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Pulsar timing limits on very low frequency stochastic gravitational radiation

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A low-frequency, stochastic gravitational radiation background can be detected through the irregularities it induces in pulsar arrival times. Here, we present an optimal statistical framework for analysis of timing data from a single pulsar, correcting an error in previous treatments. Observations of PSR B1855+09 yield an upper limit (95% confidence) of 1.0×10^{-8} or (90% confidence) 4.8×10^{-9} of the closure density at frequency 4.4×10^{-9} Hz. This result probably rules out cosmological models that use cosmic strings as seeds for galaxy formation. Combined observations of four binary pulsars yield weaker limits at frequencies as low as 10^{-12} Hz.

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I. INTRODUCTION

Since the experimental discovery of gravitational radiation [1], it has been clear that the Universe must be filled with a gravitational wave background (GWB), although neither the amplitude nor the spectrum of this radiation is known. While some fraction of the GWB must be truly primordial in origin [2,3], and some due to discrete sources such as binary stellar systems [4,5], some may also be produced in phase transitions in the early Universe [6,7] or arise during inflation [8]. Recent analyses of cosmic string models [9–16] suggest that an observationally interesting level of gravitational radiation should result from the decay of string loops if their mass per unit length were large enough for string models of galaxy formation to be viable.

Observations of millisecond and binary radio pulsars [17–23] provide the tightest current limits on the energy density per logarithmic bandwidth of the GWB. The fluctuations in pulse arrival times can be used to constrain the GWB at frequencies on the order of inverse years and higher, while the stability of binary pulsar orbits constrains the GWB at longer wavelengths. Unfortunately, recent analyses of both these effects contain statistical flaws; analyses of arrival times [22,23] underestimate their experimental sensitivity, while analyses of binary orbits [21,1], by neglecting cosmic variance, overestimate them. For both these analyses we present an optimal alternative based on the Neyman-Pearson lemma.

II. PULSAR PERIOD FLUCTUATIONS FROM GRAVITATIONAL RADIATION

A gravitational wave deforms the metric at the Earth or pulsar, perturbing the observed pulsar frequency. Because the frequencies of some pulsars can be measured with a precision better than 10^{-14}, even very small amounts of gravitational radiation can produce detectable frequency variations. If the GWB has a constant energy density per octave Ω_g relative to the closure density then the power spectrum (power per octave bandwidth) of the resulting pulse arrival time residuals will be [23]

$$S_g(f)=\frac{H_0^2}{8\pi^4}\Omega_g f^{-5}=1.34\times10^4\Omega_g h^2 f_{\mathrm{yr}^{-1}}^{-5}\mu\mathrm{s}^2\mathrm{yr}, \quad (1)$$

where H_0 represents Hubble's constant and $h=H_0/(100\ \mathrm{km\ s^{-1}Mpc^{-1}})$. In constrast, the power spectrum of "white" observation noise falls off as $1/f$. For frequencies much larger than the inverse timespan of the data, the measured root-mean-square timing residuals directly limit the quantity $\Omega_g h^2$ [17]. However, a more complicated analysis is required because the pulsar timing data are fit for the *a priori* unknown pulsar period and period derivative, a procedure that absorbs nearly all harmonic content at $f\lesssim1/T$, where T represents the timespan of the data.

Stinebring *et al.* [22] and Kaspi *et al.* [23] have outlined a technique for the rigorous analysis of pulsar timing residuals that properly accounts for this observational transfer function. Following [24,25], they calculate spectral estimators S_m for each pulsar, with $m=1,2,4,8,\ldots,$ corresponding to frequencies $f\approx m/T$. Using Monte Carlo simulations that exactly reproduce the spacing of observations in the original data, they then calculate $\langle S_m\rangle_w$, the expectation value of the mth spectral estimator if only the white measurement noise is present (the strength of this measurement noise is known from the scatter of the data within a single day). They also calculate $\langle S_m(\Omega_g h^2)\rangle_g$, the expectation of the mth spectral estimator if the dominant noise source has the form of Eq. (1). The random variable $mS_m/[\langle S_m\rangle_w+\langle S_m(\Omega_g h^2)\rangle_g]$ is then χ^2distributed with m degrees of freedom (χ_m^2). For convenience, the results of [23] are reproduced in Table I.

TABLE I. Observed and computed spectral densities of timing residuals for PSR B1855+09 and PSR B1937+21 from Ref. [23]. Values of $\langle S_m \rangle_g$ correspond to a gravitational-wave background of $\Omega_g h^2 = 10^{-7}$; values for other $\Omega_g h^2$ can be found by linear scaling.

Pulsar	m	f (yr^{-1})	S_m ($\mu\mathrm{s}^2\ \mathrm{yr}^{-1}$)	$\langle S_m \rangle_w$ ($\mu\mathrm{s}^2\ \mathrm{yr}^{-1}$)	$\langle S_m \rangle_g$ ($\mu\mathrm{s}^2\ \mathrm{yr}^{-1}$)
B1855+09	1	0.14	0.622	3.08	112.
	2	0.29	0.170	3.68	4.32
	4	0.58	0.824	2.53	0.23
	8	1.15	2.37	2.37	0.059
B1937+21	1	0.12	207	0.087	238
	2	0.24	8.84	0.071	8.03
	4	0.49	0.218	0.071	0.242
	8	0.98	0.067	0.071	0.008

By comparing the measured values of S_m to the expected values under the hypothesis H0 that arrival time variations are only due to observation noise, and under the hypothesis H1 that the arrival time variations are due to observation noise plus a given level of GWB noise, it is possible to place an upper limit on the strength of the GWB noise. Unfortunately, having obtained the spectral estimators, both [22] and [23] proceed incorrectly in calculating this limit. Their procedure is most clearly explained in App. A of [23], and we summarize it here briefly.

In order to compare the probability of obtaining a spectral estimate no larger than the observed $S_m(j)$ under the two hypotheses H0 and H1, they define a "normalized probability" $\mathscr{P}$ for a given observation which is the ratio of the probability of observing a value S_m or less given H1 to the probability of observing a value S_m or less given H0. Then, noting that the individual values of S_m are statistically independent, they define the total normalized probability $\mathscr{P}_{\mathrm{tot}}$ of the entire set of observations as the product of the individual normalized probabilities, $\mathscr{P}_{\mathrm{tot}} = \Pi_i \mathscr{P}_i$, and take $1-\mathscr{P}_{\mathrm{tot}}$ to be the confidence with which H1 can be rejected. However, this "normalized probability" is not formally a probability at all (see for example [26]); for example, note that neither the $\mathscr{P}$ nor the $\mathscr{P}_{\mathrm{tot}}$ is restricted to values less than unity. As a consequence, the quantity $1-\mathscr{P}_{\mathrm{tot}}$ does not represent a confidence level in the usual sense of being the likelihood that one has avoided an "error of the second kind" [27] and rejected H1 even though it is true.

We have reexamined the data presented by [23], using the Neyman-Pearson test. As discussed in [27], this is the optimal test for distinguishing between simple hypotheses (those with no free parameters). The problem of estimating an upper limit on the GWB can be formulated as the successive comparision of simple hypotheses, the hypothesis H1 that the arrival time variations are due to measurement error plus a GWB contribution of strength $\Omega_g h^2$, and the hypothesis H0 that the arrival times variations are due to measurement error alone. The upper limit on the strength of the GWB is determined by finding the value of $\Omega_g h^2$ for which our ability to reject H1 becomes unacceptably low.

The Neyman-Pearson test can be formulated as follows [27]. Given a random variable $\mathbf{x} = x_1, x_2, \ldots x_N$ (corresponding to N observations) which has a probability density function $f_N(\mathbf{x}|\theta)$ where θ is a parameter that distinguishes the two hypotheses; in our case $\theta=0$ for H0 and $\theta=\Omega_g h^2$ for H1. For a given pulsar we can then define a likelihood ratio

$$L = \frac{f_N(\mathbf{x}|\Omega_g h^2)}{f_N(\mathbf{x}|0)} = \frac{\prod_{m=1,2,4,8} \chi_m^2\left(\frac{m S_m}{\langle S_m \rangle_w + \langle S_m(\Omega_g h^2) \rangle_g}\right)}{\prod_{m=1,2,4,8} \chi_m^2\left(\frac{m S_m}{\langle S_m \rangle_w}\right)}. \tag{2}$$

According to the Neyman-Pearson test, if $f_N(\mathbf{x}|\theta)$ is absolutely continuous on $\mathbf{x}$, then the hypothesis H1 can be rejected at some confidence level $1-\alpha(c)$ if $L \leqslant c$. The complement of the confidence $\alpha(c)$ is the probability that H1 will be rejected (i.e., the probability the observations will yield $L \leqslant c$) even if H1 is true.

In simple cases the dependence of α on c can be calculated analytically. In practice, we determine c for a given α and $\Omega_g h^2$ via numerical integration. Using the data from [23], the residuals from PSR B1855+09 yield

$$\Omega_g h^2 < 1.0 \times 10^{-8} \quad (95\% \text{ confidence}),$$

$$\Omega_g h^2 < 4.8 \times 10^{-9} \quad (90\% \text{ confidence}).$$

Note that since the values of $\langle S_m(\Omega_g h^2) \rangle_g$ are derived using Eq. (1), these limits are strictly true only for the assumed model of a constant gravitational wave energy density per octave, but because nearly all the power of the test comes from the S_1 estimator, the limit on $\Omega_g(f)$ at $f \sim 1/T = 4.4 \times 10^{-9}$ Hz does not depend strongly on this assumption.

The 95% confidence limit of $\Omega_g h^2 < 1.0 \times 10^{-8}$ is six times tighter than the limit quoted in [23], and corresponds to a dimensionless strain [28] at 4.4×10^{-9} Hz of 3×10^{-14}. This limit is derived using only the timing residuals of PSR B1855+09 and is the best limit that pulsar timing can currently place on the GWB. As can be seen in Table I, the observed S_1 and S_2 are significantly larger for PSR B1937+21 than for PSR B1855+09. Since gravitational radiation is quadrupole in nature, and these two pulsars are close together in the sky, the GWB should contribute to the timing residuals of the two pulsars at a similar level, and the observed values of S_m for the two pulsars should be of the same order of magnitude (e.g., column six of Table I). If one assumes a GWB with $\Omega_g h^2 = 10^{-8}$, the probability of observing values of S_m as high as those obtained from the PSR

TABLE II. Orbital period variations of four binary pulsars. [a]

Pulsar	Predicted $\dot{P}_b$ $10^{-12}\mathrm{ss}^{-1}$	Observed $\dot{P}_b$ $10^{-12}\mathrm{ss}^{-1}$	$\Delta\dot{P}_b/P_b$ $10^{-10}\ \mathrm{yrs}^{-1}$
B0655+64 [b]	0	0.0(3)	0.0(1.0)
B1534+12 [c]	−0.1924	−0.175(17)	0.15(15)
B1855+09 [d]	0	0.6(1.2)	0.18(36)
B1913+16 [e]	−2.40258	−2.408(11) [f]	0.06(12)

[a] Numbers in parentheses are uncertainties in last digits quoted.
[b] Ref. [35], and unpublished data of Arzoumanian and Thorsett (1995).
[c] Ref. [35].
[d] Ref. [23].
[e] Ref. [36].
[f] Includes kinematic correction to account for differential galactic acceleration.

B1937+21 data is $<10^{-5}$, leading us to conclude that the PSR B1937+21 residuals are dominated by a noise process other than the GWB. This is not surprising, given that PSR B1937+21 is significantly younger than PSR B1855+09, and, in fact, the observed noise level for PSR B1937+21 is consistent with extrapolations from observations of slower pulsars [29]. Because of this excess noise, the timing residuals of PSR B1937+21 contain no information about the GWB not already provided by PSR B1855+09; however, a number of recently discovered millisecond pulsars appear to have a timing stability comparable or better than that of PSR B1855+09, so the limits on $\Omega_g h^2$ are likely to continue to improve.

III. BINARY PERIOD FLUCTUATIONS

In addition to the rotation of the neutron star, binary pulsar systems provide a second "clock": the binary orbit itself. Most binary pulsars have white dwarf or neutron star companions, and the orbits in these systems are expected to be very clean. Changes in the orbital period can be calculated precisely from general relativity [30] and, in many cases, such changes are negligibly small.

Gravitational waves will perturb the apparent orbital frequency, just as they do the pulsar frequency. However, because no fit for an unknown orbital period derivative $\dot{P}_b$ is required, measurements of pulsar orbits can be used to constrain waves with wavelengths as long as the distance D to the pulsar. If $\langle(\delta\dot{P}_b/P_b)^2\rangle$ is the variance of the observed $\dot{P}_b$ fluctuations, then for frequencies between D^{-1} and T^{-1} it has been shown that $\Omega_g(f)\leqslant\frac{1}{2}H_0^{-2}\langle(\delta\dot{P}_b/P_b)^2\rangle$ [21].

Order-of-magnitude limits on Ω_g have been obtained from the observed $\dot{P}_b$ for PSR B1913+16 (corrected for the general relativistic orbital decay) [21,1]. However, these limits on Ω_g are overly strong as they account for neither the observational uncertainties, nor for the fact that any contribution to $\delta\dot{P}_b/P_b$ from the GWB will be Gaussian distributed (cosmic variance). Here, we present a simple rigorous limit on the GWB for $D^{-1}<f<T^{-1}$, again using the Neyman-Pearson test. There are now three other binary pulsars with $\dot{P}_b$ measurements of similar quality to that of PSR B1913+16 (see Table II) and we use the data from all four pulsars.

TABLE III. Summary of 95% confidence limits on the energy density per octave of the stochastic gravitational background radiation.

Frequency range	Upper limit on $\Omega_g h^2$
$f\sim4.4\times10^{-9}$ Hz	1.0×10^{-8}
10^{-11} Hz$\leqslant f\leqslant4.4\times10^{-9}$ Hz	0.04
10^{-12} Hz$\leqslant f\leqslant10^{-11}$ Hz	0.5

Let H0 be the hypothesis that the measurements $x_i=\delta\dot{P}_b/P_b$ for the pulsars are only caused by the measurement errors e_i. Let H1 be the hypothesis that there is an additional variance $s_i^2(\Omega_g h^2)=2\Omega_g H_0^2$ due to the GWB. The probability density $f_i(x_i|\Omega_g h^2)$ of obtaining a given measurement x_i is standard normal, with variance $e_i^2+s_i^2(\Omega_g h^2)$, and the likelihood ratio for the Neyman-Pearson test is

$$L=\frac{f_N(\mathbf{x}|\Omega_g h^2)}{f_N(\mathbf{x}|\ 0)}=\frac{\prod_{i=1,4} f_i(x_i|\Omega_g h^2)}{\prod_{i=1,4} f_i(x_i|0)}, \tag{3}$$

where here the product runs over the four pulsars listed in Table II. This yields limits of

$$\Omega_g h^2<0.04 \quad (95\%\ \text{confidence}),$$

$$\Omega_g h^2<0.02 \quad (90\%\ \text{confidence}),$$

in the range 10^{-11} Hz$\lesssim f\lesssim10^{-9}$ Hz ($D^{-1}\lesssim10^{-11}$ for all four pulsars in Table II). Only the distant PSR B1913+16 is sensitive to waves with 10^{-12} Hz$\lesssim f\lesssim10^{-11}$ Hz. In this range, we have the limits (95%) $\Omega_g h^2\leqslant0.5$ and (90%) $\Omega_g h^2\leqslant0.1$. Note that in contrast to our results from millisecond pulsar timing, these limits are weaker than those previously claimed [21,1].

IV. DISCUSSION

We have used pulsar timing data to set new limits on the stochastic gravitational background radiation at very low frequencies. The limits are summarized in Table III. It is notable that the energy density is constrained to be lower than the critical density over the wide range 10^{-12} Hz$\leqslant f\leqslant4.4\times10^{-7}$ Hz, or almost 19 octaves in frequency. The limits at $f\sim4.4\times10^{-9}$ Hz are the the strongest limits on the current energy density in gravitational radiation at any frequency. (Limits at lower frequencies from studies of anisotropy in the microwave background measure the energy density at the epoch of last scattering [31].)

Our limits on the strength of the GWB place strong constraints on cosmic string models of the microwave backgound anisotropy or galaxy formation. Cosmic strings, one-dimensional space-time defects predicted by some grand unified theories, are characterized by a mass per unit length μ. They lose energy through the emission of gravitational radiation at a rate that depends on μ. Thus limits on the GWB constrain μ, and, therefore, the viability of certain string models. Using a simple model in which waves are emitted primarily in the fundamental mode, Bennett and Bouchet [15] found a relation between the present logarithmic gravitational wave density and the string mass per unit length:

$$\Omega_g h^2 = (0.014 \pm 0.004) G\mu/c^2. \tag{4}$$

Our limit on $\Omega_g h^2$ from PSR B1855+09 corresponds therefore to the limits $G\mu/c^2 \leqslant 1.0\times 10^{-6}$ (95%) and $G\mu/c^2 \leqslant 4.8\times 10^{-7}$ (90%). These bounds are much tighter than the limits from studies of primordial nucleosynthesis [16].

Cosmic strings are of particular interest as possible seeds for the condensation of galaxies or clusters of galaxies. Condensation around loops of string requires $G\mu/c^2 > 10^{-6}$ [32]. Models in which structure forms by accretion in the wake of moving strings have been constructed with $G\mu/c^2 \sim 4\times 10^{-6}$ [33], while a string explanation of the measured anisotropies in the microwave background radiation requires $G\mu/c^2 \approx 2\times 10^{-6}$ [34]. All of these models are difficult to reconcile with our observed limits on $\Omega_g h^2$, and cosmic string calculations that include multi-mode radiation and the effects of radiation produced during the matter era [16], are likely to increase rather than decrease the discrepancy between the GWB observations and the values of μ required by string models.

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