Lattice sum rules for the color fields

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We analyze the sum rules describing the action and energy in the color fields around glueballs, torelons, and static potentials.

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I. INTRODUCTION

In lattice gauge theory, it is possible to derive sum rules for the energy and action in the color fields around states. The technique used in [1], hereafter referred to as I, involves evaluating derivatives with respect to a parameter (β for example) of the formal expression for a correlation involving an observable of interest. For the Wilson gauge action, this yields exact relations between the β dependence of observables and the sum over a time slice of the plaquette expectation in the presence of that observable [for example, see Eqs. (1) and (23)]. These identities can be used as checks of numerical results, see [2] for an application to the glue lump state. They relate the variation with β to sums at one fixed value of β. They can also be used to investigate the β dependence of lattice quantities: so leading to evaluation of the lattice β function.

A more powerful set of relations can be derived if the β derivatives can be reexpressed using renormalization group invariance in terms of well-known quantities. For glueballs and potentials, these were also presented in I. The main conclusion is that the combination of squared color field strengths corresponding to the action (electric plus magnetic) is much larger than the combination corresponding to the energy (electric minus magnetic). This implies that the electric and magnetic field strengths are comparable. This conclusion has been a useful benchmark for models of nonperturbative QCD. Although this general conclusion was correct, the explicit results in I were in error and the correct expressions are given here.

The lattice analysis of the field strengths depends on the scale at which these fields are probed. The results can be calculated reliably by perturbation theory for very short distance scales. However, for scales appropriate for nonperturbative states, perturbation theory on the lattice in terms of the bare coupling is now known to be poorly convergent and it is worthwhile to reassess the assumptions leading to these relations.

Recently there has been a reawakening of interest in this area—partly from new accurate lattice results [3] and partly because of the realization [4,5] that the application [1] of the sum rules to static potentials was wrong.

Here we summarize the derivation of I and confirm the correction needed for the application to potentials. We extend the discussion to apply the sum rules to torelons and to analyze the transverse and longitudinal color fields separately. This enables us to explore in detail the problem of the field energy in the potential between static sources.

II. GLUEBALLS

As an example of the techniques to be used, we consider first a glueball state. Define $M(\beta)$ as the lattice observable glueball mass (in lattice units) which will depend on β the bare lattice coupling parameter, with $\beta=2N/g^2$, for the gauge sector of the SU(N) theory.

Then, for the Wilson action, the identity was derived in I that

$$\frac{dM}{d\beta}=\left\langle 1\left|\sum\Box\right|1\right\rangle-\left\langle 0\left|\sum\Box\right|0\right\rangle=\sum\Box_{1-0},\quad(1)$$

where $\Box$ is the plaquette action $(1/N)\mathrm{Tr}(1-U_\Box)$ which is summed over all $(6L^3)$ plaquettes in one time slice. The subscript $1-0$ refers to the difference of this plaquette sum in a one glueball state (1) and in the vacuum (0).

This identity can be used as it stands to check this observed plaquette difference with the left-hand side obtained as a finite difference from lattice calculations of $M(\beta)$ at two nearby values of β. A more powerful application comes from using the renormalization group invariance to relate the β dependence of M to the β dependence of the lattice spacing a. Since $M(\beta(a))/a$ is the physical continuum mass m as $a\to 0$, it must be independent of a. Hence

$$0=\frac{dM(\beta(a))/a}{da}=-\frac{M}{a^2}+\frac{1}{a}\frac{dM}{d\beta}\frac{d\beta}{da}.\quad(2)$$

Thus

$$M=\frac{d\beta}{d\ \ln a}\sum\Box_{1-0}.\quad(3)$$

Note that $d\beta/d\ln a=-11N^2/(12\pi^2)$ to lowest order in perturbation theory for an SU(N) gauge theory. Thus the plaquette action is *lowered* in the glueball surroundings compared to the vacuum.

This is one of the prototype lattice action sum rules. It relates the plaquette action around a glueball to the mass of the glueball. It is exact provided that a nonperturbative determination of the lattice β function is used.

Further relations can be derived by splitting the lattice Wilson action into several terms with different coefficients. This analysis of asymmetric lattices was incorrect in I. In order to establish clearly the correct expressions, here we use a more direct method of derivation which also has the advantage of being more general.

Consider the general case where there are different coefficients for all six orientations of plaquette:

$$\beta \sum_{i,j,i<j} \Box_{ij} \rightarrow \sum_{i,j,i<j} \beta_{ij} \Box_{ij}. \tag{4}$$

There will be four lattice spacings a_i in general. We shall need to evaluate the derivatives $\partial\beta_{ij}/\partial a_k$. At the symmetry point where $a_i = a$ for all $i = 1,4$, these derivatives fall into two classes:

$$\frac{\partial\beta_{ij}}{\partial \ln a_k} = S \quad \text{if } k = i \text{ or } j \quad \text{and}$$

$$\frac{\partial\beta_{ij}}{\partial \ln a_k} = U \quad \text{if } k \neq i \text{ or } j. \tag{5}$$

The generalization of the identities derived in I are also needed:

$$\frac{\partial M}{\partial \beta_{ij}} = \sum (\Box_{ij})_{1-0}, \tag{6}$$

where the sum is again over one time slice. Then the renormalization group invariance of the result obtained on such a lattice implies that

$$\frac{\partial}{\partial a_i} \frac{M(\beta_{jk}(a_0,a_1,a_2,a_3), \ldots)}{a_0} = 0, \tag{7}$$

where a_0 enters because the glueball correlation is conventionally determined in the time direction. Because only this time direction is privileged in this case, at the symmmetry point, we have that

$$\Box_{0j} = \Box_t \quad \text{and } \Box_{jk} = \Box_s \quad \text{for } j,k \neq 0, \tag{8}$$

where the subscript $1-0$ is implied hereon.

Applying the renormalization group invariance conditions of Eq. (7) for $i=0$ and for $i\neq 0$ gives

$$M = \sum (3S\Box_t + 3U\Box_s), \tag{9}$$

$$0 = \sum [(2U+S)\Box_t + (U+2S)\Box_s]. \tag{10}$$

Then combining Eqs. (9) and (10) yields

$$M = \sum 2(S+U)(3\Box_t + 3\Box_s), \tag{11}$$

which is the the same as Eq. (3) provided we have the consistency condition

$$2(S+U) = \frac{d\beta}{d \ln a}. \tag{12}$$

Subtracting Eqs. (9) and (10) then gives

$$M = \sum \frac{2}{3}(S-U)(3\Box_t - 3\Box_s). \tag{13}$$

This latter equation is appropriate to the energy in the color field around a glueball. In order to make it more useful, we need to estimate the combination of derivatives $S-U$.

Consider the special case, as used by Karsch [6], where $a_t = a_0$; $a_1 = a_2 = a_3 = a_s$ and $\beta_{0i} = \beta_t; \beta_{ij} = \beta_s$ where $i,j > 0$. The derivatives in this case can be related to S and U, at the symmetry point:

$$\frac{\partial\beta_t}{\partial \ln a_t} = S \quad \text{and} \quad \frac{\partial\beta_t}{\partial \ln a_s} = S + 2U, \tag{14}$$

$$\frac{\partial\beta_s}{\partial \ln a_t} = U \quad \text{and} \quad \frac{\partial\beta_s}{\partial \ln a_s} = 2S + U. \tag{15}$$

The dependence of β_s and β_t on a_t and a_s coming from the weak coupling limit of the theory [6] is that, where $\xi = a_s/a_t$,

$$\beta_t = \xi[\beta(a_s) + 2Nc_s(\xi) + \cdots],$$

$$\beta_s = \xi^{-1}[\beta(a_s) + 2Nc_t(\xi) + \cdots], \tag{16}$$

where at $\xi=1$, $c_s = c_t = 0$ and Karsch obtains $c_s' = 0.114$ for $N=2$ and $c_s' = 0.2016$ for $N=3$. Then substituting Eqs. (16) into Eqs. (14) and (15) gives the constraint

$$4N(c_t' + c_s') = -\frac{d\beta}{d \ln a}. \tag{17}$$

This is the same constraint as found by Karsch from similar consistency arguments.

Using these expressions gives

$$S = -\beta + 2Nc_s' + \frac{1}{2}\frac{d\beta}{d \ln a}, \quad U = \beta - 2Nc_s' \tag{18}$$

and thus

$$S - U = -2\beta + 4Nc_s' + \frac{1}{2}\frac{d\beta}{d \ln a}. \tag{19}$$

This implies that, as $\beta\to\infty$, the energy sum rule [Eq. (13)] becomes

$$M = \sum \frac{4}{3}\beta(3\Box_s - 3\Box_t). \tag{20}$$

The expression of Eq. (20) in I had a factor of 1 instead of 4/3, coming from an error in the evaluation of the weak coupling result for the dependence of the asymmetric β's on the a's. Note that the naive continuum expression for the energy in the color field would be obtained with a factor of 1.

In principle $S-U$ can be determined nonperturbatively by simulating a lattice with nonequal β's and determining the ratio of the lattice spacings in the four directions from the glueball correlations in those directions. Accurate data do not exist at present, although an indirect method has been used in SU(2) and substantial corrections are found [7] to the weak coupling results. This is not surprising since lattice perturbation theory in the bare coupling is now known to be poorly convergent. This nonperturbative evaluation [7] gives values of $(U-S)/(2\beta)$ of 0.66 at $\beta=2.4$ and 0.77 at $\beta=2.8$ compared to the weak coupling values of 0.85 and 0.87, respectively. It is amusing that these nonperturbative estimates are close to 0.75 which would give the naive energy relation

$$M = \sum \beta(3\Box_s - 3\Box_t). \tag{21}$$

The gluonic vacuum in QCD is known to be polarizable. It behaves like a medium and can be assigned an effective dielectric constant. Thus it is not really surprising that the naive sum of the energy in the color fields [i.e., $\Sigma\beta(3\Box_s - 3\Box_t)$] does not agree exactly with the mass. Indeed the result will depend in QCD on the scale at which the field energy is evaluated. A sensible scale would be commensurate with the glueball mass, where a nonperturbative determination of $S-U$ is needed and rough agreement is obtained between the apparent field energy and the mass. The weak coupling calculation (which shows that only 3/4 of the mass lies as apparent energy in the color fields) implies a very short distance scale of energy determination, which will probe the vacuum polarization in a different manner.

It is worth emphasizing the basic result, which was obtained in I already, that the electric and magnetic field strengths are comparable. In detail, the departure from equality is correctly given by

$$\frac{\mathscr{E}}{\mathscr{B}}=\frac{\square_t}{\square_s}=\frac{-U-2S}{2U+S}\approx 1-\frac{3}{2\beta}\frac{d\beta}{d\ \ln a}\approx 1+\frac{33N}{12\pi}\frac{g^2}{4\pi}\approx 1, \tag{22}$$

where the approximation used in estimating S and U is valid at large β.

III. POTENTIALS AND TORELONS

Having calibrated the approach on the glueball, we consider string states. The potential between static quarks is the case of greatest practical interest. Another related situation is with a closed loop of color flux encircling the periodic boundary condtions: the torelon. As discussed in I, there is some subtlety in principle in dealing with the self-energy of the static quarks. For clarity of presentation, the torelon case is considered first since the derivation is more compact.

The torelon is a closed string of color flux in the fundamental representation that encircles the periodic boundary conditions in the x direction where there are R lattice spacings in this direction. Its energy is measured on a lattice by analyzing correlations of closed Polyakov line operators at $t=0$ and $t=T$. The study of the large T behavior then gives the lattice observable $E(R,\beta)$.

The analysis of I gives, where R is kept constant,

$$\left.\frac{\partial E(R)}{\partial\beta}\right|_R=\sum\ \square_{1-0}, \tag{23}$$

where 1 now refers to the plaquette expectation value between torelon states and the sum is again over all plaquettes in a time slice. The renormalization group analysis now needs to take account of the fact that $r=Ra$ must be kept constant in taking the limit $a\to 0$. So

$$0=\left.\frac{dE(R,\beta(a))/a}{da}\right|_r=-\frac{E}{a^2}-\frac{R}{a^2}\frac{\partial E}{\partial R}+\frac{1}{a}\frac{d\beta}{da}\left.\frac{\partial E}{\partial\beta}\right|_R. \tag{24}$$

Thus,

$$E(R)+\frac{\partial E}{\partial\ \ln R}=\frac{d\beta}{d\ \ln a}\sum\ \square_{1-0}. \tag{25}$$

As pointed out by Dosch *et al.* [4], this expression differs from that in I where the term with the derivative with respect to R was omitted. The net effect of that term is, for a confining potential, to increase the effective left-hand side of the action sum rule by a factor of 2.

We now apply the general consideration of 6 couplings β_{ij} as above. The new feature is that the torelon correlator is extended in the x and t directions. Thus we need to distinguish the x (longitudinal L) and y,z (transverse P) spatial directions. The four independent types of plaquette have orientations tL, tP, LP, and PP: we label them as $\mathscr{E}_L$, $\mathscr{E}_P$, $\mathscr{B}_P$, $\mathscr{B}_L$, respectively, in a natural notation. Note that $\mathscr{E}$ here is related to the difference of the plaquette value in the torelon state and in the vacuum and so is the difference of gauge invariant combinations of electric color fields squared. Following the same steps as above we obtain three independent constraints from the invariance with respect to a_0, a_L, and a_P of $(1/a_0)E(Ra_L,\beta_{ij}(a_k))$:

$$E=\sum\ (S\mathscr{E}_L+2S\mathscr{E}_P+2U\mathscr{B}_P+U\mathscr{B}_L), \tag{26}$$

$$R\frac{\partial E}{\partial R}=\sum\ (S\mathscr{E}_L+2U\mathscr{E}_P+2S\mathscr{B}_P+U\mathscr{B}_L), \tag{27}$$

$$0=\sum\ [U\mathscr{E}_L+(S+U)\mathscr{E}_P+(S+U)\mathscr{B}_P+S\mathscr{B}_L], \tag{28}$$

where the sum is over one time slice.

Combining these equations we obtain

$$E+R\frac{\partial E}{\partial R}=\sum\ 2(S+U)(\mathscr{E}_L+2\mathscr{E}_P+2\mathscr{B}_P+\mathscr{B}_L), \tag{29}$$

$$E+R\frac{\partial E}{\partial R}=\sum\ 2(S-U)(\mathscr{E}_L-\mathscr{B}_L), \tag{30}$$

$$E-R\frac{\partial E}{\partial R}=\sum\ 2(S-U)(\mathscr{E}_P-\mathscr{B}_P). \tag{31}$$

It is also convenient to write down the combination corresponding naively to the total energy in the fields:

$$E-\frac{1}{3}\frac{\partial E}{\partial\ \ln R}=\sum\ \frac{2}{3}(S-U)(\mathscr{E}_L+2\mathscr{E}_P-2\mathscr{B}_P-\mathscr{B}_L). \tag{32}$$

Again the action sum rule [Eq. (29)] agrees with the result obtained from a symmetric lattice [Eq. (25)] with the same relationship of $S+U$ to the β function [Eq. (12)] as for the glueball case. Thus, apart from the term with a derivative with respect to R, the results for the total action [Eq. (29)] and total energy [Eq. (32)] are similar in normalization to the glueball case introduced above.

Consider, for orientation, the case where the torelon energy is a sum of a string tension piece and a string fluctuation piece:

$$E(R)=KR-f/R, \tag{33}$$

where we expect $f=\pi/3$ (this behavior of the torelon energy has been checked numerically recently [8]). Then the sum rules become

$$2KR=\sum\ 2(S+U)(\mathscr{E}_L+2\mathscr{E}_P+2\mathscr{B}_P+\mathscr{B}_L), \tag{34}$$

$$2KR=\sum\ 2(S-U)(\mathscr{E}_L-\mathscr{B}_L), \tag{35}$$

$$-2f/R=\sum\ 2(S-U)(\mathscr{E}_P-\mathscr{B}_P). \tag{36}$$

This shows that the transverse energy in the fields (where here we define energy as $\mathscr{E}-\mathscr{B}$) will be much smaller for large R than the longitudinal energy. Moreover it has the opposite sign. These sum rules provide an independent way to study the split of the total torelon energy into string tension and string fluctuation components.

Consider the sum rule for the longitudinal energy with the weak coupling ($\beta\to\infty$) value for $S-U$:

$$\tfrac{1}{2}KR=\sum \beta(\mathscr{B}_L-\mathscr{E}_L). \tag{37}$$

The right-hand side is just the naive expression for the energy in the longitudinal fields. Thus we obtain one-half of the expected semiclassical result of KR. This is somewhat surprising since the longitudinal color flux is applied explicitly and in the semiclassical limit the energy should remain relatively unaffected by quantum corrections. However, the vacuum polarization effects will be strong at the large energy scale (corresponding to $\beta\to\infty$) used to evaluate the expression.

The application to the potential between static sources follows the same steps as for the torelon. The difference is that R is now the spatial extent of a Wilson loop rather than the spatial extent of the lattice itself. The main new feature is that there will also be a self-enegry contribution in the lattice observable energy $E(R)$. This self-energy was discussed in I. As $a\to 0$ it becomes the dominant term in the energy but it is very localized spatially. Thus it is possible to separate it out, leaving just the same results as for the torelon discussion above. One way to remove the self-energy contribution, in practice, is by taking the difference of expressions for two values of R when it cancels.

The analogue of Eq. (33) for the potential energy $E(R)$ between static sources at separation R is a sum of self-energy, Coulombic and string tension terms:

$$E(R)=V_0-e/R+KR. \tag{38}$$

After removing the self-energy part (V_0), the sum rules then become the same as Eqs. (34)–(36) with f changed to e. Thus the same result applies that the transverse energy in the color fields will be much smaller for large R than the longitudinal energy (where here we define energy as $\mathscr{E}-\mathscr{B}$). This is a new result.

IV. CONCLUSIONS

We have studied the energy and action distribution in the color fields around nonperturbative states. We use the semiclassical definition of these distributions and define the appropriate difference of the plaquette combination evaluated in the nonperturbative state and in the vacuum. The lattice definition we use has energy and action determined by the plaquette. In principle it should be possible to define a quantity which characterizes the energy and action in the color fields of a state and which has a continuum limit. Such a defintion could be based, for example, on using a square Wilson loop of *fixed* physical size as $a\to 0$. This would probe the energy and action distributions at a *fixed* physical scale and so would give a result free of lattice artefacts. But, of course, such a definition would not satisfy the sum rules we have derived.

The simplest result, which was obtained in I and which has been checked in numerical studies, is that the action in the color fields is much larger than the energy. This follows because the derivatives of β_{ij} with respect to a_k on an asymmetric lattice can be expressed in terms of two independent quantities S and U which can be estimated. The correct expression for the ratio, derived here, is

$$\frac{\text{Energy}}{\text{Action}}=\frac{3(S+U)}{S-U}\approx\frac{-3}{4\beta}\,\frac{d\beta}{d\,\ln\,a}\approx\frac{g^2}{4\pi}\,\frac{33N}{24\pi}\ll 1$$

for a glueball state (where the approximation used in estimating S and U is valid at large β). For the interquark potential or for a torelon this ratio is approximately 3 times smaller still.

The naive expectation is that the spatial sum of the energy density in the color field around a state should equal the energy of the state itself. We correct the results given in I, and find that, evaluated by the semiclassical expression, the sum of the energy density in the color fields around a glueball (and torelon) is given by 3/4 (and 1/2, respectively) of the energy of the state times $2\beta/(U-S)$. Thus no nonperturbative value for $S-U$ can make both of these sums exactly equal to the energy. Evaluating the energy density sum at a large energy scale, we can use perturbation theory to obtain a field energy around a glueball (and torelon) which is 3/4 (and 1/2, respectively) of the energy of the state. These fractions are closer to one when a nonperturbative estimate at a lower energy scale is used to determine the field sums. The explanation for the departure of these relations from identity is most easily achieved by invoking the vacuum polarization effects as producing an effective dielectric constant. Moreover this effective dielectric constant must be different in the glueball (spherical) and torelon (cylindrical) geometries.

We also present sum rules for the longitudinal and transverse field energy in a string state: torelon or interquark potential. For the potential between static quarks, this implies that at large R the transverse energy in the color fields will be much smaller than the longitudinal energy. It will be interesting to explore this in lattice studies.

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