Free energy of QCD at high temperature

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Effective-field-theory methods are used to separate the free energy for a non-Abelian gauge theory at high temperature T into the contributions from the momentum scales T, gT, and g^2T, where g is the coupling constant at the scale $2\pi T$. The effects of the scale T enter through the coefficients in the effective Lagrangian for the three-dimensional effective theory obtained by dimensional reduction. These coefficients can be calculated as power series in g^2. The contribution to the free energy from the scale gT can be calculated using perturbative methods in the effective theory. It can be expressed as an expansion in g starting at order g^3. The contribution from the scale g^2T must be calculated using nonperturbative methods, but nevertheless it can be expanded in powers of g beginning at order g^6. We calculate the free energy explicitly to order g^5. We also outline the calculations necessary to obtain the free energy to order g^6.

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I. INTRODUCTION

One of the most dramatic predictions of quantum chromodynamics (QCD) is that when hadronic matter is raised to a sufficiently high temperature or density, it will undergo a phase transition to a quark-gluon plasma. One of the major thrusts of nuclear physics in the next decade will be the effort to study the quark-gluon plasma through relativistic heavy-ion collisions. For this effort to be successful, it will be important to understand the properties of the plasma as accurately as possible. The two major theoretical tools that have been used to study the quark-gluon plasma are lattice gauge theory and perturbative QCD. Lattice gauge theory has the advantage that it is a nonperturbative method and applies equally well to the quark-gluon phase and to the hadron phase. It is an effective method for calculating the static equilibrium properties of hadronic matter with zero baryon density. Unfortunately, the Monte Carlo methods used in lattice gauge theory cannot be easily applied to problems involving dynamical properties or to hadronic matter that is away from thermal equilibrium or has nonzero baryon density. These are severe restrictions, because a quark-gluon plasma that is produced in heavy-ion collisions will not be at thermal equilibrium and it may have nonzero baryon density. Furthermore, many of the most promising signatures for a quark-gluon plasma involve dynamical properties.

Perturbative QCD can help fill this gap, at least for the quark-gluon phase of hadronic matter. This method can certainly be applied to the static equilibrium properties of a quark-gluon plasma at zero baryon density, but there are no apparent obstacles to also applying it to dynamical problems, to nonequilibrium situations, or to a plasma with nonzero baryon density. Thus it is a powerful tool for studying various aspects of the quark-gluon plasma that might be probed through heavy-ion collisions. However, there are potential difficulties in applying perturbative QCD to the quark-gluon plasma. The method is based on treating the coupling constant g as a small parameter, but $g(\mu)$ is a parameter that varies rather dramatically with the momentum scale μ. In order to apply perturbative QCD, it is necessary that g be small at the scale of the typical momentum of a particle in the plasma, which is of order T or perhaps $2\pi T$. While this is necesssary, it may not be sufficient. At sufficiently high order in perturbation theory, any observable becomes sensitive to low-momentum gluons that interact with a large coupling strength g. In order to rigorously apply perturbative QCD, it is essential to be able to unravel the various momentum scales that play an important role in a problem. If low-momentum contributions are important, they must be treated using nonperturbative methods.

For a quark-gluon plasma at high temperature, there is a hierarchy of three momentum scales that play an important role in static properties. First, there is the scale T of the typical momentum of a particle in the plasma. Next, there is the scale gT associated with the screening of color-electric forces by the plasma. Finally, there is the scale g^2T associated with color-magnetic screening. Only recently has a method been developed that can systematically unravel the contributions from these various momentum scales. The method is based on the construction of effective field theories that reproduce static observables at successively longer distance scales. This effective-field-theory approach is based on an old idea called "dimensional reduction" [1,2]. According to this idea, the static properties of a (3+1)-dimensional field theory at high temperature can be expressed in terms of an effective field theory in three space dimensions. Dimensional reduction has long been used to provide insight into the qualitative behavior of field theories at high temperature [1–4]. The effective-field-theory approach makes this idea into a practical tool for quantitative calculations. In Ref. [5], we developed the effective-field-theory approach to dimensional reduction and applied it to a scalar field with a ϕ^4 interaction. We demonstrated the power of this method by using it to carry out several perturbative calculations beyond the frontiers set by previous work. A similar approach was developed independently by Farakos *et al.* [6], who applied it to the important problem of the electroweak phase transition. This method has also been applied to QCD [7], and used to resolve a long-standing problem involving the breakdown of the perturbation expansion for the free energy [8]. These ideas have also been used to determine the asymptotic behavior of the correlator of Polyakov loop operators [9] and to provide a rigorous nonperturbative definition of the Debye screening mass in non-Abelian gauge theories [10].

Once we have understood how to resolve the contributions of the various momentum scales in thermal QCD, asymptotic freedom guarantees us that perturbation theory will be under control in the high-temperature limit. At sufficiently high temperature, the running coupling constant will be small enough that calculations to leading order in g will be accurate. However, in most practical applications, such as those encountered in heavy-ion collisions, the temperature is not asymptotically large, and we must worry about higher-order corrections. The accuracy of the perturbation expansion can only be assessed by carrying out explicit perturbative calculations beyond leading order. One of the obstacles to progress in high-temperature field theory has been that the technology for perturbative calculations was not well developed. Only very recently have there been any calculations to a high enough order that the running of the coupling constant comes into play. The simplest physical observable that can be calculated in perturbation theory is the free energy, which determines all the static thermodynamic properties of the system. The running of the coupling constant first enters at order g^4. The free energy for gauge theories at zero temperature but large chemical potential was calculated to order g^4 long ago [11]. The first such calculation at high temperature was the free energy of a scalar field theory with a ϕ^4 interaction, which was calculated to order g^4 by Frenkel, Saa, and Taylor in 1992 [12]. (A technical error was later corrected by Arnold and Zhai [13].) The analogous calculations for gauge theories were carried out in 1994. The free energy for QED was calculated to order e^4 by Coriano and Parwani [14] and the free energy for a non-Abelian gauge theory was calculated to order g^4 by Arnold and Zhai [13]. The calculation of Arnold and Zhai was completely analytic, and thus represents a particularly significant leap in calculational technology. The calculational frontier has since been extended to fifth order in the coupling constant by Parwani and Singh [15] and by Braaten and Nieto [5] for the ϕ^4 field theory, by Parwani [16] and by Andersen [17] for QED, and by Kastening and Zhai [18] for non-Abelian gauge theories. In this paper, we present an independent calculation of the free energy for a non-Abelian gauge theory to order g^5 [19], verifying the result of Kastening and Zhai. In our calculation, we use effective-field-theory methods to simplify the calculation and to resolve the contributions to the free energy from the momentum scales T and gT. We also outline the calculations that are required to obtain the free energy to order g^6.

In Sec. II, we describe how effective field theories can be used to resolve the contributions to the free energy from the momentum scales T, gT, and g^2T. In Sec. III, we calculate the coefficients in the Lagrangian for the effective field theory obtained by dimensional reduction. In Sec. IV, we use the effective field theory to calculate the free energy for QCD to order g^5. In Sec. V, we outline the calculations that would be necessary to improve the accuracy to order g^6. In Sec. VI, we discuss the implications of our calculation for convergence of the perturbation expansion for the free energy. We present some conclusions in Sec. VII. In two appendixes, we tabulate the analytic expressions for all the sums and integrals that arise in our calculation.

II. SEPARATION OF SCALES IN THE FREE ENERGY

The free energy for QCD at high temperature T includes contributions from the momentum scales T, gT, and g^2T. In this section, we explain how the contributions from these three momentum scales can be unraveled by using effective-field-theory methods.

The static equilibrium properties for hot QCD are given by the free energy density F, which is proportional to the logarithm of the partition function:

$$F=-\frac{T}{V}\ln\mathcal{Z}_{\rm QCD}, \tag{1}$$

where V is the volume of space. In the imaginary-time formalism for thermal QCD, the partition function is given by a functional integral over quark and gluon fields on a four-dimensional Euclidean space. The Euclidean time τ is periodic with period $\beta=1/T$. The partition function is

$$\mathcal{Z}_{\rm QCD}=\int \mathcal{D}A_\mu(\mathbf{x},\tau)\mathcal{D}q(\mathbf{x},\tau)\mathcal{D}\bar{q}(\mathbf{x},\tau)\times\exp\left(-\int_0^\beta d\tau\int d^3x\,\mathcal{L}_{\rm QCD}\right). \tag{2}$$

The gluon fields are periodic functions of τ while the quark and antiquark fields are antiperiodic. The Lagrangian is

$$\mathcal{L}_{\rm QCD}=\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}+\bar{q}\gamma_\mu D_\mu q, \tag{3}$$

where $G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$ is the field strength and g is the gauge coupling constant. All the quark fields have been assembled into the multiple-component spinor q, and the gauge-covariant derivative acting on this spinor is $D_\mu=\partial_\mu+igA^a_\mu T^a$. The relevant quark flavors are all assumed to be massless.

In order to make our calculations as general as possible, we will express them in terms of the group-theory factors C_A, C_F, and T_F defined by

$$f^{abc}f^{abd}=C_A\delta^{cd}, \tag{4}$$

$$(T^aT^a)_{ij}=C_F\delta_{ij}, \tag{5}$$

$$\mathrm{tr}(T^aT^b)=T_F\delta^{ab}. \tag{6}$$

For an SU(N_c) gauge theory with n_f quarks in the fundamental representation, these factors are $C_A=N_c$, $C_F=(N_c^2-1)/(2N_c)$, and $T_F=n_f/2$. The dimensions of the adjoint representation and the fermion representation are $d_A=N_c^2-1$ and $d_F=N_cn_f$, respectively.

The free energy for QCD can also be calculated using an effective field theory in three space dimensions called electrostatic QCD (EQCD). This effective theory is constructed so that it reproduces static gauge-invariant correlators of QCD at distances of order $1/(gT)$ or larger. It contains an electrostatic gauge field $A^a_0(\mathbf{x})$ and a magnetostatic gauge field $A^a_i(\mathbf{x})$ that can be identified, up to field redefinitions, with the zero-frequency modes of the gluon field $A_\mu^a(\mathbf{x},\tau)$ for thermal QCD in a static gauge [3]. The free energy for thermal QCD can be written

$$F=T\left(f_E(\Lambda_E)-\frac{\ln\mathscr{Z}_{\mathrm{EQCD}}}{V}\right), \tag{7}$$

where $\mathscr{Z}_{\mathrm{EQCD}}$ is the partition function for EQCD:

$$\mathscr{Z}_{\mathrm{EQCD}}=\int^{(\Lambda_E)}\mathscr{D}A_0(\mathbf{x})\mathscr{D}A_i(\mathbf{x})\exp\left(-\int d^3x\,\mathscr{L}_{\mathrm{EQCD}}\right). \tag{8}$$

The functional integral requires an ultraviolet cutoff Λ_E. The Lagrangian for EQCD is

$$\mathscr{L}_{\mathrm{EQCD}}=\frac{1}{4}G_{ij}^aG_{ij}^a+\frac{1}{2}(D_iA_0)^a(D_iA_0)^a+\frac{1}{2}m_E^2A_0^aA_0^a+\frac{1}{8}\lambda_E(A_0^aA_0^a)^2+\delta\mathscr{L}_{\mathrm{EQCD}}, \tag{9}$$

where $G_{ij}^a=\partial_iA_j^a-\partial_jA_i^a+g_Ef^{abc}A_i^bA_j^c$ is the magnetostatic field strength with coupling constant g_E. If the fields A_0 and A_i are assigned the scaling dimension 1/2, then the operators shown explicitly in (9) have dimensions 3, 3, 1, and 2, respectively. The term $\delta\mathscr{L}_{\mathrm{EQCD}}$ in (9) includes all other local gauge-invariant operators of dimension 3 and higher that can be constructed out of A_0 and A_i. Static gauge-invariant correlation functions in full QCD can be reproduced in EQCD by tuning the gauge coupling constant g_E, the mass parameter m_E^2, the coupling constant λ_E, and the parameters in $\delta\mathscr{L}_{\mathrm{EQCD}}$ as functions of g, T, and the ultraviolet cutoff Λ_E of EQCD. The Λ_E dependence of the parameters is canceled by the Λ_E dependence of the loop integrals in the effective theory.

In order to calculate the free energy using EQCD, we must also tune the coefficient f_E of the unit operator, which was omitted from the effective Lagrangian (9) but appears as the first term in the expression (7) for the free energy. It depends on the ultraviolet cutoff Λ_E of EQCD in such a way as to cancel the cutoff dependence of the partition function for EQCD. The coefficient f_E gives the contribution to the free energy from the momentum scale T. The logarithm of the partition function for EQCD includes the remaining contributions from the smaller momentum scales gT and g^2T.

In order to further separate the contributions from the scales gT and g^2T, it is convenient to construct a second effective field theory called magnetostatic QCD (MQCD) which contains only the magnetostatic gauge field $A_i^a(\mathbf{x})$. The free energy for thermal QCD can be written

$$F=T\left(f_E(\Lambda_E)+f_M(\Lambda_E,\Lambda_M)-\frac{\ln\mathscr{Z}_{\mathrm{MQCD}}}{V}\right), \tag{10}$$

where $\mathscr{Z}_{\mathrm{MQCD}}$ is the partition function for MQCD:

$$\mathscr{Z}_{\mathrm{MQCD}}=\int^{(\Lambda_M)}\mathscr{D}A_i^a(\mathbf{x})\exp\left(-\int d^3x\,\mathscr{L}_{\mathrm{MQCD}}\right). \tag{11}$$

The functional integral requires an ultraviolet cutoff Λ_M. The Lagrangian for MQCD is

$$\mathscr{L}_{\mathrm{MQCD}}=\frac{1}{4}G_{ij}^aG_{ij}^a+\delta\mathscr{L}_{\mathrm{MQCD}}, \tag{12}$$

where G_{ij}^a is the magnetostatic field strength with coupling constant g_M. This coupling constant differs from g_E by perturbative corrections. The term $\delta\mathscr{L}_{\mathrm{MQCD}}$ includes all possible local gauge-invariant operators of dimension 5 and higher that can be constructed out of A_i^a. Gauge-invariant correlation functions in EQCD can be reproduced in MQCD by tuning the gauge coupling constant g_M and the parameters in $\delta\mathscr{L}_{\mathrm{MQCD}}$ as functions of the parameters of EQCD (g_E, m_E^2, λ_E, ...) and the ultraviolet cutoff Λ_M of MQCD. The Λ_M dependence of the parameters in the MQCD Lagrangian is canceled by the Λ_M dependence of the loop integrals in MQCD.

In order to calculate the free energy using MQCD, one must also tune the coefficient f_M of the unit operator, which was omitted from the effective Lagrangian (12) but appears as the second term in the expression (10) for the free energy. Its dependence on the ultraviolet cutoff Λ_M of MQCD is canceled by the cutoff dependence of the partition function for MQCD. The coefficient f_M gives the contribution to the free energy from the momentum scale gT. The contribution from the smaller momentum scale g^2T is contained in the logarithm of the partition function for MQCD.

By constructing the effective field theories EQCD and MQCD, we have separated the contributions from the momentum scales T, gT, and g^2T in the free energy. The general structure of the free energy is

$$F=T[f_E(T,g;\Lambda_E)+f_M(m_E^2,g_E,\lambda_E,\ldots;\Lambda_E,\Lambda_M)+f_G(g_M,\ldots;\Lambda_M)]T, \tag{13}$$

where $f_G=-\ln\mathscr{Z}_{\mathrm{MQCD}}/V$. The arbitrary factorization scales Λ_E and Λ_M separate the momentum scales T from gT and gT from g^2T, respectively. The term f_E and the parameters of EQCD (i.e., m_E^2, g_E, λ_E, ...) involve only the scale T. They can therefore be calculated using ordinary perturbation theory as power series in $g^2(2\pi T)$, where $g(2\pi T)$ is the running coupling constant at the scale of the lowest Matsubara frequency $2\pi T$. The term f_M and the parameters of MQCD (g_M, ...) involve only the scale gT. They can be calculated in EQCD as perturbation expansions in g_E^2/m_E, λ_E/m_E, and other dimensionless parameters obtained by multiplying EQCD coupling constants by appropriate powers of m_E. The leading contribution to f_M is proportional to m_E^3. The term f_G in (13) can only be calculated using nonperturbative methods, such as lattice-gauge-theory simulations of MQCD. Surprisingly, however, f_G can be expanded as a weak coupling expansion in powers of g by treating the higher dimension operators in the MQCD Lagrangian as perturbations [7]. The leading term is proportional to g_M^6.

In summary, the free energy for QCD has the general structure given in (13). The term f_E is the contribution from the scale T. It has the form T^3 multiplied by a power series in $g^2(2\pi T)$ whose coefficients can be calculated using ordi-

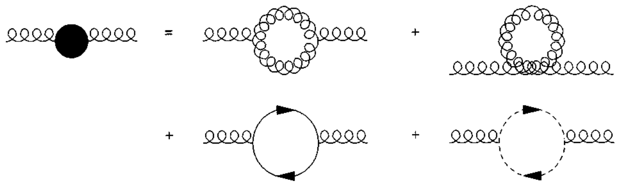

FIG. 1. One-loop Feynman diagrams for the gluon self-energy. Curly lines, solid lines, and dashed lines represent the propagators of gluons, quarks, and ghosts, respectively.

nary perturbation theory in thermal QCD. The term f_M is the contribution from the scale gT. It has the form m_E^3 multiplied by a power series in $g(2\pi T)$ whose coefficients can be calculated using perturbation theory in EQCD. The term f_G is the contribution from the scale g^2T. It has the form g_M^6 multiplied by a power series in $g(2\pi T)$ whose coefficients can be calculated using lattice simulations of MQCD.

III. PARAMETERS IN THE EQCD LAGRANGIAN

In order to calculate the free energy using the EQCD Lagrangian, the parameters in the Lagrangian (9) must be tuned as functions of g, T, and Λ_E so that EQCD reproduces the static gauge-invariant correlation functions of full QCD at distances $R \gg 1/T$. The EQCD parameters can be determined by computing various static quantities in full QCD, computing the corresponding quantities in EQCD, and demanding that they match. It is convenient to carry out these matching calculations using a strict perturbation expansion in g^2. This expansion is afflicted with infrared divergences. The divergences arise from long-range forces mediated by static gluons, which remain massless in the strict perturbation expansion. Physically, these divergences are screened by plasma effects either at the scale gT in the case of electrostatic gluons or at the scale g^2T in the case of magnetostatic gluons. The screening of electrostatic gluons can be taken into account by summing up infinite sets of higher-order diagrams

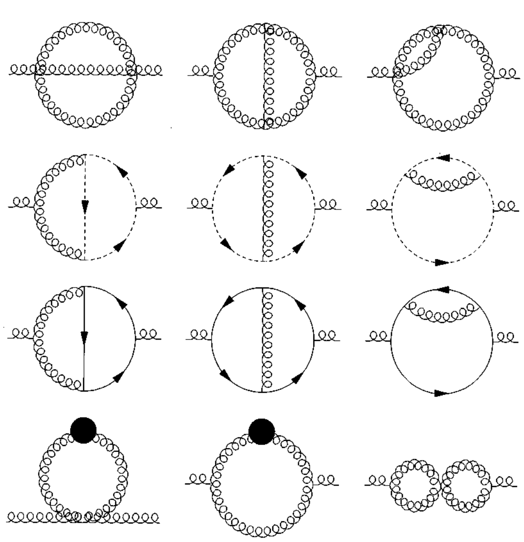

FIG. 2. Two-loop Feynman diagrams for the gluon self-energy. The solid blob represents the sum of the one-loop gluon self-energy diagrams shown in Fig. 1.

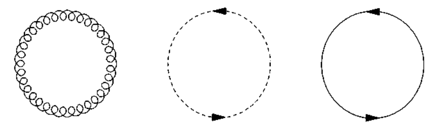

FIG. 3. One-loop Feynman diagrams for the free energy.

in the perturbation expansion, but the screening of magnetostatic gluons can only be taken into account using nonperturbative methods. Fortunately, it is not necessary to treat the effects of screening in a physically correct way in order to determine the parameters in the EQCD Lagrangian. The parameters take into account the effects of large momenta of order T, and they are therefore insensitive to the infrared effects associated with screening. We can therefore simply ignore screening and remove the infrared divergences in the strict perturbation expansion by imposing any convenient infrared cutoff. As long as we use the same infrared cutoff in EQCD and in full QCD, we can determine the EQCD parameters by matching strict perturbation expansions in the two theories. Note that we are using the strict perturbation expansion simply as a device for determining the parameters in the EQCD Lagrangian.

A. Gauge coupling constant

For the calculation of the free energy to order g^5, we require the EQCD gauge coupling constant g_E only to leading-order in g^2. At this order, we can simply read g_E off from the Lagrangian of the full theory. We substitute $A_0(\mathbf{x},\tau) \to \sqrt{T} A_0(\mathbf{x})$ in the QCD Lagrangian (3) and compare $\int_0^\beta d\tau \mathscr{L}_{\rm QCD}$ with $\mathscr{L}_{\rm EQCD}$ in (9). We find that, to leading order in g^2,

$$g_E^2 = g^2 T. \tag{14}$$

There is no dependence on the factorization scale Λ_E at this order. The coupling constant g_E could be calculated to higher-order in g^2 by matching scattering amplitudes in full QCD with the corresponding ones in EQCD.

B. Mass parameter

In this subsection, we calculate the coefficient m_E^2 of the $A_0^a A_0^a$ term in the EQCD Lagrangian to next-to-leading order in g^2. The physical interpretation of m_E is that it is the contribution to the electric screening mass $m_{\rm el}$ from large momenta of order T. The parameter m_E^2 can be determined by matching the strict perturbation expansions for the electric screening mass in full QCD and in EQCD. Beyond leading order in g, the electric screening mass becomes sensitive to magnetostatic screening and requires a nonperturbative definition [10]. However, in the presence of an infrared cutoff, $m_{\rm el}$ can be defined in full QCD by the condition that the

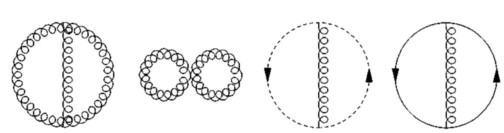

FIG. 4. Two-loop Feynman diagrams for the free energy.

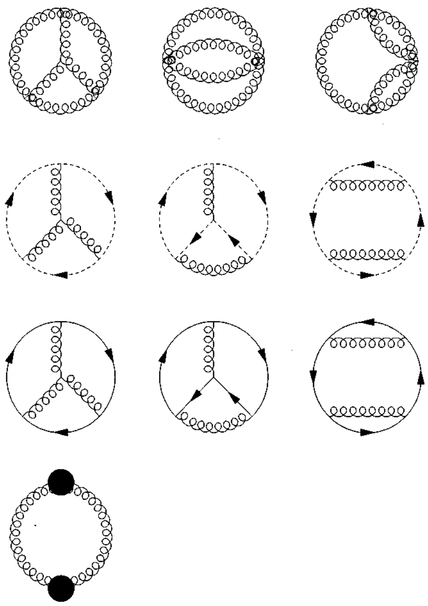

FIG. 5. Three-loop Feynman diagrams for the free energy.

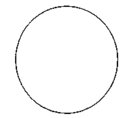

FIG. 6. One-loop Feynman diagrams for the logarithm of the partition function of EQCD.

propagator for the field $A_0^a(\tau,\mathbf{x})$ at spacelike momentum $K=(k_0=0,\mathbf{k})$ has a pole at $\mathbf{k}^2=-m_{\rm el}^2$. It is the solution to the equation

$$k^2+\Pi(k^2)=0 \quad \text{at} \quad k^2=-m_{\rm el}^2, \tag{15}$$

where $\Pi(k^2)$ is the $\mu=\nu=0$ component of the gluon self-energy tensor evaluated at $k_0=0$: $\Pi_{00}^{ab}(k_0=0,\mathbf{k})=\Pi(k^2)\delta^{ab}$. In EQCD with an infrared cutoff, the electric screening mass $m_{\rm el}$ gives the location of the pole in the propagator for the field $A_0^a(\mathbf{x})$. Denoting the self-energy function by $\Pi_E(k^2)\delta^{ab}$, $m_{\rm el}$ is the solution to

$$k^2+m_E^2+\Pi_E(k^2)= 0 \quad \text{at} \quad k^2=-m_{\rm el}^2. \tag{16}$$

By matching the expressions for $m_{\rm el}$ obtained by solving (15) and (16), we can determine the parameter m_E^2.

We calculate the electric mass $m_{\rm el}$ in the full theory using a strict perturbation expansion in g^2 and using dimensional regularization with $3-2\epsilon$ spatial dimensions to cut off both infrared and ultraviolet divergences. The self-energy function $\Pi(k^2)$ can be expanded in a loop expansion

$$\Pi(k^2)=\Pi^{(1)}(k^2)+\Pi^{(2)}(k^2)+\cdots, \tag{17}$$

with $\Pi^{(1)}(k^2)$ and $\Pi^{(2)}(k^2)$ being given by the diagrams in Figs. 1 and 2, respectively. We can simplify Eq. (15) by expanding $\Pi(k^2)$ as a Taylor expansion around $k^2=0$. This is justified by the fact that the leading-order solution to (15) gives a value of k^2 that is of order g^2T^2. The deviation of k^2 from 0 should therefore be treated as a perturbation in order to get the strict perturbation expansion for $m_{\rm el}^2$ in powers of g^2. The resulting expression for the electric screening mass to next-to-leading order in g^2 is

$$m_{\rm el}^2\approx\Pi^{(1)}(0)+\Pi^{(2)}(0)-\Pi^{(1)}(0)\frac{d\Pi^{(1)}}{dk^2}(0). \tag{18}$$

Here and below, we use the symbol $\approx$ to denote an equality that holds only in the strict perturbation expansion. The one-loop diagrams that contribute to $\Pi^{(1)}(k^2)$ are shown in Fig. 1. Evaluating this function and its first derivative at $k^2=0$ in Feynman gauge, we obtain

$$\Pi^{(1)}(0)\approx Z_g^2g^2\{2(1-\epsilon)C_A(\mathscr{I}_1-2\mathscr{J}_1)-4T_F(\tilde{\mathscr{I}}_1-2\tilde{\mathscr{J}}_1)\}, \tag{19}$$

$$\begin{aligned}\frac{d\Pi^{(1)}}{dk^2}(0)\approx g^2\Bigg\{&-2C_A\left[\mathscr{I}_2+\frac{2(1-\epsilon)(1+2\epsilon)}{3-2\epsilon}\mathscr{J}_2\right.\\&\left.-\frac{8(1-\epsilon)}{3-2\epsilon}\mathscr{K}_2\right]\\&+2\mathrm{T_F}\left[\tilde{\mathscr{I}}_2+\frac{4(1+2\epsilon)}{3-2\epsilon}\tilde{\mathscr{J}}_2-\frac{16}{3-2\epsilon}\tilde{\mathscr{K}}_2\right]\Bigg\}.\end{aligned} \tag{20}$$

The sum integrals $\mathscr{I}_n$, $\mathscr{J}_n$, $\mathscr{K}_n$ $\tilde{\mathscr{I}}_n$, $\tilde{\mathscr{J}}_n$, and $\tilde{\mathscr{K}}_n$ are defined in Appendix A. The renormalization of the coupling constant using the modified minimal subtraction ($\overline{\rm MS}$) scheme is accomplished by substituting

$$Z_g^2=1-\frac{11C_A-4T_F}{3}\frac{g^2}{(4\pi)^2}\frac{1}{\epsilon} \tag{21}$$

into the expression for $\Pi^{(1)}(0)$. The two-loop diagrams that contribute to $\Pi(k^2)$ are shown in Fig. 2. This function evaluated at $k^2=0$ is

$$\begin{aligned}\Pi^{(2)}(0)\approx g^4\{&4(1-\epsilon)C_A^2[-2\mathscr{I}_2\mathscr{J}_1+\epsilon\mathscr{I}_1\mathscr{I}_2\\&+4(1-\epsilon)\mathscr{I}_1\mathscr{J}_2]\\&+8C_AT_F[2\mathscr{I}_2\tilde{\mathscr{J}}_1-\epsilon\tilde{\mathscr{I}}_1\mathscr{I}_2-4(1-\epsilon)\tilde{\mathscr{I}}_1\mathscr{J}_2]\\&+8(1-\epsilon)C_FT_F(\mathscr{I}_1-\tilde{\mathscr{I}}_1)(\tilde{\mathscr{I}}_2-4\tilde{\mathscr{J}}_2)\}.\end{aligned} \tag{22}$$

The sum integrals in (19), (20), and (22) can be evaluated analytically using methods developed by Arnold and Zhai [13], and they are given in Appendix A. The three quantities appearing in (18) reduce to

$$\Pi^{(1)}(0)\approx Z_g^2 g^2 T^2\left\{\frac{1}{3}C_A\left[1+\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda}{4\pi T}\right)\epsilon\right]+\frac{1}{3}T_F\left[1+\left(1-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda}{4\pi T}\right)\epsilon\right]\right\}, \tag{23}$$

$$\frac{d\Pi^{(1)}}{dk^2}(0)\approx\frac{g^2}{(4\pi)^2}\left\{-\frac{5}{3}C_A\left[\frac{1}{\epsilon}-\frac{1}{5}+2\gamma+2\ln\frac{\Lambda}{4\pi T}\right]+\frac{4}{3}T_F\left[\frac{1}{\epsilon}-1+4\ln 2+2\gamma+2\ln\frac{\Lambda}{4\pi T}\right]\right\}, \tag{24}$$

$$\Pi^{(2)}(0)\approx\frac{g^4}{(4\pi)^2}T^2\left\{\frac{2}{3}C_A^2\left[\frac{1}{\epsilon}+1+2\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}+4\ln\frac{\Lambda}{4\pi T}\right]+\frac{2}{3}C_A T_F\left[\frac{1}{\epsilon}+2-2\ln 2+2\gamma+2\frac{\zeta'(-1)}{\zeta(-1)}+4\ln\frac{\Lambda}{4\pi T}\right]-2C_F T_F\right\}, \tag{25}$$

where γ is Euler's constant, $\zeta(z)$ is the Riemann zeta function, and Λ is the scale of dimensional regularization. Inserting these expressions into (18), we find that the strict perturbation expansion for $m_{\rm el}^2$ to order g^4 is

$$m_{\rm el}^2\approx\frac{1}{3}g^2(\Lambda)T^2\left\{C_A+T_F+\epsilon\left[C_A\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda}{4\pi T}\right)+T_F\left(1-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda}{4\pi T}\right)\right]+\left[C_A^2\left(\frac{5}{3}+\frac{22}{3}\gamma+\frac{22}{3}\ln\frac{\Lambda}{4\pi T}\right)+C_A T_F\left(3-\frac{16}{3}\ln 2+\frac{14}{3}\gamma+\frac{14}{3}\ln\frac{\Lambda}{4\pi T}\right)+T_F^2\left(\frac{4}{3}-\frac{16}{3}\ln 2-\frac{8}{3}\gamma-\frac{8}{3}\ln\frac{\Lambda}{4\pi T}\right)-6C_F T_F\right]\frac{g^2}{(4\pi)^2}\right\}. \tag{26}$$

Note that all the poles in ϵ have canceled. In the order-g^2 term, we have kept terms of order ϵ for later use. The expression (26) depends on Λ explicitly through logarithms of $\Lambda/4\pi T$ and implicitly through the coupling constant $g^2(\Lambda)$. The scale of the coupling constant can be shifted from the dimensional regularization scale Λ to an arbitrary renormalization scale μ by using the solution to the renormalization group equation for the running coupling constant:

$$g^2(\Lambda)=g^2(\mu)\left[1+\frac{2(11C_A-4T_F)}{3}\frac{g^2}{(4\pi)^2}\ln\frac{\mu}{\Lambda}\right]. \tag{27}$$

After making this shift in the scale of the coupling constant, the only remaining dependence on Λ occurs in the terms of order ϵ. In these terms, Λ can be identified with the factorization scale Λ_E that separates the scales T and gT.

The expression (26) for $m_{\rm el}^2$ is an expansion in powers of g^2. It does not include a g^3 term, in contrast to the expression for $m_{\rm el}^2$ that correctly incorporates the effects of the screening of electrostatic gluons [20]. This g^3 term arises because the g^4 correction includes a linear infrared divergence that is cut off at the scale gT. Since we have used dimensional regularization as an infrared cutoff, power infrared divergences such as this linear divergence have been set equal to 0.

In order to match with expression (26), we have to calculate the screening mass in EQCD using the strict expansion in g^2. Since m_E^2 is treated as a perturbation parameter of order g^2, the only scale in the self-energy function $\Pi_E(k^2)$ is k^2. After Taylor expanding in powers of k^2, there is no scale in the dimensionally regularized integrals, and so they all vanish. The solution to Eq. (16) for the screening mass is therefore trivial:

$$m_{\rm el}^2\approx m_E^2\,. \tag{28}$$

Comparing (26) and (28), we find that, in the limit $\epsilon\to 0$, the parameter m_E^2 is given by

$$m_E^2\big|_{\epsilon=0}=\frac{1}{3}g^2(\mu)T^2\left\{C_A+T_F+\left[C_A^2\left(\frac{5}{3}+\frac{22}{3}\gamma+\frac{22}{3}\ln\frac{\mu}{4\pi T}\right)+C_A T_F\left(3-\frac{16}{3}\ln 2+\frac{14}{3}\gamma+\frac{14}{3}\ln\frac{\mu}{4\pi T}\right)+T_F^2\left(\frac{4}{3}-\frac{16}{3}\ln 2-\frac{8}{3}\gamma-\frac{8}{3}\ln\frac{\mu}{4\pi T}\right)-6C_F T_F\right]\frac{g^2}{(4\pi)^2}\right\}. \tag{29}$$

At this order in g^2, there is no dependence on the factorization scale Λ_E. The order-ϵ terms in m_E^2 will also be required later in the calculation. These terms are

$$\frac{\partial m_E^2}{\partial\epsilon}\Big|_{\epsilon=0}=\frac{1}{3}g^2T^2\left\{C_A\left(2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda_E}{4\pi T}\right)+T_F\left(1-2\ln 2+2\frac{\zeta'(-1)}{\zeta(-1)}+2\ln\frac{\Lambda_E}{4\pi T}\right)\right\}. \tag{30}$$

This expression depends explicitly on the factorization scale Λ_E.

C. Coefficient of the unit operator

In this subsection, we calculate the coefficient of the unit operator f_E to next-to-next-to-leading order in g^2. The physical interpretation of f_E is that $f_E T$ is the contribution to the free energy from large momenta of order T. The parameter f_E is determined by calculating the free energy as a strict perturbation in g^2 in both full QCD and EQCD, and matching the two results.

In the full theory, the free energy has a diagrammatic expansion that begins with the one-loop, two-loop, and three-loop diagrams shown in Figs. 3, 4, and 5, respectively. Evaluating the diagrams in the Feynman gauge, we obtain

$$F \approx -(1-\epsilon) d_A \mathscr{I}_0' + 2 d_F \tilde{\mathscr{I}}_0' + d_A Z_g^2 g^2 [(1-\epsilon)^2 C_A \mathscr{I}_1^2 + 2(1-\epsilon) T_F \tilde{\mathscr{I}}_1 (\tilde{\mathscr{I}}_1 - 2\mathscr{I}_1)] + d_A g^4 \Big\{ C_A^2 (1-\epsilon)^2 \Big[2(1+\epsilon) \mathscr{I}_1^2 \mathscr{I}_2 - \frac{1}{2} \mathscr{M}_{0,0} - \mathscr{M}_{2,-2} \Big] + C_A T_F (1-\epsilon) [-8 \mathscr{I}_1 \mathscr{I}_2 \tilde{\mathscr{I}}_1 - 2\epsilon \tilde{\mathscr{M}}_{0,0} + (1+\epsilon) \mathscr{N}_{0,0} + 4 \tilde{\mathscr{M}}_{-2,2}] + T_F^2 [8(1+\epsilon) \mathscr{I}_2 \tilde{\mathscr{I}}_1^2 + 2\epsilon \mathscr{N}_{0,0} - 4 \mathscr{N}_{2,-2}] + 2 C_F T_F (1-\epsilon) [2(1-\epsilon)(\mathscr{I}_1^2 - 4 \mathscr{I}_1 \tilde{\mathscr{I}}_1 + \tilde{\mathscr{I}}_1^2) \tilde{\mathscr{I}}_2 + 2 \tilde{\mathscr{M}}_{0,0} - (1+\epsilon) \mathscr{N}_{0,0} + 2(1-\epsilon) \tilde{\mathscr{M}}_{1,-1}] \Big\}. \tag{31}$$

The symbol $\approx$ is a reminder that the strict perturbation in g^2 does not give a physically correct treatment of the screening effects of the plasma. The sum integrals in (31) are given in Appendix A. To order g^4, the renormalization of the coupling constant is accomplished in the $\overline{\text{MS}}$ scheme by substituting (21) for Z_g in the order g^2 term. The final result is

$$F \approx -\frac{\pi^2 d_A}{9} T^4 \Bigg\{ \frac{1}{5} + \frac{7}{20} \frac{d_f}{d_A} - \left(C_A + \frac{5}{2} T_F \right) \frac{g^2(\Lambda)}{(4\pi)^2} + \Bigg[C_A^2 \left(\frac{12}{\epsilon} + \frac{194}{3} \ln \frac{\Lambda}{4\pi T} + \frac{116}{5} + 4\gamma + \frac{220}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{38}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) + C_A T_F \left(\frac{12}{\epsilon} + \frac{169}{3} \ln \frac{\Lambda}{4\pi T} + \frac{1121}{60} - \frac{157}{5} \ln 2 + 8\gamma + \frac{146}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) + T_F^2 \left(\frac{20}{3} \ln \frac{\Lambda}{4\pi T} + \frac{1}{3} - \frac{88}{5} \ln 2 + 4\gamma + \frac{16}{3} \frac{\zeta'(-1)}{\zeta(-1)} - \frac{8}{3} \frac{\zeta'(-3)}{\zeta(-3)} \right) + C_F T_F \left(\frac{105}{4} - 24 \ln 2 \right) \Bigg] \left(\frac{g^2}{(4\pi)^2} \right)^2 \Bigg\}. \tag{32}$$

In EQCD, the free energy is given by the expression (7). We calculate $\ln \mathscr{Z}_{\text{EQCD}}$ using the strict perturbation expansion in which g_E^2 and m_E^2 are treated as perturbation parameters and both infrared and ultraviolet divergences are regularized using dimensional regularization. Since diagrams with massless propagators and with no external legs vanish in dimensional regularization, the only contribution to $\ln \mathscr{Z}_{\text{EQCD}}$ which does not vanish comes from the counterterm δf_E which cancels ultraviolet divergences proportional to the unit operator. The resulting expression for the free energy is simply

$$F \approx (f_E + \delta f_E) T. \tag{33}$$

The counterterm can be determined by calculating the ultraviolet divergences in $\ln \mathscr{Z}_{\text{EQCD}}$. If we use dimensional regularization together with a minimal subtraction renormalization scheme in the effective theory, then δf_E is a polynomial in g_E^2, m_E^2, and the other parameters in the Lagrangian for EQCD. The only combination of parameters that has dimension 3 and is of order g^4 is $g_E^2 m_E^2$. Thus the leading term in δf_E is proportional to $g_E^2 m_E^2$. The coefficient is determined by a two-loop calculation that is a trivial part of the three-loop calculation in Sec. IV. The result for the counterterm is

$$\delta f_E = -\frac{d_A C_A}{4(4\pi)^2} g_E^2 m_E^2 \frac{1}{\epsilon}. \tag{34}$$

When expressing this counterterm in terms of the parameters g and T of the full theory, we must take into account the fact that m_E^2 multiplies a pole in ϵ. Thus in addition to expression for m_E^2 given in (29), we must also include the terms of order ϵ which are given by (30). The counterterm (34) is therefore

$$\delta f_E = -\frac{\pi^2 d_A}{9} \left(\frac{g^2}{(4\pi)^2} \right)^2 T^3 \left[12 C_A^2 \left(\frac{1}{\epsilon} + 2 \frac{\zeta'(-1)}{\zeta(-1)} + 2 \ln \frac{\Lambda_E}{4\pi T} \right) + 12 C_A T_F \left(\frac{1}{\epsilon} + 1 - 2\ln 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} + 2 \ln \frac{\Lambda_E}{4\pi T} \right) \right]. \tag{35}$$

Note that minimal subtraction in the effective theory is not equivalent to minimal subtraction in the full theory. In addition to the poles in ϵ in (35), there are finite terms that depend on the factorization scale Λ_E.

Matching (32) with (33) and using the expression (35), we conclude that f_E to order g^4 is

$$f_E(\Lambda_E)=-\frac{\pi^2 d_A}{9}T^3\Bigg\{\left(\frac{1}{5}+\frac{7}{20}\frac{d_F}{d_A}\right)-\left(C_A+\frac{5}{2}T_F\right)\frac{g^2(\mu)}{(4\pi)^2}+\Bigg(C_A^2\Bigg[48\ln\frac{\Lambda_E}{4\pi T}-\frac{22}{3}\ln\frac{\mu}{4\pi T}+\frac{116}{5}+4\gamma+\frac{148}{3}\frac{\zeta'(-1)}{\zeta(-1)}$$

$$-\frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)}\Bigg]+C_AT_F\Bigg[48\ln\frac{\Lambda_E}{4\pi T}-\frac{47}{3}\ln\frac{\mu}{4\pi T}+\frac{401}{60}-\frac{37}{5}\ln 2+8\gamma+\frac{74}{3}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{1}{3}\frac{\zeta'(-3)}{\zeta(-3)}\Bigg]+T_F^2\Bigg[\frac{20}{3}\ln\frac{\mu}{4\pi T}+\frac{1}{3}$$

$$-\frac{88}{5}\ln 2+4\gamma+\frac{16}{3}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{8}{3}\frac{\zeta'(-3)}{\zeta(-3)}\Bigg]+C_FT_F\Bigg[\frac{105}{4}-24\ln 2\Bigg]\Bigg)\left(\frac{g^2}{(4\pi)^2}\right)^2\Bigg\}, \tag{36}$$

where $g(\mu)$ is the coupling constant in the $\overline{\rm MS}$ renormalization scheme at the scale μ. We have used (27) to shift the scale of the running coupling constant from Λ_E to an arbitrary renormalization scale μ, and we have identified the explicit factors of Λ that remain with the factorization scale Λ_E.

D. Evolution of EQCD coupling constants

The effective Lagrangian (9) for EQCD can be expressed as a sum over all local operators that respect the symmetries of the theory:

$$f_E(\Lambda_E)+\mathscr{L}_{\rm EQCD}=\sum_n C_n(\Lambda_E)\mathscr{O}_n, \tag{37}$$

where we have included the unit operator with coefficient f_E as one of the operators $\mathscr{O}_n$. The coefficients C_n are the generalized coupling constants of the effective theory. Because of ultraviolet divergences, the effective theory must be regularized with an ultraviolet cutoff Λ_E. The ultraviolet divergences in the effective theory include power ultraviolet divergences proportional to Λ_E^p, $p=1,2,\ldots$, and logarithmic divergences proportional to $\ln(\Lambda_E)$. The power divergences are artifacts of the regularization scheme and have no physical content. If they are not removed as part of the regularization procedure, they must be canceled by power divergences in the coupling constants C_n. In contrast, the logarithmic ultraviolet divergences are directly related to logarithms of T in the full theory, and therefore represent real physical effects. It is convenient to use a regularization procedure for the effective theory in which power ultraviolet divergences are automatically subtracted, such as dimensional regularization. In this case, they need not be canceled by power divergences in the coupling constants. The dimensions of a coupling constant can then only be taken up by powers of the temperature T. The coupling constant C_n must be proportional to T^{3-d_n}, where d_n is the scaling dimension of the corresponding operator $\mathscr{O}_n$. The dimensionless factor multiplying T^{3-d_n} in the coupling constant C_n can be computed as a perturbation series in $g^2(T)$, with coefficients that are polynomials in $\ln(T/\Lambda_E)$. The dependence on Λ_E is governed by a ''renormalization group equation'' or ''evolution equation'' of the form

$$\Lambda_E\frac{d}{d\Lambda_E}C_n(\Lambda_E)=\beta_n(C), \tag{38}$$

where the beta function β_n has a power series expansion in the coupling constants C_m. These equations follow from the condition that physical quantities must be independent of the arbitrary scale Λ_E. Since C_n is proportional to T^{3-d_n}, every term in the expansion of its beta function must be proportional to T^{3-d_n}. In particular, a term like $C_{m_1}C_{m_2}\cdots C_{m_k}$ can appear only if the dimensions d_{m_i} of the corresponding operators $\mathscr{O}_{m_i}$ satisfy

$$\sum_{i=1}^{k}(3-d_{m_i})=3-d_n. \tag{39}$$

The condition (39) is very restrictive, particularly if the effective Lagrangian is truncated to the super-renormalizable terms that are given explicitly in (9). It implies that the only terms that can appear in the β function for the coefficient f_E of the unit operator are $g_E^2m_E^2$, $\lambda_Em_E^2$, and a cubic polynomial in g_E^2 and λ_E. Since m_E^2, g_E^2, and λ_E are of order g^2, g^2, and g^4, respectively, the only term of order g^4 is $g_E^2m_E^2$. We can determine its coefficient by calculating the ultraviolet divergences in the strict perturbation expansion for the free energy in the effective theory. These divergences do not appear in (33), because the ultraviolet poles in ϵ have canceled against infrared poles in ϵ. We can calculate the ultraviolet divergences by using a different regularization for infrared divergences. Alternatively, since we have already calculated f_E explicitly to order g^4, we can simply differentiate (36) and use the fact that $\Lambda_E(d/d\Lambda_E)f_E$ must be proportional to $g_E^2m_E^2$. Using $g_E^2=g^2T$ and the leading-order expression for m_E^2 in (29), we find that the evolution equation is

$$\Lambda_E\frac{d}{d\Lambda_E}f_E=-\frac{d_AC_A}{(4\pi)^2}g_E^2m_E^2+O(g^6T^3). \tag{40}$$

The β function for m_E^2 must be a quadratic polynomial in g_E^2 and λ_E. The terms g_E^4, $g_E^2\lambda_E$, and λ_E^2 are of order g^4, g^6, and g^8, respectively. The coefficients of these terms can be determined by calculating the ultraviolet-divergent terms in the strict perturbation expansion for the electric screening mass in the effective theory. Alternatively, if m_E^2 is known, its β function can be determined simply by differentiating. Since the expression (29) is independent of Λ_E, we know that the coefficient of g_E^4 in the β function vanishes and the leading term must be $g_E^2\lambda_E$. Thus the evolution equation for m_E^2 is

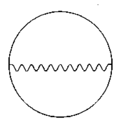

FIG. 7. Two-loop Feynman diagrams for the logarithm of the partition function of EQCD.

$$\Lambda_E \frac{d}{d\Lambda_E} m_E^2 = 0 + O(g^6 T^2). \tag{41}$$

We have not calculated the coefficient of $g_E^2 \lambda_E$ in this evolution equation, because it does not affect the free energy until order g^7.

The β functions for g_E^2 and λ_E vanish to all orders in the super-renormalizable interactions. All the nonvanishing terms in their β functions involve the coupling constants of nonrenormalizable interactions, and they are therefore suppressed by large powers of g. The evolution of these paramaters can probably be ignored for most practical purposes.

The only EQCD parameter whose evolution plays a role in the free energy to order g^6 is f_E. To this order, the solution to Eq. (40) is trivial:

$$f_E(\Lambda_E) = f_E(\Lambda_E') - \frac{d_A C_A}{(4\pi)^2} g_E^2 m_E^2 \ln\frac{\Lambda_E}{\Lambda_E'}. \tag{42}$$

IV. FREE ENERGY TO ORDER g^5

Having calculated the parameters of EQCD to the necessary order in g^2, we now use the effective theory to calculate the free energy to order g^5. The free energy is the sum of the three terms in (13), which correspond to the momentum scales T, gT, and g^2T, respectively. The term $f_E T$ is the contribution from the scale T. We have already calculated f_E to the necessary order and it is given in (36). The term $f_G T$ is the contribution from the scale g^2T, but it does not contribute until order g^6. The remaining term $f_M T$ is the contribution from the scale gT.

Through order g^5, f_M is proportional to the logarithm of the partition function for EQCD:

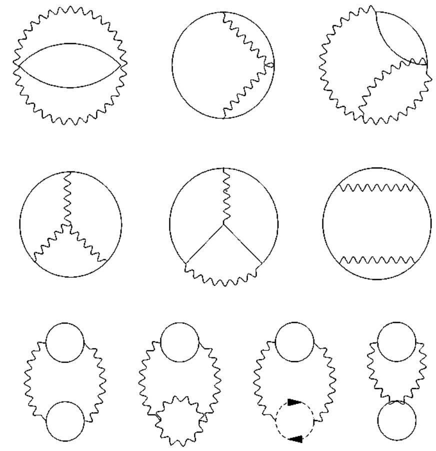

FIG. 8. Three-loop Feynman diagrams for the logarithm of the partition function of EQCD.

$$f_M = -\frac{\ln \mathcal{Z}_{\rm EQCD}}{V}. \tag{43}$$

In order to calculate this contribution using perturbation theory, we must incorporate the terms in the Lagrangian that provide electrostatic screening into the free part of the Lagrangian. The necessary screening effects are provided by the $A_0^a A_0^a$ term in the EQCD Lagrangian. Thus we must include the effects of the mass parameter m_E^2 to all orders, while treating all the other coupling constants of EQCD as perturbation parameters. The only coupling constant that is required to obtain the free energy to order g^5 is the gauge coupling constant g_E.

The contributions to $\ln\mathcal{Z}_{\rm EQCD}$ of orders g^3, g^4, and g^5 are given by the sum of the one-loop, two-loop, and three-loop diagrams in Figs. 6, 7, and 8, respectively. The solid, wavy, and dashed lines represent the propagators of the A_0 field, the A_i fields, and the associated ghosts, respectively. We evaluate these diagrams in the Feynman gauge. They can be expressed in terms of the scalar integrals defined in Appendix B. The resulting expression for the logarithm of the partition function is

$$f_M = -\frac{d_A}{2} I_0' + d_A C_A g_E^2 \left[\frac{1}{4} I_1^2 + m_E^2 J_1\right] + d_A C_A^2 g_E^4 \left[-\frac{1}{4} I_1^2 I_2 + 2 I_1 J_1 - 2 m_E^2 I_1 J_2 - m_E^2 I_1 K_2 - \frac{1}{4} M_{1,-1} - \frac{1-2\epsilon}{2} M_{0,0} + \epsilon M_{-1,1} \right.$$
$$- \frac{1-2\epsilon}{2} M_{-2,2} + 4 m_E^2 M_{1,0} + 2 m_E^2 M_{0,1} - 4 m_E^4 M_{2,0} - \frac{3}{8} N_{0,0} - \frac{1}{2} N_{1,-1} - \frac{1}{4} N_{2,-2} - 2 m_E^2 N_{1,0} - m_E^2 N_{2,-1} - m_E^4 N_{1,1}$$
$$\left. - m_E^4 N_{2,0} - \frac{1}{4} L_{1,-1} \right] + \delta f_E, \tag{44}$$

where δf_E is the counterterm associated with the unit operator of the EQCD Lagrangian. The integrals I_n, J_n, K_n, $L_{m,n}$, $M_{m,n}$, and $N_{m,n}$ can be calculated analytically using methods developed by Broadhurst [21] and they are given in Appendix B. Adding them up, we obtain

$$f_M = -\frac{d_A}{3(4\pi)} m_E^3 + \frac{d_A C_A}{4(4\pi)^2} \left(\frac{1}{\epsilon} + 4 \ln\frac{\Lambda}{2 m_E} + 3 \right) g_E^2 m_E^2 + \frac{d_A C_A^2}{(4\pi)^3} \left(\frac{89}{24} - \frac{11}{6} \ln 2 + \frac{1}{6} \pi^2 \right) g_E^4 m_E + \delta f_E, \tag{45}$$

where Λ is the scale of dimensional regularization. It can be identified with the ultraviolet cutoff Λ_E of EQCD. The ultraviolet pole in ϵ in the term proportional to $g_E^2 m_E^2$ in (45) is canceled by the counterterm δf_E, which is given in (34). Our final result is therefore

$$f_M(\Lambda_E) = -\frac{d_A}{3(4\pi)} m_E^3 \left\{ 1 + \left[-3\ln\frac{\Lambda_E}{2m_E} - \frac{9}{4} \right] \frac{C_A g_E^2}{4\pi m_E} + \left[-\frac{89}{8} + \frac{11}{2}\ln 2 - \frac{1}{2}\pi^2 \right] \left(\frac{C_A g_E^2}{4\pi m_E} \right)^2 \right\}. \tag{46}$$

The coefficient f_M in (46) can be expanded in powers of g by setting $g_E^2 = g^2 T$ and by substituting the expression (29) for m_E^2. The complete free energy to order g^5 is then $F = (f_E + f_M)T$. Note that the dependence on the arbitrary factorization scale Λ_E cancels between f_E and f_M, up to corrections that are higher-order in g, leaving a logarithm of T/m_E. This $g^4\ln(g)$ term is associated with the renormalization of f_E, and its coefficient can be determined from the evolution equation (40). There is no $g^5\ln(g)$ term in the perturbation expansion for F, and this is a consequence of the vanishing of the order-g^4 term in the β function for m_E^2.

V. OUTLINE OF CALCULATION TO ORDER g^6

The calculation of the free energy to order g^5, which was presented in the previous section, was greatly streamlined by using effective field theory to unravel the effects of the momentum scales T and gT. The same result has also been obtained by Kastening and Zhai using other methods [18]. However, the advantages of the effective-field-theory approach become more and more apparent as we go to higher order in g. In this section, we demonstrate the power of this method by outlining the calculation of the free energy to order g^6. In this case there are contributions from all three momentum scales T, gT, and g^2T.

A. Contribution from the scale g^2T

We first discuss the contribution to the free energy from the scale g^2T, which is given by the term $f_G T$ in (13). This term is proportional to the logarithm of the partition function (11) of MQCD. Treating the correction term $\delta\mathcal{L}_{\rm MQCD}$ in the MQCD Lagrangian as a perturbation, the partition function can be written

$$\mathcal{Z}_{\rm MQCD} = \int^{(\Lambda_M)} \mathcal{D}A_i^a(\mathbf{x}) \exp\left(-\int d^3x\, G^2/4 \right) \times \left\{ 1 - \int d^3x\, \delta\mathcal{L}_{\rm MQCD} + \cdots \right\}, \tag{47}$$

where $G^2 \equiv G_{ij}^a G_{ij}^a$. Taking the logarithm of both sides, we obtain

$$f_G = -\frac{\ln \mathcal{Z}_{\rm MQCD}^{(0)}}{V} + \langle \delta\mathcal{L}_{\rm MQCD} \rangle_0 + \cdots, \tag{48}$$

where $\mathcal{Z}_{\rm MQCD}^{(0)}$ is the partition function for the minimal gauge theory with action $\int d^3x G^2/4$. The subscript 0 on the expectation value $\langle \delta\mathcal{L}_{\rm MQCD} \rangle_0$ is a reminder that it is to be calculated using the minimal-gauge-theory action.

For the moment, let us consider only the first term in (48). The partition function $\mathcal{Z}_{\rm MQCD}^{(0)}$ is that of the minimal gauge theory in three dimensions. This is a super-renormalizable theory and its ultraviolet divergences have a very simple structure. By naive power counting, ultraviolet divergences in $\ln \mathcal{Z}_{\rm MQCD}^{(0)}$ can arise only from vacuum diagrams with one, two, three, or four loops or from propagator corrections with one or two loops. Ward identities guarantee that the propagator corrections are actually finite. This is related to the fact that the only gauge invariant operator with dimension lower than G^2 is the unit operator. Thus the only ultraviolet divergences are in the vacuum diagrams. The one-loop diagrams give a cubic divergence. The two-loop diagrams give a quadratic divergence proportional to g_M^2. The three-loop diagrams give a linear divergence proportional to g_M^4. Finally, the four-loop diagrams give a logarithmic divergence proportional to g_M^6. After subtraction of the power divergences, we can use dimensional analysis to determine the form of $\ln \mathcal{Z}_{\rm MQCD}^{(0)}$. Aside from the logarithmic dependence on the ultraviolet cutoff Λ_M, the only scale in the problem is g_M. By dimensional analysis, $\ln \mathcal{Z}_{\rm MQCD}^{(0)}$ must be proportional to g_M^6. Thus it must have the form

$$-\frac{\ln \mathcal{Z}_{\rm MQCD}^{(0)}}{V} = \left(a + b\ln\frac{\Lambda_M}{g_M^2} \right) g_M^6, \tag{49}$$

where a and b are pure numbers. The coefficient b can be determined by calculating the logarithmic ultraviolet divergence in the four-loop vacuum diagrams for MQCD. The coefficient a can only be calculated using nonperturbative methods. It can for example be extracted from measurements of the expectation value $\langle G^2 \rangle_0$ using lattice simulations of the pure gauge theory. A convenient expression for $\langle G^2 \rangle_0$ can be obtained by taking the logarithm of both sides of (49) and differentiating with respect to g_M^2. It is useful to first rescale the field A_i in the functional integral for $\mathcal{Z}_{\rm MQCD}^{(0)}$, so that the coupling constant appears only in the coefficient $1/g_M^2$ of the action. After subtracting the power ultraviolet divergences, we obtain the expression

$$\langle G^2 \rangle_0 = -4\left(3a - b + 3b\ln\frac{\Lambda_M}{g_M^2} \right) g_M^6. \tag{50}$$

The subscript 0 on the expectation value $\langle G^2 \rangle_0$ is a reminder that it is to be calculated using the minimal-gauge-theory action $\int d^3x G^2/4$ rather that the full action of MQCD. The expectation value $\langle G^2 \rangle_0$ can be measured on the lattice using Monte Carlo simulations of the minimal gauge theory. Once $\langle G^2 \rangle_0$ has been measured and the coefficient b has been calculated, we can determine a using the formula (50).

We now verify that the correction term in (47) from higher dimension operators in the MQCD Lagrangian can indeed be treated as a small perturbation. The lowest dimension operators in $\delta\mathcal{L}_{\rm MQCD}$ are $G^3 \equiv f^{abc} G_{ij}^a G_{jk}^b G_{ki}^c$, whose coefficient is proportional to $g^3/T^{3/2}$, and $(DG)^2$

$\equiv (D_i G_{ik})^a (D_j G_{jk})^a$, whose coefficient is proportional to g^2/T^2. Their coefficients have been calculated to leading order in g by Chapman for the case of a pure gauge theory [22]. After subtraction of power ultraviolet divergences, the only scale in the problem is g_M^2. Therefore, by dimensional analysis, $\langle G^3 \rangle_0$ must be proportional to $(g_M^2)^{9/2}$. Using $g_M^2 \approx g^2 T$ and taking into account the coefficient which is proportional to $g^3/T^{3/2}$, we find that the contribution to f_G from $\langle G^3 \rangle_0$ is of order $g^{12}T^3$. Using a similar analysis, we find that the contribution from $\langle (DG)^2 \rangle_0$ is also of order $g^{12}T^3$. Thus the effects of higher dimension operators in the MQCD Lagrangian are indeed suppressed by powers of the coupling constant g.

We have found that the contribution to the free energy from the scale $g^2 T$ can be written

$$f_G T = \left(a + b \ln \frac{\Lambda_M}{g_M^2} \right) g_M^6 T + O(g^{12} T^4). \tag{51}$$

Remarkably, the only nonperturbative calculation that is required to determine the free energy up to order g^{12} is that of the single pure number a. We also require the coupling constant g_M, which can be calculated by matching perturbative calculations in EQCD and MQCD. To calculate the free energy to order g^6, we only need g_M to leading order in g. At this order, it is given simply by $g_M^2 = g^2 T$. In summary, in order to obtain the contribution to the free energy from the scale $g^2 T$ to order g^6, all that is required are the two pure numbers a and b in (51). The number b can be calculated by evaluating four-loop diagrams in MQCD. In Ref. [7], it was assumed incorrectly that this number vanishes. The number a can be calculated using lattice simulations of the pure gauge theory in three dimensions.

B. Contribution from the scale gT

The contribution to the free energy from the scale gT is given by the term $f_M T$ in (13). The coefficient f_M can be determined by calculating the logarithm of the EQCD partition function in both EQCD and MQCD and matching the expressions. If we use dimensional regularization to cut off both infrared and ultraviolet divergences, all the loop diagrams in MQCD vanish. The expression for the logarithm of the partition function then is simply

$$-\frac{\ln \mathcal{Z}_{\rm EQCD}}{V} = f_M + \delta f_M, \tag{52}$$

where δf_M is a counterterm that cancels ultraviolet divergences in MQCD that are proportional to the unit operator. To order g^6, this counterterm is simply

$$\delta f_M = \frac{b}{2\epsilon} g_M^6, \tag{53}$$

where b is the same coefficient that appears in (51).

To determine f_M, we must match the expression (52) with the corresponding expression in EQCD, which is obtained by calculating the sum of vacuum diagrams using dimensional regularization. The resulting expression for $\ln \mathcal{Z}_{\rm EQCD}$ is a sum of polynomials in the EQCD coupling constants, such as g_E^2 and λ_E, multiplied by whatever powers of m_E are required by dimensional analysis. There are three such terms that contribute to the free energy at order g^6. The first term is $g_E^2 m_E^2$, whose coefficient has already been calculated in (46). It contributes through the next-to-leading order term in m_E^2, which is given in (29), and through the next-to-leading order term in g_E^2, which has not yet been calculated. The second term which contributes at order g^6 is proportional to g_E^6. Its coefficient is determined by calculating all four-loop vacuum diagrams that involve only the gauge coupling constant g_E. This term will have a pole in ϵ that matches that from the counterterm (53). The third term that contributes to f_M at order g^6 is proportional to $\lambda_E m_E^2$. Its coefficient is given by the single two-loop vacuum diagram that involves the A_0^4 coupling constant λ_E. This coupling constant is only required to leading order in g and has already been calculated by Nadkarni [23] and by Landsman [24].

In summary, there are three coefficients that must be calculated in order to obtain the contribution of order g^6 to the free energy from the scale gT. We need the coefficients of g_E^6 and of $\lambda_E m_E^2$ in the expression for f_M. These can be obtained by perturbative calculations in EQCD. We also need the coefficient of g^4 in the expression for the EQCD parameter g_E^2. This requires a perturbative calculation in full QCD.

C. Contribution from the scale T

The contribution to the free energy from the scale T is given by the term $f_E T$ in (13). The term f_E is obtained by matching the strict perturbation expansions for the free energy in full QCD and in EQCD. In full QCD, the contribution of order g^6 is the sum of all four-loop vacuum diagrams. If we use dimensional regularization to cut off both infrared and ultraviolet divergences, then the corresponding expression in EQCD is simply $F = (f_E + \delta f_E) T$. The counterterm δf_E includes the term proportional to $g_E^2 m_E^2/\epsilon$ given in (34) and also a term proportional to $\lambda_E m_E^2/\epsilon$. Since the counterterm is proportional to $1/\epsilon$, we need not only the value of the coupling constant λ_E at $\epsilon = 0$ but also the terms linear in ϵ. Similarly, we need the term of order ϵ in the order-g^4 correction to g_E^2.

In summary, there are several calculations that must be carried out in order to obtain the term of order g^6 in f_E. We need to calculate the four-loop vacuum diagrams in full QCD. We also need to calculate the terms of order ϵg^4 in the EQCD parameters g_E^2 and λ_E.

VI. CONVERGENCE OF PERTURBATION THEORY

We have calculated the free energy as a perturbation expansion in powers of g to order g^5. In this section, we examine the convergence of that perturbation expansion. For simplicity, we focus on the case of QCD with n_f flavors of quarks.

The effects of the momentum scale T enter into the free energy only through the coefficient f_E and the parameters in the EQCD Lagrangian. The term f_E is given in (36):

$$f_E(\Lambda_E)=-\frac{8\pi^2}{45}T^3\left\{1+\frac{21}{32}n_f-\frac{15}{4}\left(1+\frac{5}{12}n_f\right)\frac{\alpha_s(\mu)}{\pi}+\left[244.9-17.24n_f-0.415n_f^2-\frac{165}{8}\left(1+\frac{5}{12}n_f\right)\left(1-\frac{2}{33}n_f\right)\ln\frac{\mu}{2\pi T}\right.\right.$$
$$\left.\left.-135\left(1+\frac{1}{6}n_f\right)\ln\frac{\Lambda_E}{2\pi T}\right]\left(\frac{\alpha_s}{\pi}\right)^2+O(\alpha_s^3)\right\}. \tag{54}$$

The other parameters in the EQCD Lagrangian that enter into the calculation of the free energy to order g^5 are m_E and g_E, which are given by (29) and (14), respectively:

$$m_E^2=4\pi\alpha_s(\mu)T^2\left\{1+\frac{1}{6}n_f+\left[0.612-0.488n_f-0.0428n_f^2+\frac{11}{2}\left(1+\frac{1}{6}n_f\right)\left(1-\frac{2}{33}n_f\right)\ln\frac{\mu}{2\pi T}\right]\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right\}, \tag{55}$$

$$g_E^2=4\pi\alpha_s T[1+O(\alpha_s)]. \tag{56}$$

We have calculated two terms in the perturbation series for m_E^2 and three terms in the series for f_E. We can use these results to study the convergence of perturbation theory for the parameters of EQCD. We consider the case of $n_f=3$ flavors of quarks, although our conclusions will not depend sensitively on n_f. The question of the convergence is complicated by the presence of the arbitrary renormalization and factorization scales μ and Λ_E. The next-to-leading-order (NLO) correction to f_E is independent of μ and Λ_E, and is small compared to the leading-order (LO) term provided that $\alpha_s(\mu)\ll 1.1$. The NLO correction to m_E^2 and the next-to-next-to-leading-order (NNLO) correction to f_E both depend on the renormalization scale μ. One scale-setting scheme that is physically well motivated is the BLM prescription [24], in which μ is adjusted to cancel the highest power of n_f in the correction term. This prescription gives $\mu=0.93\pi T$ when applied to m_E^2 and $\mu=4.4\pi T$ when applied to f_E. These values differ only by about a factor of 2 from $2\pi T$, which is the lowest Matsubara frequency for gluons. Below, we will consider the three values $\mu=\pi T$, $2\pi T$, and $4\pi T$. For the NLO correction to m_E^2 to be much smaller than the LO term, we must have $\alpha_s(\mu)\ll$ 0.8, 3.8, and 1.4 if $\mu=\pi T$, $2\pi T$, and $4\pi T$, respectively. Based on these results, we conclude that the perturbation series for the parameters of EQCD are well behaved provided that $\alpha_s(2\pi T)\ll 1$.

The NNLO correction for f_E depends not only on μ, but also on the factorization scale Λ_E. Because the coefficient of $\ln(\Lambda_E/2\pi T)$ in (36) is so much larger than that of $\ln(\mu/2\pi T)$, the NNLO correction for f_E is much more sensitive to Λ_E than to μ. It is useful intuitively to think of the infrared cutoff Λ_E as being much smaller than the ultraviolet cutoff μ. However, these scales can be identified with momentum cutoffs only up to multiplicative constants that may be different for μ and Λ_E. Both parameters are introduced through dimensional regularization, but μ arises from ultraviolet divergences of four dimensional integrals, while Λ_E arises from infrared divergences of three-dimensional integrals. We might be tempted to set $\Lambda_E=\mu$, but then the NNLO coefficient in f_E is large. For the choice $\mu=2\pi T$, the correction to the LO term is a multiplicative factor $1-0.9\alpha_s+6.5\alpha_s^2$. The NNLO correction can be made small by adjusting Λ_E. It vanishes for $\Lambda_E=5.8\pi T$, $5.1\pi T$, and $4.5\pi T$ if $\mu=\pi T$, $2\pi T$, and $4\pi T$, respectively. We conclude that the perturbation series for f_E is well behaved if the factorization scale Λ_E is chosen to be approximately $5\pi T$. Whether this choice is reasonable can only be determined by calculating other EQCD parameters to higher order to see if the same choice leads to well behaved perturbation series.

The choice of Λ_E that makes the perturbation series for the EQCD parameters well behaved may be much larger than the largest mass scale m_E of EQCD. Perturbative corrections in EQCD will then include large logarithms of Λ_E/m_E. This problem can be avoided by using renormalization group equations to evolve the parameters of EQCD from the initial scale Λ_E down to some scale $\Lambda_E{}'$ of order m_E. The solution to the renormalization group equation for f_E is given in (42). The evolution of g_E^2 and m_E^2 occurs only at higher order in the coupling constant and therefore can be ignored.

We have carried out only one perturbative calculation in EQCD. This is the term f_M, which gives the contribution to the free energy from the scale gT. This term is given in (46):

$$f_M(\Lambda_E)=-\frac{2}{3\pi}m_E^3\left[1-\left(0.256+\frac{9}{2}\ln\frac{\Lambda_E}{m_E}\right)\frac{g_E^2}{2\pi m_E}\right.$$
$$\left.-27.6\left(\frac{g_E^2}{2\pi m_E}\right)^2+O(g^3)\right]. \tag{57}$$

We now consider the convergence of the perturbation series (46) for f_M. The size of the NLO correction depends on the choice of the factorization scale Λ_E. It is small if Λ_E is chosen to be approximately m_E. The NNLO correction in (46) is independent of any arbitrary scales. If $n_f=3$, it is small compared to the leading-order term only if $\alpha_s\ll 0.17$. Thus the perturbation series for f_M is well-behaved only for values of $\alpha_s(2\pi T)$ that are much smaller than those required for the parameters of EQCD to have well-behaved perturbation series.

Inserting (55) and (56) into (57), expanding in powers of g, and adding (54), we get the expansion for the free energy in powers of $\sqrt{\alpha_s}$:

$$F=-\frac{8\pi^2}{45}T^4\left[F_0+F_2\frac{\alpha_s(\mu)}{\pi}+F_3\left(\frac{\alpha_s(\mu)}{\pi}\right)^{3/2}+F_4\left(\frac{\alpha_s}{\pi}\right)^2\right.$$
$$\left.+F_5\left(\frac{\alpha_s}{\pi}\right)^{5/2}+O(\alpha_s^3\ln\alpha_s)\right]. \tag{58}$$

The coefficients in this expansion are

$$F_0 = 1 + \frac{21}{32} n_f, \tag{59}$$

$$F_2 = -\frac{15}{4}\left(1 + \frac{5}{12} n_f\right), \tag{60}$$

$$F_3 = 30\left(1 + \frac{1}{6} n_f\right)^{3/2}, \tag{61}$$

$$\begin{aligned} F_4 = {} & 237.2 + 15.97 n_f - 0.413 n_f^2 \\ & + \frac{135}{2}\left(1 + \frac{1}{6} n_f\right) \ln\left[\frac{\alpha_s}{\pi}\left(1 + \frac{n_f}{6}\right)\right] \\ & - \frac{165}{8}\left(1 + \frac{5}{12} n_f\right)\left(1 - \frac{2}{33} n_f\right) \ln\frac{\mu}{2\pi T}, \end{aligned} \tag{62}$$

$$\begin{aligned} F_5 = {} & \left(1 + \frac{1}{6} n_f\right)^{1/2}\left[-799.2 - 21.96 n_f - 1.926 n_f^2 \right. \\ & \left. + \frac{495}{2}\left(1 + \frac{1}{6} n_f\right)\left(1 - \frac{2}{33} n_f\right) \ln\frac{\mu}{2\pi T}\right]. \end{aligned} \tag{63}$$

The coefficient F_2 was first given by Shuryak [25]. The coefficient of F_3 was first calculated correctly by Kapusta [26]. The coefficient F_4 was calculated in 1994 by Arnold and Zhai [13]. The coefficient F_5 has also been calculated independently by Kastening and Zhai [18].

We now ask how small α_s must be in order for the expansion (58) to be well behaved. For simplicity, we consider the case $n_f = 3$, although our conclusions are not sensitive to n_f. If we choose the renormalization scale $\mu = 2\pi T$ motivated by the Brodsky-Lepagbe-Mackenzie (BLM) criterion [24], the correction to the LO result is a multiplicative factor $1 - 0.9\alpha_s + 3.3\alpha_s^{3/2} + (7.1 + 3.5\ln\alpha_s)\alpha_s^2 - 20.8\alpha_s^{5/2}$. The $\alpha_s^{5/2}$ term is the largest correction unless $\alpha_s(2\pi T) < 0.12$. We can make the $\alpha_s^{5/2}$ term small only by choosing the renormalization scale to be near the value $\mu = 36.5\pi T$ for which F_5 vanishes. This ridiculously large value of μ arises because the scale μ has been adjusted to cancel the large g^5 correction to f_M in (46). This contribution arises from the momentum scale gT and has nothing to do with renormalization of α_s. We conclude that the expansion (58) for F in powers of $\sqrt{\alpha_s}$ is well behaved only if $\alpha_s(2\pi T) \ll 1/10$. This is an order of magnitude smaller than the value required for the EQCD parameters to be well behaved. Our previous analysis indicates that this slow convergence of the expansion for F in powers of $\sqrt{\alpha_s}$ can be attributed to the slow convergence of perturbation theory at the scale gT.

VII. DISCUSSION

In this paper, we have used effective-field-theory methods to unravel the contributions to the free energy of high-temperature QCD from the scales T, gT, and g^2T. We calculated the free energy explicitly to order g^5. The calculation was significantly streamlined by using effective-field-theory methods to reduce every step of the calculation to one that involves only a single momentum scale. We also outlined the calculations that would be necessary to obtain the free energy to order g^6. It is only at this order that the full power of the effective-field-theory approach becomes evident.

The effective-field-theory approach provides an understanding of the logarithms of the coupling constant that arise in perturbation expansions in thermal field theory. These logarithms are associated with the renormalization of the parameters of effective field theories. The resulting evolution equations can be used to sum up leading logarithms of the coupling constant of the form $g^{m+2n}\ln^n(g)$ to all orders in n [5]. To the accuracy required for the calculation of the free energy to order g^5 in QCD, this resummation is trivial. The only terms of the form $g^{m+2n}\ln^n(g)$ with $m+2n \leqslant 5$ are a $g^4\ln(g)$ term associated with renormalization of the coefficient f_E. The fact that the solution (42) to the evolution equation for f_E is trivial indicates that there are no higher-order terms of the form $g^{2+2n}\ln^n(g)$ that are related to the $g^4\ln(g)$ term through the renomalization group. There are also no terms of the form $g^{3+2n}\ln^n(g)$ in the free energy. This is a consequence of the vanishing of the g_E^4 term in the β function for m_E^2. In the seemingly simpler problem of a massless scalar field with a ϕ^4 interaction, the evolution equations play a more important role [5]. There are terms in the free energy of the form $g^{3+2n}\ln^n(g)$ that can be summed up to all orders with the help of the renormalization group. The relative simplicity of the QCD case comes from the fact that the term g_E^4 in the β function for m_E^2 has a vanishing coefficient. We know of no deep reason for this coefficient to vanish.

Our explicit calculations allow us to study the convergence of the perturbation expansion for thermal QCD. They suggest that perturbation theory at the scale gT requires a much smaller value of the coupling constant than perturbation theory at the scale T. At the scale T, perturbation corrections can be small only if $\alpha_s(2\pi T) \ll 1$. Of course, even if this condition is satisfied, the perturbation expansion may break down anyway, but this is certainly a necessary condition. At the scale gT, perturbation corrections can be small only if $\alpha_s(2\pi T) \ll 1/10$. Thus, in order to achieve a given relative accuracy, the coupling constant $\alpha_s(2\pi T)$ must be an order of magnitude smaller for perturbation theory at the scale gT compared to perturbation theory at the scale T. This has important implications for calculations in thermal QCD. At extremely high temperatures, the asymptotic freedom of QCD guarantees that the running coupling constant $\alpha_s(2\pi T)$ is sufficiently small that perturbation theory will provide an accurate treatment of the effects of the scale gT as well as those of the scale T. Nonperturbative methods, such as lattice simulations of MQCD, are necessary only to calculate the effects of the scale g^2T. Of course, one can always treat the entire problem nonperturbatively by carrying out lattice simulations of full thermal QCD. However, it is probably more efficient to integrate out the scales T and gT using perturbative methods, and to reserve the nonperturbative methods only for the scale g^2T where they are essential. As the temperature is decreased, the running coupling constant increases and perturbation theory becomes less accurate. At sufficiently low temperatures, perturbation theory breaks down completely, and the entire problem must be treated nonperturbatively. This is certainly the case when the temperature is close to the critical temperature for the phase transition from the quark-gluon plasma to a hadron gas.

Our calculations suggest, however, that there is a range of temperatures in which perturbation theory at the scale gT has broken down, but perturbation theory at the scale T is reasonably accurate. In this case, one can still use perturbation theory at the scale T to calculate the parameters in the EQCD Lagrangian. Our calculations of the coefficients f_E and m_E^2 to order g^4 are therefore still useful. However, nonperturbative methods, such as lattice simulations of EQCD, are required to calculate the effects of the smaller momentum scales gT and g^2T. While one could simply treat the entire problem nonperturbatively using lattice simulations of full QCD, the effective-field-theory approach provides a dramatic savings in resources for numerical computation. The savings come from two sources. One is the reduction of the problem from a four-dimensional field theory to a three-dimensional field theory. The other source of savings is that quarks are integrated out of the theory, which reduces it to a purely bosonic problem.

We now consider briefly the implications for the study of the quark-gluon plasma in heavy-ion collisions. The critical temperature T_c for formation of a quark-gluon plasma is approximately 200 MeV. It may be possible in heavy-ion collisions to produce a quark-gluon plasma with temperatures several times T_c. At $T=350$ MeV, $\alpha_s(2\pi T)\approx 0.3$, which is small enough that perturbation theory may be reasonably convergent at the scale T, but it is certainly not convergent at the scale gT. We conclude that at the temperatures achievable in heavy-ion collisions, perturbative QCD may be accurate when applied to quantities that involve the scale T only. However, nonperturbative methods are required to accurately calculate quantities that involve the scales gT and g^2T. The most effective strategy for calculating the properties of a quark-gluon plasma at such temperatures will probably involve a combination of perturbative and nonperturbative methods. The effective-field-theory approach developed in this paper provides a systematic method for unraveling the momentum scales in the plasma and for combining perturbative and nonperturbative methods in a consistent way. This approach applies strictly only to static properties and to the case of zero baryon density. The extension to dynamical properties and to the case of nonzero baryon density remains a challenging problem.

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APPENDIX A: SUM INTEGRALS IN THE FULL THEORY

In the imaginary-time formalism for thermal field theory, the four-momentum $P=(p_0,\mathbf{p})$ is Euclidean with $P^2=p_0^2+\mathbf{p}^2$. The Euclidean energy p_0 has discrete values: $p_0=2n\pi T$ for bosons and $p_0=(2n+1)\pi T$ for fermions, where n is an integer. Loop diagrams involve sums over p_0 and integrals over $\mathbf{p}$. It is convenient to use dimensional regularization to regularize both ultraviolet and infrared divergences. We introduce a concise notation for these regularized sum integrals:

$$\sum\!\!\!\!\!\!\!\int_P \equiv \left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} T \sum_{p_0=2n\pi T} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}, \tag{A1}$$

$$\sum\!\!\!\!\!\!\!\int_{\{P\}} \equiv \left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} T \sum_{p_0=(2n+1)\pi T} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}, \tag{A2}$$

where $3-2\epsilon$ is the dimension of space and Λ is an arbitrary momentum scale. The factor $(e^{\gamma}/4\pi)^{\epsilon}$ is introduced so that, after minimal subtraction of the poles in ϵ due to ultraviolet divergences, Λ coincides with the renormalization scale in the $\overline{\rm MS}$ renormalization scheme. Below, we collect together all the sum integrals that are required to calculate the coefficient f_E to next-to-next-to-leading order in g^2 and the coefficient m_E^2 to next-to-leading order in g^2.

The one-loop bosonic sum integrals that arise in the calculation have the following forms:

$$\mathcal{I}_n \equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{(P^2)^n}, \tag{A3}$$

$$\mathcal{J}_n \equiv \sum\!\!\!\!\!\!\!\int_P \frac{p_0^2}{(P^2)^{n+1}}, \tag{A4}$$

$$\mathcal{K}_n \equiv \sum\!\!\!\!\!\!\!\int_P \frac{p_0^4}{(P^2)^{n+2}}. \tag{A5}$$

The specific sum integrals that are needed are

$$\mathcal{I}\,'_0 = \frac{\pi^2}{45} T^4 [1+O(\epsilon)], \tag{A6}$$

$$\mathcal{I}_1 = \frac{1}{12} T^2 \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[1+\left(2+2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon + O(\epsilon^2)\right], \tag{A7}$$

$$\mathcal{J}_1 = -\frac{1}{24} T^2 \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[1+2\frac{\zeta'(-1)}{\zeta(-1)}\epsilon + O(\epsilon^2)\right], \tag{A8}$$

$$\mathcal{I}_2 = \frac{1}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon}+2\gamma + O(\epsilon)\right], \tag{A9}$$

$$\mathcal{J}_2 = \frac{1}{4(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon}+2+2\gamma + O(\epsilon)\right], \tag{A10}$$

$$\mathcal{K}_2 = \frac{1}{8(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon}+\frac{8}{3}+2\gamma + O(\epsilon)\right], \tag{A11}$$

where γ is Euler's constant and $\zeta(z)$ is Riemann's zeta function. In (A6), $\mathcal{I}\,'_0$ denotes the derivative of $\mathcal{I}_n$ with respect to n evaluated at $n=0$. The one-loop fermionic sum integrals have the forms

$$\tilde{\mathcal{I}}_n \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{(P^2)^n}, \tag{A12}$$

$$\tilde{\mathcal{J}}_n \equiv \sum\!\!\!\!\!\!\int_{\{P\}} \frac{p_0^2}{(P^2)^{n+1}}, \tag{A13}$$

$$\tilde{\mathcal{K}}_n \equiv \sum\!\!\!\!\!\!\int_{\{P\}} \frac{p_0^4}{(P^2)^{n+2}}. \tag{A14}$$

The specific sum integrals that are needed are

$$\tilde{\mathcal{I}}'_0 = -\frac{7\pi^2}{360} T^4 [1 + O(\epsilon)], \tag{A15}$$

$$\tilde{\mathcal{I}}_1 = -\frac{1}{24} T^2 \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[1 + \left(2 - 2\ln 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon + O(\epsilon^2)\right], \tag{A16}$$

$$\tilde{\mathcal{J}}_1 = \frac{1}{48} T^2 \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[1 + \left(-2\ln 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon + O(\epsilon^2)\right], \tag{A17}$$

$$\tilde{\mathcal{I}}_2 = \frac{1}{(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + 4\ln 2 + 2\gamma + O(\epsilon)\right], \tag{A18}$$

$$\tilde{\mathcal{J}}_2 = \frac{1}{4(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + 2 + 4\ln 2 + 2\gamma + O(\epsilon)\right], \tag{A19}$$

$$\tilde{\mathcal{K}}_2 = \frac{1}{8(4\pi)^2} \left(\frac{\Lambda}{4\pi T}\right)^{2\epsilon} \left[\frac{1}{\epsilon} + \frac{8}{3} + 4\ln 2 + 2\gamma + O(\epsilon)\right]. \tag{A20}$$

All of the two-loop sum integrals that arise in the calculation factor into the product of 2 one-loop sum integrals. Some of the three-loop sum integrals factor into the product of 3 one-loop sum integrals. Others factor into the product of a one-loop sum integral and a two-loop sum integral. However, these sum integrals all vanish, either because the one-loop sum integral is $\mathcal{I}_0 = 0$ or $\tilde{\mathcal{I}}_0 = 0$, or because the two-loop sum integral vanishes:

$$\sum\!\!\!\!\!\!\int_{PQ} \frac{1}{P^2 Q^2 (P+Q)^2} = 0, \tag{A21}$$

$$\sum\!\!\!\!\!\!\int_{\{P\}Q} \frac{1}{P^2 Q^2 (P+Q)^2} = 0. \tag{A22}$$

The remaining three-loop sum integrals have the forms

$$\mathcal{M}_{i,j} \equiv \sum\!\!\!\!\!\!\int_{PQR} \frac{1}{P^2 Q^2 [R^2]^i [(P-Q)^2]^j (Q-R)^2 (R-P)^2}, \tag{A23}$$

$$\tilde{\mathcal{M}}_{i,j} \equiv \sum\!\!\!\!\!\!\int_{\{PQR\}} \frac{1}{P^2 Q^2 [R^2]^i [(P-Q)^2]^j (Q-R)^2 (R-P)^2}, \tag{A24}$$

$$\mathcal{N}_{i,j} \equiv \sum\!\!\!\!\!\!\int_{\{PQ\}R} \frac{1}{P^2 Q^2 [R^2]^i [(P-Q)^2]^j (Q-R)^2 (R-P)^2}. \tag{A25}$$

These sum integrals can be evaluated analytically using methods developed by Arnold and Zhai [13]. The specific integrals that are needed are

$$\mathcal{M}_{0,0} = \frac{1}{24(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{91}{15} + 8\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A26}$$

$$\tilde{\mathcal{M}}_{0,0} = -\frac{1}{192(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{179}{30} - \frac{34}{5}\ln 2 + 8\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A27}$$

$$\mathcal{N}_{0,0} = \frac{1}{96(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{173}{30} - \frac{42}{5}\ln 2 + 8\frac{\zeta'(-1)}{\zeta(-1)} - 2\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A28}$$

$$\tilde{\mathcal{M}}_{1,-1} = -\frac{1}{192(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{361}{60} + \frac{76}{5}\ln 2 + 6\gamma - 4\frac{\zeta'(-1)}{\zeta(-1)} + 4\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A29}$$

$$\mathcal{M}_{2,-2} = \frac{11}{216(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{73}{22} + \frac{12}{11}\gamma + \frac{64}{11}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{10}{11}\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A30}$$

$$\tilde{\mathcal{M}}_{-2,2} = -\frac{29}{1728(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{89}{29} + \frac{48}{29}\gamma - \frac{90}{29}\ln 2 + \frac{136}{29}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{10}{29}\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right], \tag{A31}$$

$$\mathcal{N}_{2,-2} = \frac{1}{108(4\pi)^2} T^4 \left(\frac{\Lambda}{4\pi T}\right)^{6\epsilon} \left[\frac{1}{\epsilon} + \frac{35}{8} + \frac{3}{2}\gamma - \frac{63}{10}\ln 2 + 5\frac{\zeta'(-1)}{\zeta(-1)} - \frac{1}{2}\frac{\zeta'(-3)}{\zeta(-3)} + O(\epsilon)\right]. \tag{A32}$$

APPENDIX B: INTEGRALS IN THE EFFECTIVE THEORY

The effective theory for the scale gT is an Euclidean field theory in three space dimensions. Loop diagrams involve integrals over three-momenta. It is convenient to introduce the notation $\int_p$ for these integrals. We use dimensional regularization in $3-2\epsilon$ dimensions to regularize both infrared and ultraviolet divergences. We define the integration measure

$$\int_p \equiv \left(\frac{e^{\gamma}\Lambda^2}{4\pi}\right)^{\epsilon} \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}. \tag{B1}$$

If renormalization is accomplished by the minimal subtraction of poles in ϵ, then μ is the renormalization scale in the $\overline{\mathrm{MS}}$ scheme. Below, we collect all the integrals that are needed to calculate the contribution to the free energy from the momentum scale gT to order g^5.

The nontrivial one-loop integrals that arise in the calculation have the form

$$I_n \equiv \int_p \frac{1}{[p^2+m^2]^n}. \tag{B2}$$

The specific one-loop integrals that are needed are

$$I_0' = \frac{m^3}{4\pi}\left(\frac{\Lambda}{2m}\right)^{2\epsilon}\left[\frac{2}{3}+\frac{16}{9}\epsilon+O(\epsilon^2)\right], \tag{B3}$$

$$I_1 = \frac{m}{4\pi}\left(\frac{\Lambda}{2m}\right)^{2\epsilon}[-1-2\epsilon+O(\epsilon^2)], \tag{B4}$$

$$I_2 = \frac{1}{4\pi m}\left(\frac{\Lambda}{2m}\right)^{2\epsilon}\left[\frac{1}{2}+O(\epsilon^2)\right]. \tag{B5}$$

In (B3), I_0' denotes the derivative of I_n with respect to n evaluated at $n=0$.

Some of the two-loop integrals reduce to products of one-loop integrals. The remaining two-loop integrals have the forms

$$J_n \equiv \int_{pq} \frac{1}{(p^2+m^2)[q^2+m^2]^n(p-q)^2}, \tag{B6}$$

$$K_n \equiv \int_{pq} \frac{1}{(p^2+m^2)(q^2+m^2)[(p-q)^2]^n}. \tag{B7}$$

The specific two-loop integrals that are needed are

$$J_1 = \frac{1}{(4\pi)^2}\left(\frac{\Lambda}{2m}\right)^{4\epsilon}\left[\frac{1}{4\epsilon}+\frac{1}{2}+O(\epsilon)\right], \tag{B8}$$

$$J_2 = \frac{1}{(4\pi)^2 m^2}\left(\frac{\Lambda}{2m}\right)^{4\epsilon}\left[\frac{1}{4}+O(\epsilon)\right], \tag{B9}$$

$$K_2 = \frac{1}{(4\pi)^2 m^2}\left(\frac{\Lambda}{2m}\right)^{4\epsilon}\left[-\frac{1}{8}+O(\epsilon)\right]. \tag{B10}$$

Some of the three-loop integrals reduce to the product of 3 one-loop integrals or to the product 1 one-loop integral and 1 two-loop integral. The remaining three-loop integrals have the form

$$M_{i,j} \equiv \int_{pqr} \frac{1}{(p^2+m^2)(q^2+m^2)[r^2+m^2]^i}\,\frac{1}{[(p-q)^2]^j(q-r)^2(r-p)^2}, \tag{B11}$$

$$N_{i,j} \equiv \int_{pqr} \frac{1}{(p^2+m^2)(q^2+m^2)[(q-r)^2+m^2][(r-p)^2+m^2]}\,\frac{1}{[r^2]^i[(p-q)^2]^j}, \tag{B12}$$

$$L_{i,j} \equiv \int_{pqr} \frac{1}{(p^2+m^2)[(r-p)^2+m^2]^i[q^2+m^2]^j[(q-r)^2+m^2]}\,\frac{1}{r^2(p-q)^2}. \tag{B13}$$

These integrals are special cases of more general three-loop integrals defined by Broadhurst [21]:

$$M_{i,j} = m^{1-2i-2j}\left(\frac{\Lambda}{m}\right)^{6\epsilon}\left(\frac{e^{\gamma\epsilon}\Gamma(\frac{3}{2}+\epsilon)}{(4\pi)^{3/2}}\right)^3 B_M(1,j,1,1,1,i), \tag{B14}$$

$$N_{i,j} = m^{1-2i-2j}\left(\frac{\Lambda}{m}\right)^{6\epsilon}\left(\frac{e^{\gamma\epsilon}\Gamma(\frac{3}{2}+\epsilon)}{(4\pi)^{3/2}}\right)^3 B_N(i,j,1,1,1,1), \tag{B15}$$

$$L_{i,j} = m^{1-2i-2j}\left(\frac{\Lambda}{m}\right)^{6\epsilon}\left(\frac{e^{\gamma\epsilon}\Gamma(\frac{3}{2}+\epsilon)}{(4\pi)^{3/2}}\right)^3 B_N(1,1,1,1,i,j). \tag{B16}$$

Broadhurst derived recursion equations for the integrals B_M and B_N with general arguments which can be used to reduce any of the integrals $M_{i,j}$, $N_{i,j}$, and $L_{i,j}$ to the basic integrals $M_{0,0}$ and $N_{0,0}$, together with simpler one-loop and two-loop integrals. The specific integrals that are needed in our calculation are

$$M_{0,0} = \frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{2\epsilon}-4+O(\epsilon)\right], \tag{B17}$$

$$M_{-1,1}=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[\frac{1}{4\epsilon}+2+O(\epsilon)\right], \tag{B18}$$

$$M_{-2,2}=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{4\epsilon}-\frac{3}{2}+O(\epsilon)\right], \tag{B19}$$

$$M_{1,0}=\frac{1}{(4\pi)^3 m}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[\frac{\pi^2}{12}+O(\epsilon)\right], \tag{B20}$$

$$M_{0,1}=\frac{1}{(4\pi)^3 m}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{8\epsilon}+\frac{1}{4}+O(\epsilon)\right], \tag{B21}$$

$$M_{2,0}=\frac{1}{(4\pi)^3 m^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{4}+\frac{\pi^2}{24}+O(\epsilon)\right], \tag{B22}$$

$$N_{0,0}=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{\epsilon}-8+4\ln2+O(\epsilon)\right], \tag{B23}$$

$$N_{1,-1}=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}[2-4\ln2+O(\epsilon)], \tag{B24}$$

$$N_{2,-2}=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}[-3+4\ln2+O(\epsilon)], \tag{B25}$$

$$N_{1,0}=\frac{1}{(4\pi)^3 m}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}[\ln2+O(\epsilon)], \tag{B26}$$

$$N_{2,-1}=\frac{1}{(4\pi)^3 m}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[\frac{1}{3}-\frac{1}{3}\ln2+O(\epsilon)\right], \tag{B27}$$

$$N_{2,0}=\frac{1}{(4\pi)^3 m^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[-\frac{1}{24}-\frac{1}{12}\ln2+O(\epsilon)\right], \tag{B28}$$

$$N_{1,1}=\frac{1}{(4\pi)^3 m^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}\left[\frac{1}{4}-\frac{1}{4}\ln2+O(\epsilon)\right]. \tag{B29}$$

We also require the sum of the integrals $M_{1,-1}$ and $L_{1,-1}$, which is simpler to calculate than the individual integrals:

$$M_{1,-1}+L_{1,-1}=-M_{0,0}+2I_1J_1=\frac{m}{(4\pi)^3}\left(\frac{\Lambda}{2m}\right)^{6\epsilon}[2+O(\epsilon)]. \tag{B30}$$

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