S-matrix approach to two-pion production in e^+e^- annihilation and τ decay

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Based on the *S*-matrix approach, we introduce a modified formula for the $\pi^\pm$ electromagnetic form factor which describes very well the experimental data in the energy region $2m_\pi \leqslant \sqrt{s} \leqslant 1.1$ GeV. Using the CVC hypothesis we predict $B(\tau^- \to \pi^- \pi^0 \nu_\tau) = (24.75 \pm 0.38)\%$, in excellent agreement with recent experiments.

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I. INTRODUCTION

The processes $e^+e^- \to \pi^+\pi^-$ and $\tau^- \to \pi^-\pi^0\nu_\tau$ provide a clean environment for a consistency check of the conserved vector current (CVC) hypothesis [1]. Actually, the measurement of the $\pi^\pm$ electromagnetic form factor in e^+e^- annihilation is used to predict [2] the dominant hadronic decay of the τ lepton, namely, $\tau^- \to \pi^-\pi^0\nu_\tau$. The weak pion form factor involved in τ decay is obtained by removing the (model-dependent) $I=0$ contribution (arising from isospin violation and included via ρ-ω mixing) from the measured pion electromagnetic form factor.

In a previous paper [3] we applied the *S*-matrix approach to the $e^+e^- \to \pi^+\pi^-$ data of Ref. [4] and determined the pole parameters of the ρ^0 resonance. In particular, we fitted the data of Ref. [4] by assuming a constant value for the strength of the ρ-ω mixing parameter and using different parametrizations to account for the nonresonant background. As a result, the pole position of the scattering amplitude was found to be insensitive to the specific background chosen to fit the experimental data [3].

The purpose of this Brief Report is twofold. We first argue that the pole position in $e^+e^- \to \pi^+\pi^-$ is not modified by taking the ρ-ω mixing parameter as a function of the center-of-mass energy, as already suggested in recent papers [5]. Then we propose a new parametrization for the scattering amplitude of $e^+e^- \to \pi^+\pi^-$, based on the *S*-matrix approach, which looks very similar to the Breit-Wigner parametrization with an energy-dependent width. This results in an improvement in the quality of the fits (with respect to Ref. [3]) while the pole position and ρ-ω mixing parameters remain unchanged (as it should be). Finally, we make use of CVC to predict the $\tau^- \to \pi^-\pi^0\nu_\tau$ branching ratio, which is found to be in excellent agreement with recent experimental measurements.

II. ENERGY-DEPENDENT ρ-ω MIXING

We start by giving a simple argument to show that the pole position will not be changed if we choose the ρ-ω mixing parameter to be $m^2_{\rho\omega}(s) \propto s$ [namely, $m^2_{\rho\text{-}\omega}(0)=0$], where $\sqrt{s}$ is the total center-of-mass energy in $e^+e^- \to \pi^+\pi^-$.

Let us consider Eq. (7) of Ref. [3] and replace[1] $y \to y's/s_\omega$, where $s_V = m_V^2 - i m_V \Gamma_V$. This yields the following expression for Eq. (7) of Ref. [3]:

$$F_\pi(s) = \frac{A}{s-s_\rho}\left(1+\frac{y's}{s_\omega}\frac{m_\omega^2}{s-s_\omega}\right)+B(s)$$
$$= \frac{A'}{s-s_\rho}\left(1+y''\frac{m_\omega^2}{s-s_\omega}\right)+B(s), \qquad (1)$$

where A and $B(s)$ denote the residue at the pole and nonresonant background terms, respectively. The second equality above follows from the approximations

$$A' \equiv A\left(1+y'\frac{m_\omega^2}{s_\omega}\right) \approx A(1+y'),$$

$$y'' \equiv \frac{y'}{1+y' m_\omega^2/s_\omega} \approx \frac{y'}{1+y'}$$

i.e., by neglecting small imaginary parts of order $y'\Gamma_\omega/m_\omega \approx 10^{-5}$ [3]. Thus, since introducing $m^2_{\rho\omega} \propto s$ is equivalent to a redefinition of the residue at the pole and of the ρ-ω mixing parameter, we conclude that the pole position would not be changed if we take a constant or an energy-dependent ρ-ω mixing parameter.

III. ELECTROMAGNETIC PION FORM FACTOR

Next, we consider a new parametrization for the pion electromagnetic form factor. This parametrization is obtained by modifying the pole term in the following way:

$$s - m_\rho^2 + i m_\rho \Gamma_\rho \theta(\tilde{s}) \to D(s)$$
$$\equiv [1 - i x(s)\theta(\tilde{s})][s - m_\rho^2 + i m_\rho \Gamma_\rho \theta(\tilde{s})], \qquad (2)$$

[1] In the vector meson dominance model, y is related to the usual ρ-ω mixing strength through $y = m^2_{\rho\omega} f_\rho/(m_\rho^2 f_\omega) \simeq -2\times 10^{-3}$ [3].

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where $\theta(\tilde{s})$ is the step function, with argument $\tilde{s}=s-4m_\pi^2$.

Observe that if we choose

$$x(s)=-m_\rho\left(\frac{\Gamma_\rho(s)-\Gamma_\rho}{s-m_\rho^2}\right), \tag{3}$$

then Eq. (2) becomes

$$D(s)=s-m_\rho^2+m_\rho\Gamma_\rho x(s)\,\theta(s-4m_\pi^2)+im_\rho\Gamma_\rho(s) \tag{4}$$

which, when inserted in (1), looks very similar to a Breit-Wigner form with an energy-dependent width, which we will choose to be

$$\Gamma_\rho(s)=\Gamma_\rho\left(\frac{s-4m_\pi^2}{m_\rho^2-4m_\pi^2}\right)^{3/2}\frac{m_\rho}{\sqrt{s}}\,\theta(s-4m_\pi^2) \tag{5}$$

with the obvious identification $\Gamma_\rho=\Gamma(m_\rho^2)$.

Using Eq. (2) we are led to modified expressions for Eqs. (8), (9), and (15) of Ref. [3]: namely,

$$F_\pi^{(1)}(s)=\left(-\frac{am_\rho^2}{D(s)}+b\right)\left(1+\frac{ym_\omega^2}{s-s_\omega}\right), \tag{6}$$

$$F_\pi^{(2)}(s)=-\frac{am_\rho^2}{D(s)}\left(1+\frac{ym_\omega^2}{s-s_\omega}\right)+b, \tag{7}$$

$$F_\pi^{(4)}(s)=-\frac{am_\rho^2}{D(s)}\left(1+\frac{ym_\omega^2}{s-s_\omega}\right)\left[1+b\left(\frac{s-m_\rho^2}{m_\rho^2}\right)\right]^{-1}. \tag{8}$$

Using Eqs. (6)–(8), we have repeated the fits to the experimental data of Barkov *et al.* [4] in the energy region $2m_\pi\leqslant\sqrt{s}\leqslant 1.1$ GeV. As in Ref. [3], the free parameters of the fit are m_ρ, Γ_ρ, a, b, and y. The results of the best fits are shown in Table I.

From a straightforward comparison of Table I and the corresponding results in Ref. [3] [see particularly, Eqs. (10), (11), (16) and Table I of that reference], we observe that the quality of the fits is very similar. Furthermore, the pole position, namely the numerical values of m_ρ and Γ_ρ, and of the ρ-ω mixing parameter y, is rather insensitive to the new parametrizations (as it should be). The major effect of the new parametrizations is observed in the numerical values of a (the residue at the pole) and b (which describes the background).

TABLE I. Best fits to the pion electromagnetic form factor of Ref. [4], using Eqs. (6)–(8).

	m_ρ (MeV)	Γ_ρ (MeV)	a	b	$y(10^{-3})$	χ/d.o.f.
$F_\pi^{(1)}$	756.74± 0.82	143.78± 1.16	1.236± 0.008	−0.239± 0.013	−1.91± 0.15	0.998
$F_\pi^{(2)}$	756.58± 0.82	144.05± 1.17	1.237± 0.008	−0.240± 0.013	−1.91± 0.15	1.008
$F_\pi^{(4)}$	757.03± 0.76	141.15± 1.18	1.206± 0.008	−0.193± 0.009	−1.86± 0.15	0.899

An interesting consequence of the results in Table I is an improvement in the value of $F_\pi(0)$, which should be equal to 1 (the charge of π^+). Indeed, from Eqs. (6)–(8) and Table I we obtain

$$\begin{aligned}F_\pi^{(1)}(0)&=a+b\\&=0.997\pm0.015\ \ (0.962\pm0.020),\\F_\pi^{(2)}(0)&=a+b\\&=0.997\pm0.015\ \ (0.960\pm0.017),\\F_\pi^{(4)}(0)&=\frac{a}{1-b}\\&=1.011\pm0.010\ \ (0.987\pm0.013),\end{aligned} \tag{9}$$

where the corresponding values obtained in Ref. [3] are shown in brackets. An evident improvement is observed.

Let us close the discussion of this new parametrization with a short comment: using $F_\pi^{(4)}(s)$ (with imaginary parts and y set to zero) we are able to reproduce very well the data of Ref. [6] in the space-like region $-0.253\ \text{GeV}^2\leqslant s\leqslant -0.015\ \text{GeV}^2$.

IV. PREDICTION FOR $\tau^-\to\pi^-\pi^0\nu_\tau$

Finally, using the previous results of the pion electromagnetic form factor, we consider the decay rate for $\tau^-\to\pi^-\pi^0\nu_\tau$. As is well known [2], the CVC hypothesis allows us to predict the decay rate for $\tau^-\to(2n\pi)^-\nu_\tau$ in terms of the measured cross section in $e^+e^-\to(2n\pi)^0$. Since for the $\tau^-\to\pi^-\pi^0\nu_\tau$ case the kinematical range extends up to $\sqrt{s}=m_\tau$, let us point out that we have verified that our parametrizations for $F_\pi(s)$ reproduce very well the data of $e^+e^-\to\pi^+\pi^-$ in the energy region from 1.1 GeV to m_τ.

The decay rate for $\tau^-\to\pi^-\pi^0\nu_\tau$ at the lowest order is given by [2]

$$\begin{aligned}\Gamma^0(\tau^-\to\pi^-\pi^0\nu_\tau)=&\frac{G_F^2|V_{ud}|^2m_\tau^3}{384\pi^3}\int_{4m_\pi^2}^{m_\tau^2}ds\left(1+\frac{2s}{m_\tau^2}\right)\\&\times\left(1-\frac{s}{m_\tau^2}\right)^2\left(\frac{s-4m_\pi^2}{s}\right)^{3/2}\\&\times|F_\pi^{I=1}(s)|^2,\end{aligned} \tag{10}$$

where V_{ud} is the relevant Cabibbo-Kobayashi-Maskawa mixing angle. In the above expression we have neglected isospin breaking in the pion masses. The form factor $F_\pi^{I=1}(s)$ in Eq. (10) is obtained from Eqs. (6)–(8) by removing the $I=0$ contribution due to ρ-ω mixing (namely, $y=0$).

According to Ref. [7], after including the dominant short-distance electroweak radiative corrections the expression for the decay rate becomes

$$\Gamma(\tau^-\to\pi^-\pi^0\nu_\tau)=\left(1+\frac{2\alpha}{\pi}\ln\frac{M_Z}{m_\tau}\right)\Gamma^0(\tau^-\to\pi^-\pi^0\nu_\tau). \tag{11}$$

We have not included the effects of long-distance electromagnetic radiative corrections, but we expect that they would not exceed 2.0%.

In order to predict the branching ratio, we use Eqs. (6)–(8) with $y=0$, the results of Table I, and the following values of fundamental parameters (Refs. [7,8]):

$$m_\tau = 1777.1 \pm 0.5 \;\; \text{MeV},$$

$$G_F = 1.16639(2) \times 10^{-5} \;\; \text{GeV}^{-2},$$

$$|V_{ud}| = 0.9750 \pm 0.0007.$$

With the above inputs we obtain

$$B(\tau^- \to \pi^- \pi^0 \nu_\tau) = \left(\frac{\tau_\tau}{2.956 \times 10^{-13} \;\; \text{s}} \right) \times \begin{cases} (24.66 \pm 0.26)\% & \text{from Eq. (6)}, \\ (24.62 \pm 0.26)\% & \text{from Eq. (7)}, \\ (24.96 \pm 0.32)\% & \text{from Eq. (8)}, \end{cases} \quad (12)$$

or, the simple average,

$$B(\tau^- \to \pi^- \pi^0 \nu_\tau) = (24.75 \pm 0.38)\%, \quad (13)$$

which is in excellent agreement with recent experimental measurements and other theoretical calculations (see Table II). Equation (13) includes the errors (added in quadrature) coming from the fit to $e^+e^- \to \pi^+\pi^-$ and the 1% error in the τ lifetime [8]: $\tau_\tau = (295.6 \pm 3.1) \times 10^{-15}$ s.

TABLE II. Summary of recent experimental measurements (Expt.) and theoretical results (Th.) for the $\tau^- \to \pi^- \pi^0 \nu_\tau$ branching ratio. The errors in the first entry arise from use of $e^+e^- \to \pi^+\pi^-$ data, the τ lifetime, and radiative correction effects [9], respectively.

Reference	$B(\tau^- \to \pi^- \pi^0 \nu_\tau)$ (in %)
Th. [9]	$24.58 \pm 0.93 \pm 0.27 \pm 0.50$
Th. [10]	24.60 ± 1.40
Th./Expt. [11]	24.01 ± 0.47
Expt. [8]	25.20 ± 0.40
Expt. [12]	25.36 ± 0.44
Expt. [13]	25.78 ± 0.64

In summary, based on the *S*-matrix approach we have considered a modified parametrization for the $\pi^\pm$ electromagnetic form factor, which describes very well the experimental data of $e^+e^- \to \pi^+\pi^-$ in the energy region from threshold to 1.1 GeV. The pole position of the *S*-matrix amplitude is not changed by this new parametrization. Using CVC, we have predicted the $\tau^- \to \pi^- \pi^0 \nu_\tau$ branching ratio, which is found to be in excellent agreement with experiment.

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