Threshold corrections in orbifold models and superstring unification of gauge interactions

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The string one loop renormalization of the gauge coupling constants is examined in Abelian orbifold models. The contributions to string threshold corrections independent of the compactification moduli fields are evaluated numerically for several representative examples of orbifold models. We consider cases with standard and nonstandard embeddings as well as cases with discrete Wilson line background fields which match reasonably well with low energy phenomenology. The gap separating the observed grand unification scale $M_{\rm GUT}\simeq 2\times 10^{16}$ GeV from the string unification scale $M_X\simeq 5\times 10^{17}$ GeV is discussed on the basis of standardlike orbifold models. We examine one loop gauge coupling constant unification in a description incorporating the combined effects of moduli-dependent and -independent threshold corrections, an adjustable affine level for the hypercharge group factor, and a large mass threshold associated with an anomalous U(1) mechanism.

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I. INTRODUCTION

The idea that particle physics at the Fermi scale descends from string physics at the Planck scale has gained increasing credibility since the proposal in 1985 to compactify the anomaly-free $E_8\times E_8'$ heterotic string on a six-dimensional (6D) Calabi-Yau manifold [1]. Indeed, the circumstancial evidence gathered from the precision tests of the electroweak interaction [2,3], the high energy unification of the standard model gauge interactions [4], and the implications of a heavy top quark [5] have given faith in the supersymmetric (as opposed to the composite) Higgs boson option as a viable extension of the standard model of electroweak interactions. On the other hand, the highly selective search of a semirealistic 4D superstring model, within the Calabi-Yau [6,7], the orbifold [8,9], or the free fermionic [10,11] approaches, which realizes the standard model as its low energy limit, has been well rewarded. It may well be that a realistic model becomes soon reachable through what has been termed a discrete fine-tuning [12] among the simplest classes of free orbifold [13], fermionic [14], or $N=2$ direct product [15] superconformal field theories. The main reason, however, for the interest in a superstring-inspired $N=1$ supergravity lies in the remarkable organizational principle that string theory provides in constructing the basic (Kähler metric, superpotential, gauge functions) components of the effective locally supersymmetric σ model [16].

String theory does not only explain the gravitational, gauge, and Yukawa interactions, it also makes simple definite predictions about their classical level unification at the string scale [17]. The low energy field theory action, obtained by integrating out the massive string modes, is described by an infinite series expansion in powers of the inverse string scale which is the unique free parameter of string theory. The effective Lagrangian comprises local interactions of increasing dimensionalities involving the gravitational, gauge, matter, and moduli supermultiplet fields, subject to the strong constraints imposed by supersymmetry [18], gauge (world sheet) symmetries [19], global (compactification space) symmetries [20], and modular (string duality) symmetries [21]. As to the nontrivial quantum level of string theory, this is intimately related with the mechanisms which determine the vacuum expectation values (VEV's) of the (flat potential) moduli fields. As is well known, the expansion parameters which control the quantum string topological and σ model perturbation theories are themselves identified with the inverse VEV's of the external spacetime S-dilaton field and the internal spacetime overall T-dilaton modulus [22]. Further, to fully fix the entire set of coupling constants entering the effective σ model Lagrangian, one must postulate perturbative or nonperturbative (such as, for instance, hidden sector gaugino condensation inducing spontaneous local supersymmetry breaking) mechanisms in order to stabilize the various moduli fields which parametrize the continuous families of string vacua. Fortunately, exact statements can still be made for certain terms in the superpotential and gauge functions which are protected by perturbative nonrenormalization theorems resulting from characteristic holomorphicity properties of superstring theory [23,24]. Also, certain semiclassical nonperturbative effects, such as the σ model world sheet instantons [25], are usefully constrained through the modular symmetries [26].

One of the most serious challenges for superstring phenomenology is the S-dilaton VEV problem. The argument, spelled out some time ago [27,28], that this is likely to settle at intermediate values inducing a strongly coupled string theory, seems to upset, on phenomenological grounds, the attractive proposal that the observed high energy extrapolation of the standard model gauge coupling constants reflects a perturbative string unification. A related difficulty resides in the order of magnitude mismatch between the observed unification scale and the string scale [4]. The general hope, of course, is that these problems could be surmounted by the variety of mechanisms that string theory reserves in stock. It is significant, however, that the various anticipated effects

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which might influence the quantum string vacuum [threshold corrections, S-T dilatons mixing, anomalous U(1) group factor, strongly anisotropic compactification space [29], etc.] have all in common a sensitivity to the string loop contributions.

One of the first attempts to deal with the string loop renormalization of the gauge coupling constants was undertaken by Minahan [30]. Evaluating the one string loop contribution to the three-point correlator of the gauge boson vertex operators in orbifold models and taking a suitable limit of large string and compactification scales enabled him to identify the familar (β function) logarithmic divergence which renormalizes the gauge coupling constant. As is well known, the string loop perturbation theory is ultraviolet finite, and so the only source of divergences in need of renormalization should be those originating in the infrared cutoff which is introduced to separate the massless from the massive string modes. The first systematic discussion of the low energy matching of string theory to field theory was provided by Kaplunovsky [31], using the effective world sheet σ model background field approach [32]. Based on a specific regularization procedure in string theory which parallels the conventional Pauli-Villars field theory regularization, he identified two additive contributions to the renormalization of the inverse squared gauge coupling constants, the so-called string threshold corrections: a universal correction, associated with (back-reaction) gravitational effects and string scale massive oscillator modes, and a genuine compactification correction, associated with compactification scale massive string modes. The latter correction contains, along with a moduli-independent component (D-term masses), a moduli-dependent component (F-term masses) which plays a crucial role in the mechanism of cancellation of the σ model anomalies affecting the modular symmetries at the level of the supergravity effective action [33–37]. Because the moduli-dependent component arises from the $N=2$ spacetime supersymmetric subspaces of the Hilbert space of states only, general results about its structure can be inferred by using the highly constrained framework of the (2,2) world sheet superconformal theories [38].

Our main interest in the first half of this paper will be focused on the moduli-independent threshold corrections which we shall analyze on the basis of the Kaplunovsky formula [31]. It is appropriate to mention at this point that regarding, as we do, the compactification and the universal back-reaction effects as separate corrections may prove artificial and unjustified. Clearly, the problem resides in the dependence on the infrared regularization scheme. This observation has recently guided a proposal by Kiritsis and Kounnas [39,40] to implement a consistent string infrared cutoff by working with a curved (rather than flat) external spacetime superconformal field theory [41]. The formalism developed in [40] generalizes that of Kaplunovsky [31] and suggests that the universal back-reaction correction, associated with the various background fields, is indeed entangled with the moduli-independent compactification correction. The second half of the present paper will be devoted to a phenomenological discussion of the standard model gauge coupling constants unification. To set the stage for these applications, we shall expose in the following in a more concrete way the main physical motivations for undertaking these studies in the present context.

In superstring unification, as in grand unification theory (GUT), the high energy extrapolation of the standard model renormalized gauge coupling constants is described by a one loop scale evolution of familiar form:

$$\frac{(4\pi)^2}{g_a^2(\mu)}=\frac{(4\pi)^2k_a}{g_X^2}+2b_a\ \ln\frac{\mu}{M_X}+\tilde{\Delta}_a(M_i,\bar{M}_i). \qquad (1)$$

The index $a=3,2,1$ labels the $SU(3)\times SU(2)\times U(1)$ group factors G_a, b_a are the β function slope parameters associated with the low energy modes, $\beta_a(g)=-b_a g_a^3/(4\pi)^2+\cdots)$, and $M_i,\bar{M}_i$ are the compactification moduli fields. The superstring case is, however, distinguished by three important features [31].

(i) Tree level relations [17] involving the gauge and gravitational interactions:

$$g_X^2=k_a g_a^2=\frac{4\kappa^2}{\alpha'}=\frac{32\pi}{\alpha' M_P^2}. \qquad (2)$$

In addition to the string theory expansion parameter g_X (or 4D dilaton VEV $\langle S\rangle=1/g_X^2$) which is specified by the ratio of the string mass scale $M_S=2/\sqrt{\alpha'}$ to the phenomenological Planck mass $M_P=\sqrt{8\pi}/\kappa=1.22\times10^{19}$ GeV, as exhibited in Eq. (2), three extra free [positive integers for non-Abelian group factors and positive rational numbers for Abelian U(1) factors] parameters k_a are introduced into Eq. (2), corresponding to the levels of the affine Lie algebras for the gauge group factors G_a in the underlying string theory.

(ii) An improved unification scale M_X defined in Eq. (1) as the matching scale between the field and string theories renormalized coupling constants at which these obey most closely the tree level relations, Eq. (2). For the field theory coupling constants in the dimensional reduction with modified minimal subtraction $\overline{\mathrm{DR}}$ regularization scheme [42], one has [31]

$$M_X=\frac{e^{(1-\gamma)/2}}{4\pi\sqrt[4]{27}}g_X M_P=\frac{e^{(1-\gamma)/2}}{\sqrt{2\pi}\ \sqrt[4]{27}}M_S \simeq g_X 5.27\times10^{17}\ \text{GeV}. \qquad (3)$$

The field theory (FT) convention in use here is related to the string theory (ST) one as $g_a^{\mathrm{FT}}=\sqrt{2}g_a^{\mathrm{ST}}$, corresponding to the normalization of the Lie algebra generators, $\mathrm{Tr}_R(Q_a^2)=\frac{1}{2}c(R)$, where $c(R)=l(R)$ is the Dynkin index of representation R.

(iii) Threshold corrections accounting for the contributions of the infinite set of massive string states at the string (M_S) and compactification (M_C) scales, integrated out by matching the field and string theory scattering amplitudes. These corrections are represented in Eq. (1) by the functions $\tilde{\Delta}_a(M_i,\bar{M}_i)$ depending upon the structure of the string mass spectrum and the other characteristic parameters of the compactified space manifold, such as the VEV's of the compactification moduli fields, $M_i=T_i,U_i$ [33]. Specifically, M_X is defined as the choice of scale which minimizes the threshold corrections contributions. Of course, the perturbative character of formula (1) implies that the size of $\tilde{\Delta}_a$ should be comparable to that of two-loop effects, so that $\tilde{\Delta}_a \sim 1$.

For a quantitative test of superstring unification based on Eq. (1) and for a proper identification of the fundamental parameters M_X and g_X, it is essential to understand the structure and size of threshold corrections. Thus, an additive decomposition such as described by the following ansatz, $\tilde{\Delta}_a = k_a Y - b_a \Delta$, may be exploited to introduce an effective unification scale and coupling constant,

$$M_X \rightarrow M'_X = M_X e^{\Delta/2}, \quad g_X \rightarrow g'_X = \frac{g_X}{\left(1+\frac{Y g_X^2}{(4\pi)^2}\right)^{1/2}}, \qquad (4)$$

so defined as to incorporate the contributions from the above two components Y and Δ.

The toroidal compactification orbifold models prove very helpful in obtaining information on $\tilde{\Delta}_a$. The contributions from compactification modes admit here a natural additive decomposition into a moduli-dependent component arising from the chiral mass F terms and a moduli-independent component arising from the vector mass D terms [34]. As is well known, the moduli-dependent contributions play an essential role in the cancellation of σ model anomalies affecting the target space duality symmetry [36]. These can be represented by general formulas involving the automorphic functions of the compactication manifold accompanied by model-dependent coefficients. On the other hand, the moduli-independent contributions carry only an implicit dependence on the compactification manifold and on the gauge group embedding of its point and space (discrete Wilson lines) symmetry groups. In spite of several attempts in the literature to estimate numerically the size of both components of threshold corrections [31,43–46] (orbifold models [31,45], fermionic heterotic [43,44], and type II [46] models), one is still lacking a clear physical understanding of their magnitude. Our main goal in this paper is to present results for the moduli-independent threshold corrections through an extensive numerical study based on a sample of orbifold models. A recent work by Dienes and Faraggi [47,48], which appeared while the present paper was being completed, pursues a similar goal to ours based on the fermionic models.

The main physical motivation for this paper is, however, the wide gap that separates the improved string unification scale $M_X = 0.216 M_S \simeq 5\times 10^{17}$ GeV, assuming $g_X \sim 1$, from the observed grand unification scale, $M_{\rm GUT} \simeq 2\times 10^{16}$ GeV, as determined by extrapolating the gauge coupling constants up from their experimentally determined values at the Z-boson mass [4]. The implications of this order of magnitude discrepancy in scales have been emphasized on several occasions [12,49]. The conflict for superstring unification can be resolved in three different ways. One can, of course, always postulate the existence of low energy (10^3 GeV), intermediate energy (10^{10}–10^{13} GeV), or string scale energy matter thresholds entering as vector representations of the color and/or of the electroweak interactions groups. As was discussed within specific superstring models in Refs. [47,48,50] and in a model-independent way in Refs. [51,10,52], such additional multiplets must be very few ($\leqslant 2$) in number. In the semirealistic orbifold models [8,9] also, the extra chiral-antichiral generations get reduced to very small numbers, once the anomalous U(1)-breaking mechanism is turned on. The second possibility is to postulate [53] large string threshold corrections such that after becoming equal and joining together at the observed scale $M_{\rm GUT}$, the gauge coupling constants follow diverging flows up to M_X. A matching of the one loop extrapolated values of $g_a(M_X)$ with their predicted values, as obtained by adjusting the moduli-dependent threshold corrections, can be successfully achieved in terms of wide classes of solutions for the modular weights of massless modes consistent with the anomaly cancellation constraints [36,53]. The third and final possibility is to postulate [54] an affine level parameter for the weak hypercharge group $U(1)_Y$ somewhat lower than the standard grand unification group value $k_1 = \frac{5}{3}$. With such an enhanced starting value for $[k_1 \alpha_1^2(m_Z)]^{-1}$ one achieves a delayed joining of the gauge coupling constants flows which can easily raise up the unification scale by one order of magnitude. While either of the last two possibilities is well motivated by itself and appears sufficient to rescue a superstring grand desert scenario, there remains certain unsatisfactory points. Thus, the rather large VEV's for the moduli fields requested in the first possibility, $\langle T\rangle = 10$–30, induce an order of magnitude gap between the compactification and string scales that might harm the consistency of a weakly coupled superstring (cf. next paragraph). These VEV's are also much larger than the values ($\langle T\rangle \simeq 2$) favored on the basis of the gaugino condensation models for broken local supersymmetry [55]. On the other hand, no known semirealistic orbifold examples of low (point group) order [9] seem to exist for which the hypercharge group level parameter comes as low as the value $k_1 \simeq 1.4$ favored in the third possibility. A simple argument is developed by Dienes and Faraggi in Refs. [47,48] which shows that for any (orbifold or fermionic) model which realizes a direct compactification to the standard model group with the low energy quark-lepton spectrum, requiring a correctly normalized hypercharge imposes the bound $k_1 \geqslant 5/3$. The preceding bound can, however, be evaded by considering suitable simple extensions of the Z_N abelian orbifolds [56–58]. Thus, as demonstrated in [56], for the $Z_N \times Z_M$ orbifold models, the constraints from the standard model spectrum tolerate wide intervals of variations, $k_1 \approx 1$–2. The Abelian Z_N orbifolds with Wilson lines can also evade the above bound, as exemplified in [57], where a semirealistic Z_8-I orbifold standard model is constructed which has $k_1 = 61/384$.

A generic feature of standardlike orbifold models is the occurrence of a rich spectrum of charged massless modes appearing on side of the requested (quark and lepton) chiral families in vector representations of the color and weak groups. In fact, the matter representations of the observable sectors group factors are generally sizable enough so that the corresponding β function parameters β_a arise with either small negative values or large positive values. This suggests that a first stage of slow or nonasymptotically free scale evolution may well take place from M_X down to some scale where the extra modes pair up by acquiring mass and decouple. As is well known [27,28], in order for the 4D low energy effective theory to be weakly coupled, so as not to invalidate the use of Eq. (1) ($g_X \approx g_d M_C^3 < 1$, $g_d = 10$-dimensional gauge coupling constant), and in order to avoid dealing with a strongly coupled 10D theory ($g_d M_S^3 < 1$), one must require that the compactification and unification scales retain a magnitude comparable to the string scale, $M_X \simeq M_C \simeq M_S$. The second restriction can be relaxed by allowing, for instance, for an anisotropic compactification manifold (large radius in one out of the six compactified dimensions) in which a weakly coupled effective theory, $g_X \lesssim 1$, could remain compatible with a strongly coupled string theory (large g_d) [29]. Assuming the above near equality of scales, then a natural identification for the decoupling scale of the extra matter is the mass scale, denoted M_A, which is induced by a nonvanishing Fayet-Iliopoulos D-term contribution to some apparently anomalous U(1) group factor occurring upon compactification [59]. This suggestion is not new, of course, and appears in several places in the specialized literature. The idea is to cancel the nonvanishing one loop string contributions to the D-term scalar potential of an apparently anomalous U(1) factor by judiciously lifting the VEV's of certain scalar fields while restoring a stable supersymmetric vacuum. We shall carry out an analysis of the one loop gauge coupling constant unification which combines together the above ideas of adjustable moduli VEV's and k_1 level parameters together with that of an adjustable intermediate scale M_A, while describing the scale evolution in the interval from M_X to M_A on the basis of orbifold models predictions.

The paper contains four sections. In Sec. II, we discuss in wide outline the basic formalism involved in the one loop string renormalization of the gauge coupling constants as applied to orbifold models. None of the results discussed in this section is new, our main intent being to provide a concrete, encapsulated presentation of the relevant formalism. In Sec. III, we present numerical results for the moduli-independent threshold corrections for a sample of representative orbifold models. In Sec. IV, we examine the viability of superstring unification in an extended picture including threshold corrections and an additional string size energy scale associated with an anomalous U(1) symmetry. In Sec. V, we summarize the main conclusions.

II. ONE LOOP STRING RENORMALIZATION

A. Threshold corrections to gauge coupling constants

We consider the class of low energy supersymmetric theories descending from 4D heterotic string theories with a nonsemisimple gauge group $\Pi_a G_a$. The genus zero (unity) world sheet (with Wick-rotated Euclidean metric) of the conformal field theory is a sphere (torus) parametrized by planar coordinates $\bar{z} = e^{-2\pi i \bar{\zeta}}$, $z = e^{2\pi i \zeta}$, with corresponding cylindrical coordinates given for the sphere by $\bar{\zeta} = \sigma - it$, $\zeta = \sigma + it$, $\sigma \in [0,1]$, $t \in [-\infty, \infty]$ and for the torus by $\zeta = \sigma + \tau t$, $\bar{\zeta} = \sigma + \bar{\tau} t$, $\sigma, t \in [0,1]$, where the torus modular parameter is denoted by $\tau = \tau_1 + i\tau_2$. The right-moving Ramond-Neveu-Schwarz (RNS) superstring is built with 20 spacetime and spin fields $X^\mu(\bar{z})$, $\psi^\mu(\bar{z})$ ($\mu = 0, \ldots, 9$), associated with $D = 4$ external dimensions of the flat spacetime ($\mu = 0, \ldots, 3$) and $d - D = 10 - D = 6$ internal dimensions ($\mu = 4, \ldots, 9$) of the compactification space manifold, represented in a complex basis as $X_R^i, X_R^{\bar{i}}$, $\psi^i = e^{i\phi_i}$, $\psi^{\bar{i}} = e^{-i\phi_i}$ ($i = 1,2,3$), where the complex scalar fields $\phi_i(z)$ are coordinates of the SO(6) group Cartan torus. This is tensored by a left-moving bosonic string built with 26 fields $X^\mu(z)$ ($\mu = 0, \ldots, 25$), comprising D external space coordinates and $26 - D$ internal space coordinates which are distributed into 6 compactified space coordinates $X_L^i, X_L^{\bar{i}}$ and 16 gauge coordinates of the $E_8 \times E_8'$ Cartan torus $F^I, F'^I (I = 1, \ldots, 8)$, generating the currents $J_a(z)$ of the affine Lie algebras G_a of levels k_a. At certain places, we refer to these coordinates globally as F^I ($I = 1, \ldots, 16$) and also by using their fermionic representation in terms of complex 2D Weyl spinors $(\lambda^\alpha, \lambda^{\bar{\alpha}}) = e^{\pm i F^I}$ [$I = 1, \ldots, 16$, $\alpha = 1, \ldots, 8$]. Of course, the above covariantly quantized string theory must be supplemented with the anticommuting conformal ghost fields $c^z(z,\bar{z}), b_{zz}(z,\bar{z})$ and the commuting superconformal spinor ghost fields $\gamma(\bar{z}), \beta_z(\bar{z})$ [60].

The one loop string threshold corrections in the approach of Kaplunovsky [31] are described by the general formula

$$\tilde{\Delta}_a \equiv k_a Y_0 + \Delta_a, \quad \Delta_a = -\int_F \frac{d^2\tau}{\tau_2}(k_a B_a(q,\bar{q}) - b_a), \tag{5}$$

where one has decomposed the total contribution, denoted $\tilde{\Delta}_a$, into a universal contribution $k_a Y_0$, independent of the gauge group factor (except for the coefficient k_a), arising from the (back-reaction) gravitational interactions and oscillator excitations modes, and a contribution solely due to the massive compactification modes, denoted Δ_a. The latter component is expressed as a deformed partition function integrated over the inequivalent (even representations of the modular group) complex structures of the genus 1 world sheet, with an integrand

$$B_a(q,\bar{q}) = -2\,\mathrm{Tr}(Q_s^2 Q_a^2 q^{L_0 - 1} \bar{q}^{\bar{L}_0 - 1/2})$$

$$= -\frac{1}{2} \sum_{\mathrm{even}(\bar{\alpha},\bar{\beta})} \left[(-1)^{2\bar{\alpha} + 2\bar{\beta}} \frac{1}{\eta^2(\tau)\bar{\eta}^2(\bar{\tau})} 2\bar{q} \frac{d}{d\bar{q}} \left(\frac{\bar{\vartheta}\begin{bmatrix}\bar{\alpha}\\ \bar{\beta}\end{bmatrix}(\bar{\tau})}{\bar{\eta}(\bar{\tau})} \right) \right] 2\,\mathrm{Tr}[(-1)^{2\bar{\beta}F} Q_a^2 q^{L_0 - 22/24} \bar{q}^{\bar{L}_0 - 9/24}], \tag{6}$$

where the first factor represents the partition function of the external theory inserted with the operator $Q_s^2=(-\frac{1}{12}+\chi^2)$, where χ denotes the 4D helicity or chirality vertex operator and we have introduced the familiar Dedekind function $\eta(\tau)=q^{1/24}\Pi_n(1-q^n)$ and the Jacobi theta function $\vartheta[{\theta \atop \phi}](\tau)$ whose definition is stated explicitly in Eq. (18) below. The overbars stand for complex conjugation: namely, $\bar{\vartheta}[{\theta \atop \phi}](\bar{\tau})=\{\vartheta[{\theta \atop \phi}](\tau)\}^\star$, $\bar{\eta}(\bar{\tau})=\{\eta(\tau)\}^\star$.

We have accounted in Eq. (6) for the contributions from the ghost fields in the familiar way [16], which simply amounts to canceling out the determinantal factors associated with the time and (string) longitudinal components of the space-time and spin fields. The extra numerical factor of 2 in front of the trace in Eq. (6) reflects the change from string theory to field theory normalization conventions for the gauge coupling constants. The second trace factor in Eq. (6) (with $F=$ fermion number operator, L_0, $\bar{L}_0$=conformal dimensions operators) corresponds to the partition function for the internal conformal field theory characterized by the central charges for the (L,R) sectors $(c,\bar{c})=(22,9)$, inserted with the square Q_a^2 of any one of the gauge group generators for subgroup G_a. The integral over the world sheet torus complex modular parameter $\tau=\tau_1+i\tau_2$, with $q=e^{2\pi i\tau}$, $\bar{q}=e^{-2\pi i\bar{\tau}}$, extends over one modular group SL(2,Z) fundamental domain, for which we consider the standard choice $F=[|\tau_1|\leqslant\frac{1}{2}, \ |\tau|\geqslant 1]$. Infrared convergence of the integral, Eq. (5), is ensured by the subtraction of $b_a=\lim_{\tau_2\to\infty}k_aB_a$, where $b_a=\frac{1}{6}\Sigma_\alpha[-c(R_\alpha^S)-2c(R_\alpha^F)+11c(R_\alpha^V)]$ (S=complex scalar, F=Weyl or Majorana fermion, V=vector) represents the summed contributions to the β function slope parameters from the massless string modes α belonging to the representation R_α.

The summation in Eq. (6) over the subset of even spin structures of the right-moving sector, $(\bar{\alpha},\bar{\beta})=[(0,0),(0,\frac{1}{2}),(\frac{1}{2},0)]$ where $\bar{\alpha},\bar{\beta}=0=$ NS(A) (Neveu-Schwarz, antiperiodic) or $\frac{1}{2}$=R(P) (Ramond, periodic), is performed by insertion of the familiar Gliozzi-Scherk-Olive (GSO) projection phase factors leading to the supersymmetric string [16,61].

For a comparison with Eqs. (5) and (6), we record the corresponding formulas for the one string loop cosmological constant,

$$\Lambda=\int_F\frac{d^2\tau}{\tau_2^3}Z(q,\bar{q}),$$

and the partition function

$$Z(q,\bar{q})=\mathrm{Tr}(q^{L_0-1}\bar{q}^{\bar{L}_0-1/2})=\frac{1}{2}\sum_{(\bar{\alpha},\bar{\beta})}\left[(-1)^{2\bar{\alpha}+2\bar{\beta}}\frac{\bar{\vartheta}\left[{\bar{\alpha}\atop\bar{\beta}}\right](\bar{\tau})}{\eta^2(\tau)\bar{\eta}^3(\bar{\tau})}\right]\mathrm{Tr}[(-1)^{2\bar{\beta}F}q^{L_0-22/24}\bar{q}^{\bar{L}_0-9/24}]\ . \tag{7}$$

B. Specialization to orbifolds

To express the second internal space factor in Eq. (6) for orbifolds, we recall first that the projection (modding) with respect to the orbifold point symmetry is achieved by summing over the (space and time) twisted subsectors (g,h) by using [62–64]

$$\mathrm{Tr}(\cdots)=\frac{1}{|G|}\sum_g\sum_{h;[g,h]=0}\chi(g,h)\mathrm{Tr}_g(h\cdots), \tag{8}$$

where $|G|$ is the orbifold point group order and $\chi(g,h)$ are degeneracy factors. For toroidal compactication, all fields are free so that the torus partition function is obtained by associating to a complex coordinate field $X(\sigma,t)$ of given chirality, a factor $1/(\pi\sqrt{2\tau_2})$ (flat case) or $(1-e^{2\pi iv})/\eta(\tau)$ [untwisted case with time twist $X(\sigma,t+1)=e^{2\pi iv}X(\sigma,t)$] or

$$\eta(\tau)/\vartheta\left[{\frac{1}{2}+v\atop\frac{1}{2}+v}\right]$$

[space twisted case $X(\sigma+1,t)=e^{2\pi iv}X(\sigma,t)$] and to a fermionic Majorana-Weyl field, obeying the twisted boundary conditions

$$\psi(\sigma+1,t)=-e^{2\pi i\theta'}\psi(\sigma,t),$$
$$\psi(\sigma,t+1)=-e^{-2\pi i\phi'}\psi(\sigma,t), \tag{9}$$

a fermionic determinant factor

$$\left[\vartheta\left[{\theta'\atop\phi'}\right]/\eta(\tau)\right]^{1/2}.$$

The zero modes are associated a factor $q^{p_L^2/2}\bar{q}^{p_R^2/2}$ summed over the winding modes spanning the compactification manifold lattice Λ_6 with basis vectors e_a^i and over the Kaluza-Klein momentum modes spanning its dual lattice $\Lambda^\star$ with basis vectors $e_i^{\star a}$ [cf. Eq. (21) below].

We recall next that a torus R^6/Λ^6, defined by $X^i\equiv X^i+2\pi n^a e_a^i$, having a point symmetry group P of automorphisms of the lattice Λ^6, defines an Abelian orbifold endowed with a space symmetry group $G=P\times\Lambda^6$. The space group action on the string theory fields is described in terms of rotations θ^k and translations $u_{k,f}$ together with their associated gauge group shift embedding elements described by translations V^I and Wilson lines translations a_a^I $(I=1,\ldots,16,\ a=1,2,3)$. The space group $G=\{g_p\}=\{\beta_p,w_p\}$ composition laws read $g_1g_2=(\beta_1\beta_2,\beta_1w_2+w_1)$, $g_p^{-1}=(\beta_p^{-1},-\beta_p^{-1}w_p)$. For definiteness, we shall specialize henceforth to the case of an Abelian point group $P=Z_N$ and discuss as we proceed certain of the generalizations encountered in later applications.

The string Hilbert space of states consists of the untwisted sector ($k=0$) and the twisted ($k=1,\ldots,N-1$) sectors. The twisted sectors g^k are distinguished by the boundary conditions $(X(\sigma+1,t),\psi(\sigma+1,t))=g^k\ (X(\sigma,t),-(-1)^{2\bar{\alpha}}\psi(\sigma,t))$. They are organized into conjugacy classes of the space group with representative elements $g^k=[\theta^k,u_{k,f}]$ and their associated classes $\{g^k\simeq g'g^kg'^{-1}=(\theta^k,u_k),g'=g^p\in Z_N\}$, where the set of shift vectors $u_k=[\theta^p u_{k,f}+(1-\theta^k)u]$ ($u\in\Lambda_6,p=0,\ldots,N-1$), span lattice cosets (labeled by the index f) with representative elements $u_{k,f}$. The compactified space coordinates $X^i=X^i_L+X^i_R=x^i+i\pi tp^i+2\pi\sigma w^i+\cdots$ (units $2\alpha'=1$) admit the (zero and oscillators) modes expansion

$$(X^i_L(z),X^i_R(\bar{z}))=\frac{x^i}{2}-\frac{i}{2}(p^i_L\ln z,p^i_R\ln\bar{z})+\frac{i}{2}\sum_{m_i}\left(\frac{\alpha^{Li}_{m_i}}{m_i}z^{-m_i},\frac{\alpha^{Ri}_{m_i}}{m_i}\bar{z}^{-m_i}\right). \tag{10}$$

In twisted sectors, the string center-of-mass coordinates x^i are not arbitrary real parameters but rather must satisfy $g^k x=x+\hat{u}_{k,f}+u$, ($\hat{u}_{k,f},u\in\Lambda_6$). Therefore, each of the g^k twisted sectors splits into subsets which can be classified in terms of the corresponding set of fixed points of the space group, $f^{(k)i}$, defined as $\theta^k f^{(k)}=f^{(k)}+\hat{u}_{k,f}$ where $\hat{u}^i_{k,f}=m^a_{k,f}e^i_a$ ($m^a=$ integers) are translation vectors of the 6D toroidal lattice Λ_6 determined by the condition that they return the rotated fixed point $\theta^k f$ back to its original position, so that $f=(1-\theta^k)^{-1}\hat{u}_{k,f}+u$. Specifically, the k-twisted sector fixed points $f^{(k)}_\alpha$ are distinguished by a label α running over the number of fixed points. The lattice vectors $\hat{u}_{k,f}$ identify with the lattice coset representatives $u_{k,f}$ introduced above only for prime orbifolds. For simply twisted sectors $k=1$ or $k=N-1=-1(\mathrm{mod}N)$, the fixed points $f^{(k)}$ and conjugacy classes $u_{k,f}$ are in one-to-one correspondence, so that $f^{(k)}$ faithfully label these classes and $\hat{u}_{k,f}=u_{k,f}$. This property holds true for all the twisted sectors in the prime orbifolds $Z_{3,7}$. For the multiply twisted sectors, the full set of fixed points $f^{(k)}_\alpha$ decomposes into disjoint subsets $\{f^{(k)}_A,f'^{(k)}_A,\ldots\}$, where the fixed points within each subset (labeled by A) are related as $\theta^{p_A}f^{(k)}_A=f'^{(k)}_A\neq f^{(k)}_A$ for $p_A<k$, and hence are in one-to-one correspondence with the same conjugacy classes $u_{k,f}$. The cases involving nontrivial subsets $[f^{(k)}_A]$, comprising more than one fixed point, arise only for the nonprime ($N=1$) orbifolds $Z_{4,6,8,12}$ and for the direct product orbifolds $Z_N\times Z_M$.

The orbifold space group elements can now be expressed as

$$g^k=\{\theta^k,u_{k,f}=m^a_{k,f}e_a;k\tilde{V}^I=kV^I+m^a_{k,f}a^i_a\},$$

$$\theta^k=\mathrm{diag}(\theta^k_i)=\mathrm{diag}(e^{2\pi ikv_i})\ \left[\sum_i v_i=0\right]. \tag{11}$$

The orbifold group action on fields [Eq. (12) below] and state vectors [Eq. (13) below] reads, in obvious notation,

$$g^kX^i_{L,R}=\theta^k_iX^i_{L,R}+2\pi m^a_{k,f}e^i_a,\quad g^kF^I=F^I+2\pi(kV^I+m^a_{k,f}a^I_a),\quad g^k\psi_i=\theta^k_i\psi_i, \tag{12}$$

$$\begin{aligned}&g^h[(\alpha^i_{-n_i})^{p_i}(\alpha^{\bar{j}}_{-m_j})^{q_j}]\binom{L}{R}|p_R,r^i\equiv\alpha^i+kv^i\rangle_R\otimes|p_L,P^I\equiv W^I+k\tilde{V}^I\rangle_L\\&=e^{2\pi ikh(v\cdot r+\tilde{V}\cdot P)\mp2\pi ih(n_i+m_j)}[(\alpha^i)^{p_i}_{-n_i}(\alpha^{\bar{j}})^{q_j}_{-m_j}]\binom{L}{R}|p_R,r^i\rangle_R\otimes|p_L,P^I\rangle_L.\end{aligned} \tag{13}$$

The above used correspondence between Wilson line translation vectors and the noncontractible loops $u_{k,f}$ refers to Abelian orbifolds with Abelian gauge embeddings. Abelian orbifolds with non-Abelian shift gauge embeddings can be constructed by extending the definition of Wilson lines to class-dependent shift vectors $k\tilde{V}^I\to V^I_{k,f}$ derived from a gauge embedding matrix of general form [65]. Non-Abelian orbifolds with non-Abelian gauge embeddings are discussed in Ref. [66].

The internal space oscillator operators $(\alpha^i_{n_i},\alpha^{\bar{j}}_{m_j})\binom{L}{R}$, where $i,\bar{j}$ are complex conjugate bases indices [given by the familiar linear combinations of real basis indices $\mu=(1+i2)/\sqrt{2},(1-i2)/\sqrt{2},\ldots$], enter with the moddings $n_i\in Z\mp\theta_i$, $m_j\in Z\pm\theta_j$, where Z designates the set of integers. The translation vectors $\alpha^i=n_i,(n_i+\frac{1}{2})$ [$n_i\in Z$, $\Sigma_in_i\in2Z+1$(odd integers)] are elements of the SO(6) group weight lattice Γ_6 and $W^I=n^I$, $(n^I+\frac{1}{2})$, [$n^I\in Z$, $\Sigma^8_{I=1}n^I\in2Z$ (even integers)] are elements of the $E_8\times E_8'$ group weight lattice Γ_{8+8}. The translation vectors v^i and V^I, a_a^I with respect to these lattices must obey $Nv^i\in\Gamma_6$, $NV^I\in\Gamma_{8+8}$, $Nm^a a_a^I\in\Gamma_{8+8}$ as well as the level matching (modular invariance under T^N) conditions $N[(kV^I+m^a_{k,f}a^I_a)^2-(kv^i)^2]\in 2Z$.

With the above rules in hand, we can now quote the following more explicit formula derived from Eq. (6):

$$B_a(q,\bar q)=-2\frac{1}{|G|}\sum_{m,n}\chi(m,n)\epsilon(m,n)\frac{1}{2}\sum_{\text{even}(\bar\alpha,\bar\beta)}\left[(-1)^{2\bar\alpha+2\bar\beta}\frac{1}{\eta^2(\tau)\bar\eta^2(\bar\tau)}2\bar q\frac{d}{d\bar q}\left(\frac{\bar\vartheta\begin{bmatrix}\bar\alpha\\ \bar\beta\end{bmatrix}(\bar\tau)}{\bar\eta(\bar\tau)}\right)\right]$$
$$\times\prod_{i=1,3}\left[\frac{\bar\vartheta\begin{bmatrix}\bar\alpha+mv_i\\ \bar\beta+nv_i\end{bmatrix}(\bar\tau)}{\bar\eta(\bar\tau)}\right]\prod_{i=1,3}\left[\frac{\bar\eta(\bar\tau)}{\bar\vartheta\begin{bmatrix}\frac{1}{2}+mv_i\\ \frac{1}{2}+nv_i\end{bmatrix}(\bar\tau)}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}+mv_i\\ \frac{1}{2}+nv_i\end{bmatrix}(\tau)}\right]$$
$$\times\frac{1}{4}\frac{1}{\eta^{16}(\tau)}\left[\sum_{\alpha,\beta;\alpha',\beta'}\eta(m,n;\alpha,\beta;\alpha',\beta')\prod_{I=1}^{8}Q_a^{I2}\vartheta\begin{bmatrix}\alpha+m\tilde V_I\\ \beta+n\tilde V_I\end{bmatrix}(\tau)\right.$$
$$\left.\times\prod_{I=1}^{8}Q_a'^{I2}\vartheta\begin{bmatrix}\alpha'+m\tilde V_I'\\ \beta'+n\tilde V_I'\end{bmatrix}(\tau)\right]\left[\sum_{\Lambda_6,\Lambda_6^\star}q^{p_L^2/2}\bar q^{p_R^2/2}\right],\tag{14}$$

where the second and third factors, recognizable by the brackets, are contributed by the internal space coordinates and spinors, the fourth factor by the gauge coordinates, and the last (fifth) factor by the compactified space zero modes. The numerical factors appearing in denominators account for the averaging over the timelike spin structures. The corresponding formula for the partition function $Z(q,\bar q)$ can be obtained from Eq. (14) by removing the overall numerical factor -2, the logarithmic derivative operator, $2\bar q(d/d\bar q)$, and the factors $(Q_a^I)^2$, $(Q_a'^I)^2$ from inside the internal theory trace.

C. Classification of threshold corrections

The generalized GSO orbifold projection, which selects the singlet states with respect to the orbifold space symmetry group, is represented by the sum over the various twisted orbifold subsectors $(g,h)=(m,n)$, performed jointly with the sum over the spin structures (α,β), (α',β') for the fermionized fields associated with the gauge degrees of freedom.

The summations over the twisted subsectors $(g,h)=(m,n)$ and the spin structures for the gauge fermions degrees of freedom, $(\alpha,\beta),(\alpha',\beta')$, are weighted by phase factors $\epsilon(m,n)$ and $\eta(m,n;\alpha,\beta;\alpha',\beta')$ which are determined by the requirement that $Z(q,\bar q)$ be invariant under the modular SL(2, Z) group, generated by $S:\tau\to-\frac{1}{\tau}$ and T: $\tau\to\tau+1$. The elements of the set of twisted (g,h) subsectors are mixed together under the action of the modular group according to the transformation law [62,67] $\tau\to(a\tau+b)/(c\tau+d)$, $(g,h)\to(h^c g^d,h^a g^b)$, $(a,b,c,d\in Z, ad-bc=1)$. [For Z_N orbifolds, $S:(m,n)\to(N-n,m)$, $T:(m,n)\to(m,m+n)$.] The entire set of twisted subsectors can be organized into disjoint subsets (orbits) of subsectors which close under the modular group action. The interorbit phase factors $\eta(m,n,\dots)$ are fixed uniquely by the requirement of modular invariance. The intraorbit (discrete torsion) phase factors $\epsilon(m,n)$ are independently fixed by the constraints derived from higher string loops modular invariance and unitarity [68]. These constraints define $\epsilon(m,n)$ as the solutions to the equations

$$\epsilon^N(m,n)=1,\quad \epsilon(0,n)=\epsilon(N,n)=1,$$
$$\epsilon(n,m)=\epsilon^{-1}(m,n).\tag{15}$$

The additional freedom that might be present when the factors $\epsilon(m,n)$ are nontrivial phases serves then to label distinct string theories constructed from the same orbifold. Orbifolds with no (g,h) fixed 2D torus (i.e., not simultaneously fixed by both space g and time h twists) possess one modular orbit only. Orbifolds having one simultaneous (g,h) fixed 2D torus possess several modular orbits which are in correspondence with the distinct $N=2$ suborbifolds of the initial orbifold.

The multiplicity factors $\chi(m,n)=\chi(g,h)$ count, for twisted subsectors, the number of distinct degenerate subsectors associated with fixed points of the orbifold point group which are simultaneously invariant under both g and h. (Useful information on these factors is provided in Refs. [69–71].) For untwisted sectors $(m=0)$, there occurs corresponding nontrivial factors $\chi(1,h)$ from the projection on oscillator states symmetric with respect to the orbifold point group. These can be explicitly calculated from the formula $\chi(1,\theta^n)=\Pi_i|-2i\sin(\pi n v_i)|^2=|\det'(1-\theta^n)|$, where the product and determinant are understood to extend over the rotated 2D tori planes.

In the presence of Wilson lines, an additional summation must be included over the independent Wilson lines a_a satisfying the property $\theta^k a_a \neq a_a$ and over the independent noncontractible loop parameters labeled by the integers m^a, spanning restricted finite intervals. The overall sum over twisted subsectors in Eq. (14) is then replaced as follows: $\Sigma_{m,n} = \Sigma_m \Sigma_{a_a} \Sigma_{n,m^a_{m,f}}$. For the Abelian direct product orbifolds $Z_N \times Z_M$, $(M = pN,\ p \in Z)$, straightforward extensions of the above rules apply in which one deals with pairs of generators (θ_1, θ_2), shift vectors (v_1, v_2), (V_1, V_2), and twisted subsectors $(g_1 g_2; h_1 h_2) = (m_1 m_2; n_1 n_2)$, setting the discrete torsion phase factor in accordance with the above constraints, Eq. (15), to take the discrete set of values [9,68] $\epsilon(m_1 m_2; n_1 n_2) = e^{2\pi i k (m_1 n_2 - m_2 n_1)/N}$ $(k = 0, \dots, N-1)$.

Turning to the interorbit phases $\eta(m,n;\alpha,\beta;\alpha',\beta')$, we note that these depend, of course, on the conventions adopted for the fermionic determinants. A complex left-moving chiral (Weyl) fermion field with the spin structure prescribed by the boundary conditions

$$\psi(\sigma+1,t) = -e^{2\pi i(\theta+\alpha)}\psi(\sigma,t), \quad \psi(\sigma,t+1) = -e^{-2\pi i(\phi+\beta)}\psi(\sigma,t), \tag{16}$$

contributes a Weyl-Dirac operator determinantal factor

$$\det_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \frac{\vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\tau)}{\eta(\tau)}, \tag{17}$$

while the corresponding case of a right-moving chiral fermion is obtained by simply taking the complex conjugate expression. The carefully chosen phase convention [68,72]

$$\eta(\tau)\det_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\nu=0|\tau) = e^{-i\pi\theta(\phi+2\beta)}\vartheta\begin{bmatrix}\alpha+\theta\\ \beta+\phi\end{bmatrix}(\nu=0|\tau),$$

$$\vartheta\begin{bmatrix}\theta'\\ \phi'\end{bmatrix}(\nu|\tau) = \sum_{n\in Z} q^{(n+\theta')^2/2} e^{2\pi i(n+\theta')(\nu+\phi')}, \tag{18}$$

where the equation in the second line exhibits the conventional definition for the Jacobi θ function, incorporates the appropriate relative phases which ensure the modular invariance of the partition function. Thus, in the notation of Eq. (14), the interorbit phases are identified with the products of phase factors, $\Pi e^{-i\pi\theta(\phi+2\beta)} \times \Pi e^{+i\pi\bar{\theta}(\bar{\phi}+2\bar{\beta})}$, extending over the left- and right-moving spinor fields. The combined extended GSO phase factor from the fermionic determinants can also be written, for the bosonic representation of the gauge degrees of freedom, in the form

$$\eta(m,n;\alpha,\beta,\alpha',\beta') \to \Delta^n_{f,m} = \exp^{(2\pi i\{nP^I\cdot\tilde{V}^I - (1/2)\,mn(\tilde{V}^I)^2 - [nr^i\cdot v^i - (1/2)\,mn(v^i)^2]\})}, \tag{19}$$

where we use the notation specified in Eq. (13). Equivalently, if one were instead to substitute in Eq. (14),

$$\vartheta\begin{bmatrix}\alpha+\theta\\ \beta+\phi\end{bmatrix} \to \vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix},$$

using the definition for the θ function specified by Eq. (18), the modular invariance constraints would then take the simple form of unit phases for the whole set of interorbit phases, $\eta(m,n;\alpha,\beta;\alpha',\beta') = 1$. The most direct way to establish the above result for the extended GSO projection in orbifolds is by operating on each of the individual terms in the sum over twisted subsectors in Eq. (14) repeatedly with the modular group generators S,T in an appropriate order so as to span the various orbits, until one hits back on the starting individual term. For a more elegant proof, one can follow the same steps as in [72] involving the use of the identities relating the fermionic and bosonic representations of θ functions and of the Poisson formula by which one transforms the summation over the compactification lattice to that over its dual. Combining in this way the fourth (gauge sector) and fifth (zero modes) factors in Eq. (14) yields an equivalent representation for the product of these factors in terms of a manifestly modular invariant sum over an even, self-dual (shifted) (22,6)-dimensional Lorentzian lattice:

$$Z = \sum_{w\in\Lambda_6,\, p\in\Lambda_6^\star,\, W\in\Gamma_{8+8}} q^{P_L^2/2}\bar{q}^{P_R^2/2}, \tag{20}$$

$$P_{L,R} = [p_{L\mu}, P_I = W_I + kV_I + A_I^\mu w_\mu; p_{R\mu}], \quad p_\mu^{L,R} = \pm G_{\mu\nu}w^\nu + \frac{1}{2}(p_\mu - k_\mu),$$

$$k_\mu = 2B_{\mu\nu}w^\nu + W_I A_\mu^I + \frac{1}{2}A_{I\nu}w^\nu A_\mu^I \quad (p^2 = p_\mu G^{\mu\nu} p_\nu), \tag{21}$$

where $w^\mu=\frac{1}{2}G^{\mu\nu}(p_{L\nu}-p_{R\nu})=m^a e_a^\mu$, $p_\mu=p_{L\mu}+p_{R\mu}=n_a e_\mu^{\star a}$ (m^a,n_a=winding and momentum modes integers), $G^{\mu\lambda}G_{\lambda\nu}=\delta_\nu^\mu$, and the basis vector norms $\Sigma_\mu(e_a^\mu)^2$ identify with the compactification radii squared, r_a^2, along the various periods of the 6D compactification torus. The background metric and antisymmetric tensor fields $(G_{\mu\nu},B_{\mu\nu})=(G_{ab},B_{ab})e_\mu^{\star a}e_\nu^{\star b}$ and the Wilson line vector field $A_\mu^I=a_a^I e_\mu^{\star a}$ represent the generalized coupling constants of the world sheet σ model of the heterotic string whose action (specialized to the superconformal gauge) is reproduced below, for definiteness:

$$S=-\frac{1}{4\pi\alpha'}\int\int d\sigma dt\{\sqrt{h}h^{\alpha\beta}(\partial_\alpha X^\mu\partial_\beta X^\nu+i\bar{\psi}_R^\mu\rho_\alpha\nabla_\beta\psi_R^\nu)G_{\mu\nu}(X)$$
$$+\epsilon^{\alpha\beta}[\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X)+\partial_\alpha X_L^\mu\partial_\beta F_I A_\mu^I(X)]-\alpha'\sqrt{h}R^{(2)}D(X)\},\tag{22}$$

where $\nabla_\alpha\psi^\nu=\partial_\alpha\psi^\nu+\Omega_{\lambda\mu}^\nu\partial_\alpha X^\lambda\psi^\mu$ ($\Omega=$ generalized spin connection with respect to the metric and torsion tensors) and $D(X)=-\frac{1}{2}\ln S(X)$ denotes the dilaton field. The σ model background fields in orbifolds, as in toroidal manifolds, are X-independent constants, due to the vanishing curvature tensor.

The charge generators Q_a in Eq. (6) identify with the zero mode components of the Lie algebra G_a gauge current vertex operators, $Q_a=J_a^0\equiv\int(d^2z/2\pi i)J_a(z)$. The allowed currents are chosen among the linear combinations of the vertex operators $\{i\partial F^I(z),e^{iP_IF^I(z)}\}$, invariant under the orbifold group. Any choice of component Q_a^α $[\alpha=1,\dots,\dim(G_a)]$ is admissible since all the components squared $Q_a^{\alpha 2}$ contribute equally to the trace over string states. It is easiest to work with the Cartan subalgebra generators because of the simpler structure of their representation as linear combinations of the momentum operators, $Q_a=Q_{aI}\int(d^2z/2\pi i)i\partial F^I$, with coefficients Q_{aI} such that $Q_{ai}=\Sigma_I Q_{aI}E_i^I$ $[E_i^I,E_I^{\star i}$ $(i=1,\dots,16)$ are the moving orthogonal frames basis and its dual for the Γ_{8+8} torus] represent the directions (flat components) in the $E_8\times E_8'$ weight lattice invariant with respect to the orbifold group subject to the invariance constraints $Q_{aI}V^I$, $Q_{aI}a_b^I\in Z$. The weight lattice vector components representing the eigenvalues of the Cartan subalgebra operators, Q_a^α $[\alpha=1,\dots,\mathrm{rank}(G)]$, for the momentum eigenstates $|P^I=W^I+k\tilde{V}^I\rangle$, are given by the scalar products $\{Q_a^\alpha\cdot P=Q_{aI}^\alpha P^I\}$. These relations can be used to explicitly determine the Q_{aI}, their absolute normalization being fixed by reference to the normalization condition $\mathrm{Tr}(Q_aQ_b)=\frac{1}{2}c(R)\delta_{ab}$ for the associated matrices.

For non-Abelian subgroup factors, the gauge group shift embedding case, to which we have limited our considerations here, always leads to unit levels $k_a=1$. For Abelian subgroups, the parameters k_a, which are still called levels for convenience of language, depend on the normalization of the corresponding charge operators Q_a and specified by [9] $k_a=2\Sigma_I(Q_a^I)^2$.

The insertion of the charge squared operators is accounted for, in the notations introduced in Eq. (14), by replacing the θ function factors (denoted ϑ_I for short) by modified ones using the rule

$$\prod_I \vartheta_I Q_a^{I2}\to\sum_{I\neq J=1}^{8} Q_a^IQ_a^J\vartheta_I'\vartheta_J'\prod_{K\neq I,J}\vartheta_K+\sum_{I=1}^{8}(Q_a^I)^2\vartheta_I''\prod_{K\neq I}\vartheta_K,\tag{23}$$

where the primed and double-primed θ functions are defined in terms of the sum representation given in Eq. (18) by inserting linear and quadratic powers of the lattice momenta according to the prescriptions

$$\vartheta^I=\sum_{P^I}q^{P^{I2}/2},\quad \vartheta'^I=\sum_{P^I}P^Iq^{P^{I2}/2},$$
$$\vartheta''^I=2q\frac{d}{dq}\vartheta^I=\sum_{P^I}P^{I2}q^{P^{I2}/2},\tag{24}$$

using self-evident shorthand notation. More directly, using the dependence on the variable ν exhibited in Eq. (18), one can write $(\vartheta'^I,\vartheta''^I)(\nu|\tau)=(\partial/\partial(2\pi i\nu),\partial^2/\partial(2\pi i\nu)^2)\vartheta^I(\nu|\tau)$. Note that the precise definition of the 4D chirality operator, introduced after Eq. (6), reads, in this notation,

$$\chi^2=2q\frac{d}{dq}\ln\frac{\vartheta}{\eta}+\frac{1}{12}=+\frac{1}{12}+\frac{\vartheta''}{\vartheta}+2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

The rules in Eqs. (23) and (24) follow directly from a consideration of the bosonic representation of the partition function, as described above in the paragraph preceding Eq. (21).

One expects that the solution for the phases $\eta(m,n;\alpha,\beta;\alpha',\beta')$ yielding a modular invariant partition function $Z(q,\bar{q})$ should also automatically make the product $\tau_2B_a(q,\bar{q})$ modular invariant. This result is indeed true, although the way it is achieved is rather subtle. As discussed in detail by Dienes and Faraggi [48], the contributions arising from the square terms $(Q_a^I)^2$ in Eq. (23) introduce modular symmetry-breaking terms via the double primed quantities ϑ_I''. These left-moving sector modular anomaly terms are, however, precisely canceled by corresponding right-moving sector modular anomaly terms arising from the helicity charge operator squared Q_s^2.

Turning now to the threshold corrections as calculated from Eq. (14), we note that the Δ_a have a natural additive decomposition in terms of moduli-dependent and -independent contributions which we associate with the first and second terms in the formula

$$\Delta_a(M,\bar{M})=\delta_a+\Delta_a^{(m)}(M,\bar{M}).$$

This separation arises when one classifies contributions according to the number $N=4,2,1$ of space-time supersymmetries which are realized in terms of disjoint subspaces of the Hilbert space of states [33]. There exists a one-to-one correspondence between the supersymmetry irreducible representation spaces and the spaces of states of suborbifolds which are constructed from subgroups of the full point symmetry group, themselves identified with the modular orbits. The $N=4,2,1$ supersymmetries are then associated with the suborbifolds leaving fixed 3, 1, or 0 2D tori, respectively. The $N=4$ supersymmetric subsector arises from the purely toroidal, trivial orbit, $(g,h)=(1,1)$, which is clearly absent in orbifolds, due to the projection. The moduli-dependent terms originate from $N=2$ suborbifolds (one fixed 2D torus) subsectors and the moduli-independent ones from the $N=1$ suborbifolds (no fixed 2D torus) subsectors. The $N=1,2$ orbits generally contribute to both b_a or Δ_a, while the $N=4$ toroidal subsector $(g,h)=(1,1)$ (three fixed 2D tori) contributes to neither.

The moduli-dependent $N=2$ contributions arise necessarily from subsectors having nonvanishing momenta $p_{L,R}$. Indeed, a nontrivial zero mode factor different from unity occurs only for twisted subsectors (m,n) with a simultaneously fixed 2D torus. For this case, the factors in the partition function in Eq. (14) multiplying the zero mode factor combine into the product of a constant function of $\bar{\tau}$ times a holomorphic function of τ which, being a nonsingular modular invariant function, must therefore also reduce to a constant independent of τ. The modular integral over the zero mode factor can then be expressed by a general formula involving automorphic functions for the moduli fields associated with the fixed 2D torus. For decomposable 6D tori, one finds [33]

$$\Delta_a^{(m)}(T,\bar{T})=\sum_{i=1}^{3}\sum_{G^i}\tilde{b}_a'^{\,i}[\ln[(T_i+\bar{T}_i)|\eta(T_i)|^4]+\ln[(U_i+\bar{U}_i)|\eta(U_i)|^4]], \tag{25}$$

where the sum over G^i runs over the distinct $N=2$ suborbifolds G' or modular orbits of the point group, $T_{i\bar{i}}=T_i$, $U_{i\bar{i}}=U_i$ designate the [(1,1) and (1,2) harmonic forms] diagonal compactification moduli fields, and the coefficients $\tilde{b}_a'^{\,i}=\hat{b}_a'^{\,i}|G^i|/|G|$ denote the associated massless mode β function slope parameters $\hat{b}_a'^{\,i}$ multiplied by the ratios of point group orders. The dependence on the Dedekind function reflects the target space duality symmetry under the SL (2,Z) modular group. The model-dependent coefficients $\tilde{b}_a'^{\,i}$ can also be identified in terms of the massless modes properties by means of the formula [36] $\tilde{b}_a'^{\,i}\equiv b_a'^{\,i}-k_a\delta_{\rm GS}^i$, with $b_a'^{\,i}=\frac{1}{2}[c(G_a)-\Sigma_{R^\alpha}(1+2n_\alpha^i)c(R_\alpha)]$, where n_α^i are the massless modes modular weights and $\delta_{\rm GS}^i$ the coefficients of the anomaly canceling Green-Schwarz counterterm. The splitting $b_a'^{\,i}=\tilde{b}_a'^{\,i}+k_a\delta_{\rm GS}^i$ exhibits the characteristic property of the mechanisms responsible for the cancellation of the σ model duality symmetry anomalies (proportional to $b_a'^{\,i}$), which involve both the threshold corrections ($\tilde{b}_a'^{\,i}$) and a gauge-group-independent Green-Schwarz counterterm corresponding to a one loop redefined dilaton field [35] $S+\bar{S}\rightarrow S+\bar{S}+\Sigma_i[2\delta_{\rm GS}^i/(4\pi)^2]\ln(T_i+\bar{T}_i)$. For nondecomposable tori, the target space modular symmetry is lowered to subgroups of PSL(2,Z). Similar expressions to Eq. (25) continue to hold, differing by a nontrivial dependence on the sets of allowed moduli, in particular, involving rescalings such as $T_i\rightarrow T_i/3$ or $T_i/4$ [73].

The moduli-independent contributions δ_a are associated with the vanishing of all components of the momentum and winding modes $p_{L,R}^\mu$, which therefore results in a trivial zero mode factor equal to unity. No analytic simplification for the modular parameter τ integral is known to exist in this case, for which one must resort to a numerical evaluation. This task is the subject of next section and represents the main new result reported in this paper.

III. NUMERICAL RESULTS

Before presenting the results we digress to describe how we deal with the numerical integration over the complex parameter τ. The two-dimensional modular integral can be separated in two different ways:

$$\int_F d^2\tau f(\tau_1,\tau_2)=\int_0^{1/2}d\tau_1\int_{(1-\tau_1^2)^{1/2}}^{\infty}d\tau_2[f(\tau_1,\tau_2)+f(-\tau_1,\tau_2)]=\int_{\sqrt{3}/2}^{\infty}d\tau_2\int_{{\rm Re}(1-\tau_2^2)^{1/2}}^{1/2}d\tau_1[f(\tau_1,\tau_2)+f(-\tau_1,\tau_2)]. \tag{26}$$

The general structure of the integrand function $B_a(q,\bar{q})$, as follows from Eq. (6), is that of a doubly infinite sum of terms involving products of functions of $q,\bar{q}$ reading, schematically,

$$B_a(\tau,\bar{\tau})=\sum_{\lambda,\mu} c_a(\lambda,\mu)\phi_\lambda(q)\phi_\mu(\bar{q})=\sum_{E_L,E_R} w_a(E_L,E_R)q^{E_L}\bar{q}^{E_R}. \tag{27}$$

The power indices $E_{L,R}$ in the second equation identify with the squared masses or conformal dimensions for the physical spectrum,

$$E_L\equiv h_L-1=N_L+\frac{P^{I2}}{2}+E_0-1,$$

$$E_R\equiv h_R-\frac{1}{2}=N_R+\frac{r^{i2}}{2}+E_0-\frac{1}{2}\quad\left[E_0=\frac{1}{2}\sum_i [kv_i](1-[kv_i]),\quad 0<[kv_i]<1\right], \tag{28}$$

where $h_{L,R}$ are the conformal dimensions, $N_{L,R}$ the oscillator operators eigenvalues, P^I,v^i are defined in Eq. (13), and E_0 the vacuum energy shift induced by the orbifold moddings. The projection on the modular group invariants is an essential element here in canceling the terms in the second sum of Eq. (27) with negative powers of

$$\binom{q}{\bar{q}}=e^{\pm 2\pi i\tau_1-2\pi\tau_2},$$

thus leading to nonsingular expansions with powers identified with the left and right sectors physical spectrum squared masses, $E_{L,R}$. [The diagonal elements of the coefficients block, $w_a(E_L,E_R)$, identify with the density of string states of fixed mass, weighted by the squares of their gauge and helicity charges.]

The functions of τ_2 obtained upon integration over τ_1, as exhibited by the second equation in Eq. (26), have discontinuous derivatives at $\tau_2=1$. This is seen clearly in Fig. 1 where the τ_2 integrand is plotted for certain orbifold models to be discussed below. This figure illustrates certain generic features that are encountered in all the other cases. The main items are (i) untwisted sector contributions smaller in comparison with the twisted sector ones and nearly independent of τ_2 for $\tau_2>1$, (ii) twisted sectors contributions exponentially convergent for $\tau_2>1$, and (iii) constant limiting values for k_aB_a reached at the rather early values $\tau_2\approx 2$–3, which are to be identified with the β function slope parameters b_a associated with the charged massless modes for the various sectors.

To explain these results, we note first that when the power indices $E_{L,R}$ take integral values, as is always the case for the untwisted sector contributions, then the Fourier τ_1 integral extends (for $\tau_2\geqslant 1$) over one period of the integrand, namely, $\int_{-1/2}^{1/2}d\tau_1 e^{2\pi i(E_L-E_R)\tau_1}$, and so selects in the expansion given in Eq. (27) the terms $E_L=E_R+n, n\in Z$. The leading contributions to the τ_2 integral at large τ_2 is dominated by the charged modes with a minimal value of (E_L+E_R), as follows from the dependence $\int d\tau_2 e^{-2\pi\tau_2(E_L+E_R)}$. This explains the rapid convergence to a nearly flat behavior with variable τ_2 for $\tau_2>1$ of the untwisted sector contributions to $B_a(q,\bar{q})$. On the other hand, when $E_{L,R}$ take fractional values, $E_{L,R}=k_{L,R}/N$, $k_{L,R}\in Z$, as is the case for the twisted sectors contribution, then the τ_1 integral is nontrivial over the entire τ_2 interval and does not anymore select $E_L-E_R\in Z$. The τ_2 integral still involves a sum of exponentially decreasing terms of the form $e^{-2\pi\tau_2(k_L+k_R)/N}$.

As is well known, the modular invariance under T^N leads to the restriction $N(E_L-E_R)\in Z$ for the physical spectrum which implies for the supersymmetric Abelian orbifolds that each E_L level is matched by an E_R level [13,68]. One should not be misled, however, by the above left-right level matching property and deduce that the double sum in Eq. (27) actually degenerates into a single sum restricted to $E_L=E_R$. If true, this property would have made the τ_1 integral trivial and dispensed with the need to evaluate it for each case.

Once the constant parts in the full integrand, which are identified with the massless mode contributions given by b_a, are removed, the subtracted integrands $(k_aB_a-b_a)$ are fastly convergent functions. An infrared cutoff at, say, $\tau_2=2$–3 is more than sufficient to retain the dominant part of the quadrature. Nevertheless, the projections involved in the summations over the orbifold subsectors cause strong cancellations which adversely affect the accuracy of the final results. The most appropriate way to organize calculations here would be to express analytically the integrands in power expansions in $q,\bar{q}$ prior to the numerical integration. This procedure is the one adopted in Refs. [45,46,48]. However, in order to deal with a variety of orbifold examples of increasing complexity with respect to the orbifold group order or the inclusion of Wilson lines, the implementation of this procedure would obviously require the use of symbolic programming. For our limited purposes in this work we shall sacrifice a high numerical accuracy of the results and choose instead, as in Ref. [43], to perform all calculations by means of brute force numerical programming.

The numerical integrations are carried out in the order indicated by the second equation in (26); namely, the τ_1 integral is performed first, successively for the two τ_2 intervals $\sqrt{3}/2\leqslant\tau_2\leqslant 1$ and $1\leqslant\tau_2\leqslant\infty$, and the τ_2 integrals next, after subtracting the numerically determined asymptotic values, $k_aB_a-(k_aB_a)_{\tau=\infty}$. The τ integrals are evaluated in succession for the untwisted and the various twisted sectors (labeled by g or m). The GSO (physical subspace) projections for each g sector, represented in Eq. (14) by the summations over h (n integers) and over the independent Wilson lines a_a and the associated winding numbers (m^a integers), are carried out inside the τ integrals, namely, at fixed $\tau=\tau_1+i\tau_2$. This ordering is, of course, important and necessary since the summation over h is essential in projecting

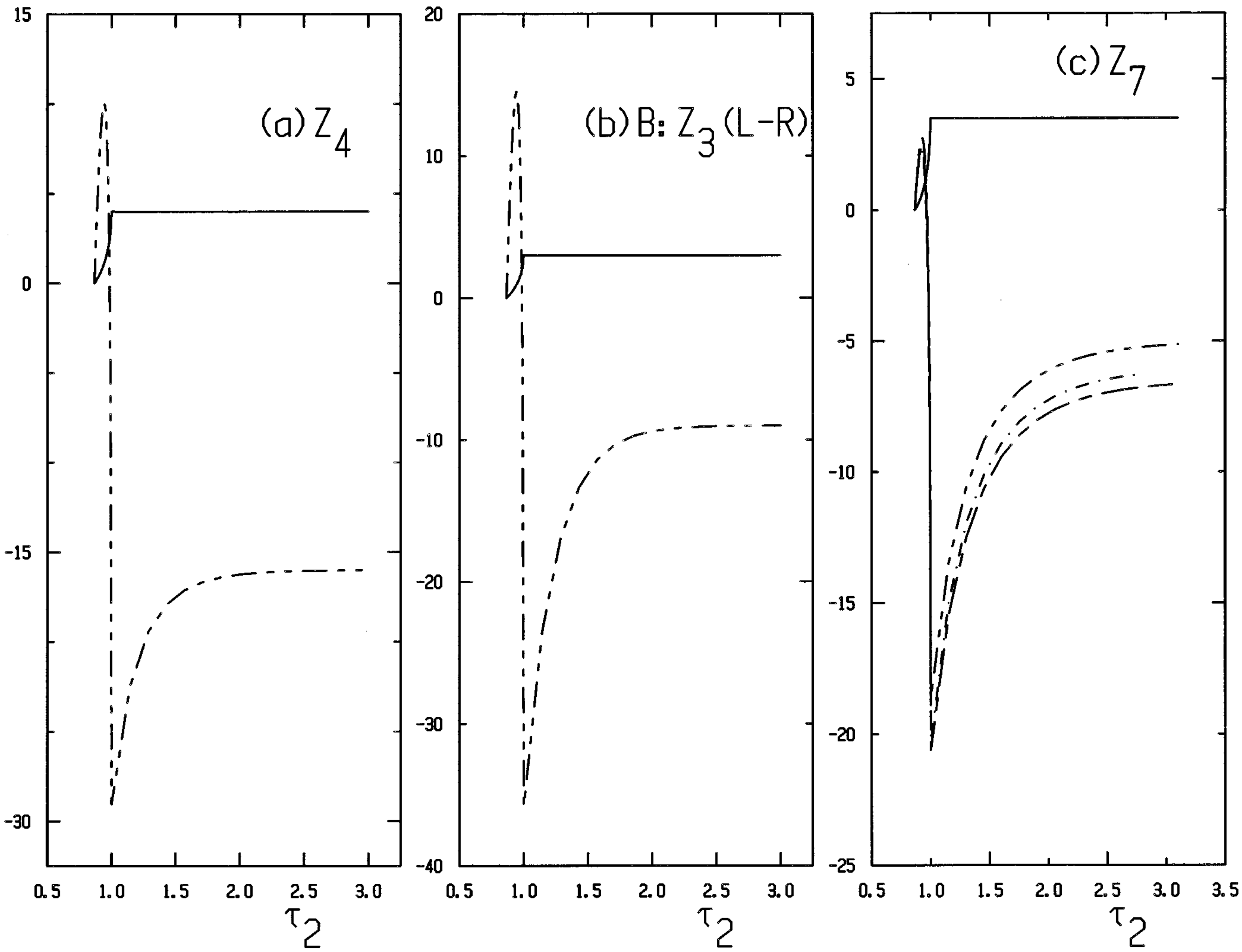

FIG. 1. The threshold function $-k_aB_a(\tau)$, integrated over τ_1, is plotted as a function of τ_2 for three cases: (a) the group SU(2) of the nonstandard embedding Z_4 orbifold, in Table II; (b) the group factor $SU(3)_c$ of the case B Z_3 orbifold model in Table III; (iii) the first SU(2) on line group factor of the Z_7 orbifold model in case D Table III. The untwisted sector contributions are drawn as solid lines and those of twisted sectors are drawn as long-dashed-short-double-dashed (θ), long-dashed–short-dashed (θ^2), and short-dashed (θ^3) lines.

out the unphysical tachyonic terms (negative powers of $q,\bar{q}$). It is also helpful in reducing the rounding off errors caused by the severe cancellations taking place in the projection.

We convinced ourselves by various cross-checks that one could maintain a numerical accuracy with relative errors of order 10^{-2} for most of the cases to be considered below, and specifically for all the orbifolds Z_N ($N \leqslant 12$). The rounding errors worsen with increasing orbifold order and increasing numbers of Wilson lines. The cross-checks involve the following verifications: (i) smoothness of the τ_2 integrands as a function of τ_2 for each g sector after the h projections on the physical subspaces of the Hilbert space of states are carried out, a feature which is apparent on the results in Fig. 1; (ii) convergence of the integrals k_aB_a for each g sector to the expected value of the β function slope parameters b_a, which are independently calculated in terms of the known massless spectrum of the model; (iii) independence of the results for non-Abelian group factors on the choice of a specific charge generator Q_a^I within the Cartan basis; (iv) stability of the numerical results against variations by about 30% in the number of integration points used in the numerical quadratures over τ_1 and τ_2, with respect to an average number of integration points of 7 and 15, respectively. Further, the expected vanishing for each twisted subsector of the contributions to the cosmological constant Λ, which originates from the right-moving supersymmetry [30], is systematically verified at the level of 10^{-7}.

Let us quote here useful results concerning the inputs for some of the orbifold parameters. Details regarding the gauge symmetry groups and the massless spectra can be found by consulting the results in Refs. [9,69,70]. For the $Z_{3,7}$ prime orbifolds, the degeneracy factors $\chi(g,h)$ count the number of simultaneously fixed points by g and h. Thus, for twisted sectors, $\chi(g,h)=27,9,3,1$ $(g\neq 1)$, independently of $(h=1,\ldots,\theta^N)$, for the Z_3 orbifolds with 0,1,2,3 inequivalent Wilson lines, respectively. For the Z_7 orbifolds, $\chi(g,h)=7(1)$, independently of $(h=1,\ldots,\theta^N)$, where the first (second) numbers refer to cases without (with) one Wilson line. (The reduction of the degeneracy factors in the presence of Wilson lines, reflecting the distinguishability of subsets of twisted subsectors, is compensated by the summation over the winding numbers $m^a_{m,f}$.) For the nonprime orbifolds, where $\chi(g,h)$ count the number of conjugacy classes, we shall restrict here to cases with no Wilson lines. Quoting from Ref. [70], one has, for the Z_4 orbifold, $\chi(\theta,\theta^{0,1,2,3})=(16)^4$, $\chi(\theta^2,\theta^{0,1,2,3})=(16,4)^2$; for the Z_6-I orbifold, $\chi(\theta,\theta^{0,\ldots,5})=(3)^6$, $\chi(\theta^2,\theta^{0,\ldots,5})=(27,3)^3$, $\chi(\theta^3,\theta^{0,\ldots,5})=(16,1,1)^2$; for the Z_8-I orbifold, $\chi(\theta^{[1,3]},\theta^{0,\ldots,7})=(4)^8$, $\chi(\theta^2,\theta^{0,\ldots,7})=(16,4)^4$, $\chi(\theta^4,\theta^{0,\ldots,7})=(16,2,4,2)^2$; for the $Z_{12}-I$ orbifold, $\chi(\theta^{[1,2,5]},\theta^{0,\ldots,11})=(3)^{12}$, $\chi(\theta^3,\theta^{0,\ldots,11})=(4,1,1)^4$, $\chi(\theta^4,\theta^{0,\ldots,11})=(27,3,3,3)^3$, $\chi(\theta^6,\theta^{0,\ldots,11})=(16,1,1,4,1,1)^2$, where the exponents indicate the number of repetitions of the associated patterns. In the $Z_3\times Z_3$ orbifold with one Wilson line associated with the first factor, as in the example presented below, $\chi(g,h)=[3,3,3,9,3,3,3,9]$ for $g=[\theta_1,\theta_1^2,\theta_2,\theta_1\theta_2,\theta_1^2\theta_2,\theta_2^2,\theta_1\theta_2^2,\theta_1^2\theta_2^2]$, independently of $h=\theta_1^{n_1}\theta_2^{n_2}$. Note that $\chi(1,h)=\chi(h,1)$ and $\chi(\theta^m,h)=\chi(\theta^{N-m},h)$.

The $N=2$ subtwisted sectors associated with given (g,h) simultaneously fixed planes consist in the Z_4 orbifold case of a single modular orbit $\mathcal{O}$ of (g,h) sectors given by $\mathcal{O}=\{(1,\theta^2),(\theta^2,1),(\theta^2,\theta^2)\}$. The other nonprime Z_N orbifolds in Tables I and II all possess single $N=2$ modular orbits which are constructed analogously by including all distinct (g,h) subsectors, with g and h running over the Z_N elements leaving a given two-torus fixed. Thus, for the Z_6-I orbifold, the relevant element is θ^3; for Z_8-I, it is θ^4; and for Z_{12}-I, the relevant elements are $\theta^3,\theta^6,\theta^9$. The $Z_3\times Z_3$ orbifold possesses three orbits $\mathcal{O}_i$, associated with the three fixed planes, given by

$$\mathcal{O}_1=\{(1,\theta_2^{1,2}),(\theta_2,\theta_2^{0,1,2}),(\theta_2^2,\theta_2^{0,1,2})\},\quad \mathcal{O}_2=O_1[\theta_2\rightarrow\theta_1],$$

$$\mathcal{O}_3=\left\{(1,\theta_1\theta_2^2),(1,\theta_1^2\theta_2),\left[\begin{pmatrix}\theta_1\theta_2^2\\ \theta_1^2\theta_2\end{pmatrix},1\right],\quad \left[\begin{pmatrix}\theta_1\theta_2^2\\ \theta_1^2\theta_2\end{pmatrix},\theta_1\theta_2^2\right],\quad \left[\begin{pmatrix}\theta_1\theta_2^2\\ \theta_1^2\theta_2\end{pmatrix},\theta_1^2\theta_2\right]\right\}. \tag{29}$$

We present our results for cases corresponding to the standard embedding (2,2) orbifolds in Table I. Results for cases corresponding to nonstandard embedding (0,2) orbifolds are presented in Table II. Details concerning the gauge group and the massless spectra can be found in Refs. [64,36]. Finally, to elucidate the role of discrete Wilson lines, threshold corrections results for semirealistic orbifold models (cases A–E) having three chiral matter generations are presented in Table III. Cases A–C refer to Z_3 orbifolds. Up to extra U(1) factors, the observable sector gauge group for case A [9] coincides with the standard model gauge group, while that for case B, also due to Font *et al.* [9], is a left-right chirally symmetric gauge group extension $SU(3)_c\times SU(2)_L\times SU(2)_R$ and that of case C, due to Kim and Kim [74], is an intermediate unification gauge group $SU(3)_c\times SU(3)_w$. Case D in Table III refers to a Z_7 orbifold model with gauge group $SU(3)\times SU(2)\times U(1)$, proposed by Katsuki *et al.* [70]. Case E in Table III refers to a $Z_3\times Z_3$ [9] orbifold with an observable sector gauge group $SU(3)_c\times SU(2)_L\times SU(2)_R\times SU(2)$.

One of the first calculations of the moduli-independent threshold corrections was that attempted by Kaplunovsky [31] for the simplest case of standard embedding orbifolds. He reported a small gauge-group-dependent term $\Delta=-[(\Delta_a/k_a)-(\Delta_b/k_b)]/[(b_a/k_a)-(b_b/k_b)]\simeq0.07$. The Z_3 orbifold case with two Wilson lines, designated in Table III as case A, was recently considered by Mayr *et al.* [45]. Assuming tentatively the decomposition $\delta_a=-b_a\Delta+k_aY$, with the first component proportional to the factor group slope parameters and the second to the affine levels, these authors found $\Delta\simeq0.079$, $Y\simeq4.41$. As for the comparison with the existing estimates made in fermionic constructions of 4D superstrings, this is not very enlightening in the context of the present work because the threshold corrections in the models discussed in Ref. [43] $[\Delta(SU(5))-\Delta(U(1))=-24]$ and in Ref. [46] $[\Delta(SU(3))-\Delta(U(1))=-2.5]$ arise from moduli-dependent contributions in $N=2$ sectors only. In a recent systematic study, Dienes and Faraggi [48] report results for several new cases. They indicate, in particular, that the above-quoted threshold corrections in the flipped SU(5) case [46] must be reduced by an approximate factor of 3. Let us note here that the models obtained in the fermionic construction refer to specific points in the moduli space for which one lumps together the moduli-dependent and -independent contributions.

Several conclusions can be drawn from the results in Tables I–III. Our results for the Z_3 orbifold cases essentially reproduce those available from previous works [31,45]. Further, the two-component decomposition $\delta_a=-b_a\Delta+k_aY$ excellently fits all our Z_3 orbifold cases, including the nonstandard embedding and Wilson line models, with the same values for the parameters, namely, within the numerical accuracy of our results, $\Delta\simeq0.068$, $Y\simeq3.4$. Because of the additional difference effects involved in extracting Δ, a liberal estimate of the relative precision on the parameter Δ is 10%. The independence with respect to the shift embedding and Wilson lines is a remarkable feature which, as emphasized in [45], presumably originates in a general, deep principle of string theory, which still remains mysterious.

The threshold corrections δ_a significantly increase in magnitude with increasing Z_N orbifold order, roughly as N, while always retaining the same positive sign. For the other prime orbifold, namely, Z_7, the two-component formula again fits well with our results in Tables I and III, but only if we restrict ourselves to the non-Abelian group factors, for which we find $\Delta\simeq0.20$, $Y\simeq15$. The formula loses its validity if one includes the Abelian U(1) factors, since a pairwise comparison of the various group factors leads to the ranges

TABLE I. Threshold corrections for the $Z_{3,4,7}$ orbifolds with standard gauge embeddings. The entries in the first line are the rotation angles θ_i $(i=1,2,3)$ and the shift vectors v^i $(i=1,2,3)$, V^I $(I=1,\ldots,8)$. The second and subsequent columns correspond to the gauge group factors in the observable and hidden (primed) sectors. The successive line entries list the group factors, the charge generators components Q_a^I, the levels k_a, the β function slope parameter, b_a or, for the nonprime orbifolds with $N=2$ suborbifolds, the pairs $(b_a^{N=1},\hat{b}_a')$, such that $b_a=b_a^{N=1}+\hat{b}_a'$, and the moduli-independent threshold corrections δ_a.

Orbifold $Z_3: \theta_i=(113)/3,\ v_i=(11-2)/3,\ V^I=(1120^5)/3$				
Group	SU(3)	E_6	E_8'	
Q_a	$(1–10^6)$	$(0^31^20^3)$	$(1^20^6)'$	
k_a	1	1	1	
b_a	-72	-72	90	
δ_a	8.31	8.31	-2.76	
Orbifold $Z_4: \theta_i=(112)/4,\ v_i=(11-2)/4,\ V^I=(1120^5)/4$				
Group	SU(2)	E_6	U(1)	E_8'
Q_a	$(1–10^6)$	$(0^31^20^3)$	$\frac{1}{2}(1^220^5)$	$(0^21^20^4)'$
k_a	1	1	3	1
$(b_a^{N=1},\hat{b}_a')$	$(-12,-42)$	$(-36,-42)$	$(-162,-94.5)$	(60,30)
δ_a	12	9.1	10.4	-1.21
Orbifold Z_6-I : $\theta_i=(114)/6,\ v_i=(11-2)/6,\ V^I=(11-20^5)/6$				
Group	SU(2)	E_6	U(1)	E_8'
Q_a	$(1–10^6)$	(0^41-10^2)	(1^2-20^5)	$(1^20^6)'$
k_a	1	1	12	1
$(b_a^{N=1},\hat{b}_a')$	$(-28,-28)$	$(-38,-28)$	$(-1998,-1008)$	(70,20)
δ_a	13.5	11.5	337	-1.1
Orbifold $Z_7: \theta_i=(124)/7,\ v_i=(12-3)/7,\ V^I=(12-30^5)/7$				
Group	E_6	$U(1)_1$	$U(1)_2$	E_8'
Q_a	$(0^31^20^3)$	(-1010^5)	$(-1-2-10^5)$	$(110^6)'$
k_a	1	4	12	1
b_a	-36	-208	-1398	90
δ_a	22.7	78.2	805	-2.24
Orbifold Z_8-I : $\theta_i=(125)/8,\ v_i=(12-3)/8,\ V^I=(12-30^5)/8$				
Group	E_6	$U(1)_1$	$U(1)_2$	E_8'
Q_a	(0^41-10^2)	$(01-10^5)$	$(2-1-10^5)$	$(110^6)'$
k_a	1	4	12	1
$(b_a^{N=1},\hat{b}_a')$	$(-33,-21)$	$(-180,-84)$	$(-1301,-756)$	(75,15)
δ_a	17.5	62	601	-1.65
Orbifold Z_{12}-I : $\theta_i=(147)/12,\ v_i=(14-5)/12,\ V^I=(14-50^5)/12$				
Group	E_6	$U(1)_1$	$U(1)_2$	E_8'
Q_a	(0^31-10^3)	$(01-10^5)$	$(2-1-10^5)$	$(110^6)'$
k_a	1	4	12	1
$(b_a^{N=1},\hat{b}_a')$	$(-39,-28)$	$(-188,-112)$	$(-1398,-101)$	(70,20)
δ_a	21	65	657	-3.25

of variation, $\Delta \simeq 0.6$ to -0.2, for the results in Table I, and $\Delta \simeq 0.1–1$ for the results in Table III. However, these predictions are not very precise, since one expects larger numerical inaccuracies to occur in Δ for the Abelian factors due to the stronger cancellation effects there. We have also considered the standardlike Z_7 models proposed by Casas *et al.* [75]. Performing the calculations for example 2 of this paper, with $7V^I=(0^21^331^2)(0^8)'$, $7a^I=(3^2\text{-}1^3\text{-}2\text{-}3\text{-}2)(51^7)'$, gives for the corresponding gauge group factors $SU(2)\times SU(3)\times SO(15)'$ the results $b_a=(-12,-7,20)$, $\delta_a=(8.5,9.6,4.4)$, which leads to analogous conclusions to those reached for the model of Katsuki *et al.* [70].

For the nonprime orbifolds, the two-component formula ceases again having a universal validity, even when applied

TABLE II. Threshold corrections for orbifolds $Z_{3,4}$ with nonstandard gauge embeddings. For each case, the first line gives the shift vectors V^I, V'^I $(I=1,\dots,8)$. The second and subsequent columns correspond to a selection of the gauge group factors in the observable and hidden (primed) sectors. The successive line entries list the group factors, the charge generators components Q_a^I, the levels k_a, the β function slope parameters b_a or, for the nonprime orbifolds with $N=2$ suborbifolds, the pairs $(b_a^{N=1}, \hat{b}'_a)$, such that $b_a = b_a^{N=1} + \hat{b}'_a$, and the moduli-independent threshold corrections δ_a.

Orbifold $Z_3: V_I=(1120^5)/3,\ V'_I=(1120^5)'/3$					
Group	SU(3)	E_6	SU(3)$'$	E'_6	
Q_a	(1010^5)	$(0^3 1^2 0^3)$	$(1010^5)'$	$(0^6 1^2)'$	
k_a	1	1	1	1	
b_a	-72	9	-45	-9	
δ_a	8.28	2.73	8.28	2.73	
Orbifold $Z_3: V_I=(110^6)/3,\ V'_I=(20^7)'/3$					
Group	E_7	$U(1)_1$	$U(1)'_2$	SO(14)$'$	
Q_a	$(1-10^6)$	$(1^2 0^6)$	$(10^7)'$	$(01^2 0^5)'$	
k_a	1	4	2	1	
b_a	36	-432	-90	-18	
δ_a	0.9	42.8	9.77	4.52	
Orbifold $Z_3: V_I=(1^4 20^3)/3,\ V'_I=(20^7)'/3$					
Group	SU(9)	SO(14)$'$	U(1)$'$		
Q_a	$(1-10^6)$	$(01^2 0^5)'$	$(10^7)'$		
k_a	1	1	2		
b_a	-18	9	-99		
δ_a	4.58	2.74	10.1		
Orbifold $Z_4: V_I=(1120^5)/4,\ V'_I=(220^6)'/4$					
Group	SU(2)	E_6	U(1)	SU(2)$'$	E'_7
Q_a	$(1-10^6)$	$(0^3 1^2 0^3)$	$(1^2 20^5)$	$(1^2 0^6)'$	$(0^2 1^2 0^4)'$
k_a	1	1	12	1	1
$(b_a^{N=1}, \hat{b}'_a)$	$(-12,-42)$	$(12,-42)$	$(-2163,-1512)$	$(-84,30)$	$(12,30)$
δ_a	0.98	3.74	427	14.6	3.74

to a fixed orbifold case. It seems natural here to identify the coefficient b_a with the $N=1$ sectors slope parameter $b_a^{N=1}$, since δ_a also arises from these sectors only. Certain regularities do appear, however. Applying the formula $\delta_a = -b_a^{N=1}\Delta + k_a Y$ to the Z_4 orbifold in Table I, we find that the observable sector gauge groups agree with the prediction: $\Delta \simeq -0.12$, $Y \simeq 14$. By contrast, fitting with the two-component decomposition simultaneously the observable and hidden sector gauge groups leads to inconclusive fits. The overall variations for the parameters cover the range, $\Delta \simeq -0.12$ to $+0.2$, $Y \simeq 10$ to 14. The corresponding predictions for Table II show now variations for both observable and hidden sectors separately, yielding the ranges $\Delta \simeq -0.12$ to $+0.2$, $Y \simeq 2$. We note here that with the alternative identification of the coefficient b_a to the total, summed $N=1,2$ sector slope parameter, regularities are altogether absent.

For the $Z_{6,8,12}$ orbifolds, in the lattice realization designated by the suffix I, the results shown in Table I reveal similar features to those of the Z_4 case. Thus, restricting ourselves to the observable sector only, one satisfactorily fits the numerical results based on the two-component formula with $\Delta = -(0.2, 0.15, 0.59)$, $Y=(19,22,44)$ for the $Z_{6,8,12}$-I orbifolds, respectively. Application to the mixed observable and hidden sector corrections yields instead the domains of variations, $\Delta \simeq (0.15\text{–}0.22)$, $Y \simeq (9\text{–}12)$, when going from Z_6-I to Z_{12}-I. The alternative lattice realizations associated with $Z_{6,8,12}$-II lead to qualitatively similar results. Thus, for the Z_8-II standard embedding orbifold [characterized by the shift vector $v_i = (13-4)/8$], we find, for the same order of gauge group factors as for the Z_8-I case in Table I, $b_a = (-37.5,\ -220,\ -1135,\ 60)$, $\delta_a = (19,\ 49,\ 589,\ -2.6)$, which results in the range of variation, $\Delta \simeq -0.4$ to $+0.2$.

Finally, the results for our single $Z_3 \times Z_3$ orbifold case in Table III, indicate rather small values for δ_a. These can be well fitted with the two-component formula, with the parameters $\Delta \simeq 0.03 \sim 0.1$, $Y \simeq 0.54$. We have also examined for the $Z_3 \times Z_3$ orbifold the effect of the discrete torsion factor $\epsilon(m_1, m_2, n_1, n_2) = e^{2\pi i p(m_1 n_2 - m_2 n_1)/N}$ $(p=0,\dots,N)$. The results in Table III refer to the case $p=0$. The spectrum and the slope parameters $b_a^{N=1}$ are known [9] to depend on the torsion. The changes in the present case are light; in particu-

TABLE III. Threshold corrections for a selection of three-generation orbifold models with two Wilson lines (cases A–C) and one Wilson line (case D). For Z_3 orbifolds, the winding number parameters attached to the Wilson lines take the values $m_{k,f}=0,\pm 1$. Case A is a standard model group Z_3 orbifold model studied by Font *et al.* [9], (Sec. 4.2): $3V^I=(1^4 2000)(20^7)'$, $3a^I_{1,2}=(0^7 2)(0110^5)'$, $3a_{3,4}=(11121011)\times(110^6)'$. Case B is a left-right group Z_3 orbifold model Z_3 orbifold model studied by Font *et al.* [9] (Sec. 4.3): $3V^I=(1^4 2000)(20^7)'$, $3a^I_1=(0^7 2)(00110^4)'$, $3a^I_3=(1^3 21^3 0)(110^6)'$. Case C is an intermediate unification group Z_3 orbifold model examined by Kim and Kim [74]: $3V^I=(11211200)(0^8)'$, $3a^I_1=(0^3 11211)(1^4 0^4)'$, $3a^I_3=(0^7 2)(1^8)'$. Case D is a standard model group Z_7 orbifold model due to Katsuki *et al.* [70]: $7V^I=(2^3 0^5)(110^6)'$, $7a^I_1=(32-152111)(4220^5)'$. Case E is an intermediate unification group $Z_3\times Z_3$ orbifold model with one Wilson line from Font *et al.* [9] (Sec. 5): $3v^i_1=(1,0,-1)$, $3v^i_2(0,1,-1)$; $3V^I_1=(2110^5)(110^6)'$, $3V^I_2=(020^6)(0-1111000)'$; $3a^{(1)I}_1=(0^5 11-2)(0^5 11-2)'$. (The indices 1,2 refer to the two Z_N factors.) The successive line entries list the group factors, the charge generators components Q^I_a, the levels k_a, the β function slope parameters b_a or, for the nonprime orbifolds with $N=2$ suborbifolds, the pairs $(b^{N=1}_a,\hat{b}'_a)$, such that $b_a=b^{N=1}_a+\hat{b}'_a$, and the moduli independent threshold corrections δ_a.

A: Z_3	SU(3)	SU(2)	U(1)$_1$	U(1)$_2$	U(1)$_4$	U(1)$_5$	U(1)$_Y$	U(1)$'_4$	U(1)$'_6$	SO$'_{10}$
Q_a	$(0^3 1^2 0^3)$	$(10-10^5)$	$(1^3 0^5)$	$(0^3 1-10^3)$	$(0^6 10)$	$(0^7 1)$		$(10^7)'$	$(110^6)'$	$(0^6 1-1)'$
k_a	1	1	6	4	2	2	$\frac{11}{3}$	2	4	1
b_a	-9	-18	-216	-104	-30	-16	-71.5	-14	-68	18
δ_a	4.01	4.62	45.3	20.7	5.44	4.49	16.3	2.65	18.2	2.16

B: Z_3	SU(3)	SU(2)^L	SU(2)^R	U(1)$_1$	U(1)$_2$	U(1)$_3$	U(1)$_4$	SU(2)$'$	SO(8)$'$	U(1)$'$
Q_a	$(10-10^5)$	$(0^5 1-10)$	$(0^3 1^2 0^3)$	$(1^3 0^5)$	$(0^3 1-10^3)$	$(0^5 1^2 0)$	$(0^7 1)$	$(0^2 1-110^4)'$	$(0^4 10^3)'$	$(10^7)'$
k_a	1	1	1	6	4	4	2	1	1	2
b_a	-6	-15	-15	-216	-100	-100	-12	-24	6	-26
δ_a	3.76	4.37	4.37	44.9	20.2	20.2	4.17	5.0	1.27	5.12

C: Z_3	SU(3)	SU(3)	U(1)$_1$	U(1)$_2$	U(1)$_3$	U(1)$_4$	U(1)$_5$	SO(12)$'$	U(1)(5)$'$	U(1)(6)$'$
Q_a	(1010^5)	$(0^3 1010^2)$	(1^2-10^5)	$(0^3 1^2-10^2)$	(0^6-10)	(0^7-1)	$(12-10^5)$	$(1-10^6)'$	$(-1^4 0^4)'$	$(0^4-1^4)'$
k_a	1	1	6	6	2	2	12	1	8	8
b_a	-18	-18	-310	-257	-31.5	-25.5	-1176	27	-464	-400
δ_a	4.58	4.58	51.3	47.6	5.50	5.10	200	1.51	85.2	81

D: Z_7	SU(2)	SU(3)	U(1)$_1$	U(1)$_2$	U(1)$_3$	U(1)$_4$	U(1)$_5$	E$'_6$	U(1)(6)$'$	U(1)(7)$'$
Q_a	$(0^3 1^2 0^3)$	$(0^6 1-1)$	(10^7)	(010^6)	$(0^2 10^5)$	$(0^3 1-10^3)$	$(0^5 1^3)$	$(0^5 1-10)'$	$(1^2 0^6)'$	$(1-120^5)'$
k_a	1	1	2	2	2	4	6	1	4	12
b_a	-14	-14	-20	-22	-23	-100	-217	36	-92	-788
δ_a	11	11	11	12	12	46	108	0.70	48	397

E: $Z_3\times Z_3$	SU(2)	SU(2)^L	SU(2)^R	SU(3)	U(1)$_1$	U(1)$_2$	U(1)$_3$	SU(3)$'$	SO(6)$'$	U(1)$'$
Q_a	(1010^5)	$(0^3 1^2 0^3)$	$(0^3 1-10^3)$	$(0^5 10-1)$	(010^6)	$(10-10^5)$	$(0^5 1^3)$	$(0^5 10-1)'$	$(0^2 1-10^4)'$	$(1^2 0^6)'$
k_a	1	1	1	1	2	4	6	1	1	4
$\begin{pmatrix} b^{N=1}_a \\ \hat{b}'_a \end{pmatrix}$	$\begin{pmatrix} 2 \\ -24 \end{pmatrix}$	$\begin{pmatrix} -10 \\ -24 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -36 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -13 \end{pmatrix}$	$\begin{pmatrix} -3 \\ -56 \end{pmatrix}$	$\begin{pmatrix} -42 \\ -112 \end{pmatrix}$	$\begin{pmatrix} -12.5 \\ -270 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -13 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -13 \end{pmatrix}$	$\begin{pmatrix} -15 \\ -126 \end{pmatrix}$
δ_a	0.50	0.85	0.60	0.60	0.60	6	6.35	0.60	0.60	3.75

lar, the $N=2$ slope parameters are unchanged. Thus, for the choice $p=1$, we find for the same ten group factors of case E in Table III, $b^{N=1}_a=(-1,5,-2,-2,-2,-62,-6,-2,-2,-59)$, $\delta_a=(0.49,0.31,0.59,0.59,0.82,6.2,4.6,0.59,0.59,5.0)$. These results again fit nicely the two-component formula with identical ranges for the parameters as in the above $p=0$ case.

IV. UNIFICATION AND ANOMALOUS U(1) SCALE

A. Threshold corrections

In this section we examine the viability of the perturbative superstring unification within the orbifold approach. Let us first discuss the implications of the results obtained in Sec. III for the moduli-independent threshold corrections. We restrict to the standardlike Z_3 orbifold models given in Table III. Since δ_a are generally of positive sign, it follows that the moduli-independent threshold corrections will always result in reduced effective unified coupling constant and enhanced (reduced) unification scale, depending on whether the β function slope parameters b_a are positive (negative) or, equivalently, gauge (matter) dominated. Assuming the simple formula $\delta_a=-b_a\Delta+k_aY$, then, as already noted in connection with Eq. (4), one can absorb the string threshold corrections into an effective unification scale M_X' and an effective string coupling constant g_X'. Using the numerical values for b_a and δ_a in Table III, we find very small moduli-independent corrections to the unification scale and the coupling constant, which attain at most a 10% enhancement and a 5% reduction, respectively.

Identifying the string moduli-independent threshold corrections obtained here, $\delta_a/4\pi\simeq 0.4$, tentatively with a corresponding field theory threshold correction of typical structure [76], $\delta(4\pi/g_a^2)=\pm O(1)\ln(M_HM_X)$, yields for the ratio of the average heavy particle mass to unification mass, $M_H/M_X\simeq \frac{1}{2}$. Thus, one checks that these contributions are of the same order of magnitude as the two loop field theory renormalization corrections [77]. We conclude therefore that the moduli-independent threshold corrections should mildly affect the high energy extrapolation of the gauge coupling constants. More quantitatively, one can estimate the corrections to the weak angle and color coupling constant by means of the formulas [36]

$$\sin^2\theta_W(m_Z)=\frac{k_2}{k_1+k_2}+\frac{\alpha(m_Z)}{4\pi}\frac{k_1}{k_1+k_2}\left[A\ln\frac{m_Z^2}{M_X^2}+\Delta_A\right], \tag{30}$$

$$\alpha_s^{-1}(m_Z)=\frac{k_3}{k_1+k_2}\left[\frac{1}{\alpha(m_Z)}+\frac{B}{4\pi}\ln\frac{m_Z^2}{M_X^2}+\frac{\Delta_B}{4\pi}\right], \tag{31}$$

where we use the notation $A=-(b_1k_2/k_1-b_2)$, $B=-[b_1+b_2-b_3(k_1+k_2)/k_3]$, $\Delta_A=-(\Delta_1k_2/k_1-\Delta_2)$, $\Delta_B=-[\Delta_1+\Delta_2-\Delta_3(k_1+k_2)/k_3]$. Evaluating the threshold corrections for case A in Table III, by using $k_1=11/3$ and $\Delta_{1,2,3}=(16.3,4.62,4.01)$, yields

$$\delta(\sin^2\theta_W(m_Z))\simeq(-5\sim+8)\times10^{-5},\quad \alpha_s^{-1}(m_Z)\simeq-(4\text{–}5)\times10^{-2}\quad [\delta\alpha_s(m_Z)\simeq(5\text{–}8)\times10^{-4}],$$

where we have set $\alpha^{-1}(m_Z)=127.9\pm0.1$. We see that the corrections are rather small and lie well inside the present experimental uncertainties on these parameters [77], $\alpha_s(m_Z)=0.120\pm0.010$, $\sin^2\theta_W(m_Z)=0.2324\pm0.0006$. The extreme smallness of the effect here is due to the cancellation of the level-dependent component k_aY in δ_a in the linear combinations appearing in $\Delta_{A,B}$. In fact, these cancellation effects are the cause for the large uncertainties in the results above. If we used the two-component formula, so that $\Delta_A=-A\Delta$, $\Delta_B=-B\Delta$, and set k_1,b_a at the vaues prescribed in the minimal supersymmetric model, we would find instead $\delta(\sin^2\theta_W)\simeq-2\times10^{-4}$, $\delta\alpha_s\simeq3\times10^{-4}$.

Turning to the moduli-dependent corrections $\Delta_a^{(m)}$, we note that these are generically of opposite sign with respect to δ_a and so have an opposite effect on the effective unification parameters. These contributions become sizable only to the extent that large moduli VEV's and large ratios $\tilde{b}_a'/b_a$ are used, as is clearly demonstrated in the approximate formula, valid for large VEV's,

$$M_X'\simeq M_X\left[\frac{e^{\frac{\pi(T+\bar{T})}{6}}}{T+\bar{T}}\right]^{\frac{\tilde{b}_a'}{2b_a}}. \tag{32}$$

To estimate the corrections in Eqs. (17), one can use the approximate formulas

$$\Delta_{A,B}\simeq\begin{pmatrix}A'\\B'\end{pmatrix}\left[\ln(2T_R)-\frac{\pi}{3}T_R\right],$$

where

$$\begin{pmatrix}A'\\B'\end{pmatrix}=\begin{pmatrix}A-\delta A\\B-\delta B\end{pmatrix},$$

such that

$$\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}28/5\\20\end{pmatrix}$$

for the minimal supersymmetric standard model and $\delta A,\delta B$ depend on the modular weight parameter assignments. The solutions reported in Refs. [36,53] give

$$\begin{pmatrix}A\\B\end{pmatrix}\simeq\begin{pmatrix}4\text{–}16\\24\text{–}40\end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix}A'\\B'\end{pmatrix}\simeq\begin{pmatrix}2-10\\0\text{–}20\end{pmatrix}.$$

In order for the corrections to $\sin^2\theta_W$ and α_s to reach an order of magnitude higher than those found above from the moduli-independent corrections, one needs, at least, $T_R=\mathrm{Re}(T)\sim10$.

B. Standardlike superstring unification scenario

We shall now present an extended analysis of the string unification picture in which the coupling constant scale evolution proceeds through an intermediate threshold at M_A induced by an anomalous U(1) mechanism. A two-stage scale evolution is considered: an initial short evolution from M_S to M_A, described by the slope parameters b_a^A set at the values predicted in the orbifold models, followed by a wide scale evolution from M_A to m_Z described by the minimal supersymmetric standard model slope parameters. The relevant formula reads

$$\frac{(4\pi)^2}{g_a^2(\mu)}=k_a\left(\frac{(4\pi)^2}{g_X^2}+\tilde{Y}\right)+2b_a\ln\frac{\mu}{M_A}+2b_a^A\ln\frac{M_A}{M_X}+\Delta_a^{(m)}(T,\bar{T}). \tag{33}$$

We regard the five parameters (g_X, k_1, T, $\tilde{Y}\equiv Y_0+Y, M_A$), which enter explicitly Eq. (33), as adjustable parameters. Note that M_X has a fixed linear dependence on g_X which is specified by Eq. (3). The moduli-independent contribution has been incorporated here through the level-dependent component k_aY only. Thus, we set $\Delta=0$. Incorporation of the slope-dependent component, $-b_a\Delta$ could be achieved by modifying the relationship between the string unification scale and coupling constant, using $M_X=5.27\times10^{17}g_Xe^{\Delta/2}$ GeV. The compactification scale M_C can be tentatively identified in order of magnitude with the average of the inverse radii of the 6D torus periods by writing

$$M_C=\frac{2\pi}{R}\simeq\frac{M_S}{2}\left(\frac{C_{\rm orb}}{T}\right)^{1/2}\simeq\frac{2\sqrt{C}_{\rm orb}M_X}{\sqrt{T}}, \tag{34}$$

where $R=2\pi r$, the average circumference, and the average moduli field VEV's $\langle T\rangle=T$ are related as $T=C_{\rm orb}R^2/\alpha'(2\pi)^2$, with $C_{\rm orb}$ a calculable constant of order unity [71], determined by the requirement that the target space duality transformation acts like $T\rightarrow1/T$. For the simple T^6 torus, $C_{\rm orb}=1$, while for, say, the Z_3 orbifold, $C_{\rm orb}=\sqrt{3}/4$. One concludes from Eq. (34) that $M_C/M_X\simeq1/\sqrt{T}$.

A rough order of magnitude estimate for the anomalous U(1) Higgs mechanism scale M_A can be obtained by imposing the condition of a vanishing D-term scalar potential [59] $-D_A/g_A^2=\Sigma_\alpha Q_A^\alpha|\phi_\alpha|^2+g_Xc_A/4\alpha'\sqrt{k_A}$ for a group factor $\mathrm{U}_A(1)$ distinguished by the index A. We recall that the triangle anomaly coefficient c_A is defined as [19] $48\pi^2c_A=\mathrm{Tr}(Q_A)=4\mathrm{Tr}(Q_A^3)$, where the traces extend over the massless modes. This enters the Green-Schwarz counterterm through the substitution for the dilaton field, $S+\bar{S}\rightarrow S+\bar{S}+c_AV_A$, whose function is to cancel the various $\mathrm{U}_A(1)$ group factor (gauge and gravitational) triangle anomalies by assigning to the gauge vector and dilaton chiral supermultiplet fields the transformation laws $V_A\rightarrow V_A-\Lambda_A-\Lambda_A^\star$, $S\rightarrow S+c_A\Lambda_A$. The predicted magnitude for the scale is

$$M_A\simeq\langle\phi\rangle=\frac{M_P}{\sqrt{8\pi}}\frac{g_X}{\sqrt{2}}\left[-\frac{g_X\mathrm{Tr}(Q_A)}{192\pi^2Q_A\sqrt{k_A}}\right]^{1/2}. \tag{35}$$

Using tentatively for the model-dependent ratio the estimate $-\mathrm{Tr}(Q_A)/(Q_{A\alpha}\sqrt{k_A})\simeq10$, one obtains $M_A\simeq1.2g_X^{3/2}\times10^{17}$ GeV, which indicates that M_A should be of the same order of magnitude as M_X.

We use the known experimental values of the gauge coupling constants at the Z-boson mass, namely, $g_1^2(m_Z)=0.127$, $g_2^2(m_Z)=0.425$, $g_3^2(m_Z)=1.44$, as inputs to determine via Eq. (33) three among the above-quoted adjustable parameters. We choose these to be g_X, $\tilde{Y}$, M_A. This choice is motivated by the fact that the dependence on these parameters in Eq. (33) can be made linear by means of an obvious change of variables. The solutions for g_X, $\tilde{Y}$, M_A are determined as a function of the remaining free parameters, namely, T and k_1, and the sets of slope parameters, b_a^A, $\tilde{b}_a'$. For a solution to be acceptable it must comply with the perturbation theory constraints that g_X and Y be of order unity and with the obvious inequalities between scales, $M_A/M_X<1$, $M_C/M_X<1$, which we shall eventually supplement by the inequality $M_A/M_C<1$, reflecting the assumption that the mechanism inducing the scale M_A is a consequence of compactification.

We shall present the results of numerical applications only for case A in Table III, setting $b_a=(-11,-1,3)$, corresponding to the minimal supersymmetric standard model, $b_a^A=(-71.5,-18,-9)$, as obtained from Table III, and $b_a'=(18,8,6)$, $\delta_{\rm GS}=7$, where the choice of slope and Green-Schwarz parameters $b_a'=\Sigma_ib_a'^i$, $\delta_{\rm GS}=\Sigma_i\delta_{\rm GS}^i$ for the moduli-dependent threshold corrections is based on the solutions reported in Ref. [36] (see also Ref. [78]). Regarding k_1 as a free parameter when this is predicted to be 11/3 and including moduli-dependent threshold corrections in a case (such as the Z_3 orbifold) where these are absent is certainly liable to criticism. It may also be objected that since k_1 and M_A take fixed values once one chooses a given orbifold model, it is not justified to consider these as free parameters. The answer is that we are really studying here a class of models having similar characteristics with respect to the massless spectrum. The dramatic rise in the number of solutions for orbifold models involving two or three Wilson lines [75,79] may be invoked as a plausibility argument to justify some freedom in choosing the hypercharge and the anomalous gauge coupling constant normalizations. Further, since the type of orbifold model appears to have a marginal influence on the size of threshold corrections, as we have concluded in Sec. II, we hope that these shortcomings do not affect the consistency of our procedure.

Our main purpose is to explain the nontrivial interplay between the various parameters which are most significant for string phenomenology. Choosing the particular subset k_1,T as our free parameters while adjusting the others $(\tilde{Y},M_A,g_X)$ to the inputs $g_a(m_Z^2)$ ($a=3,2,1$) is only a technical convenience. Let us first discuss some qualitative features of the solutions and, in particular, the correlations among the parameters. The dependence on $\tilde{Y}$ and g_X shows clearly that any change in g_X can be compensated by a negative contribution to $\tilde{Y}$. A decrease of k_1 widens the distance between the quantities $(g_a^2k_a)^{-1}$ and so can be compensated by decreasing M_A/M_X or g_X. Finally, because the functional dependence on M_A and g_X in Eq. (33) involves a logarithm of these quantities, one expects a strong sensitivity of the parameters on the inputs.

The results are displayed in Fig. 2. These represent a continous two-parameter (k_1,T) family of solutions for g_X, $\tilde{Y}$, M_A consistent with a high energy extrapolation of the gauge

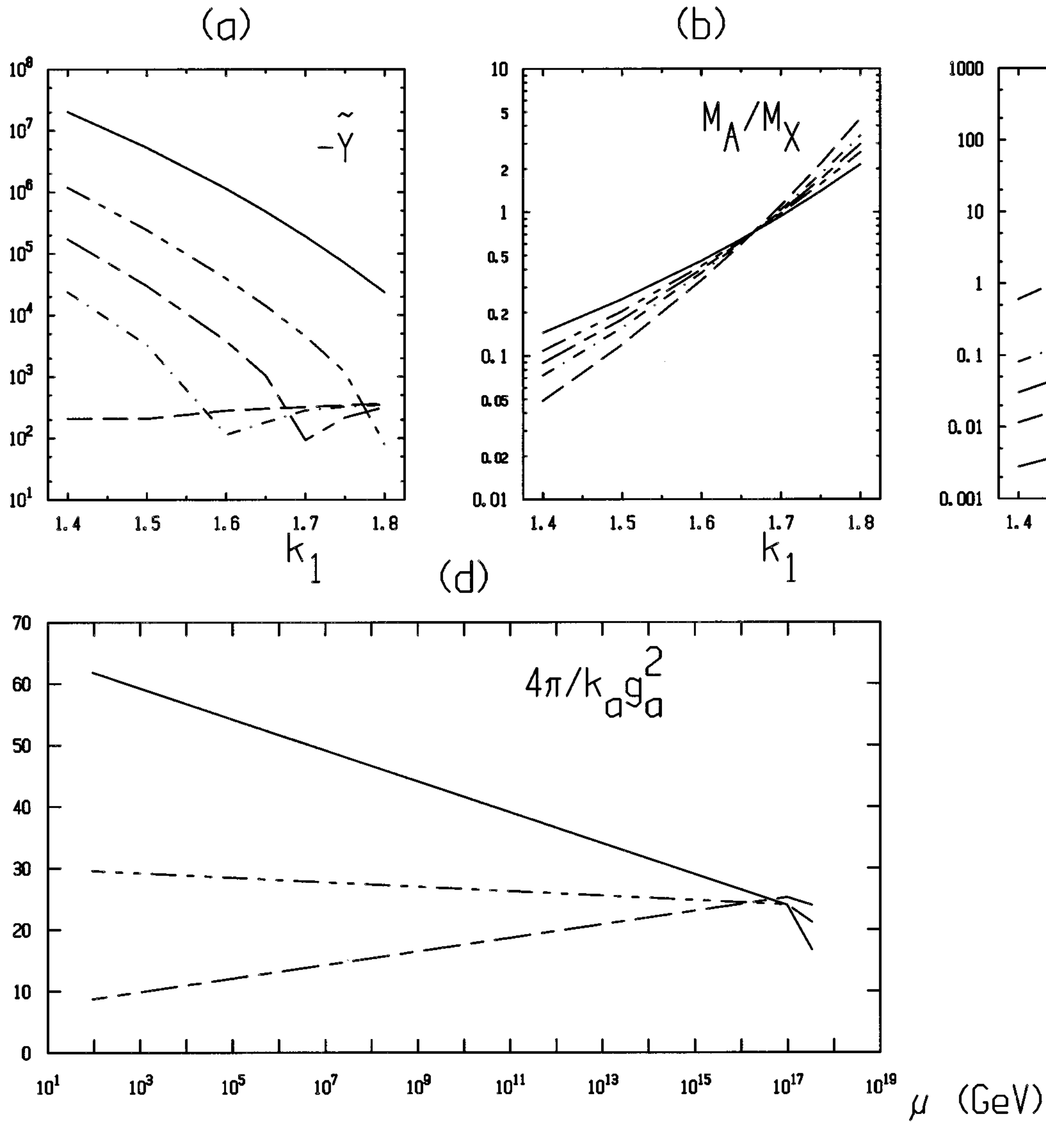

FIG. 2. One loop renormalization group analysis of superstring unification parameters based on high energy extrapolation of the gauge coupling constants starting from their experimental values at m_Z. The solutions for $-\tilde{Y}$ (a), M_A/M_X (b), and g_X (c) are plotted as a function of k_1 for a discrete set of values of the moduli VEV, $T=1$ (solid line), 10 (long-dashed–short-double-dashed line), 15 (long-dashed–short-dashed line), 20 (dash-dotted line), 30 (dashed line). The slope discontinuities exibited by $\tilde{Y}$ in (a) arise because of the changes of sign of $\tilde{Y}$ in this semilogarithmic plot. (For the $T=30$ curve, $\tilde{Y}>0$.) We display in (d) graphs of the gauge coupling constant $[4\pi/g_a^2 k_a,\ (a=3,2,1)]$ variation with renormalization scale for the particular solution characterized by the values $k_1=1.6$, $T=20$, yielding the solution $\tilde{Y}=-114$, $M_A/M_X=0.38$, $g_X=0.63$.

coupling constants joining roughly at the common value $4\pi/g_a^2 k_a \simeq 25$. The physical constraints on $\tilde{Y}$, g_X, M_A select a reduced domain for the free parameters, $k_1 \in (1.4,1.8)$, $T \in (1,30)$. The variations with respect to these parameters are monotonic. For fixed T, increasing k_1 leads to a rapidly (algebraically) increasing $\tilde{Y}$ from large negative to positive values and to less rapidly increasing M_A/M_X and g_X. Strong variations are also found for the T dependence. However, as T increases past $T \approx 25$, $\tilde{Y}$ becomes positive and nearly independent of k_1. The values of k_1 on the lower side, $k_1 < 1.4$, are excluded by the constraints on $\tilde{Y}$ and those on the higher side, $k_1 > 1.8$, by the constraints on g_X and M_A/M_X.

A wide class of solutions occurs with $g_X \ll 1$ and $-\tilde{Y} \gg 10^3$, independently of T and k_1. These arise through an obvious compensation effect of the moduli-independent corrections with g_X, as is apparent in Eq. (33). Although the Y_0 component of $\tilde{Y}$ remains uncalculated so far, it appears unlikely that this can much exceed the component Y which was evaluated in Sec. IV to be of $O(1)$. In fact, since a large Y_0 is only possible for a strongly coupled string theory involving a large g_X, the above must be regarded as an inconsistent class of solutions. (However, because the generic dependence on coupling constant of nonperturbative effects is expected to be less suppressed in string theory than in field theory [29], e^{-c/g_X} versus $e^{-(4\pi)^2/g_X^2}$, one could possibly achieve large Y_0 with not too large g_X.) In the following, we shall restrict ourselves to the conventional framework where one assumes a smooth connection between string theory and its low energy limit and hence retains the constraints $g_X \sim 1$, $|\tilde{Y}| \sim 10$.

Examining the variation of the solutions with k_1 in Figs. 2(a)–2(c), we see that these are very rapid, especially that of $\tilde{Y}$. The condition $\tilde{Y} \sim 1$ can be satisfied only through a very careful fine-tuning of k_1 for fixed T or of T for fixed k_1. This is possible only in cases where $\tilde{Y}$ changes sign in the relevant intervals of k_1, T. The moduli-dependent corrections are quite essential to achieve a high energy extrapolation consistent with superstring unification. Incorporating the threshold M_A provides solutions with reduced T. The constraints on $\tilde{Y}$ and g_X require $15 < T < 30$ and $1.5 < k_1 < 1.8$. Incorporating the constraint $M_A/M_X < 1$ restricts this interval to $1.5 < k_1 < 1.7$. (Narrower intervals would be imposed if one also sets lower bounds, say, $M_A/M_X > 10^{-1}$ and $g_X > 10^{-1}$.) If one takes into account the additional constraint $M_A/M_C < 1$, this would lead to the stronger bound, $M_A/M_X \simeq M_A/M_C \sqrt{T} < 1/\sqrt{T}$, which would select the narrower interval $1.5 < k_1 < 1.6$.

For concreteness, we show in Fig. 2(d) the scale evolution of the gauge coupling constants for one particular solution as determined by the above procedure. One should not be disturbed by the large value of $|\tilde{Y}|$ used here, since the nearby solution determined with a carefully tuned value of k_1 or T so as to give $\tilde{Y} \sim 1$ would yield nearly identical flows for the gauge coupling constants. This figure illustrates one of the characteristic implications of string unification, namely, that the simultaneous equality at some scale of the extrapolated coupling constants has no special significance. The picture depicted in Fig. 2(d) is rather generic. The most favorable situation corresponds then to an approximate joining of the coupling constants flows at a large scale near 5×10^{16} GeV, which is to be identified with the anomalous $U_A(1)$ scale M_A, associated with the decoupling of the extra quarks or leptons modes. In the string unification picture, the joining scale M_A can be made larger than $M_{\rm GUT}$ because of the slightly reduced normalization of the hypercharge group coupling constant and of the spread of the coupling constants at M_X which is related to the moduli-dependent threshold corrections.

Let us comment briefly on the sensitivity of the solutions to the slope parameters. [Our procedure would obviously break down for $b_a^A \approx b_a$, as this would make the linear system of equations, Eq. (33), singular.] The slope parameters b_a^A determine the variation of the coupling constants from M_X to M_A. The choice of b_a^A is correlated to that of the moduli-dependent slope parameters $\tilde{b}_a'$, since the latter determine the amount by which the coupling constants are spread at M_X. Consider first the case of fixed $\tilde{b}_a'$. Increasing T implies a wider spread of the coupling constants at M_X which should therefore be compensated by larger slopes b_a^A in order to catch up with the extrapolated coupling constants up to 10^{16} GeV. Rather than showing new plots, we only mention here that if one performs a uniform reduction of the slopes b_a^A by, say, a factor 2, the solutions would rule out the entire domain in k_1, T except for a narrow region around $T=15$, $k_1=1.7$. Conversely, enhancing the slopes b_a^A by, say, a factor 2 ameliorates the initial picture without changing qualitatively the character of solutions. One concludes therefore that the cases involving negative slope parameters b_a^A with large absolute values (richer matter spectra), which are generic in orbifold model building, are more favorable for unification.

The choice of $\tilde{b}_a' = b_a' - k_a \delta_{\rm GS}$ is also quite sensitive. Rather than performing an exhaustive study we have considered two other cases obtained from Ref. [36] and further motivated in Ref. [78]. Applying the above procedure of solution for these cases, we found a significantly worsened picture. The first case, characterized by $b_a' = (7.5, 2.5, 1.50)$, $\delta_{\rm GS} = 2.5$, admits solutions only for large values of $T > 30$ and correspondingly large $k_1 > 1.8$. It improves slightly if reduced values are used for the slopes b_a^A. The second case, characterized by $b_a' = (-4.67, 4, 5)$, $\delta_{\rm GS} = 6$, admits no solutions at all, mainly on account of an incompatibity between the constraints on Y and M_A/M_X. One concludes therefore that negative or small values for the $N=2$ slope parameters $\tilde{b}_a'$ do not constitute a favorable option.

Having focused so far on standardlike compactification models, we briefly discuss the other two possible classes of superstring models. The first refers to compactification models with grand unified groups SU(5) [43] or SO(10) [65] [up to extra U(1) factors], with a flipped assignment for the matter fields with respect to the standard GUT basis or with a regular GUT assignment involving higher affine levels, $k > 1$ [80]. A perturbative weak coupling scenario assuming a smooth evolution from $M_{\rm GUT}$ to M_X can be analyzed in the manner described above either by setting the parameters b_G, $\tilde{b}_G'$, and Y at values specified by the models or by imposing appropriate constraints on them. It should not be difficult to obtain satisfactory solutions for g_X and M_A by following a procedure similar to that used above. An alternative strong coupling scenario could also be envisaged [65] if the slope b_G takes a large (gauge-dominated) positive value and g_X is large so as to lead to GUT group G with renormalization group invariant scale comparable to the string scale, $\Lambda_G = M_X' e^{-8\pi^2 k_G / b_G g'_X{}^2}$. Although such a scenario forbids a smooth connection from string theory to the low energy field theory, it still provides a prediction for the GUT scale, namely, $M_{\rm GUT} \simeq \Lambda_G$.

The second class of compactification models corresponds to intermediate unification on a semisimple electroweak gauge group. One interesting example is case C in Table III where the gauge symmetry at compactification, $SU(3)_c \times SU(3)_w \times U(1)_{P_3}$, breaks down to the standard model group at an anomalous U(1) scale according to $SU(3)_w \times U(1)_{P_3} \to SU(2)_w \times U(1)_Y$, where $Y = T_{8w} + P_3/3$. Using the information supplied in Ref. [74], we find a level parameter $k(P_3) = \frac{1}{3}$. This implies a normalization of the hypercharge coupling constant such that $k_1 = 1 + \frac{1}{27} = \frac{28}{27}$. Although this falls well below the favorable interval of k_1 values specified above, it is nevertheless interesting that the situation for case C is exactly opposite to that found above for case A. However, as already discussed in the Introduction, although one can derive a bound $k_1 \geq \frac{5}{3}$ for minimal standard-like models, this is evaded for $Z_N \times Z_M$ orbifolds or for Z_N orbifolds with Wilson lines [56–58]. As we have just demonstrated, intermediate unification of hypercharge in a non-Abelian group factor provides a further viable option.

V. CONCLUSIONS

Our results confirm those obtained in previous works [31,45] and extend these with new predictions covering a large sample of orbifold cases. The moduli-independent components of the threshold corrections are positive and of typical size $\delta_a/4\pi\sim 1$, which is therefore quite comparable to those for gauge field theories in spite of the fact that infinitely many massive states are integrated out for superstrings. We find that δ_a are nearly insensitive to the gauge group embedding and to the discrete Wilson lines, but that they increase with the point group order of the Z_N orbifolds, roughly according to a linear power law $\delta_a \sim N$. The two-parameter decomposition $\delta_a=-b_a^{N=1}\Delta+k_a Y$, suggested in previous investigations [45], fits very well the Z_3 orbifold models, with the predictions $\Delta\simeq 0.068$, $Y\simeq 3.3$, independently of the embedding of the point and space groups in the gauge group, but has a restricted applicability for the higher order orbifolds Z_7 as well as the nonprime ones. Combining pairwise the various observable group factors, one still finds for the nonprime orbifolds certain regularities, with the following domains of variations for the parameters: $\Delta\simeq -0.6$ to $+0.2$, $Y\simeq 50$–10.

In order for the large value of the predicted string unification scale M_X not to conflict with observations, one needs both moduli-dependent threshold corrections (with associated compactification scale $M_C/M_X\simeq 1/\sqrt{T}\approx 0.3$) as well as a weak hypercharge group level parameter varying in the narrow interval $k_1=1.4$–1.7. The information that the moduli-independent corrections δ_a are 1–10 is useful in providing stronger correlations among the parameters relevant to string phenomenology. Postulating an anomalous U(1) mechanism at a scale $0.1<M_A/M_X<1$ significantly eases the above constraints on slope parameters while raising the bound on the allowed values of M_C. The resulting picture is intermediate between a delayed joining of the coupling constant flows, due to the smaller value of k_1, and of a continued flow beyond crossing, consistent with the moduli-dependent threshold corrections. Our analysis emphasizes the need of constructing orbifold models combining the property of a low value for the hypercharge group level parameter along with the usual desirable features, namely, three chiral families, low rank gauge group, and $N=2$ subsectors.

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