Stabilization of the Skyrmion

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We examine the effects of treating the linear scale of the Skyrmion as a quantum variable. It is found that when angular momentum also is included, the soliton is stable and there is no need for the artificial quartic term traditionally used. These results involve a careful treatment of the constrained degrees of freedom and the quantization procedure and differ from those of earlier work using a less fundamental approach. Acceptable values are found for the masses and other static properties of the nucleon, Δ(1232), and Roper resonance.

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I. INTRODUCTION

A well known model of the baryons, based on the large N_c limit of QCD [1,2], is to treat baryons as Skyrmions [3], i.e., solitons of the pion field based on the Lagrangian

$$L=\frac{f_\pi^2}{4}\int \mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)d^3r, \tag{1.1}$$

where

$$U=\exp[i\tau\cdot\phi(\mathbf{r}/R,t)],$$

with ϕ the pion field and τ the isospin matrices. The soliton solutions for which $U\to 1$ at infinity fall into distinct topological classes. Those with the topological constant $B=1$ are identified as single baryons.

In classical mechanics, the static soliton solutions of this Lagrangian are unstable against radial collapse because of the Derrick instability [4], that is the energy collapses to zero as the length scale R decreases. Actually, Skyrme's Lagrangian contains an additional term quartic in U which gives stable solutions. However, there has been considerable interest in the simple quadratic Lagrangian (1.1) and it has been suggested that solitons may be stable if the scale $R(t)$ is treated as a quantum variable [5].

We have examined this question from first principles and this paper shows that these baryonic solitons are indeed stable provided that the collective coordinates describing rotations and vibrations are both quantized.

II. GENERAL FORMALISM

In the one-baryon sector, the minimum-energy, maximum-symmetry configuration of the classical soliton is the "hedgehog"

$$U_c=\exp[iF(\sigma)\tau\cdot\hat{\mathbf{r}}], \tag{2.1}$$

where $\sigma=\mathbf{r}/R$ and R is a scale constant.

To examine the properties of the baryon, one must determine the "profile function" F. In classical dynamics, one determines F from the Euler-Lagrange equation, a satisfactory procedure when the quartic term is present, but not otherwise [6].

Rotation is introduced by taking

$$U=A(t)U_cA^\dagger(t), \tag{2.2}$$

where A is a matrix introducing collective rotation as in the work of Adkins, Nappi, and Witten [7]. Spin and isospin are quantum variables and the appropriate spin states are projected to describe a baryon. Then one obtains an altered Euler-Lagrange equation which has certain unphysical characteristics [8].

However, Verschelde and co-workers have shown in a series of papers [9–12] that it is essential in deriving the profile equation determining F to maintain in the calculations the complete number of degrees of freedom in the basic Lagrangian (1.1). This is done by keeping the small nonhedgehog portion of the field and by using the appropriate quantal dynamical principle to form the profile equation. We extend Verschelde's work by including radial vibrations.

The basic Eq. (1.1) gives the Lagrangian density

$$\mathscr{L}=\frac{f_\pi^2}{2}[\dot{\phi}^i g_{ij}(\phi)\dot{\phi}^j-\partial_k\phi^i g_{ij}(\phi)\partial_k\phi^j], \tag{2.3}$$

with ϕ^i the ith-component of the isospin field. The metric of the isospin space is

$$g_{ij}(\phi)=S\delta_{ij}+(1-S)\frac{\phi^i\phi^j}{\phi^2}, \tag{2.4}$$

with $S=(\sin^2\phi/\phi^2)$ and $\phi^2=\phi^i\phi^i$.

The explicit dependence on vibrational and rotational coordinates appears in the expression for ϕ^i:

$$\phi^i(\mathbf{r},t)=U_{ij}(\alpha)[\phi_s^j(\sigma)+\eta^j(\sigma,t)], \tag{2.5a}$$

where

$$\phi_s^i(\sigma)=F(\sigma)\hat{\sigma}^i=\frac{F\sigma^i}{\sigma}, \tag{2.5b}$$

where U_{ij} is now the usual matrix function of Euler angles $\boldsymbol{\alpha}$. As the scale variable R undergoes quantal oscillations, the field vibrates radially.

It is seen that $U\phi_s$ is the radially vibrating and rotating hedgehog. The quantity η^i is a general isospin field (referred to body-fixed axes) with the same number of degrees of freedom as ϕ^i. In order to discuss the Skyrmion, we consider the limit as $\eta\to 0$, but it is important to do so at the final stage of the calculation after the profile equation has been found. There are now four redundant variables since α and R are additional degrees of freedom. Thus one must introduce constraints and use an appropriate quantization procedure. For this, we use the Dirac formalism [13].

In this picture, η is to represent a pion field other than the Skyrmion, hence it is natural to constrain η to be orthogonal to ϕ_s. In Eq. (2.5a), η is shown with time dependence more general than only that contained in the scale variable $R(t)$ in the denominator of σ; this implies that η may behave quite generally, neither having scale vibrations nor rotations synchronous with the hedgehog. For the purposes of this study, however, it is sufficient to assume the rigid gauge, so called because it requires η to adapt immediately to the collective motions [11]. This implies removing the explicit time dependence of η in Eq. (2.5a) leaving time only in σ.

Comparisons of ''soft'' and rigid gauges have been made [12,14]. Use of the rigid gauge implies an adiabatic assumption that more general ''nonrigid'' changes in η are of a much higher energy scale (shorter time scale) than that of the collective motion. Thus the rigid gauge is to be used for ground state properties, not for highly excited states or dynamical processes such as pion scattering or Δ decay [10,15].

Also, since we are describing odd parity states with total angular momentum 1/2 or 3/2, η contains spherical harmonics of order one only.

One obtains the momenta conjugate to ϕ^i, η^j, R, and α_a by noting that Eq. (2.5a) gives

$$\dot{\phi}^i = \frac{\partial U_{ij}}{\partial \alpha_a}\,\dot{\alpha}_a[\phi_s^j(\sigma)+\eta^j] + U_{ij}\left(-\frac{F'}{R}\,\sigma^j\dot{R}+\dot{\eta}^j\right), \tag{2.6}$$

where

$$F' = \frac{dF}{d\sigma},$$

and, since

$$\pi_\phi^i = \frac{\partial \mathscr{L}}{\partial \dot{\phi}^i},$$

$$\pi_\eta^i = \frac{\partial \mathscr{L}}{\partial \dot{\eta}^i} = \pi_\phi^j U_{ji} = U_{ij}^{-1}\pi_\phi^j, \tag{2.7}$$

$$P = \frac{\partial L}{\partial \dot{R}} = -\int d^3\mathbf{r}\;\pi_\phi^i U_{ij}\left[\frac{F'\sigma^j}{R}\right] = -\int d^3\mathbf{r}\;\pi_\eta^j\,\frac{F'}{R}\,\sigma^j, \tag{2.8}$$

$$P_a = \frac{\partial L}{\partial \dot{\alpha}_a} = \int d^3\mathbf{r}\;\pi_\eta^i U_{ij}^{-1}\,\frac{\partial U_{jk}}{\partial \alpha_a}\,(\phi_s^k+\eta^k). \tag{2.9}$$

Equations (2.8) and (2.9) are conditions connecting the momenta, reflecting the fact that there are four superfluous degrees of freedom. We also need the four conditions on the coordinates. As noted above, we wish η to be orthogonal to the rotational [9,11] and vibrational zero modes, that is

$$\Omega_R \equiv \int d^3\sigma\;\eta^i F'\sigma^i = 0, \tag{2.10a}$$

$$\Omega_a \equiv \int d^3\sigma\;\eta^i g_{ij}(\phi_s)\epsilon_{ajk}\phi_s^k = 0. \tag{2.10b}$$

In Eq. (2.10a), the metric g_{ij} does not appear because $g_{ij}(\phi_s)\sigma^j = \sigma^i$.

In dealing with the angular variables, it is more convenient to use the body-fixed angular momenta I_a rather than the P_a conjugate to Euler angles. Using explicit forms of U as the angular momentum D functions, Eq. (2.9) becomes

$$I_a = -\int d^3\mathbf{r}\;\pi_\eta^i\epsilon_{aij}(\phi_s^j+\eta^j). \tag{2.11}$$

It is also desirable to remove from π_η any portion that depends on the collective coordinates and arrange that the constraints (2.8) and (2.11) not contain P or I_a. This is achieved by writing

$$\pi_\eta^i = \pi_l^i + \pi_t^i, \tag{2.12}$$

insisting that

$$X_R \equiv \int d^3\sigma\;\pi_t^i F'\sigma^i = 0, \tag{2.13a}$$

$$X_a \equiv \int d^3\sigma\;\pi_t^i\epsilon_{aij}F\sigma^j = 0 \tag{2.13b}$$

and finding that

$$\pi_l^m = \frac{1}{\alpha}\,G^m P - \mathscr{I}_a t_a^m, \tag{2.14}$$

$$G^m = -R^{-2}F'\sigma^m + R\,\delta_a t_a^m, \tag{2.15}$$

$$\delta_a = \epsilon_{aij}\int d^3\sigma\;F'\sigma^i\eta^j, \tag{2.16}$$

$$t_a^m = R^{-3}\Lambda_{ab}^{-1}\epsilon_{bmj}\hat{\sigma}^j S_s F, \tag{2.17}$$

$$\Lambda_{ab} = \epsilon_{amk}\epsilon_{bml}\int d^3\sigma\;S_s\phi_s^l(\phi_s^k+\eta^k), \tag{2.18}$$

$$\mathscr{I}_a = I_a + \tilde{I}_a, \tag{2.19}$$

$$\tilde{I}_a = R^3\epsilon_{aij}\int d^3\sigma\;\pi_t^i\eta^j, \tag{2.20}$$

$$\alpha = \int d^3\sigma\;\sigma^2 F'^2. \tag{2.21}$$

Our coordinate system is now $\eta^i(\sigma), R, \alpha_a, \pi_t^i(\sigma), P, I_a$ with the constraint functions $\Omega_R, \Omega_a, X_R, X_a$.

In the Dirac method for constrained systems [13] with constraint functions C_i, one forms the matrix of the Poisson brackets of the constraints

$$\Delta_{ij} = [C_i, C_j]. \tag{2.22}$$

For any two functions A and B, the Dirac brackets are

$$[A,B]_D = [A,B] - [A,C_i]\Delta_{ij}^{-1}[C_j,B]. \tag{2.23}$$

To quantize, the commutator of two operators is put equal to i times the Dirac brackets, not the Poisson brackets.

Here, the constraints are the eight-vector $(X_R,\Omega_R,X_a,\Omega_a)$. One sees, for example, that

$$[\Omega_R,X_R]=\int d^3\sigma\,\frac{\delta X_R}{\delta\pi_t^i}\,\frac{\delta\Omega_R}{\delta_\eta^i}=\alpha,$$

and in this way one finds that

$$\Delta^{-1}=\begin{bmatrix} 0 & -1/\alpha & 0 & 0 & 0 & 0 & 0 & 0\\ 1/\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/\Lambda & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\Lambda & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\Lambda\\ 0 & 0 & -1/\Lambda & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1/\Lambda & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1/\Lambda & 0 & 0 & 0 \end{bmatrix}, \tag{2.24}$$

where

$$\Lambda=\frac{2}{3}\int d^3\sigma\ S_sF^2=\frac{8\pi}{3}\int d\sigma\ \sigma^2\sin^2F. \tag{2.25}$$

The Dirac brackets

$$[\eta^i(\sigma),\pi_t^j(\sigma')]=R^{-3}\bigg(\delta_{ij}\delta^3(\sigma-\sigma') -\frac{1}{\alpha}F'(\sigma)\sigma^iF'(\sigma')\sigma'^j -\frac{1}{\Lambda}\epsilon_{mik}\phi_s^k(\sigma)\epsilon_{mjl}\phi_s^l(\sigma')S_s(\sigma')\bigg). \tag{2.26}$$

The Dirac brackets of η^j and π_t^i with the collective coordinates and momenta are zero and the Dirac brackets of the collective coordinates and momenta are the same as the Poisson brackets.

That the new coordinate system is canonical may be verified by showing that with Eqs. (2.5) and (2.6), the Dirac brackets and the constraint equations, one finds

$$[\phi^i(\mathbf{r}),\pi_\phi^i(\mathbf{r}')]=\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}'). \tag{2.27}$$

III. THE HAMILTONIAN

The Lagrangian (2.3) gives the classical Hamiltonian

$$H=\frac{1}{2f_\pi^2}\int d^3\mathbf{r}\ \pi_\phi^i g_{ij}^{-1}(\phi)\pi_\phi^j +\frac{f_\pi^2}{2}\int d^3\mathbf{r}\ \partial_k\phi^i g_{ij}(\phi)\partial_k\phi^j. \tag{3.1}$$

To obtain the corresponding quantum Hamiltonian in the new coordinates, one symmetrizes the expressions involving noncommuting factors to ensure Hermiticity, recalling that commutators are given by Dirac brackets. The noncommuting pairs in H are I_a and $U(\alpha)$, P and R, and π_t^i and η^j.

Many earlier papers have not proceeded in this way but have first obtained the zero-order collective Hamiltonian not involving η and then quantized, ordering operators only at this "collective theory" level. But it is crucial to retain η until one has the profile equation and this requires that both H and π_ϕ be Hermitian at the full "field theory" level. In a detailed account of this work [16], we show that the following symmetric ordering is the only way to achieve this dual Hermiticity.

Equations (2.7) and (2.14) become

$$\pi_\phi^i(\sigma)=\frac{1}{2}\{U_{im},\pi_l^m\}+U_{im}\pi_t^m, \tag{3.2}$$

$$\pi_l^m=\frac{1}{2\alpha}\{P,G^m\}-\frac{1}{2}\{t_a^m,\mathscr{I}_a\}, \tag{3.3}$$

where $\{\}$ denotes an anticommutator. Using

$$[U_{ij},I_a]=i\epsilon_{ajk}U_{ik} \tag{3.4}$$

and taking $\pi_\eta=U^{-1}\pi_\phi$, we find

$$\pi_\eta^m=\pi_t^m+\pi_l^m+\frac{i}{2}\epsilon_{a\nu m}t_a^\nu=\pi_t^m+\tilde{\pi}_l^m. \tag{3.5}$$

Notice the additional quantum term in π_η compared with Eq. (2.12). Using these expressions and the Dirac commutators based on (2.26) to expand to first order in η and π_t, one finds, after lengthy algebra, that

$$H=H_{\rm coll}+H_{\rm intr}, \tag{3.6}$$

where $H_{\rm coll}$ depends only on the collective coordinates and $H_{\rm intr}$ is of first order in η or π_t. There are, in fact, no terms of first order in π_t and

$$f_\pi^2 H_{\rm coll} = f_\pi^4 b R + \frac{1}{2\Lambda} R^{-3}\left(\mathbf{I}^2 - \frac{3}{4}\right) + \frac{1}{2\alpha}\left[P^2 R^{-1} - iPR^{-2} + \frac{3}{2} R^{-3}\left(\lambda_1^2 - \lambda_1 - \lambda_2 + \frac{1}{6}\xi - \frac{4}{3}\right)\right], \tag{3.7}$$

$$\begin{aligned} f_\pi^2 H_{\rm intr} = & -f_\pi^4 R \int d^3\sigma\ \eta^i \hat{\sigma}^i \left(F'' + 2\frac{F'}{\sigma} - \frac{\sin 2F}{\sigma^2}\right) + \frac{1}{2\alpha\Lambda} I_a \epsilon_{ank}\left\{P, \frac{1}{R^2}\int d^3\sigma\ F' S_s \eta^n \hat{\sigma}^k\right\} \\ & - \frac{1}{5\Lambda^2 R^3} I_a I_b \left[\delta_{ab}\int d^3\sigma\left(\mathscr{F} + 2\frac{d}{dF}(\mathscr{F}F)\right)\eta^i\hat{\sigma}^i - \int d^3\sigma\left(3\mathscr{F} + \frac{1}{2}\frac{d}{dF}(F\mathscr{F})(\eta^a\hat{\sigma}^b + \eta^b\hat{\sigma}^a)\right)\right] \\ & + \frac{1}{\Lambda^2 R^3}\int d^3\sigma\ \eta^i\hat{\sigma}^i\left[\frac{1}{4}\frac{d}{dF}(F\mathscr{F}) + \mathscr{F}S_s + \mathscr{F}\left(-\Omega_1 + \frac{\Lambda}{\alpha}\left(\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 - \lambda_1^2\right)\right)\right], \end{aligned} \tag{3.8}$$

where

$$S_s = (\sin F/F)^2, \quad \mathscr{F} = F S_s$$

and

$$b = \int d^3\sigma\left[\frac{1}{2}(F')^2 + \mathscr{F}F/\sigma^2\right], \tag{3.9a}$$

$$\tilde{\Lambda} = \int d^3\sigma\ \mathscr{F}F = \frac{3}{2}\Lambda, \tag{3.9b}$$

$$\lambda_1\tilde{\Lambda} = \int d^3\sigma\ \mathscr{F}F'\sigma, \tag{3.9c}$$

$$\lambda_2\tilde{\Lambda} = \int d^3\sigma\ \mathscr{F}F''\sigma^2, \tag{3.9d}$$

$$\Omega_1\tilde{\Lambda} = \int d^3\sigma\ \mathscr{F}^2, \tag{3.9e}$$

$$\Omega_2\tilde{\Lambda} = \int d^3\sigma\ \mathscr{F}^2(F'\sigma/F), \tag{3.9f}$$

$$\alpha = \int d^3\sigma (F'\sigma)^2, \tag{3.9g}$$

$$\xi\alpha = \int d^3\sigma (F''\sigma^2)^2. \tag{3.9h}$$

In expanding H to first order in η, one encounters an integral of the form $\int d^3\sigma\ f(\sigma)\eta^i\hat{\sigma}^i\hat{\sigma}^a\hat{\sigma}^b$ and, since η^i has only first order spherical harmonics, $\hat{\sigma}^i\hat{\sigma}^a\hat{\sigma}^b$ is replaced by its $l=1$ part, viz. $(1/5)(\delta_{ia}\hat{\sigma}^b + \delta_{ib}\hat{\sigma}^a + \delta_{ab}\hat{\sigma}^i)$.

The second and third terms in $H_{\rm coll}$ due to rotation and vibration involve a moment of inertia Λ and a vibrational inertial parameter α. The presence of λ_1 and λ_2 requires both rotation and vibration. This rotation-vibration coupling involves the third term in π^i_η given by Eq. (3.5) which in turn comes from symmetrizing the rotational part of π^i_ϕ. We find that these λ_i terms provide a positive binding contribution to the potential term in $H_{\rm coll}$ which plays an important role in the stability of the soliton.

The rotational term in $H_{\rm coll}$ has the factor $\mathbf{I}^2 - 3/4$. Unlike the rigid rotor Hamiltonian of Adkins, Nappi, and Witten [7], there is the renormalizing term $-3/4$. This purely quantum effect from symmetrization was noted in a paper [11] by Verschelde and Verbeke which also displays, for rotation only, an explicit form of H consistent with the appropriate terms of Eqs. (3.7) and (3.8).

It is helpful to cast $H_{\rm coll}$ in a simpler form by using the dimensionless variable

$$q = (f_\pi b R)^{3/2} \tag{3.10}$$

for which the momentum relationship is

$$P = \frac{3}{4} f_\pi b\{q^{1/3}, P_q\}. \tag{3.11}$$

This gives the Schrödinger equation

$$\frac{f_\pi}{2m}\left[-\frac{d^2}{dq^2} + \frac{\nu}{q^2} + 2mq^{2/3}\right]\psi = E\psi, \tag{3.12}$$

where

$$m = \frac{4\alpha}{9b^3}, \tag{3.13}$$

$$\nu = \frac{20}{9} u \frac{\alpha}{\Lambda} + \frac{2}{3}(\lambda_1^2 - \lambda_1 - \lambda_2) + \frac{1}{9}\xi - \frac{25}{36}, \tag{3.14}$$

with

$$u = \frac{1}{5}\left[I(I+1) - \frac{3}{4}\right]. \tag{3.15}$$

The state of a Skyrmion is specified by the spin and i spin $|I=J, I_3, J_3\rangle$ and by a vibrational eigenstate of Eq. (3.12).

IV. THE PROFILE EQUATION AND STABILITY

As Verschelde [9] first emphasized, the equation of motion for the static field, viz., $\dot{\pi}_\phi = 0$, should be cast in the form

$$\langle [H, \pi^i_\phi] \rangle = 0, \tag{4.1}$$

where all degrees of freedom are retained and the expectation value is in a stationary state of the collective Hamiltonian. This equation leads to the condition for the profile function F. Thus the profile and the parameters of the Skyrmion such as the inertial coefficients vary from state to state.

The value of the commutator in (4.1) must be found by using the appropriate Dirac brackets such as in Eq. (2.26). Thus, using Eqs. (2.7) and (3.5), Eq. (4.1) is

$$\langle[H_{\rm coll}+H_{\rm intr},U_{im}(\tilde{\pi}_l^m+\pi_t^m)]\rangle=0. \tag{4.2}$$

Since the expectation value is in a state of $H_{\rm coll}$, the commutator with $H_{\rm coll}$ does not contribute to (4.2). There is further simplification because we require in (4.2) only terms that are independent of η and π_t. Since $H_{\rm intr}$ is of order η and η commutes with $U(\boldsymbol{\alpha})$, $[H_{\rm intr}, U_{im}]\ \pi_\eta$ can be ignored. Also, the only term in $\tilde{\pi}_l^m$ in Eq. (3.3) which does not commute with $H_{\rm intr}$ is $\{\tilde{I}_a, t_a^m\}$, but this term is of second order in η and π_t and therefore its commutator with $H_{\rm intr}$ remains proportional to η. Also, one sees that, for any f,

$$\langle\{P,f(R)\}\rangle=\frac{3}{2}f_\pi b\int dq\,\frac{d}{dq}\,(\psi^2 f q^{1/3})=0,$$

since $\psi=0$ at both limits. Consequently, the only part of (4.2) which contains zero order terms gives

$$\langle U_{im}[H_{\rm intr},\pi_t^m]\rangle=0. \tag{4.3}$$

Equation (2.25) and the properties of the matrix U allow us to write

$$\left[\int d^3\rho\ f(\rho)\,\eta^k\hat{\rho}^l,\pi_t^m(\sigma)\right]=\frac{i}{R^3}\left\{\delta_{km}f(\sigma)\hat{\sigma}^l-\frac{1}{3\alpha}\,\delta_{lk}F'\sigma\hat{\sigma}^m\int d^3\rho\ fF'\rho-\frac{1}{3\Lambda}\,(\delta_{mk}\hat{\sigma}^l-\delta_{ml}\hat{\sigma}^k)FS_s\int d^3\rho\ fF\right\}, \tag{4.4}$$

$$\left[\int d^3\rho\ f\eta^k\hat{\rho}^k,\pi_t^m(\sigma)\right]=i\,\frac{\hat{\sigma}^m}{R^3}\left(f-\frac{1}{\alpha}\,F'\sigma\int d^3\rho\ fF'\rho\right), \tag{4.5}$$

$$\langle U_{im}\hat{\sigma}^m\rangle=\delta_{i3}\hat{\sigma}^3\langle U_{33}\rangle=\Gamma_i, \tag{4.6}$$

$$\langle U_{im}\{I_m,I_a\}\hat{\sigma}^a\rangle=\Gamma[2I(I+1)-1]. \tag{4.7}$$

With these results, Eq. (4.3) gives the profile equation

$$\left\langle\frac{q^{-4}}{\Lambda^2}\,[S_s^2F+C_1S_sF-u\ \sin 2F+C_2F'\sigma]+\frac{q^{-4/3}}{b^6}\left[\frac{b^3}{\alpha}\,F'\sigma-b^2\left(F''+2\,\frac{F'}{\sigma}-\frac{\sin 2F}{\sigma^2}\right)\right]\right\rangle=0, \tag{4.8}$$

$$C_1=2u-\Omega_1-\left(\frac{\Lambda}{2\alpha}\right)(2\lambda_1^2-\lambda_1-\lambda_2), \tag{4.9a}$$

$$C_2=\left(\frac{3\Lambda^2}{4\alpha^2}\right)\lambda_1(2\lambda_1^2-\lambda_1-\lambda_2)+\left(\frac{3\Lambda}{2\alpha}\right)(\Omega_1\lambda_1-\Omega_2-5u). \tag{4.9b}$$

The first term of $H_{\rm coll}$ in Eq. (3.7) is the classical energy $f_\pi^2 bR$ which tends to zero as the scale variable R goes to zero. This is the Derrick instability.

If one includes also the rotational energy,

$$E=f_\pi\left(f_\pi bR+\frac{5}{2}\,\frac{u}{\Lambda}\,\frac{1}{f_\pi^3R^3}\right). \tag{4.10}$$

Thus for $I=J=1/2$, i.e., $u=0$, the energy is simply the classical result. This differs from the earlier papers and is due to the "renormalization" of the angular momentum term, noted by Verschelde [11] and due to the appropriate ordering of the quantum operators and the inclusion of the intrinsic degrees of freedom of the field. However, for the $\Delta(1232)$, $u=3/5$ and the minimum energy of the soliton occurs for $f_\pi R=(9/2b\Lambda)^{1/4}$ and is $(4/3)f_\pi(9b^3/2\Lambda)^{1/4}$. This energy is of a reasonable order of magnitude. Its precise value depends on the form of F. The equation for F is Eq. (4.8) but with q taking the single specific value $(9b^3/2\Lambda)^{3/8}$ and without terms due to vibration, i.e., as $\alpha\to\infty$,

$$(2u-\Omega_1)S_sF+S_s^2F-u\ \sin 2F-\left(\frac{9\Lambda}{2b}\right)\left(F''+\frac{2F'}{\sigma}-\frac{\sin 2F}{\sigma^2}\right)=0. \tag{4.11}$$

The form of F at large distances was considered at one time to be sinusoidal [8] but Verschelde [9] suggested that it has exponential decay when η is properly included. Equation (4.11) confirms Verschelde's result since at large distances, as $F\to 0$, $S_s\to 1$, one gets

$$F''+\frac{2F'}{\sigma}-\frac{2F}{\sigma^2}-\frac{2b}{9\Lambda}\,(1-\Omega_1)F=0. \tag{4.12}$$

Recalling that $S_s\leqslant 1$, the definitions in Eq. (3.9) show that $\Omega_1<1$. Thus the asymptotic solution for F is $e^{-\mu\sigma}/\sigma$ where $\mu>0$.

When the scale vibration is included in the theory, the energy of the system is the eigenvalue of $H_{\rm coll}$ given by the Schrödinger equation (3.12). The coefficients of this equation are scale invariant and depend only on the functional form of $F(\sigma)$. Hence the expectation value of R (i.e., the scale) and the energy are determined and there is no Derrick instability. The various static properties of the nucleonic states are also fixed.

For this conclusion to be valid, there must be a solution for F, simultaneously satisfying the Schrödinger equation (3.12) and the profile equation (4.8).

The profile equation

$$\int dq\ \psi^2(q)g(\sigma,q)=0 \tag{4.13}$$

is an integral equation which must hold for all values of the "parameter" x, writing $\sigma=xq^{-2/3}$, $x=f_\pi br$. The wave function ψ given by (3.12) behaves at small q as q^t and at large q as $\exp(-\gamma q^{4/3})$ where

$$2t=1+(1+4\nu)^{1/2} \tag{4.14a}$$

and

$$\gamma = 3(m/8)^{1/2} = (\alpha/2b^3)^{1/2}. \tag{4.14b}$$

Indeed, using

$$\psi = q^t \exp(-\gamma q^{4/3})$$

with γ as a variational parameter gives results very near numerical solutions of a similar equation [17].

We wish to study the asymptotic form of F for large σ. For large values of x, σ is also large except when q is large of order $x^{3/2}$. But when q is large, ψ^2 is exponentially small and, therefore, we can choose a value of x large enough that

$$\int_0^Q dq\ \psi^2 g \gg \int_Q^\infty dq\ \psi^2 g, \tag{4.15}$$

where Q is the large value of q at which the asymptotic approximation for $F(\sigma)$ begins to fail. Of course, in neglecting the contribution for $q > Q$, we note that the functional $g(\sigma,q)$ is finite as $\sigma \to 0$.

Now, since $S_s \to 1$ at large σ, the profile equation for large r becomes

$$\langle q^{-4}[A_1F + A_2\sigma F'] + q^{-4/3}[A_3\sigma F' + A_4(F'' + 2F'/\sigma - 2F/\sigma^2)]\rangle = 0, \tag{4.16}$$

where

$$A_1 = (1 + C_1 - 2u)/\Lambda^2,$$

$$A_2 = C_2/\Lambda^2,$$

$$A_3 = 1/(\alpha b^3),$$

$$A_4 = -1/b^4.$$

To examine possible asymptotic forms of F, we consider the very general function

$$F \sim \sigma^{-N} \exp(-k\sigma^n). \tag{4.17}$$

One sees that the terms in (4.16) are of the type

$$\int dq\ \psi^2 q^a \sigma^{-M} \exp(-k\sigma^n),$$

which for q large (but still $<Q$) is proportional to

$$\int dq\ q^s e^{-\phi}, \tag{4.18}$$

with

$$\phi = 2\gamma q^{4/3} + kx^n q^{-2n/3},$$

$$s = a + 2M/3 + \tau.$$

τ is introduced to allow for any power-dependent factor in ψ.

We study these integrals for large values of x. One can show that the integrand in (4.18) has a pronounced relatively narrow maximum near q_0, the value for which $\phi(q)$ is minimum. One finds

$$q_0 = (\kappa x)^{3v}, \tag{4.19}$$

where

$$\kappa = (nk/4\gamma)^{1/n} \quad \text{and} \quad v = n/(2n+4).$$

For large x, q_0 is large but still less than Q in (4.15), since $\sigma(q_0) = xq_0^{-2/3} \propto x^{4v/n}$ and thus the asymptotic form of F is still valid.

We expand the integrand in (4.18) in a power series in $\zeta = q - q_0$ and, since the rth derivative $\phi^{(r)}(q_0)$ is proportional to $x^{(4-3r)v}$, the ratio of successive terms is of order ζx^{-3v}. Thus, if $|\zeta| \ll x^{3v}$, only leading terms of the series are needed. In this way, one finds that the maximum of the integrand is at $\zeta = (s/C)\kappa^{-3v}x^{-v}$ and that the integrand has one-half its maximum value for $\zeta = (2\ln 2/C)^{1/2}x^v$, where $C = x^{2v}\phi''(q_0)$ is a finite number independent of x. Both these values of ζ are small compared with q_0 and are also well within the range for validity for the series.

We thus see that the integrand is well represented by a narrow Gaussian centered on q_0 and that

$$\int dq\ q^s e^{-\phi} \simeq (2\pi/C)^{1/2} q_0^s \exp[-\phi(q_0)] x^v = (2\pi/C)^{1/2} \kappa^{3sv} \exp(-\phi_0) x^{(3s+1)v}. \tag{4.20}$$

This allows us to find the relative orders of magnitude for large x of the four terms in the profile equation (4.16). The results are

$$\langle q^{-4}F\rangle \propto x^{s_0 - 12v},$$

$$\langle q^{-4}\sigma F'\rangle \propto x^{s_0 - 8v},$$

$$\langle q^{-4/3}\sigma F'\rangle \propto x^{s_0},$$

$$\langle q^{-4/3}(F'' + 2F'/\sigma - 2F/\sigma^2)\rangle \propto x^{s_0 + 8v - 2},$$

where $s_0 = (2N+1)v - N + \tau$. Since $v = 1/4$ if $n = 2$, we see that the third term is dominant for $0 < n < 2$, the fourth for $n > 2$ and the third and fourth are of the same order for $n = 2$. Since each term of the profile equation is nonzero, there is therefore no solution for $n > 0$, unless perhaps for $n = 2$. However, when $n = 2$, both third and fourth terms are negative and there is therefore no solution. [The coefficients of the leading integrals of form (4.20) are $-nkA_3$ and $n^2k^2A_4$, respectively.]

We conclude that the asymptotic form of F must have $n = 0$, i.e., it is simply σ^{-N}. In this case, the first three terms of (4.16) are all of order x^{s_0} and the fourth term is negligible. The equation is

$$(A_1 - NA_2)\langle q^{-4}F\rangle = NA_3\langle q^{-4/3}F\rangle. \tag{4.21}$$

If $A_1 - NA_2 > 0$, there may be a solution.

If $\Lambda \to \infty$, i.e., if one ignores rotation, $A_1 - NA_2 = 0$. We conclude that stability requires both rotation and vibration. Chepilko *et al.* [18] found otherwise, but their work quantized the theory only at the collective level, that is without taking account of the η degrees of freedom.

There might still be no solution if F were such that α and Λ simultaneously became infinite for an F that satisfied the equations. The danger to be considered is that F at large σ is proportional to σ^{-N} with $N\rightarrow 3/2$, when α and Λ are logarithmically divergent. We shall now show that such an F does not simultaneously satisfy the Schrödinger equation (3.12) and the profile equation.

If α and Λ are large because of the behavior of F at large σ, it is only the contributions from large σ that need be considered in the defining integrals (3.9). Thus, in the integrals, we put $S_s=1$, $\mathscr{F}=F$ and find by simple integration that in this limit $\Omega_1=1$, $\Omega_2=\lambda_1=-3/2$, $\lambda_2=6-2\alpha/3\Lambda$, and

$$C_1=2u-\frac{4}{3}, \tag{4.22a}$$

$$C_2=-\left(\frac{3}{4}+\frac{15}{2}u\right)\left(\frac{\Lambda}{\alpha}\right), \tag{4.22b}$$

$$\frac{\alpha}{\Lambda}=\frac{3}{2}N^2, \tag{4.22c}$$

$$\xi=(N+1)^2. \tag{4.22d}$$

Again, consider the profile equation for large r. As before, because of the exponential behavior of ψ, we may take r large enough to ignore the region where $q>Q$ and so use the asymptotic form of $F(\sigma)$. Using Eq. (4.22) and $N=3/2+\epsilon$, the profile equation (4.21) becomes

$$\left\langle \frac{1}{\Lambda^2}q^{-3+2\epsilon/3}\left(\frac{10}{3}u-\frac{2}{9}\epsilon\right)-\frac{3}{2\alpha b^3}q^{-1/3}\right\rangle=0. \tag{4.23}$$

We write the wave function as

$$\psi=q^t\exp(-\gamma q^{4/3})f(q), \tag{4.24}$$

where f is well-behaved at all points, and we find

$$\langle q^a\rangle=(2\gamma)^{-3a/4}\frac{I\left(\frac{3}{2}t-\frac{1}{4}+\frac{3}{4}a\right)}{I\left(\frac{3}{2}t-\frac{1}{4}\right)}, \tag{4.25}$$

where

$$I(\mu)=\int dx\, x^{\mu}e^{-x}f^2. \tag{4.26}$$

Using Eqs. (4.14) and (3.14), the profile equation for $I=J=1/2$ to order ϵ is

$$27\epsilon I\left(-1+\frac{13}{3}\epsilon\right)+8I(1)=0 \tag{4.27}$$

and for $I=J=3/2$, it is

$$243I(1.52)-8I(3.52)=0. \tag{4.28}$$

As $\epsilon\rightarrow 0$, $I(-1+\epsilon)$ is of order $1/\epsilon$, so the first term of (4.27) is finite, but both terms are positive. In (4.28), the terms are of opposite sign, but although the values of the integrals depend on f, it is reasonable, as noted above, to consider that $I(\mu)$ is approximately the gamma function $\Gamma(\mu+1)$. With this approximation, the left side of (4.28) is $172\,\Gamma(2.52)\gg 0$. Thus in neither case is the equation satisfied.

To summarize, we have found that N near 3/2 does not provide a solution and therefore the danger of a collapsing soliton does not arise. This analysis shows that, for stability, one needs both rotation and vibration and suggests that a form factor with asymptotic behavior σ^{-N} should be sought.

V. NUMERICAL SOLUTIONS AND STATIC PROPERTIES

The profile equation (4.8) and the vibrational Schrödinger equation (3.12) must be solved consistently to determine a profile $F(\sigma)$, since all parameters are functionals of F. A strictly numerical iterative procedure is not feasible, in part because Eq. (3.12) is actually an infinite number of conditions, one for each value of r. We therefore constructed analytic forms for F containing arbitrary parameters to be determined by how well the profile equation is satisfied for all r. Equation (4.8) contains eight terms which must cancel to give a sum of zero. An arbitrary function F will give a nonzero residue. Divide this residue by the sum of the absolute values of the eight terms and call the result $\omega(r)$. If $\omega(r)$ is zero, F satisfies the equation for that value of r, and if $\omega(r)$ is small, F is a good approximation. For our approximate functions, we evaluated $\omega(r)$ for about 20 values of r and accepted functions for which all $\omega(r)\lesssim 0.005$. Amongst the "accepted" functions, we chose as best those which minimized the sum of the $\omega(r)$ over the various values of r.

If one finds an $F(\sigma)$ which satisfies the profile equation, one can form an infinite number of others by defining a new function $F_1(\sigma)=F(\beta\sigma)$. Writing $\tau=\beta\sigma$ and noting $dF_1/d\sigma=\beta\, dF(\tau)/d\tau$, one sees from Eq. (3.9) that $b_1=b/\beta$, $\Lambda_1=\Lambda/\beta^3$, $\alpha_1=\alpha/\beta^3$ and the other coefficients in Eq. (4.8) are unchanged. Also when the argument of F_1 is σ, the value of r for a fixed q is $r_1=\sigma q^{2/3}/f_\pi b_1$ and when the argument of F is $\beta\sigma$, the value of r for the same q is $r=\beta\sigma q^{2/3}/f_\pi b=r_1$. So, in fact, $F_1(r,q)=F(r,q)$ and so F_1 is also a solution. However, as a function of σ, it is different and has a different slope at the origin, i.e., $F_1'(0)=\beta F'(0)$.

The form we have found most successful is

$$F(\sigma)=2\arctan[g(\sigma)], \tag{5.1a}$$

$$g(\sigma)=-\frac{2}{F'}\left[\frac{1}{\sigma}\exp(-c\sigma)+\frac{c}{1+d\sigma^N}\right], \tag{5.1b}$$

where N, F', c, d are adjustable constants. Note that $F(0)=\pi$, $F(\infty)=0$, and $F'(0)$ is the parameter F'. Also $F''(0)=0$ which agrees with a power series expansion of Eq. (4.8) for small r. For a given N, an equivalent function F_1 has $F_1'=\beta F'$, $c_1=\beta c$, $d_1=\beta^N d$.

Our procedure was to choose values for N, F', and c and then to find d such that the functionals λ_i satisfy the asymptotic form of the profile equation, that is Eq. (4.21) with $F=\sigma^{-N/3}$. One then determines if this set of values $\{N,F',c,d\}$ gives an acceptable F with small $\Sigma\omega(r)$. Because of the equivalent solutions, F' is essentially arbitrary. For $N=2$, there are acceptable solutions for a short range of

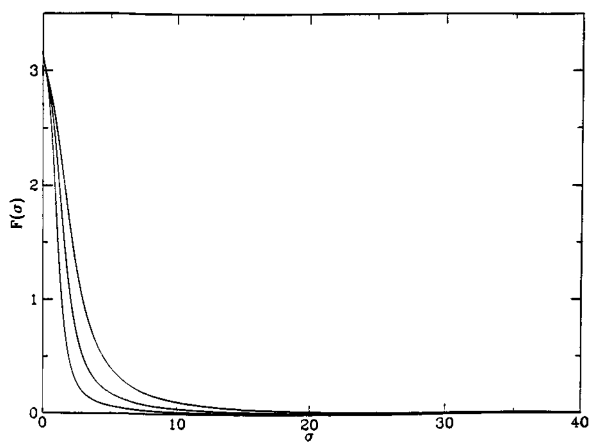

FIG. 1. The profile function $F(\sigma)$ of the nucleon, Δ, and Roper resonance from top to bottom, respectively, with common slope -0.6 at origin.

c; with $F'=-0.4$, the range is $1.0 \leqslant c \leqslant 1.4$. However, these solutions differ from each other by less than 3% at any value of σ. Indeed, when graphed on a reasonable scale, the appearance is simply that of a rather thick curve. Values of N significantly different from 2 did not give acceptable profiles.

For the nucleon, the best profile we found $\{N,F',c,d\}=\{2.0, -0.40, 1.20, 0.5056\}$ has an average $\omega(r)$ of about 0.003. In view of narrow ranges of N and c found for acceptable solutions and the insensitivity of the profile to the value of c within the range, it may be that this profile is quite close to a precise unique solution.

The profile found for $I=J=3/2$ is $\{2.0, -0.60, 1.40, 2.074\}$. Taking $I=J=1/2$ with the first excited state of H_{coll} to describe the Roper resonance N^* (1440) as a radial oscillation, we find a good profile with the set $\{1.9, -0.75, 1.95, 10.328\}$. These profiles are shown in Fig. 1, with a common slope $F'=-0.6$. Figure 2 shows the vibrational wave functions of Eq. (3.12).

We have calculated the masses of the nucleon, Δ and Roper resonance, and the isoscalar mean-square radius, isoscalar and isovector magnetic moments and axial vector coupling constant for the nucleon using

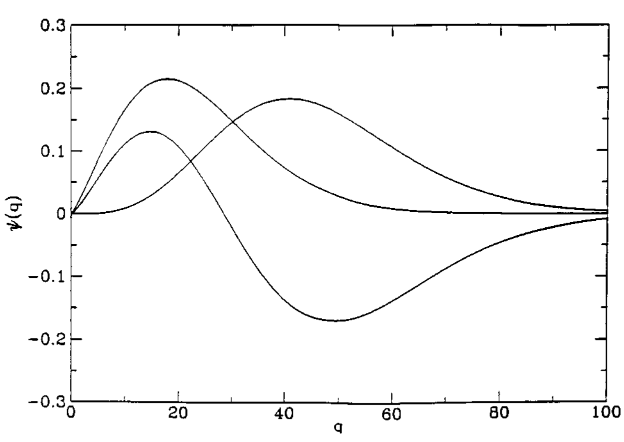

FIG. 2. The vibrational wave functions of nucleon, Δ, and Roper resonance. The Roper resonance has a node and the Δ is shifted to the right with respect to the nucleon.

TABLE I. Static properties: the values in column 3 are the results of this work; those in column 4 are for the standard Skyrmion with $f_\pi=93$ MeV, $g=5.5$, and $m_\pi=0$.

	Experiment	Vibration-rotation	Skyrmion
$M(N)$ (GeV)	0.939	0.921	1.34
$M(\Delta)$ (GeV)	1.232	1.51	1.77
$M(N^*)$ (GeV)	1.440	1.48	
$\langle r^2\rangle_0^{1/2}$ (fm)	0.79	0.46	0.41
$\mu_0(\mu_N)$	0.88	0.81	0.55
$\mu_1(\mu_N)$	4.70	1.69	3.09
g_A	1.23	0.27	0.60

$$\langle r^2\rangle_0=\left(\frac{\Lambda\theta}{f_\pi^2 b^3}\right)\langle q^{4/3}\rangle, \tag{5.2a}$$

$$\mu_0=\frac{M\theta}{3f_\pi}\langle q^{-2/3}\rangle, \tag{5.2b}$$

$$\mu_1=\left(\frac{2M\Lambda}{3f_\pi b^3}\right)\langle q^2\rangle, \tag{5.2c}$$

$$g_A=\left(\frac{\chi}{3}\right)\langle q^{4/3}\rangle, \tag{5.2d}$$

where

$$\theta=-\left(\frac{b}{2\pi^2\Lambda}\right)\int d^3\sigma\, \mathscr{F}F'F, \tag{5.3a}$$

$$\chi=-b^{-2}\int d^3\sigma\left(F'+\frac{1}{\sigma}\sin 2F\right). \tag{5.3b}$$

M is the observed mass of the proton, μ_0 and μ_1 are in units of nuclear magneton, and $f_\pi=93$ MeV. The results are in Table I, where they are compared with experiment and with values for the Skyrmion with the quartic term [7].

It is seen that the results for the quantally stabilized soliton are reasonable except for μ_1 and g_A. For a more realistic description of baryons, one must include additional features in the Lagrangian such as vector meson fields. In particular, the inclusion of the ρ meson improves the value of g_A [19].

VI. SUMMARY

The question as to whether quantization will remove the Derrick instability from the simple Skyrme Lagrangian with no quartic term has been answered in the affirmative, provided one includes both rotational and vibrational collective coordinates as quantal variables.

Our analysis extends Verschelde's work on rotation. It requires careful attention to all degrees of freedom of the pion field and use of the Dirac procedure for quantizing a system with constraints. The resulting stability and static properties are also sensitive to care in ordering operators to ensure Hermiticity of observables at what we have called the field theory level. The profiles F that determine the radial shape are a result of the competing effects of classical binding, rotation, and vibration. It is satisfying that the profiles differ somewhat for the different baryons.

Basically, both the radial extent of the soliton and its orientation in space are variables which should be treated quantally. We see that when this is done, there is no need for the quartic term either to give stability or to obtain reasonable baryon properties. It would seem then that the most satisfactory way to develop the Skyrme soliton model is to abandon the quartic term and use the formalism of this work made more realistic by introducing interactions with the other meson fields.

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