the radial synodic perturbation.

C. Better orbital tests of the equivalence principle?

Nordtvedt [25] has suggested that the resonance at $m=m_{\rm cr}$ could perhaps be useful to improve the precision of equivalence-principle tests. The idea would be to put an artificial satellite in an orbit close to the resonant orbit $(m=m_{\rm cr})$. From our numerical estimates, the resonant orbit has a sidereal period $T_{\rm cr}=m_{\rm cr}(1+m_{\rm cr})^{-1}T'$ (where $T'=2\pi/n'=1$ yr), i.e., $T_{\rm cr}=1.959\,03$ month. The corresponding "bare" semimajor axis $a_0\equiv(Gm_0/n^2)^{1/3}$ is $1.682\,55 a_0$(Moon). Though interesting, this suggestion is not without difficulties. First, our results suggest that one must be careful to use a *subcritical* orbit $(m<m_{\rm cr})$ as supercritical orbits are exponentially unstable (real characteristic multiplier >1) and one needs many orbits to decorrelate the externally-driven synodic effect from the (nearly degenerate) natural orbital frequency for radial perturbations. Second, the fact that all the parallactic perturbations [proportional to $m_3^{\rm grav}/a'^k$ with $k\geq 4$, i.e., to $m^2(\tilde{a}/a')^{k-3}$] develop also pole singularities at $m=m_{\rm cr}$, probably implies that the orbit be-

[8]The value $m=-1$ corresponds to very wide retrograde orbits $0<-n\ll n'$.

[9]It is to be noted that, thanks to the nearly geometric progression of the coefficients in many of the power series of Appendix B, a simpler (though less general) alternative to Padé approximants would be simply to factorize $(1-m/m_{\rm cr})^{-1}$.

comes unstable slightly below the ideal Hill value (3.1).[10] Moreover, the blow up of the parallactic perturbations amplify already large synodic effects which are known only with finite accuracy. One might worry that the finite accuracy with which Newtonian parallactic perturbations can be accounted for could limit the precision of an improved equivalence-principle test based on ranging to a near-resonant satellite. To investigate this point we have included the octupole-tide perturbation F_3, Eq. (2.29d), i.e., we added the contribution W_3, Eqs. (2.48), in our Hill-Brown iteration program. Our explicit results are given in Appendix B. The form of the radial perturbation is

$$(\delta_\pi r)_{\rm synodic}=\frac{15}{16}\frac{m_2-m_1}{m_2+m_1}\left(\frac{m_1+m_2}{m_1+m_2+m_3}\right)^{2/3}a' P(m)\cos\tau, \tag{3.4a}$$

$$P(m)=m^{7/3}[\kappa(m)]^{-2/3}Q(m), \tag{3.4b}$$

$$Q(m)=1+\frac{22}{5}m+\frac{215}{16}m^2+\frac{57\,599}{960}m^3+\frac{917\,401}{2880}m^4 + \frac{230\,247\,737}{138\,240}m^5+\frac{14\,206\,254\,151}{1\,658\,880}m^6+\cdots. \tag{3.4c}$$

The numerical value of the coefficient giving the scale of $\delta_\pi r$ in Eq. (3.4a) is $\delta_\pi r\simeq 28\,716.38P(m)\cos\tau$ km. The Padé approximant of the series $Q(m)$ is given in Appendix B. For rough orders of magnitude estimates we can approximate (when $0<m<m_{\rm cr}$) $P(m)$ by $P(m)\simeq m^{7/3}(1-m/m_{\rm cr})^{-1}$. By comparison, the coefficient entering the Nordtvedt effect (2.60a) can be roughly approximated by $C(m)\simeq\frac{3}{2}m(1-m/m_{\rm cr})^{-1}$. Let us define the *amplification factor* of the Nordtvedt effect as the ratio $A(m)\equiv C(m)/C(m_{\rm Moon})$ where $C(m)$ is the coefficient in Eq. (2.60a). The amplification factor in the synodic parallactic oscillation (3.4a) will be $B(m)\equiv P(m)/P(m_{\rm Moon})\simeq(m/m_{\rm cr})^{4/3}A(m)$. For an artificial satellite ($m_1\ll m_2$), one expects from Eq. (3.4a) that the main uncertainty in the theoretical value of $\delta_\pi r$ will come from the Earth/Sun mass ratio: m_2/m_3. The current fractional uncertainty on this ratio is $\epsilon_9\times10^{-9}$ with $\epsilon_9\simeq2$ according to the results of Ref. [53] on m_2 (LLR data alone give $\epsilon_9\simeq10$ [27]). The corresponding uncertainty in $\delta_\pi r$ is $0.0073\epsilon_9 B(m)$ cm. Therefore, the use of a higher orbit, amplifying the Nordtvedt effect by a factor $A(m)$, will entail a correspondingly increased uncertainty on the synodic parallactic radial oscillation:

$$\delta_\pi r\simeq 0.0073\epsilon_9(m/m_{\rm Moon})^{4/3}A(m)\cos\tau\ \ {\rm cm}. \tag{3.5}$$

For moderately large amplification factors $A(m)$, this may be small enough not to limit the accuracy of an improved equivalence-principle test. On the other hand, the problem might cure itself by the fact that the $\cos3\tau$ parallactic effects will also be amplified, thereby allowing the ratio m_2/m_3 which enters their coefficients to become measurable with increased precision. But things might get complicated because, as one approaches the resonance, several frequencies become close to one another and one needs long data span to resolve the various frequencies and measure separately their Fourier coefficients. Moreover, the real motion of an artificial satellite beyond the Moon's orbit will be very complex because of the combined gravitational effects of the Earth and the Moon. Finally, such a satellite would need to be endowed with a very high-performance, drag-free system to compete with the Moon which is, naturally, drag free to a high precision.

In view of the difficulties associated with near-resonant, lunar-type orbits, it is worth thinking about other possibilities.[11] Let us list some possibilities: artificial (drag-free) satellites around outer planets would be interesting in that the basic dimensionful scale factor in the synodic effect (2.60a) is a', the semimajor axis of the considered planet around the Sun. That would give a factor 5 for Jupiter and a factor 10 for Saturn. In either case, one would need far enough satellites (i.e., m big enough) to have a coefficient $C(m)$ at least comparable to the lunar value (3.3). A second possibility is to use retrograde orbits which are always stable (in the Hill approximation). However, Fig. 1 shows that they give, at best, a factor $C(-1)=-0.267\,706$. An equivalence-principle mission consisting of a pair of artificial satellites around an outer planet (one prograde, one retrograde), with a laser link between the satellites, could improve by a significant factor upon the LLR experiment. In addition to an improved scale factor a', the advantage of being around an outer planet is that the radiation pressure from the Sun is much smaller, so that the requirements on the drag-free system are much less stringent. Other advantages concern the theoretical value of the parameter $\bar{\delta}_{12}$, Eq. (2.3): on the one hand, the composition-dependent contribution $\hat{\delta}_1-\hat{\delta}_2$ can be expected [according to Eq. (3.7) below] to be appreciably increased if one uses proof masses of high-Z material orbiting around a (low-Z) outer planet, and, on the other hand, as emphasized by J. G. Williams (private communication), the η-dependent contribution to $\bar{\delta}_{12}$ will be greatly increased in view of the much larger gravitational self-energies of the outer planets.

Finally, let us note that, for simplicity, we have restricted our attention to circular, coplanar orbits. If one considers satellite orbits with arbitrary inclinations and eccentricities, they will exhibit a rich spectrum of resonances to external equivalence-principle-violating perturbations. For instance,

[10]Unpublished calculations of Hénon (private communication) for a small but nonzero mass ratio $\mu=m_0/(m_0+m_3)=10^{-6}$ show that the topology of the loss of stability of Hill's prograde orbits is different from the ideal Hill case ($\mu=0$) and the same as for generic values of $\mu\neq0$. The difference takes place in a region of fractional size 10^{-3} ($\sim\mu^{1/2}$?) which suggests that the actual $m_{\rm cr}$ is roughly 0.1% smaller than the value (3.1).

[11]Let us note in passing that, because of tidal dissipation, the Moon itself is, kindly, slowly receding toward higher orbits. However, even under the overoptimistic assumption that the present rate of energy dissipation continues to apply in the future, the increase in the semimajor axis of the Moon will be only $\simeq23\%$ in 6 billion yr (which is the expected lifetime of the Sun) [54].

Ref. [36] has discussed the resonances (''small divisors'') exhibited by (circular) low-orbit artificial Earth satellites with arbitrary inclinations. [The location in the (a,I) plane of these families of resonances will be found in Fig. 3 of Ref. [55].] We think that the circular, coplanar, high-orbit resonances discussed here are the most promising ones for improved tests of the equivalence principle.

D. Theoretical significance of orbital tests of the universality of free fall

As we mentioned above, the Lunar Laser Ranging experiment is sensitive, through the synodic effect (2.60), to the sum of two physically independent contributions

$$\bar{\delta}_{12}=\hat{\delta}_1-\hat{\delta}_2+\eta\left(\frac{E_1^{\rm grav}}{m_1c^2}-\frac{E_2^{\rm grav}}{m_2c^2}\right)=\hat{\delta}_{12}+4.45\,\eta 10^{-10}. \tag{3.6}$$

The first contribution, $\hat{\delta}_{12}\equiv\hat{\delta}_1-\hat{\delta}_2$, is essentially equivalent to what Newton and Laplace had in mind when they proposed orbital tests of the universality of free fall: bodies of different internal compositions could fall differently. The second contribution, proportional to $\eta=4\bar{\beta}-\bar{\gamma}$, was discovered by Nordtvedt who was considering deviations from Einstein's theory within the class of metrically-coupled theories of gravity (see, e.g., [20] for a review). Actually, from the perspective of modern unified theories, the class of metrically-coupled theories of gravity seems quite *ad hoc*. For instance, string theory does suggest the possibility that there exist long-range scalar fields contributing to the interaction between macroscopic bodies and thereby modifying the standard predictions of general relativity. However, all the scalar fields present in string theory have composition-dependent couplings for very basic reasons (for a discussion of general theoretical alternatives to Einstein's theory and the types of composition-dependent couplings they might exhibit see [37]).

Recently, a mechanism has been proposed by which some of the scalar fields of string theory might survive in the macroscopic world as very weakly coupled, long-range fields [38] (see also [56]). In the model of Ref. [38], the surviving scalar field(s) modify the observational consequences of general relativity in several ways: (i) they violate the ''weak equivalence principle'' ($\hat{\delta}_A\neq 0$) because of the composition-dependence of their couplings to matter, (ii) they modify the post-Newtonian $[O(1/c^2)]$ effects in essentially the way which is parametrized by the Eddington parameters[12] $\bar{\beta}$ and $\bar{\gamma}$, and (iii) they induce a slow time variability of all the coupling constants of nature: $G,\alpha,\alpha_{\rm weak},\ldots$. The point we want to emphasize here, because we think it is generic, is that all those modifications of general relativity are related, because they derive ultimately from the couplings of the same field. In particular, the first term on the right-hand side of Eq. (3.6) is related to the second. The results of Ref. [39] about metrically-coupled tensor-scalar theories show that a convenient measure of the basic coupling strength of the scalar fields to matter is the Eddington parameter $\bar{\gamma}$. In more generic tensor-scalar theories, which (slightly) violate the weak equivalence principle, one can define an *effective* Eddington parameter $\bar{\gamma}$ by neglecting the fractionally small composition dependence of the body-dependent quantity $\bar{\gamma}_{AB}$ measuring the scalar coupling between bodies A and B [38]. In the model of the latter reference, the effective $\bar{\gamma}$ can be defined as the (universal) scalar coupling strength between ideal bodies whose masses are purely of QCD origin. [Let us recall that, in the real world, the masses of atoms are predominantly of QCD origin, with small additional contributions due to lepton masses and electroweak interactions.]

More precisely, in the model of Ref. [38], we have, for an individual atom labeled by A, the link

$$\hat{\delta}_A\simeq -0.943\times 10^{-5}\,\bar{\gamma}(E/M)_A, \tag{3.7}$$

where $E\equiv Z(Z-1)/(N+Z)^{1/3}$ is associated to the electrostatic interaction energy of the nucleus of the atom, and where M denotes the mass of A in atomic mass units. We believe that the structure of this link between $\hat{\delta}_A$ and $\bar{\gamma}(E/M)_A$[13] is generic in string-derived models, because it follows from a basic physical feature of the massless scalar fields (''moduli'') present in string theory, namely, that they define the values of the gauge coupling constants. Even the magnitude of the numerical coefficient should be somewhat generic. Indeed, its analytical expression $-\frac{1}{2}a_3\alpha\lambda_\alpha/\lambda_{u_3}$ (in the notation of [38]) shows that it is determined by basic physical facts or assumptions: fractional smallness of electrostatic nuclear contributions ($a_3\alpha\simeq 0.770\times 10^{-3}$), unification of gauge coupling constants ($\lambda_\alpha\simeq 1$), and QCD confinement $\{\lambda_{u_3}\simeq\ln[\Lambda_{\rm string}/({\rm a.m.u.})]\simeq 40.8\}$.

We have also the model-dependent link $\bar{\beta}\simeq -10.2\kappa\bar{\gamma}$, where the dimensionless theory parameter κ is expected to be of order unity. [$\kappa>0$ denotes here the curvature of a coupling function around a minimum and should not be confused with the notation $\kappa(m)$ used above.] These links indicate that, in the Earth-Moon case (but not necessarily in other cases involving bodies with stronger self-energies), the gravitational binding contribution to $\bar{\delta}_{12}$ is numerically negligible compared to the composition-dependent term $\hat{\delta}_{12}$. Indeed, using Eq. (3.7) and the compositional difference between the Earth and the Moon (i.e., the difference between an Earth iron core of mass $0.32m_2$ and a silica-dominated Moon [57,28]), we find $\hat{\delta}_{12}\simeq 0.32(\hat{\delta}_{\rm silica}-\hat{\delta}_{\rm Fe})\simeq 3.67\times 10^{-6}\bar{\gamma}$, while the gravitational binding energy contribution is $4.45\,\eta\times 10^{-10}=-4.45(40.8\kappa+1)\times 10^{-10}\bar{\gamma}$. From the point of view advocated here, the conclusion is that LLR data give us a very precise test of the *weak* equivalence principle. The loss of a Nordtvedt-type direct test of the combination $\eta=4\bar{\beta}-\bar{\gamma}$ is compensated by the theoretically expected link $\delta_{12}\simeq\hat{\delta}_{12}\simeq 3.67\times 10^{-6}\bar{\gamma}$ which gives an extremely good limit on the effective Eddington parameter of the considered scalar model. More precisely, the observa-

[12]This comes from a feature of their couplings which is deeply rooted into the structure of QCD and the consequences it has for the mass of atoms; see pp. 550–553 of [38].

[13]We neglect here contributions proportional to the ratios (baryon number)/(mass) and (neutron excess)/(mass) which tend to be subdominant, even for moderate Z differences [38].

tional limit[14] $\bar{\delta}_{12}=(-3.2\pm4.6)\times10^{-13}$, recently derived from LLR data [27,16], translates into the following observational constraint[15] on $\bar{\gamma}$:

$$\bar{\gamma}=(-0.9\pm1.3)\times10^{-7}. \tag{3.8}$$

The recent laboratory tests of the weak equivalence principle give comparable results. Using the experimental limit $\hat{\delta}_{\mathrm{Be\ Cu}}=(-1.9\pm2.5)\times10^{-12}$ [28] and the theoretical formula (3.7) (which yields $\hat{\delta}_{\mathrm{Be\ Cu}}=2.41\times10^{-5}\bar{\gamma}$), we find

$$\bar{\gamma}=(-0.8\pm1.0)\times10^{-7}. \tag{3.9}$$

Impressive as these limits may seem, Ref. [38] gives a motivation for pushing equivalence-principle tests further because this reference estimates that a natural range for the coupling parameter $\bar{\gamma}$ is $10^{-19}\lesssim-\bar{\gamma}\lesssim10^{-10}$. [Note, however, that if the theory parameter κ is of order 1/40 (which corresponds, in the notation of [56], to $\kappa\sim1$) larger values of $-\bar{\gamma}$, of order 10^{-7}, are expected, in agreement with [56].] In this connection, let us mention that the LLR CERGA team plans to improve the precision of the ranging down to the 2–3 mm level for normal points [C. Veillet (private communication)]. Extracting $\bar{\delta}_{12}$ at this level will necessitate to improve the modeling of the solar radiation pressure effects which are currently believed to contribute a synodic range oscillations of approximately 0.3 cm [25]. If this can be done, the LLR experiment will reach the level $\bar{\delta}_{12}\sim5\times10^{-14}$, corresponding to the level $\bar{\gamma}\sim10^{-8}$. It seems that significant progress in testing the equivalence principle will require space missions: either a low Earth orbiting artificial satellite dedicated to testing the weak equivalence principle, as the STEP (Satellite Test of the Equivalence Principle) mission, or, possibly, some type of orbital test such as the one suggested in Ref. [25] and the ones suggested above.

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[14]This was obtained from partial derivatives of the numerically integrated equations of motion, and therefore independently of theoretical estimates of the value of the coefficient $C(m)$ in Eq. (2.60a).

[15]We do not take into account here the theoretical constraint that $\bar{\gamma}<0$ in all scalar models.

APPENDIX A: TRADITIONAL LUNAR PERTURBATION THEORY

As a check on the lowest orders of the Hill-Brown calculations presented in the text, we have also investigated the mixing between equivalence-principle-violation effects and tidal effects by means of the traditional lunar perturbation theory of de Pontécoulant [26,42]. The equations of motion corresponding to the Lagrangian $\hat{L}=\frac{1}{2}\mathbf{v}^2+\mu/r+R$ read

$$\frac{d^2\mathbf{r}}{dt^2}+\mu\frac{\mathbf{r}}{r^3}=\frac{\partial R}{\partial\mathbf{r}}. \tag{A1}$$

Here, $\mathbf{r}\equiv\mathbf{x}_{12}\equiv\mathbf{x}_1-\mathbf{x}_2$ is the position vector of the Moon with respect to the Earth (in an inertial, nonrotating, coordinate system), $\mu\equiv G_{12}(m_1+m_2)$, and $R=R_1+R_2+R_3+\cdots$ is the total potential perturbing the Keplerian motion of the Moon around the Earth. [This corresponds to Eqs. (2.21) and (2.22).] We consider the coplanar problem for which it is enough to solve for the radius $r\equiv|\mathbf{r}|$ and the longitude v (polar angle). Decomposing the acceleration into radial and longitudinal components leads to

$$\ddot{r}-r\dot{v}^2=-\mu r^{-2}+\partial R/\partial r, \tag{A2a}$$

$$d(r^2\dot{v})/dt=\partial R/\partial v. \tag{A2b}$$

de Pontécoulant's method uses the longitudinal equation (A2b) but replaces the radial one (A2a) by the ''virial'' equation dealing with the second time derivative of r^2. The basic equations are then written as

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2)-\frac{\mu}{r}+\frac{\mu}{a_c}=P, \tag{A3a}$$

$$\dot{v}-\frac{h_c}{r^2}=\frac{1}{r^2}\int dt\frac{\partial R}{\partial v}, \tag{A3b}$$

where a_c and h_c are some integration constants and where the transformed source term in the radial equation is

$$P\equiv r\frac{\partial R}{\partial r}+2\int dt\left(\frac{d}{dt}\right)_1 R=r\frac{\partial R}{\partial r}+2R+2n'\int dt\frac{\partial R}{\partial v}. \tag{A4}$$

In the first form of P, $(d/dt)_1$ denotes a time derivative taking into account only the variability due to the time-dependence of the coordinates of the Moon: $(d/dt)_1=\dot{r}\partial/\partial r+\dot{v}\partial/\partial v$. The second form of P is obtained by taking into account the time dependence of R upon the Sun's coordinates, and assumes that the Sun moves on a circular orbit ($\dot{r}'=0$, $\dot{v}'=n'$). It is very useful to notice that if $R=\Sigma_p R_p$, where each contribution $R_p(r,v-v')$ has a radial dependence $\propto r^p$, then $P=\Sigma_p P_p$ with

$$P_p=(p+2)R_p+2n'\int dt\frac{\partial R_p}{\partial v}. \tag{A5}$$

Note that the use of the suffix p is consistent with the notation R_1,R_2,R_3 of Eqs. (2.22).

First-order perturbation theory is very easy. Let us consider a general term $R_p^{(q)}=Ar^p\cos q(v-v')$, perturbing the

zeroth-order (circular) solution $r=a,\ v_0=nt+\varepsilon$ (with zeroth-order integration constants $a_c=a,\ h_c=na^2$, and the link $n^2a^3=\mu$). Inserting the perturbed solution $r=a+\delta r$, $v=v_0+\delta v$ into Eqs. (A3) (with perturbed integration constants $a_c=a+\delta a,\ h_c=h+\delta h$) yields

$$\frac{d^2}{dt^2}(\delta r)+n^2\delta r=\tilde{p}Aa^{p-1}\cos[q(n-n')(t-t_0)]+n^2\delta a, \tag{A6a}$$

$$\frac{d}{dt}(\delta v)+2\frac{n}{a}\delta r=\frac{1}{(n-n')}Aa^{p-2}\cos[q(n-n')(t-t_0)] +a^{-2}\delta h, \tag{A6b}$$

where

$$\tilde{p}\equiv p+2+2\frac{n'}{n-n'}=p+2+2m. \tag{A7}$$

Here, as in the text, we use $m\equiv n'/(n-n')$ as the small expansion parameter of perturbation theory. The solution of Eqs. (A6) is [with $\tau\equiv v_0-v_0'=(n-n')(t-t_0)$]

$$\delta r=\frac{\tilde{p}}{n^2-q^2(n-n')^2}Aa^{p-1}\cos q\tau+\delta a, \tag{A8a}$$

$$\delta v=-\frac{1}{q(n-n')}\left[\frac{2\tilde{p}n}{n^2-q^2(n-n')^2}-\frac{1}{n-n'}\right]Aa^{p-2}\sin q\tau +(\delta n)t+\delta\epsilon, \tag{A8b}$$

where $\delta n=-2na^{-1}\delta a+a^{-2}\delta h$. Equations (A6)–(A8) have been written assuming $q\neq 0$. They take a different form when $q=0$. It is traditional [42] to keep n fixed throughout the approximation process (and therefore numerically equal to the observed mean motion), and to define a by $n^2a^3\equiv\mu$. Then δh is computed in terms of δa so that $\delta n=0$. Finally, one must make use of the original radial equation (A2a) to determine δa.

We are especially interested in the case where the perturbing potential R is the sum of the quadrupole tide R_2, Eq. (2.26b), and of a term with frequency $q=1$:

$$R=R_2+R_p=n'^2r^2\left[\frac{1}{4}+\frac{3}{4}\cos2(v-v')\right]+Ar^p\cos(v-v'). \tag{A9}$$

In the case of the equivalence-principle-violating term (2.26a), the perturbing term in $\cos(v-v')$ has $p=1$, while the octupolar tide (2.26c), for which $p=3$, contains a perturbing term in $\cos(v-v')$ (that we focus on) and a term in $\cos3(v-v')$. At the fractional order $O(m^2)$ beyond the first-order solution (A8), the $\cos3(v-v')$ term mixes with the quadrupole tides $\propto\cos2(v-v')$ to generate the frequency $q=1$. As we work here only at the fractional order $O(m)$ beyond the first-order solution, we do not need to study the effect of the $\cos3(v-v')$ term. The first-order solution corresponding to Eq. (A9) reads

$$r=a+\delta^0r+\delta^2r+\delta^1r, \tag{A10a}$$

$$v=nt+\epsilon+\delta^2v+\delta^1v, \tag{A10b}$$

where the superscripts on δ indicate the values of the frequency q. To sufficient accuracy for our purpose we have, from Eqs. (A8),

$$\delta^0r+\delta^2r=-m^2a\left(\frac{1}{6}+\cos2\tau\right)+O(m^3), \tag{A11a}$$

$$\delta^2v=\frac{11}{8}m^2\sin2\tau+O(m^3), \tag{A11b}$$

$$\delta^1r=C_r^{\text{first order}}\frac{A}{(n-n')^2}a^{p-1}\cos\tau, \tag{A11c}$$

$$\delta^1v=C_v^{\text{first order}}\frac{A}{(n-n')^2}a^{p-2}\sin\tau, \tag{A11d}$$

with

$$C_r^{\text{first order}}=\frac{p+2+2m}{2m+m^2}=\frac{p+2}{2m}\left[1+\frac{2-p}{2(2+p)}m+O(m^2)\right], \tag{A12a}$$

$$C_v^{\text{first order}}=-2\frac{(p+2+2m)(1+m)}{2m+m^2}+1 =-2\frac{p+2}{2m}\left[1+\frac{4+p}{2(2+p)}m+O(m^2)\right]. \tag{A12b}$$

Note that the constant term $-m^2a/6$ in Eq. (A11a) depends on $\delta a=a_c-a$ and must be determined by having recourse to Eq. (A2a). Note also that the small denominator present when $q=1$, $n^2-(n-n')^2=(n-n')^2[(1+m)^2-1] =(n-n')^2[2m+m^2]$, causes the synodic effects to be of order $O(m^{-1}A)$ instead of the usual order $O(A)$ valid when $q\neq 1$. One of the effects of this small denominator is to have a simple, approximate link between the radial and longitudinal synodic oscillations: $C_v^{\text{first order}}=-2C_r^{\text{first order}}\times[1+O(m)]$. Another effect is that only the *leading* terms in Eqs. (A12) are correct. The $O(m)$ fractional corrections are modified by higher iterations as we are going to see.

When proceeding to the next iteration, several effects must be taken into account. On the left-hand side of Eq. (A3a) one must keep the terms of order $(\delta r)^2$, while on the right-hand side one must include the change δP of the source term $P(r,v-v')$ induced by the first-order solution (A11). This leads to the following equation for $\delta r=r-a$:

$$a\left(\frac{d^2}{dt^2}+n^2\right)\delta r=P(r_0,v_0-v_0')+\delta P_{\text{eff}}, \tag{A13a}$$

$$\delta P_{\text{eff}}=\delta P+\left(n^2-\frac{1}{2}\frac{d^2}{dt^2}\right)(\delta r)^2. \tag{A13b}$$

When computing the synodic effects with fractional accuracy $1+O(m)$, the computation of δP is simplified by sev-

eral circumstances. Because of the amplification $O(m^{-1}A)$ of the first-order synodic effects, one finds that it is enough to compute in Eq. (A5) the change of the first contribution, $(p+2)R_p$, for $p=2$ and under the synodic variations $\delta^1 r$, $\delta^1 v$. This yields

$$\begin{aligned}(\delta P)_{\rm synodic} &= 4(\delta R_2)_{\rm synodic}\\ &= \left\{\left(\delta^1 r\frac{\partial}{\partial r}+\delta^1 v\frac{\partial}{\partial v}\right)\right.\\ &\quad\left.\times n'^2 r^2[1+3\cos 2(v-v')]\right\}_{\rm synodic}\\ &= 11\, n'^2 a^2 B\cos\tau[1+O(m)],\end{aligned} \tag{A14}$$

where $B\equiv(2m)^{-1}(p+2)(n-n')^{-2}Aa^{p-2}$ denotes the leading value of the fractional synodic range oscillation ($\delta^1 r/a\simeq B\cos\tau$, $\delta^1 v\simeq -2B\sin\tau$). We need also to extract the synodic piece of $(\delta r)^2=(\delta^0 r+\delta^1 r+\delta^2 r)^2$ coming from the mixing between $\delta^0 r+\delta^2 r\simeq -m^2 a(\frac{1}{6}+\cos 2\tau)$ and $\delta^1 r\simeq aB\cos\tau$: $(\delta r)^2_{\rm synodic}\simeq -\frac{4}{3}a^2 m^2 B\cos\tau[1+O(m)]$. Finally, the synodic piece of the second-order effective source term for de Pontécoulant's radial equations (A13) is obtained as

$$\begin{aligned}\delta^1 P_{\rm eff} &= (\delta P)_{\rm synodic}+\left[n^2+\frac{1}{2}(n-n')^2\right](\delta r^2)_{\rm synodic}\\ &= 9n'^2 a^2 B\cos\tau[1+O(m)].\end{aligned} \tag{A15}$$

The corresponding solution reads

$$\left(\frac{\delta^1 r}{a}\right)^{\rm second\ order}\simeq 9\frac{n'^2}{n^2-(n-n')^2}B\cos\tau\simeq\frac{9}{2}mB\cos\tau. \tag{A16}$$

When turning to the longitude equation (A3b), one finds also some simplifications: the change of the source term $\propto\int dt\,\partial R/\partial v$ is of order $m^2\times m^{-1}A=O(mA)$. The corresponding term in the solution is not amplified by a small denominator and is therefore negligible compared to the precision $m\times m^{-1}A=O(A)$ we are aiming for. It is, therefore, sufficient to integrate the equation $d/dt(\delta^1 v)^{\rm second\ order}\simeq -2na^{-1}(\delta^1 r)^{\rm second\ order}$. The final result can be very simply expressed by saying that the second iteration leads to multiplying the first-order synodic perturbations (A11c) and (A11d) by the common factor $1+\frac{9}{2}m+O(m^2)$.

In conclusion, the mixing between the quadrupole tide R_2 and some synodic-frequency perturbation potential (in which we factorize an effective gravitational mass of the Sun, $Gm'\equiv n'^2 a'^3$),

$$(R_p)_{\rm synodic}=\beta\frac{Gm'}{a'}\left(\frac{r}{a'}\right)^p\cos(v-v'), \tag{A17}$$

leads, when neglecting nonlinearities in the dimensionless parameter β (which should not be confused with its post-Newtonian homonym), to the synodic oscillations

$$\frac{\delta^1 r}{a}=C_r\beta m^2\left(\frac{a}{a'}\right)^{p-2}\cos\tau, \tag{A18a}$$

$$\delta^1 v=C_v\beta m^2\left(\frac{a}{a'}\right)^{p-2}\sin\tau, \tag{A18b}$$

where

$$\begin{aligned}C_r &= \left[1+\frac{9}{2}m+O(m^2)\right]C_r^{\rm first\ order}\\ &= \frac{2+p}{2m}\left[1+\frac{10+4p}{2+p}m+O(m^2)\right],\end{aligned} \tag{A19a}$$

$$\begin{aligned}C_v &= \left[1+\frac{9}{2}m+O(m^2)\right]C_v^{\rm first\ order}\\ &= -2\frac{2+p}{2m}\left[1+\frac{11+5p}{2+p}m+O(m^2)\right].\end{aligned} \tag{A19b}$$

The two cases of interest are: (i) a hypothetical violation of the equivalence principle in which [comparing Eq. (A17) with Eq. (2.26a)]

$$p=1\,,\quad \beta_{\rm ep}=\bar{\delta}_{12}\,, \tag{A20}$$

and, (ii) the octupolar tide ("parallactic effects"), Eq. (2.26c), with

$$\begin{aligned}R_3 &\simeq \frac{1}{8}(X_2-X_1)\frac{Gm'}{a'}\left(\frac{r}{a'}\right)^3\\ &\quad\times[3\cos(v-v')+5\cos 3(v-v')],\end{aligned} \tag{A21}$$

whose synodic piece has

$$p=3\,,\quad \beta_{\rm par}=\frac{3}{8}(X_2-X_1). \tag{A22}$$

The results (A18)–(A22) agree with the (much more accurate) Hill-Brown-type results given in the text and in Appendix B.

APPENDIX B: HILL'S EQUATIONS—MORE ON THE ITERATIVE SCHEME OF SOLUTION AND NUMERICAL RESULTS

In this appendix, we explain in detail the iterative scheme we employed for solving Hill's equations (2.47) with the source terms (2.46), and also with the parallactic perturbation (2.48). We also give tables of the obtained solution for several physically interesting quantities. Obviously, one can envisage several iterative methods for solving the considered equations. We do not claim that the scheme we adopted is the optimal one, but we found it suitable from the point of view of memory and computing time requests. Thanks to our use of the dedicated algebraic manipulator MINIMS, we could obtain the solution to a very high order in the formally small parameters. In what follows, we shall present the solution for the perturbation of Hill's variational orbit related to the equivalence principle violation. Exactly the same scheme applies in the case of the parallactic perturbations.

Keeping the notation of Sec. II D, notably $L(w,\bar{w})$ for the linear operator (2.46), we have to solve

$$L(w,\bar{w})=-\hat{\lambda}\zeta^{-1}-\frac{3}{2}m^2\zeta^{-2}-\frac{3}{2}m^2\bar{w}\zeta^{-2}+\kappa(m)Q(w,\bar{w}), \tag{B1}$$

where

$$Q(w,\bar{w})=(1+w)^{-1/2}(1+\bar{w})^{-3/2}-1+\frac{1}{2}w+\frac{3}{2}\bar{w}. \tag{B2}$$

The nonlinear function Q can be written as

$$Q(w,\bar{w})=\sum_{k\geqslant 2}\chi_k[w^k+(2k+1)\bar{w}^k]$$

$$+\sum_{j,k\geqslant 1}(2k+1)\chi_j\chi_k w^j\bar{w}^k. \tag{B3}$$

Here,

$$\chi_k\equiv\begin{pmatrix}-1/2\\ k\end{pmatrix}$$

and

$$(2k+1)\chi_k=\begin{pmatrix}-3/2\\ k\end{pmatrix}$$

are binomial coefficients.

As in Sec. II D, we look for a formal power series solution of Eq. (B1),

$$w=w^{(1)}+w^{(2)}+w^{(3)}+\cdots, \tag{B4}$$

and similarly for the complex conjugate. The superscripts on the consecutive terms in (B4) refer to corresponding orders in the combined formal small parameter $(\hat{\lambda}+m^2)$. Keeping track of the orders in this formal small parameter, we decompose the nonlinear source Q as

$$Q(w,\bar{w})=Q^{(2)}+Q^{(3)}+\cdots, \tag{B5}$$

where the individual terms include symbolically

$$Q^{(i)}=\sum_{ja+kb=i}(\text{coefficient})(w^{(a)})^j(\bar{w}^{(b)})^k. \tag{B6}$$

For any particular value of i in (B6), $Q^{(i)}$ is given by a finite number of terms which depend only on the knowledge of $w^{(a)}$ for $a<i$. Although the procedure of breaking $Q(w,\bar{w})$ into a sum of equal-order terms $Q^{(i)}$ might seem laborious, it is relatively easy to be programed using a well suited algebraic manipulator such as MINIMS. One can introduce a formal index which conserves the order of a particular term and manipulate it as any other variable.

The heart of our iteration scheme consists of the following infinite system of differential equations

$$L(w^{(1)},\bar{w}^{(1)})=-\hat{\lambda}\zeta^{-1}-\frac{3}{2}m^2\zeta^{-2}, \tag{B7}$$

$$L(w^{(2)},\bar{w}^{(2)})=-\frac{3}{2}m^2\bar{w}^{(1)}\zeta^{-2}+\kappa Q^{(2)}(w^{(1)}),$$

$$L(w^{(3)},\bar{w}^{(3)})=-\frac{3}{2}m^2\bar{w}^{(2)}\zeta^{-2}+\kappa Q^{(3)}(w^{(1)},w^{(2)}),$$

$$\ldots=\ldots,$$

$$L(w^{(k)},\bar{w}^{(k)})=-\frac{3}{2}m^2\bar{w}^{(k-1)}\zeta^{-2}$$

$$+\kappa Q^{(k)}(w^{(1)},w^{(2)},\ldots,w^{(k-1)}),$$

$$\ldots=\ldots.$$

It is easy to verify that the generic form of the terms in the right-hand sides of Eqs. (B7) reads $W_k\zeta^k+W_{-k}\zeta^{-k}$ as presented in (2.50). The unique inversion of the linear operator L on the left-hand sides of Eqs. (B7) is given by formulas (2.51) and (2.52). Notice also that suppressing the equivalence-principle-violation term [$\hat{\lambda}=0$ in (B7)], one recovers a system of equations for constructing the usual variational periodic orbit.

In the preceding scheme, we consider $\hat{\lambda}$ and m^2 as two comparable ''small'' parameters. However, in practice, the order of the $\hat{\lambda}$ parameter associated with the studied violation of the equivalence principle is numerically much smaller than m^2 (which can be as large as one). As a result, we restrict the generality of our solution by keeping only *the first order* in the parameter $\hat{\lambda}$. This truncation allows a clear separation in the interpretation of the odd- and even-power terms in the ζ variable of the final solution for w (and $\bar{w}$): (i) the even-power terms ($\propto\zeta^{2i}$) never contain the perturbation parameter $\hat{\lambda}$ and fully reconstruct Hill's variational solution, (ii) the odd-power terms ($\propto\zeta^{2i+1}$) are all of the first order in $\hat{\lambda}$ (but they are coupled to the ''background'' variational terms through an infinite series of powers of m^2). We thus *simultaneously* obtain Hill's variational solution and its $\hat{\lambda}$ perturbation by filtering the various powers of ζ. This is a particularly important circumstance, because the series giving the variational solution enters the definition of several studied quantities such as the radial or longitudinal perturbations of the lunar orbit by the equivalence-principle-violation terms [see Eqs. (2.58)].

Once the iterative scheme is set up and the numerical program debugged, we can obtain the solution of our problem up to an arbitrary order. The limits of the solution are then given mainly by the computer power. A minor limit comes from the fact that the MINIMS algebraic manipulator works with double precision real coefficients (16 digits) [49]. During the manipulation of the series, one thereby accumulates round-off errors. However, we have checked that this restriction is not significant for our work.[16]

[16]We have performed a lower order solution in using the modified version of the distributed MINIMS manipulator which accepts the

TABLE I. Coefficients w_{jk} of the double series giving Hill's variational curve: $w=\Sigma w_{jk}\zeta^j m^k$; j labels the rows and k the columns.

	2	3	4	5	6
−6	–	–	–	–	0.005208333333333333
−4	–	–	0.0000000000000000	0.03593750000000000	0.1245833333333333
−2	−1.187500000000000	−1.666666666666667	−1.194444444444444	−0.5185185185185185	−0.8265365788966050
0	–	–	0.6210937500000000	−0.6770833333333333	−0.6892361111111111
2	0.1875000000000000	0.5000000000000000	0.5833333333333333	0.3055555555555556	−0.1615849247685185
4	–	–	0.0976562500000000	0.4182291666666667	0.8484722222222222
6	–	–	–	–	0.06778971354166667

In the rest of this appendix we shall present tables of the numerical coefficients achieved by the previous algorithm for different series introduced in the main text of the paper and related to physical quantities.

Let us start with our solution for the variational curve ($\hat{\lambda}=0$). Tables I and II give the coefficients w_{ij} of the double series expansion of w: $w=\Sigma w_{jk}\zeta^j m^k$, where $j=0,\pm 2,\pm 4,\ldots$, $k=2,3,4,\ldots$, and $|j|\leqslant k$ (j labels the rows and k the columns). Contrary to the method of Ref. [18], the ζ-independent term is not fixed to unity. However, because of the choice of the fiducial semimajor axis $\tilde{a}$, defined in Eq. (2.36), it starts only at the power m^4.

We then give the lunar orbit perturbations due to a hypothetical equivalence-principle violation (terms linear in $\hat{\lambda}$).

Table III gives the coefficients c_k of the series in powers of m giving w_{-1}, i.e., the coefficient of ζ^{-1} in the Laurent expansion of $\delta_\lambda w(\zeta)$, after factorization of the leading term $\frac{9}{4}(\hat{\lambda}/m)$ [see Eq. (2.56b)]. The second column, p_k, gives the numerical value of $c_k m^k$ (in %) for the lunar orbit ($m=m_{\rm Moon}=0.080\,848\,937\,5\ldots$). The last column, r_k, gives the ratio c_{k-1}/c_k of the successive coefficients of the series (the same structure is conserved also for Tables IV–VI and VIII–IX).

Table IV gives the coefficients c_k of the series in powers of m giving w_1, i.e., the coefficient of ζ in the Laurent expansion of $\delta_\lambda w$, after factorization of the leading term $-\frac{3}{4}(\hat{\lambda}/m)$ [see Eq. (2.56a)].

quadruple precision (32 digits) for the coefficients of the series and compared it with the double precision one.

Some general properties of the presented results, already commented upon in the main text of the paper and in Appendix C, are (i) a fast ("geometriclike") growth of the coefficients c_k of the series with a surprisingly rapid approach of the $r_k(=c_{k-1}/c_k)$ ratio to the $m_{\rm cr}$ value of about $0.195\,103\,996\,6\ldots$, and (ii) a very substantial contribution of the nonlinear terms ($k\geqslant 1$) in the series for the lunar value of Hill's parameter $m_{\rm Moon}$.

Tables V and VI give the coefficients of the power series $S(m)$ and $S'(m)$ related, respectively, to the radial and longitudinal perturbations (with synodic periodicity) of the lunar orbit due to a hypothetical violation of the equivalence principle. The coefficients of the Padé approximants of those series are given in Table VII. More precisely, we denote $N_8(m)\equiv\Sigma_0^8 a_k m^k$ and $D_8(m)\equiv\Sigma_0^8 b_k m^k$, the Padé approximants for the $S(m)$ series [see Eq. (3.2)], and $N'_8(m)\equiv\Sigma_0^8 a'_k m^k$ and $D'_8(m)\equiv\Sigma_0^8 b'_k m^k$, the Padé approximants for the $S'(m)$ series. The denominator polynomials $D_8(m)$ and $D'_8(m)$ have $0.195\,103\,996\,7\ldots$ and $0.195\,103\,996\,6\ldots$, respectively, as real roots.

Table VIII gives the coefficients of the $S''(m)$ series yielding the equivalence-principle-violation perturbation of the lunar orbit with one-third of the synodic period ("3τ"). This series is defined by

$$(\delta_\lambda r)_{\rm third\ synodic}=C''(m)\,\bar{\delta}_{12}a'\cos 3\tau, \tag{B8}$$

with

$$C''(m)=\frac{51}{32}m^3\left(1+\sum_{k\geqslant 1}c''_k m^k\right)\equiv\frac{51}{32}m^3 S''(m). \tag{B9}$$

TABLE II. Continuation of Table I.

	7	8	9	10	11
−10	–	–	–	0.002728271484375000	0.02314830588154192
−8	–	0.003743489583333333	0.02753602458235899	0.09640929637552359	0.2138120808825399
−6	0.03477027529761905	0.1040162627551020	0.1880280591858106	0.2403927412241086	0.2416131234195372
−4	0.1956215277777778	0.2153863326461227	0.2955465183221726	0.3769043026659738	−0.1199168976723571
−2	1.205252137988683	7.122740269204389	15.70605739258259	19.78110897697562	14.66018287215882
0	−1.605902777777778	−0.5700574333285108	0.3975170245386445	1.670582431118184	7.457045029767965
2	−1.034723427854938	−3.070661490483539	−6.516200630679869	−9.530344734021654	−7.910677339637028
4	1.038887731481481	0.7041097601996523	−0.5710030937052194	−4.063905605879704	−11.57027457856626
6	0.3898297991071429	1.087330552012472	1.926893073875549	2.301395728392179	1.149297956477343
8	–	0.05397033691406250	0.3869401996638499	1.360005033209030	3.095895579274965
10	–	–	–	0.04656556447347005	0.3988932608444074

TABLE III. Coefficients of m^k in the lunar orbit perturbation w_{-1}. The mark $<$ signifies that the value is smaller than 0.001%.

k	c_k	p_k	r_k
0	1.000000000000000	–	–
1	5.111111111111111	41.323	0.1956521739
2	25.52777777777778	16.686	0.2002176279
3	129.5277777777778	6.845	0.1970834227
4	663.1076388888889	2.833	0.1953344679
5	3400.509837962963	1.175	0.1950024174
6	17434.56978202160	0.487	0.1950440923
7	89366.97811374742	0.202	0.1950896198
8	458049.7173103071	0.084	0.1951032273
9	2347711.432406117	0.035	0.1951047778
10	12033105.92287766	0.014	0.1951043602
11	61675313.45886662	0.006	0.1951040902
12	316115045.4115870	0.002	0.1951040115
13	1620238694.871019	0.001	0.1951039970
14	8304487501.579596	$<$	0.1951039958
15	42564415192.16274	$<$	0.1951039962
16	218162702733.8958	$<$	0.1951039965
17	1118186744062.939	$<$	0.1951039966

Note that the third-synodic effect (2.8) is $O(m^2)$ smaller than the synodic effect (2.60). For instance, if we assume $\hat{\delta}_{12}=0$, the result (2.9) gives numerically $(\delta_\lambda r)_{\text{third synodic}} \simeq 6.62\,\eta\,\cos 3\tau$ cm for the lunar orbit. Although there is probably no practical use of this higher frequency excitation (and of the others with frequencies 5τ, 7τ, ...), two points are to be mentioned: (i) a significantly smaller amplitude of the effect, and (ii) the persistence of the pole singularity at $m=m_{\text{cr}}$ for these odd multiples of the basic synodic frequency as discussed in Appendix C.

Table IX lists the coefficients of the $Q(m)$ series giving the radial parallactic inequality of the lunar motion

TABLE IV. Coefficients of m^k in the lunar orbit perturbation w_1. Notation as in Table III.

k	c_k	p_k	r_k
0	1.000000000000000	–	–
1	6.000000000000000	48.509	0.1666666667
2	29.62500000000000	19.365	0.2025316456
3	147.5000000000000	7.795	0.2008474576
4	749.9427083333333	3.204	0.1966816910
5	3843.245659722222	1.328	0.1951326495
6	19711.47459129051	0.551	0.1949750457
7	101057.1821729118	0.228	0.1950526837
8	517989.9317152261	0.095	0.1950948773
9	2654938.425605692	0.039	0.1951043108
10	13607759.81112155	0.016	0.1951047389
11	69746091.35545381	0.007	0.1951042639
12	357481499.9590677	0.003	0.1951040581
13	1832261194.517921	0.001	0.1951040065
14	9391202741.857013	$<$	0.1951039973
15	48134343317.40093	$<$	0.1951039963
16	246711211454.4727	$<$	0.1951039964
17	1264511316090.325	$<$	0.1951039966

TABLE V. Coefficients of the $S(m)$ series yielding the radial perturbation, with synodic ("τ") period, of the variational curve due to an equivalence principle violation.

k	c_k	p_k	r_k
0	1.000000000000000	–	–
1	4.666666666666667	37.730	0.21428571429
2	22.10416666666667	14.449	0.21112158341
3	111.0416666666667	5.868	0.19906191370
4	569.0742187500000	2.431	0.19512686221
5	2920.624565972222	1.001	0.19484675483
6	14976.54921694155	0.418	0.19501318519
7	76767.66017493731	0.173	0.19508930170
8	393469.7706768071	0.072	0.19510434065
9	2016707.919972300	0.030	0.19510498609
10	10336561.13767503	0.012	0.19510433819
11	52979731.33709500	0.005	0.19510406861
12	271546096.6185635	0.002	0.19510400627
13	1391801807.779853	0.001	0.19510399764
14	7133640668.991231	$<$	0.19510399701
15	36563272849.28028	$<$	0.19510399680
16	187404017729.6833	$<$	0.19510399666
17	960533976540.2227	$<$	0.19510399664

$[Q(m)\equiv 1+\Sigma_{k\geq 1} q_k m^k]$ as defined in Eqs. (3.4) of the text. Similarly to the treatment of the equivalence-principle-violation lunar perturbation, we improved on our solution by using Padé approximants. Table X yields the coefficients of the corresponding polynomials. We also computed the corresponding lunar parallactic inequality in longitude. As a partial check on our results we have compared the latter with the result by Deprit, Henrard, and Rom [58]. When substituting the current recommended values of the mass constants and

TABLE VI. Coefficients of the $S'(m)$ series yielding the longitudinal perturbation, with synodic ("τ") period, of the variational curve due to an equivalence principle violation.

k	c_k	p_k	r_k
0	1.000000000000000	–	–
1	5.333333333333333	43.119	0.18750000000
2	26.39583333333333	17.254	0.20205209155
3	132.9166666666667	7.024	0.19858934169
4	678.3172743055553	2.898	0.19595058493
5	3478.588686342593	1.202	0.19499783834
6	17838.85192117573	0.498	0.19500070418
7	91447.83453394253	0.206	0.19507134326
8	468721.9555645895	0.086	0.19510038616
9	2402405.984973695	0.035	0.19510522305
10	12313419.50072995	0.015	0.19510469735
11	63112018.54675127	0.006	0.19510419385
12	323478806.4318246	0.003	0.19510402936
13	1657981435.159388	0.001	0.19510399789
14	8497936880.202027	$<$	0.19510399507
15	43555934597.05303	$<$	0.19510399579
16	223244707500.3778	$<$	0.19510399635
17	1144234415604.679	$<$	0.19510399657

TABLE VII. Coefficients of the Padé approximants of order eight of the radial and longitudinal perturbation series, with synodic ("τ") period, of the variational curve due to an equivalence principle violation.

k	a_k	b_k	a'_k	b'_k
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	−4.679659654425508	−9.346326321092175	−2.948114664716526	−8.281447998049859
2	9.706375765630793	31.21839859739427	2.472472675594219	20.24436199852680
3	−12.09649460259615	−62.23126666796116	−0.035120215085467699	−22.32599642537073
4	9.264485113438459	78.37781038863183	−0.8074761600241968	6.322888291078106
5	−0.8206525542778040	−59.42770492795888	1.129701196880901	4.768342316076093
6	−4.562346877625465	5.529574170471975	1.916287015273362	13.88318783117331
7	0.6450458113816286	30.28114298289518	−3.678767273115525	−38.01361281338056
8	0.9269455883562498	−30.33613801897244	−3.906968891383218	34.97077854522613

the Earth semimajor axis, we obtain 125.″438 for the amplitude, in the main lunar problem, of the parallactic inequality in longitude when truncating our series to the power m^7. This value is to be compared with 125.″4201 reported in Ref. [58]. We believe that the origin of the minor discrepancy between those results lays in the slightly different values of the astronomical constants employed by Deprit, Henrard, and Rom at the end of the 1960s.[17]

APPENDIX C: CHARACTERISTIC MULTIPLIERS, COMMENSURABILITIES, AND INSTABILITY

Let us first recall the basic concept of characteristic multipliers. The small perturbations around a periodic orbit in the restricted three-body problem can be described in terms of a two-dimensional Poincaré map: this is the application connecting two successive intersections of the trajectory in phase space with a two-plane transversal to the orbit. (One works, say, in the three-dimensional reduced phase space corresponding to a fixed value of the Jacobi integral; see [60] for a catalog of such Poincaré maps in the case of the Hill problem.) For infinitesimal perturbations, the Poincaré map reduces to a linear transformation of the plane, leaving fixed the origin which corresponds to the reference periodic orbit. The two eigenvalues (λ_1,λ_2) of the infinitesimal Poincaré map (a 2×2 matrix) are the characteristic multipliers. From the Hamiltonian nature of the dynamics, it follows that these multipliers are either of the form $(e^{i\alpha},e^{-i\alpha})$ or $(\varepsilon e^{\beta},\varepsilon e^{-\beta})$ with $\varepsilon=\pm1$ [61]. The first case means generally (apart from the exceptional cases where $\alpha=2\pi/3$, or where $\alpha=2\pi/4$ and some inequality is not satisfied) that the periodic orbit is (quasi-)stable. The second case means that the periodic orbit is unstable. A useful quantity for studying the possible loss of stability is half the sum of the multipliers: $a\equiv\frac{1}{2}(\lambda_1+\lambda_2)$, which is either $\cos\alpha$ (in which case $|a|\leqslant1$) or $\varepsilon\cosh\beta$ (in which case $|a|\geqslant1$). The loss of sta-

TABLE VIII. Coefficients of the $S''(m)$ series yielding the radial perturbation, with third-synodic ("3τ") period, of the variational curve due to an equivalence principle violation.

k	c''_k	p_k	r_k
1	1.000000000000000	–	–
2	6.450980392156863	52.155	0.15501519757
3	32.60457516339869	21.312	0.19785506665
4	163.0646514161220	8.618	0.19994876192
5	828.6388478122731	3.540	0.19678615340
6	4240.758469846889	1.465	0.19539873674
7	21736.94075865620	0.607	0.19509454053
8	111428.4766430605	0.252	0.19507527531
9	571159.5105168029	0.104	0.19509169434
10	2927510.871437882	0.043	0.19510073083
11	15004916.03283528	0.018	0.19510344910
12	76907280.99332377	0.007	0.19510397246
13	394186033.1895487	0.003	0.19510402327
14	2020389145.708269	0.001	0.19510401450
15	10355446785.91400	<	0.19510400541
16	53076547791.36041	<	0.19510400011
17	272042338525.2278	<	0.19510399771

TABLE IX. Radial perturbation of the variational curve due to parallactic terms with synodic ("τ") period.

k	q_k	p_k	r_k
0	1.000000000000000	–	–
1	4.400000000000000	35.574	0.2272727273
2	13.43750000000000	8.783	0.3274418605
3	59.99895833333333	3.171	0.2239622216
4	318.5420138888889	1.361	0.1883549288
5	1665.565227141204	0.575	0.1912515996
6	8563.762388478974	0.239	0.1944898926
7	43904.44785527987	0.099	0.1950545516
8	225048.4315368017	0.041	0.1950888862
9	1153503.664577334	0.017	0.1950998843
10	5912167.013677382	0.007	0.1951067454
11	30302311.46598434	0.003	0.1951061397
12	155313062.3650712	0.001	0.1951047195
13	796051992.0171008	<	0.1951041690
14	4080141094.383684	<	0.1951040353
15	20912646936.88289	<	0.1951040013
16	107187180609.8796	<	0.1951039930
17	549384862217.9330	<	0.1951039935

[17]For completeness, let us mention that, when including the effects of e, e', and I, the amplitude of the parallactic inequality becomes 124.″98812 [59].

TABLE X. Coefficients of the Padé approximants of order eight of the radial and longitudinal perturbations of the variational curve due to the parallactic terms with synodic (''τ'') period.

k	a''_k	b''_k	a'''_k	b'''_k
0	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
1	−3.053328223352915	−7.453328223352915	−2.946473061279729	−8.146473061279729
2	−3.844661609919813	15.51248257283301	−4.603878697382873	20.29528122127172
3	5.918301506191725	−22.18148214630207	7.845106685708085	−30.39194483230758
4	3.507936007628144	21.30738850763350	1.021511391279373	33.43540948735330
5	13.74620469848220	−4.042478627609922	15.68678071178852	−19.25251169116597
6	2.377354880592943	−26.42346152043986	0.230401974593673	−10.98565280233657
7	−25.90435993273240	19.02504889156572	−27.85325375859802	18.16643796715844
8	−40.39610428715375	−28.47566056885599	−36.65716185443595	−10.99350748347175

bility can occur (apart from the above-mentioned exceptional cases) only when a crosses the values ± 1. The stability of all the periodic orbits in Hill's problem has been studied by Hénon [52]. We are interested here in the families ''g'' and ''f'' of periodic orbits which correspond, respectively, to prograde and retrograde lunar-type orbits. Hénon found that retrograde orbits are stable for all values of the Jacobi integral (i.e., $-1<m<0$), while prograde orbits are stable only for close enough orbits, $0<m<m_{\rm cr}$ with[18]

$$m_{\rm cr}=0.195\,103\,996\,6\ldots. \tag{C1}$$

For this value a crosses the value $+1$.

Let us translate this result in terms of the perigee precession of a perturbed Hill orbit. Perturbations of Hill's orbit can be described in terms of the (isoenergetic) *normal* displacement $q=(\dot X\,\delta Y-\dot Y\,\delta X)/(\dot X^2+\dot Y^2)^{1/2}$. This variable satisfies ''Hill's equation''

$$\frac{d^2q(\tau)}{d\tau^2}+\Theta(\tau)q(\tau)=\sigma(\tau), \tag{C2}$$

where $\Theta(\tau)=\theta_0+2\Sigma_1^\infty\theta_j\cos 2j\tau$ is periodic with period π. The source term $\sigma(\tau)$ on the right-hand side of Eq. (C2) is zero for free perturbations (i.e., corresponding to adding some ''eccentricity'' to Hill's ''circularlike'' variational orbit), and nonzero when one perturbs Hill's Hamiltonian $H_{\rm Hill}=(\text{kinetic terms})+F_0+F_2$ (e.g., by adding the F_1 perturbation we are mainly concerned with, or the parallactic terms $F_3+\cdots$). Perigee precession is described by the general homogeneous solution of Eq. (C2) ($\sigma=0$). The latter general homogeneous solution can be written as a linear combination of complex solutions of the form $q(\tau)=\zeta^{\rm c}\Sigma_j b_j\zeta^{2j}$ and of their complex conjugates (we recall that $\zeta=e^{i\tau}$). On the one hand, the quantity c is linked to the usual rate of perigee advance $d\varpi/dt$ (in the nonrotating frame) by [18]

$$\mathrm{c}=\frac{dl}{d\tau}=(1+m)\left(1-\frac{1}{n}\frac{d\varpi}{dt}\right) \tag{C3}$$

(where $l=nt+\epsilon-\varpi$ is the ''mean anomaly''). On the other hand, the quantity c is directly linked to the characteristic multipliers. Indeed, when $\tau\to\tau+2\pi$, $q(\tau)$ gets multiplied by $e^{2\pi i\mathrm{c}}$, while $\bar q(\tau)$ gets multiplied by $e^{-2\pi i\mathrm{c}}$. Therefore, the half-sum of the multipliers is simply

$$a=\cos(2\pi\mathrm{c}). \tag{C4}$$

By using the perturbative series giving the perigee precession, $n^{-1}(d\varpi/dt)=\frac{3}{4}m^2+\frac{177}{32}m^3+\cdots$ (which has been computed to high accuracy in Ref. [58]; see also [19]), one can check that the crossing of $a=+1$, found by Hénon, corresponds, when m increases from 0 to $m_{\rm cr}$, to c increasing from one to a slightly higher value ($\simeq 1.1$) and then decreasing to reach the value of one at $m=m_{\rm cr}$. From the smoothness of the variation of the characteristic multipliers, and therefore of a, with m we deduce that, beyond $m=m_{\rm cr}$, $\mathrm{c}(m)$ goes through a quadratic branch point $\mathrm{c}(m)-1\sim(m_{\rm cr}-m)^{1/2}$ and becomes complex.[19]

Finally, the important information for our purpose is that when m increases up to $m_{\rm cr}$, the quantity $\mathrm{c}(m)-1\sim(m_{\rm cr}-m)^{1/2}$ is such that both functions $\cos 2\pi\mathrm{c}(m)\times(=a)$ and $\cos\pi\mathrm{c}(m)$ cross smoothly (without branch points or discontinuities of derivatives) their corresponding limiting values $\cos 2\pi=+1$ and $\cos\pi=-1$.

Let us now consider Hill's way of solving Eq. (C2). By inserting $q(\tau)=\zeta^{\rm c}\Sigma_j b_j\zeta^{2j}$ into (C2), one gets an infinite system of linear equations for the coefficients b_j. When written in a suitably normalized way, the determinant of this infinite system (which depends on c), say $\Delta(\mathrm{c})$, is a well-defined quantity (Hill's determinant). A homogeneous solution ($\sigma=0$) exists only for the values of c for which $\Delta(\mathrm{c})=0$. On the other hand, if we consider the case where there is a

[18] In Hénon's fourth paper [52], he gives the value $\pi m_{\rm cr}=T/2=0.612\,94$. The more precise value (C1) was privately communicated to us by Hénon, and also follows independently from our results in Appendix B (study of the geometriclike growth of various m series and of the zeros of Padé denominators).

[19] The combined facts that $m_{\rm cr}$ is rather small and that $\mathrm{c}(m)$ has a quadratic branch point at $m=m_{\rm cr}$ ''explains'' the notoriously bad convergence of the perturbation series giving $d\varpi/dt$. Rewriting this series in terms of better-behaved quantities, such as $\cos(2\pi\mathrm{c})$ or $\cos(\pi\mathrm{c})$, improves more its convergence than that by the ''Euler transformations,'' $m\to m/(1+\alpha m)$, which have been traditionally used [62].

source term on the right-hand side of Eq. (C2) of the form $\sigma(\tau)=\zeta^{c_\sigma}\Sigma_j\sigma_j\zeta^{2j}$, the corresponding inhomogeneous solution $q_{\rm inhom}(\tau)$ will have the same form as $\sigma(\tau)$, but will contain, as for usual finite determinants, a factor $1/\Delta(c_\sigma)$. This factor will become infinite when c_σ tends to one of the free perigee precession values for which $\Delta(c)=0$. The analysis of the determinant $\Delta(c)=0$ [42,18] shows that, as a function of c, it is a linear function of $\cos\pi c$. We conclude that the forced perturbation $q_{\rm inhom}(\tau)$ will contain the factor $[\cos\pi c_\sigma-\cos\pi c(m)]^{-1}$, where $c(m)$ denotes the free perigee precession value. From our analysis above, we know that when m crosses $m_{\rm cr}$, $\cos\pi c$ crosses smoothly -1. The final conclusion is that the source terms for which $\cos\pi c_\sigma=-1$, i.e., $c_\sigma=\pm1,\pm3,\pm5,\ldots$, generates normal displacements of Hill's orbit which have pole singularities $\propto(m-m_{\rm cr})^{-1}$ as m crosses the value (C1) found in linear stability analyses. Moreover, it is easily checked that the addition of a perturbing potential, say F_p, to Hill's potential F_0+F_2, generates a source term $\sigma(\tau)$ in Hill's variational equation (C2) which is a linear combination of F_p and $\mathrm{Re}[Du(\partial F_p/\partial u)]$ with real coefficients of the form $k_0+2\Sigma_j k_j\cos2j\tau$. More precisely

$$\sigma(\tau)=-2\varphi\{\varphi^2\Phi F_p+\mathrm{Re}[Du(\partial F_p/\partial u)]\}, \tag{C5}$$

where

$$\varphi^{-2}(u,\bar u)=-DuD\bar u, \tag{C6a}$$

$$\Phi(u,\bar u)=mDuD\bar u-2\mathrm{Re}[Du(\partial F_{\rm Hill}/\partial u)], \tag{C6b}$$

with $F_{\rm Hill}\equiv F_0+F_2$.

Therefore, if $F_p=\mathrm{Re}[\zeta^{c_p}\Sigma_j f_j\zeta^{2j}]$, σ will have the form $\sigma(\tau)=\mathrm{Re}[\zeta^{c_p}\Sigma_j\sigma_j\zeta^{2j}]$. In other words, $c_\sigma=\pm c_p$, so that the perturbing potentials $F_1,F_3,F_5,\ldots$ generate source terms with $c_\sigma=\pm1$, $c_\sigma=\pm3$, $c_\sigma=\pm5,\ldots$, respectively.

APPENDIX D: LAPLACE ON THE EQUIVALENCE PRINCIPLE

In the first volume of his *Traité de Mécanique Céleste* [63] (presented to the French Academy of Sciences in 1799), Laplace lists a series of facts suggesting that gravity is proportional to the masses. This list (which is probably inspired by a corresponding list in the *Principia*, although Laplace does not mention Newton here) contains Newton's argument that the motion of satellites would be very sensitive to a violation of the universality of free fall, but does not quantify it. As far as we are aware, the only quantitative work of Laplace on this idea [5] is contained in the last book of the last volume of the *Traité de Mécanique Céleste* which was presented to the Académie des Sciences on 16 August 1825 [64].

The fact that Laplace was then 76 year old (he died a year and a half later on 5 March 1827) may explain why this work of Laplace contains some strange leaps of reasoning.[20] It seems plausible to us that Laplace, when writing this chapter, was using previous notes of his which contained more detailed calculations and more consistent reasonings. Anyway, the aim of this appendix is to show that the final limit he quotes on a possible violation of the equivalence principle,

$$|\bar\delta_{12}|<\frac{1}{3\,410\,000}\simeq2.9\times10^{-7}, \tag{D1}$$

is a very reasonable (and slightly pessimistic) bound, which can be derived in a logically clear way using only the information Laplace had in hand.

Laplace's new idea (compared to Newton) was to use the "parallactic inequality" in the longitude of the Moon as a sensitive test of the equivalence principle.[21] "Parallactic inequality" means the coefficient of the synodic term, $\sin\tau$ in the expression of the lunar longitude v as a function of time.[22] We have computed this coefficient, say A, in Appendices A and B for the Main Problem (i.e., neglecting eccentricities and inclinations). Its theoretical expression reads

$$A^{\rm th}=A_{\rm par}+A_{\rm ep}, \tag{D2}$$

where the normal "parallactic" contribution reads

$$A_{\rm par}=\frac{15}{8}(X_2-X_1)m\frac{\tilde a}{a'}S_{\rm par}(m), \tag{D3}$$

while the "equivalence principle" contribution reads

$$A_{\rm ep}=3m\bar\delta_{12}\frac{a'}{\tilde a}S_{\rm ep}(m). \tag{D4}$$

Here, $S_{\rm par}(m)=1+\frac{26}{5}m+\cdots$, $S_{\rm ep}(m)=1+\frac{16}{3}m+\cdots$, are slowly converging series in powers of $m=n'/(n-n')$. In the third volume of his *Traité de Mécanique Céleste*, Laplace computes $A_{\rm par}$ with particular care, pushing the calculation to fifth order in m inclusively.[23] He was therefore entitled to considering that the theoretical error on $A_{\rm par}$ was negligible compared to the observational uncertainties in $A^{\rm obs}$. Note that $A_{\rm par}$ is proportional to the inverse distance to the Sun, i.e., to the "solar parallax" $\pi_S\equiv R_E/a'$, where R_E is the equatorial radius of the Earth (hence the name "parallactic inequality"). The result of Laplace can be expressed as

$$A_{\rm par}\simeq14.3\pi_S, \tag{D5}$$

where both $A_{\rm par}$ and π_S are expressed in sec of arc.[24]

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[20]Basically, he mentions that his lunar theory and combined (selected) lunar and solar parallax data agree to about 1.2% and then goes on to admit 1/8=12.5% as fractional upper limit on the theory-observation agreement.

[21]Note that as early as 1753, T. Mayer had used the theory of the parallactic inequality to infer the value of the solar parallax [65].

[22]Beware that Laplace was actually working with the inverse function: $t=t(v)$.

[23]We have checked the first orders of his result and found them to agree with ours. Note that Laplace includes the effect of eccentricities and inclinations that we neglect.

[24]Beware that Laplace, in his volume III, uses (Révolution oblige) "decimal seconds," i.e., 10^{-6} of a right angle.

On the observational side, Laplace had in hand both lunar data and data on the solar parallax. The two phenomenological ''lunar tables'' he was using (one by Mason and one by Burg) gave for the ''observed'' value of the synodic inequality in longitude, $A^{\rm obs}_{\rm Mason}=116''.68$ and $A^{\rm obs}_{\rm Burg}=122''.38$ [66]. As for the solar parallax, many scientific expeditions had taken advantage of the passages of Venus in front of the Sun in 1761 and, especially, in 1769 to measure π_S (see [67] for a detailed text by a contemporary of Laplace, or [68] for a more exhaustive historical treatment). The published results ranged between $\pi_S^{\rm obs}=8''.43$ (Planman) and $\pi_S^{\rm obs}=8''.80$ (Pingré) [65,68]. The comparison between the theoretical results (D2) and (D5) and the observational results on A and π_S gives a value for a possible equivalence-principle-violation contribution

$$A_{\rm ep}=A^{\rm obs}-14.3\pi_S^{\rm obs}. \tag{D6}$$

Worst-case limits on $A_{\rm ep}$ are obtained by taking the extreme values on the right-hand side of (D6) (e.g., $A^{\rm max}_{\rm ep}=A^{\rm obs}_{\rm Burg}-14.3\pi_S^{\rm Planman}$).[25] This yields

$$-9''.2<A_{\rm ep}<1''.8. \tag{D7}$$

On the other hand, the theoretical result (D4) [using $a'/\tilde{a}\simeq 39\underline{1}$ and $S_{\underline{\rm ep}}(m)\simeq 1.72$ (see Appendix B)] reads $A_{\rm ep}\simeq 163\bar{\delta}_{12}\simeq 3.36\bar{\delta}_{12}\times 10^7$ in sec of arc, so that we get the following worst-case bounds on $\bar{\delta}_{12}$:

$$-2.7\times 10^{-7}<\bar{\delta}_{12}<0.54\times 10^{-7}. \tag{D8}$$

From this point of view, the final bound quoted by Laplace, Eq. (D1), seems very reasonable and consistent with the observational uncertainties in his time. Note, however, that Laplace never quotes a precise theoretical formula for $A_{\rm ep}$. He only says (and uses) the fact that the synodic amplitude A is proportional to its source term in the perturbing function. This neglects the leading ''p dependence'' of C_v $\propto 2+p$ in Eq. (A19b) which says that $A_{\rm ep}/A_{\rm par}$ $\propto 3/5\times S_{\rm ep}(m)/S_{\rm par}(m)$. In Laplace's published analysis the lacking (unfavorable) factor 3/5 is effectively compensated by his overpessimistic estimate of the fractional uncertainty on A: $1/8=12.5\%$.[26]

Let us end this section by raising a historical question. Though Laplace was fully aware of the scientific interest of the bound (D1), and of the fact that it was (at the time) more precise than the bounds obtained from ground tests of the universality of free fall, his successors in celestial mechanics seem (as far as we know) to have lost interest in the issue. However, near the end of the nineteenth century, especially after the theoretical work of Hansen, Delaunay, Hill, and Brown (who improved the computation of $A^{\rm th}_{\rm par}$), and after many improvements on the observational side, it should have been possible to obtain more stringent limits on $\bar{\delta}_{12}$. For instance, Delaunay computed $A^{\rm th}_{\rm par}$ to m^7 which corresponds to a truncature error of $0''.3366/125''.4201\simeq 2.7\times 10^{-3}$ [58] (for the principal part of A; see also Appendix B). This is negligible (when added in quadrature) compared to the observational error on π_S at the time. For instance, the laboratory measurements of the velocity of light by Foucault and Cornu gave values ranging between $\pi_S=8''.834$ and $\pi_S=8''.881$, the passage of Venus across the solar disk in 1874 gave a range $8''.76-8''.88$, [65] and the recommended value starting in 1896 was $8''.80$. This suggests that a reasonable upper bound on the uncertainty of π_S at that time was $0''.08$ (i.e., 0.9%). (By comparison, the modern value is $\pi_S=8''.794148$ [51].) On the other hand, the observational error on $A^{\rm obs}$ at the time was $<\frac{1}{2}(125''.46-124''.70)=0''.38$, i.e., $<0.3\%$ (see [65], p. 533). This has a negligible effect on the derivation of a bound on $\bar{\delta}_{12}$.[27] Using Eq. (D6) with $A_{\underline{\rm ep}}\simeq 3.36\bar{\delta}_{12}\times 10^7$ sec of arc, this leads to the bound $|\bar{\delta}_{12}|<14.3\delta\pi_S^{\rm obs}/(3.36\times 10^7)\simeq 3.4\times 10^{-8}$, which is slightly better than the value obtained by Eötvös in 1890 ($|\bar{\delta}_{AB}|<5\times 10^{-8}$ [7]). We do not wish to take too seriously such *a posteriori* derivations of limits on $\bar{\delta}_{12}$, but we consider this as an interesting example of historical eclipse of a deep concept, which has been rejuvenated, within a new theoretical and observational context, only in the last decades.[28]

[25] Such a worst-case approach seems appropriate to a pre-least-squares-law period. Before Gauss' theory of measurement errors, scientists quoted only ''central values'' for measured quantities.

[26] As for the subleading dependence on p, i.e., the ratio $S_{\rm ep}(m)/S_{\rm par}(m)\simeq 1.72/1.60\simeq 1.08$, Laplace's experience with similar factors in many terms of lunar theory might have suggested to him that he did not need to worry about it.

[27] Let us note for completeness that the determinations of the mass of the Moon at the time were accurate enough to estimate with negligible error the mass ratio factor X_2-X_1 entering $A_{\rm par}$.

[28] For completeness, let us note that Poincaré kept alive this concept by mentioning Laplace's result in one of his popular books [69].

[1] G. Galilei, *Discorsi e dimostrazioni matematiche intorno à due nuove scienze* (Elzevir, Leiden, 1638); French translation by M. Clavelin, *Discours concernant deux sciences nouvelles* (A. Colin, Paris, 1970); see p. 61: ''... it came to my mind that, if the resistance from the medium were totally suppressed, all bodies would go down with the same speed.''; see also p. 70 where Galilei claims to have verified the universality of free fall, to better than 0.1%, by pendulum experiments. The pages quoted above (61,70) correspond, respectively, to p. 116 and p. 129 of the Edizione nazionale, and to p. 72 and p. 85 of the English translation by H. Crew and A. de Salvio, *Dialogues concerning two new sciences* (Dover, New York, 1954).

[2] Though this issue has a long prehistory (starting, say, with J. Philoponos, 5th Century A.D.), even G.B. Benedetti who came closest to stating clearly the universality of free fall (in his *Resolutio* of 1553, and his *Demonstratio* of 1554) did not quite

succeed in doing so. See *Mechanics in Sixteenth-Century Italy; Selections from Tartaglia, Benedetti, Guido Ubaldo and Galileo*, translated and annotated by S. Drake and I.E. Drabkin (University of Wisconsin Press, Madison, 1969).

[3] I. Newton, *Philosophiae Naturalis Principia Mathematica* (London, 1687); English translation by F. Cajori, *Newton's Principia* (University of California Press, Berkeley, 1934).

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