Varieties of vacua in classical supersymmetric gauge theories

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We give a simple description of the classical moduli space of vacua for supersymmetric gauge theories with or without a superpotential. The key ingredient in our analysis is the observation that the Lagrangian is invariant under the action of the complexified gauge group G^c. From this point of view the usual D-flatness conditions are an artifact of the Wess-Zumino gauge. By using a gauge that preserves G^c invariance we show that every constant matter field configuration that extremizes the superpotential is G^c gauge equivalent (in a sense that we make precise) to a unique classical vacuum. This result is used to prove that in the absence of a superpotential the classical moduli space is the algebraic variety described by the set of all holomorphic gauge-invariant polynomials. When a superpotential is present, we show that the classical moduli space is a variety defined by imposing additional relations on the holomorphic polynomials. Many of these points are already contained in the existing literature. The main contribution of the present work is that we give a careful and self-contained treatment of limit points and singularities.

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I. INTRODUCTION

Recently, significant progress has been made in understanding the structure of four-dimensional supersymmetric gauge theories. Building on earlier work [1,2] and using arguments based on symmetry, holomorphy, and weak-coupling limits, it has been possible to reach remarkable conclusions about the nonperturbative structure of these theories [3,4]. Particularly striking results have been achieved in $N=2$ theories using these methods [5]. One of the main goals of this recent work has been to understand the structure of the moduli spaces of vacua in supersymmetric gauge theories.

In Ref. [1] it was noted that the classical space of vacua of the models studied there could be described in terms of holomorphic gauge-invariant polynomials in the matter fields. Specifically, the classical space of vacua can be parametrized by a set of holomorphic polynomials subject to polynomial constraints. This description of the moduli space of vacua is the starting point for many of the recent advances in understanding nonperturbative effects in supersymmetric gauge theories. For example, when the quantum moduli space differs from the classical one, it can still be described in terms of the gauge-invariant polynomials, but with constraints that differ from those in the classical theory [3]. In many interesting cases, the classical and quantum moduli spaces are identical, showing the absence of chiral symmetry breaking in strongly coupled gauge theories [3]. When the classical and quantum spaces are identical, there are often dual descriptions of the theories [4]; the identity of the moduli spaces in the dual descriptions constitutes one of the main pieces of evidence for this duality. To understand all of these

features of supersymmetric gauge theories, the formulation of the classical and quantum moduli spaces of vacua in terms of gauge invariant polynomials is essential.

It is therefore clearly desirable to have a precise understanding of why the classical moduli space can be described in terms of gauge invariant polynomials. However, in the literature this description is used on a case-by-case basis, with little insight given as to its general applicability. The purpose of this paper is to give a simple but rigorous proof that the moduli space of vacua can always be precisely described in this way. Some of the results we obtain are contained in the existing literature [6–8]. The main contribution of the present work is that we give a complete and general proof of the desired result in a unified framework. Also, our work properly takes into account "fine points" such as sets of measure zero and singularities. As we will discuss, these points are physically important because they are often associated with qualitative features such as enhanced gauge symmetry.

Our point of departure is the observation that a supersymmetric gauge theory with gauge group G is invariant under the complexified gauge group G^c. From this point of view, the usual D-flatness conditions can be viewed as a G^c gauge artifact. By using a gauge in which G^c invariance is preserved, we show that in the absence of a superpotential *every* constant value of the matter fields is G^c gauge equivalent (in an extended sense that we make precise) to a solution of the D-flatness conditions. This gives the result that the space of classical vacua is

$$\mathcal{M}_0 = \mathcal{F} \| G^c, \tag{1.1}$$

where $\mathcal{F}$ is the space of all constant matter field configurations and the quotient denoted by $\|$ identifies any G^c orbits that have common limit points. This gives a manifestly holomorphic description of the space $\mathcal{M}_0$. In fact, we can use this result to prove (using elementary results of algebraic geometry) that the space of vacua can be described by the set of all gauge-invariant holomorphic polynomials. These poly-

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nomials form an algebra generated by a finite number of monomials subject to (finitely many) defining constraints, as in Ref. [1]. That is, $\mathcal{M}_0$ is an algebraic variety.

These results generalize simply to the case where a superpotential is present. In that case every constant field configuration that extremizes the superpotential is G^c gauge equivalent (in the extended sense) to a classical vacuum and the space of classical vacua is given by Eq. (1.1), where $\mathcal{F}$ is the space of stationary points of the superpotential. This space $\mathcal{F}$ is by definition an algebraic variety, which is sufficient to show that $\mathcal{M}_0$ is a variety in this case as well.

This paper is organized as follows. In Sec. II we derive our principal results on the structure of the space of vacua; in Sec. III we give several illustrative examples. Section IV contains a discussion of related work and our conclusions. In the Appendix, we give a simple proof that the space $\mathcal{M}_0$ is a variety.

II. CLASSICAL VACUA

A. Quotient space

The Lagrangian of a supersymmetric gauge theory can be written[1]

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\,\Phi^\dagger e^V \Phi + \left(\frac{1}{4e^2} \int d^2\theta\,\mathrm{tr}(W^\alpha W_\alpha) \right.$$
$$\left. + \int d^2\theta W(\Phi) + \mathrm{H.c.} \right), \quad (2.1)$$

where Φ are chiral matter fields transforming in some (in general reducible) representation of the gauge group G, V is a vector superfield taking values in the Lie algebra of G, and $W(\Phi)$ is a superpotential. This Lagrangian is invariant under a large group of gauge transformations:

$$\Phi \mapsto g\Phi, \quad e^V \mapsto g^{-1\dagger} e^V g^{-1}, \quad (2.2)$$

where $g = e^{i\Lambda}$ and Λ is a chiral superfield in the Lie algebra of G. In particular, Λ can be a complex scalar, so this includes G^c transformations. Conventionally, one fixes Wess-Zumino gauge, which breaks G^c invariance leaving only "ordinary" G gauge invariance. We will instead use a gauge in which V takes the form

$$V_A = C_A - \theta\sigma^\mu\bar{\theta}v_{\mu A} + i\theta\theta\bar{\theta}\bar{\lambda}_A - i\bar{\theta}\bar{\theta}\theta\lambda_A + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D_A, \quad (2.3)$$

where A is a G adjoint index. This leaves a residual G^c gauge freedom. It is straightforward to derive the D-flatness conditions in this gauge, which read

$$\frac{\partial}{\partial C_A}(\phi^\dagger e^C \phi) = 0, \quad (2.4)$$

where ϕ is the scalar component of Φ.

This immediately shows that any ϕ that satisfies the D-flatness conditions (2.4) for some C is G^c gauge equivalent to the field $\hat{\phi} = e^{C/2}\phi$, which satisfies

$$0 = \frac{\partial}{\partial \hat{C}_A}(\hat{\phi}^\dagger e^{\hat{C}} \hat{\phi})|_{\hat{C}=0} = \frac{\partial}{\partial \hat{C}_A} \nu(e^{\hat{C}/2}\hat{\phi})|_{\hat{C}=0}, \quad (2.5)$$

where

$$\nu(\phi) \equiv \phi^\dagger \phi. \quad (2.6)$$

Equation (2.5) is just the usual D-flatness condition in Wess-Zumino gauge. Since $\nu(\phi)$ is G invariant, we see that the fields that satisfy these D-flatness conditions are precisely those for which $\nu(\phi)$ is stationary with respect to G^c.[2] The set of points for which this condition is satisfied lie on closed G orbits (since G is compact) that we will refer to as D orbits.

We consider now the case where the superpotential vanishes, and show that *every* constant field configuration ϕ_0 is G^c gauge equivalent to a solution $\hat{\phi}$ of the Wess-Zumino gauge D-flatness condition Eq. (2.5). To make our results precise, we need a slightly generalized notion of G^c gauge equivalence. We say that two constant field configurations ϕ_1 and ϕ_2 are G^c equivalent in the *extended* sense if there is a sequence $\{g_n\}$ of elements in G^c such that

$$\lim_{n\to\infty} g_n \cdot \phi_1 = \phi_2. \quad (2.7)$$

In order for this to define an equivalence we must also impose the same condition with the roles of ϕ_1 and ϕ_2 reversed; we must also impose transitivity; i.e., ϕ_1 and ϕ_2 are equivalent if there is a ϕ_3 that is equivalent to both ϕ_1 and ϕ_2. We call the set of all fields that are equivalent in this sense to a field ϕ the *extended G^c orbit* of ϕ. These definitions are physically sensible because any gauge-invariant function takes the same value on all the field configurations in an extended orbit, so that the points of such an orbit are physically indistinguishable.

With these definitions, the result to be proven can be concisely stated: every extended G^c orbit contains a D orbit. This immediately implies that the space of classical vacua is given by

$$\mathcal{M}_0 = \mathcal{F} \| G^c, \quad (2.8)$$

where $\mathcal{F}$ is the space of all constant matter field configurations, and the extended quotient by G^c is defined using the equivalence defined above. This result is intuitively satisfying since it is closely analogous to the result for nonsupersymmetric theories that (in a theory with no potential) every constant field configuration lies in a gauge equivalence class of vacua.

The proof of this assertion is extremely simple. Fix an arbitrary ϕ_0. Since the function $\nu(\phi)$ is positive semidefinite and is less than or equal to $\nu(\phi_0)$ only on a compact ball

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[1]We use the conventions of Wess and Bagger [6].

[2]Similar arguments have been discussed recently by N. Arkani-Hamed, H. Georgi, and J. March-Russell (private communications).

in ϕ space, it must take on a minimum value at some point in the closure of the ordinary G^c orbit that contains ϕ_0. Thus, there is a $\hat{\phi}$ such that

$$\hat{\phi} = \lim_{n \to \infty} g_n \cdot \phi_0, \qquad (2.9)$$

which minimizes ν on the closure of the orbit. Clearly, $\hat{\phi}$ lies in the extended G^c orbit containing ϕ_0, and thus $\hat{\phi}$ is in a D orbit.

This result makes it intuitively clear why the space of classical vacua can be parameterized by the set of gauge-invariant holomorphic polynomials in the fields ϕ, as advocated in Ref. [1]. Such polynomials are constant on extended G^c orbits, and it seems natural that there are "enough" polynomials to distinguish any two distinct extended orbits. In the Appendix, we show that this intuition can be made rigorous using fairly elementary results from algebraic geometry. We prove that the space $\mathcal{M}_0$ has as coordinates a set of gauge-invariant polynomials subject to finitely many defining relations. In the language of algebraic geometry, $\mathcal{M}_0$ is the algebraic variety defined by the ring of all invariant polynomials on $\mathcal{F}$.

The argument above can be extended immediately to the case where there is a superpotential present. In that case, the fields must extremize the superpotential

$$R_j(\phi) \equiv \frac{\partial W(\phi)}{\partial \phi_j} = 0 \qquad (2.10)$$

as well as satisfying the D-flatness conditions. It is easy to see that if any point in an extended G^c orbit satisfies Eq. (2.10) then all other points in that extended orbit also satisfy this equation. We can thus simply restrict ϕ to satisfy Eq. (2.10) and proceed as above. The result is that the space of vacua is given by Eq. (2.8), where $\mathcal{F}$ is the space of fields that extremize the superpotential. (See also Ref. [6].)

It is now straightforward to describe the classical moduli space of vacua in theories with a superpotential as a variety. The results proven in the Appendix show that the moduli space can be parametrized by the gauge-invariant polynomials on the set of fields that extremize the superpotential. This means that in addition to the defining relations, there are extra relations on the polynomials stating that any gauge-invariant combination of the R's defined in Eq. (2.10) with the ϕ's must vanish. We will give an example of this construction in Sec. III.

B. Observations on orbit structure

We now collect some observations about the structure of extended orbits. The main results of this paper do not depend on these observations, but we include them to clarify the significance of the extended G^c orbits. We first show that there is exactly one D orbit in every extended orbit. This shows that the classical moduli space can be precisely identified with the set of solutions to the Wess-Zumino gauge D-flat conditions with points in the same G orbit identified, and provides a simple connection between our approach and the conventional treatment. We then discuss the relationship between extended orbits and points of enhanced symmetry. We show that in any extended orbit that contains more than

one ordinary G^c orbit, points in the G^c orbit that contains the D orbit have more gauge symmetry than points in other orbits of the same extended orbit.

To show that there is a unique D orbit in every extended orbit, we begin by showing that every stationary point $\hat{\phi}$ of $\nu(\phi)$ on an ordinary G^c orbit O lies in a D orbit which is a global minimum of ν in O. Along any exponential curve

$$\phi(t) = e^{tC/2}\phi_0, \qquad (2.11)$$

because ν is positive semidefinite we have[3]

$$\frac{\partial^2}{\partial t^2} \nu(\phi(t)) = \nu(C\phi(t)) \geq 0. \qquad (2.12)$$

Equation (2.12) can vanish for finite t only if $C\phi(t) = 0$, which is only possible when $\phi(t) = \phi_0$ for all t. Every element of G^c can be written in the form

$$g = e^C \cdot u, \qquad (2.13)$$

where C is Hermitian and $u \in G$. Thus, every point in O can be reached by an exponential curve starting at a point in the same D orbit as ϕ_0, and ν is monotonically increasing along every such curve. This proves that the D orbit is a global minimum of ν in O. In fact, because the D orbit is compact, it is not hard to see that the set of points in O where ν is less than or equal to any fixed number x is a compact set. This implies that any limit of a sequence in O which does not lie in O would have a divergent value of ν, which implies that O is a closed orbit containing all its limit points.

We cannot immediately conclude from this that $\hat{\phi}$ minimizes ν on the extended orbit X containing O, since there are in general directions in X which do not correspond to G^c transformations.[4] We can, however, use the fact that the action of G^c is algebraic (see Appendix) to show that every extended orbit contains a unique D orbit. We have shown that every ordinary G^c orbit which contains a D orbit is closed. The proof of statement (i) in the Appendix shows that for any two disjoint closed sets which are invariant under G^c, there exists a gauge invariant polynomial which takes different values on the two sets. Thus, two distinct closed orbits cannot lie in a single extended orbit. This clearly implies that each extended orbit contains a unique D orbit.

Note that the above proof does not hold when the group G is not compact. A simple example is an Abelian theory with relatively irrational charges.[5] In this case, the gauge group G is not compact, and a single extended orbit contains multiple D orbits.

We now discuss the connection between extended orbits and enhanced gauge symmetry. On any ordinary G^c orbit, the invariant subgroup of G^c is the same (up to conjugation) at all points on the orbit. However, in an extended orbit X the

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[3]We thank H. Georgi for this observation.

[4]As an example of the type of difficulty which may arise, we mention that there are instances where a point $\hat{\phi}$ is the limit of a sequence of points $g_n \cdot \phi_0$, and yet there is no exponential curve $e^{tC}\phi_0$ that approaches $\hat{\phi}$.

[5]We thank A. Nelson for suggesting this example.

G^c orbit containing the D orbit contains points with more gauge symmetry than the points in other orbits in X. This can be seen intuitively by noting that if a sequence of points in one ordinary G^c orbit O approaches a point in another orbit $\hat{O}$ then the direction in which the limit is approached corresponds to an extra invariance of the limit point. Because of the complications mentioned above it is easier to make this result precise using algebraic arguments. As noted in the Appendix, every orbit can be written as a finite union and intersection of algebraic sets. This implies that in the situation above, since $\hat{O}$ must be contained in the closure of O, the dimension of $\hat{O}$ must be strictly smaller than that of O.

Based on this one might suppose that every extended orbit corresponds to a vacuum with enhanced gauge symmetry. However, in theories with no flat directions there is a single extended G^c orbit which contains multiple ordinary G^c orbits, but there is clearly no extra gauge symmetry. One might also conjecture that one can identify points with extra gauge symmetry from the singularity structure of the resulting variety, but we will give examples which show that this is not possible.

III. EXAMPLES

A. Supersymmetric QED

Our first example is supersymmetric QED, a theory with gauge group $G = U(1)$, a matter field Q with charge 1, and a matter field $\tilde{Q}$ with charge -1. We use this simple example to illustrate the structure of the extended G^c orbits. The classical moduli space in this case is parametrized by

$$A \equiv Q\tilde{Q} \tag{3.1}$$

so the moduli space can be identified with the set of all complex numbers $\mathbf{C}$.

To understand the G^c orbit structure, note that $U(1)^c$ is simply the multiplicative group of nonzero complex numbers. The action of G^c in this case is therefore

$$(Q,\tilde{Q}) \mapsto (\alpha Q, \alpha^{-1}\tilde{Q}) \tag{3.2}$$

with $\alpha \neq 0$. The extended orbit corresponding to a value $A \neq 0$ is the set of points

$$(Q,\tilde{Q}) = (q, A/q) \tag{3.3}$$

with $q \neq 0$, which all lie on an ordinary G^c orbit. On the other hand, the extended orbit with $A = 0$ contains three ordinary G^c orbits:

$$(Q,\tilde{Q}) = (q,0),\ (0,\tilde{q}),\ \text{or}\ (0,0) \tag{3.4}$$

with $q,\tilde{q} \neq 0$. The orbit $(0,0)$ is a limit point of the other two orbits. Note that the point $A = 0$ is a point of enhanced symmetry, but the moduli space is completely nonsingular there.

The structure of the classical moduli space in this theory is extremely simple, but it illustrates many of the features we have described above. Generic extended G^c orbits ($A \neq 0$) contain a single ordinary G^c orbit which contains a single D orbit. At points of enhanced symmetry ($A = 0$), the extended orbit contains multiple ordinary G^c orbits, of which only one contains a D orbit. In this case, the G^c orbit which

contains the D orbit has enhanced gauge symmetry, while the other orbits do not. In this extended orbit the orbit containing a D orbit is closed and contains all its limit points, while the remaining orbits contain curves approaching the D orbit.

B. A U(1)×U(1) model

We now consider a chiral theory with a more interesting classical moduli space. The gauge group is $U(1) \times U(1)$, and the matter fields have charges

$$Q \sim (2,0),\quad R \sim (-2,1),\quad S \sim (1,-1),\quad T \sim (-1,0). \tag{3.5}$$

The gauge-invariant polynomials are generated by

$$A = QR^2S^2,\quad B = QT^2,\quad C = QRST, \tag{3.6}$$

which satisfy the defining relation

$$AB = C^2. \tag{3.7}$$

This simple two-dimensional variety is an example of a *quadratic surface*. The only singular point on this variety is the point $A = B = C = 0$. [This can be seen by noting that when $A \neq 0$ the variables (A,C) are good coordinates and when $B \neq 0$, (B,C) are good coordinates.]

This classical moduli space has a one-parameter family of nontrivial extended orbits. For every $B \neq 0$, there is an extended orbit with coordinates $A = C = 0$ which contains three ordinary G^c orbits

$$(Q,R,S,T) = (B/t^2,0,s,t),\ (B/t^2,r,0,t),\ \text{or}\ (B/t^2,0,0,t). \tag{3.8}$$

where $r,s,t \neq 0$. The orbit with $R = S = 0$ contains a D orbit which has enhanced gauge symmetry. (The second $U(1)$ is unbroken.) Note that the variety is not singular on the vacua corresponding to these orbits. It is also amusing to note that the extended orbit structure is not symmetric under interchanging A and B, even though the variety is. This again illustrates that the presence of enhanced gauge symmetry cannot in general be detected from the structure of the variety.

C. Supersymmetric QCD with $N_F = N$

Our final example illustrates how one can obtain a simple description of the moduli space in the presence of a superpotential. Consider supersymmetric QCD, $SU(N)$ gauge theory with chiral superfields Q^{aj} ($j = 1, \ldots, N_F$; $a = 1, \ldots, N$) in the fundamental representation and chiral superfields $\tilde{Q}_{ak}$ ($k = 1, \ldots, N_F$) in the antifundamental representation. We consider here the special case $N_F = N > 2$. According to the discussion above (or from Ref. [1]), the classical space of vacua can be parametrized by the variables

$$M^j{}_k \equiv Q^{aj}\tilde{Q}_{ak},$$

$$B \equiv \frac{1}{N!}\epsilon_{a_1 \cdots a_N}\epsilon_{j_1 \cdots j_N}Q^{a_1 j_1} \cdots Q^{a_N j_N}, \tag{3.9}$$

$$\tilde{B} \equiv \frac{1}{N!} \epsilon^{a_1 \cdots a_N} \epsilon^{k_1 \cdots k_N} \tilde{Q}_{a_1 k_1} \cdots \tilde{Q}_{a_N k_N},$$

subject to the constraint

$$B\tilde{B} = \det(M). \tag{3.10}$$

We wish to consider the theory in the presence of a superpotential

$$W = bB + \tilde{b}\tilde{B} \tag{3.11}$$

with $b, \tilde{b} \neq 0$. (We do not add a mass term.) According to the discussion in the main part of the paper, the moduli space in the presence of the superpotential is given by imposing the additional constraints that all gauge-invariant polynomials which can be constructed from

$$R_{aj} \equiv \frac{\partial W}{\partial Q^{aj}}, \quad \tilde{R}^{ak} \equiv \frac{\partial W}{\partial \tilde{Q}_{ak}} \tag{3.12}$$

vanish. We must therefore impose

$$R_{aj}Q^{ak} = 0, \tag{3.13}$$

$$\tilde{R}^{aj}\tilde{Q}_{ak} = 0, \tag{3.14}$$

$$R_{aj}\tilde{R}^{ak} = 0, \tag{3.15}$$

$$\epsilon^{a_1 \cdots a_N} R_{a_1 j_1} \cdots R_{a_r j_r} \tilde{Q}_{a_{r+1} k_{r+1}} \cdots \tilde{Q}_{a_N k_N} = 0,$$
$$(r = 1, \ldots, N), \tag{3.16}$$

$$\epsilon_{a_1 \cdots a_N} \tilde{R}^{a_1 k_1} \cdots \tilde{R}^{a_r k_r} Q^{a_{r+1} j_{r+1}} \cdots Q^{a_N j_N} = 0,$$
$$(r = 1, \ldots, N). \tag{3.17}$$

Expressed in terms of $M^j{}_k$, B, and $\tilde{B}$, Eqs. (3.13) and (3.14) give

$$B = \tilde{B} = 0. \tag{3.18}$$

The left-hand sides of Eqs. (3.16) and (3.17) for $r>1$ have nonzero baryon number and therefore vanish when expressed in terms of the M's and B's when $B = \tilde{B} = 0$. For $r=1$, we obtain

$$\epsilon_{j_1 \cdots j_N} M^{j_2}{}_{k_2} \cdots M^{j_N}{}_{k_N} = 0, \tag{3.19}$$

$$\epsilon^{k_1 \cdots k_N} M^{j_2}{}_{k_2} \cdots M^{j_N}{}_{k_N} = 0. \tag{3.20}$$

Equation (3.15) gives the constraint

$$\epsilon_{j_1 \cdots j_N} \epsilon^{k_1 \cdots k_N} M^{j_2}{}_{k_2} \cdots M^{j_N}{}_{k_N} = 0, \tag{3.21}$$

which is clearly implied by Eqs. (3.19) and (3.20) above. Thus, the classical moduli space is the space of M's subject to Eqs. (3.19) and (3.20). To understand the meaning of these constraints, note that we can use the $U(N) \times U(N)$ global symmetry of the model to diagonalize M. It is then easy to see that Eqs. (3.19) and (3.20) impose the same constraint,

namely that the rank of M be at most $N-2$. This is therefore the defining constraint of the classical moduli space.

IV. DISCUSSION

We have shown that in classical supersymmetric gauge theories, every matter field ϕ that extremizes the superpotential is related by a (limit of a) complex gauge transformation to a vacuum. Furthermore, we have proven that the space $\mathcal{M}_0$ of classical vacua has a natural structure as an algebraic variety.

There is a related approach to describing the classical space of vacua that follows from the observation that the usual gauge-fixed D-flatness equations precisely describe the symplectic reduction of $\mathcal{F}$ by G. This point of view was used by Witten [7] to discuss $N=2$ Abelian gauge theories in two dimensions. The symplectic quotient of a complex space by G is closely related to the holomorphic quotient by G^c, which is the natural domain of geometric invariant theory. Our result in II B connecting the space of extended orbits to the space of D orbits makes this connection precise for the cases of physical interest. The approach taken in the present paper has the virtue that the quotient space structure emerges naturally and directly as a result of the underlying complexified gauge symmetry. Furthermore, the explicit description of the structure of extended orbits allows us to rigorously describe the quotient space as an algebraic variety without the application of sophisticated mathematical theorems.

Several aspects of the picture that we have presented in this paper have also been considered by others. A closely related argument for the existence of fields minimizing the D-term potential appears in Ref. [6]. A local holomorphic description of the space of vacua was given in Ref. [9]. During the completion of the present work, we learned that similar work has been done independently by N. Arkani-Hamed, H. Georgi, and J. March-Russell (private communications).

It should be emphasized that the descriptions of $\mathcal{M}_0$, both as an extended quotient space and as an algebraic variety, give the precise structure of the space of vacua including isolated special points and singularities. This is important, since such "fine points" often have physical significance. For example, we have shown that there is a close connection between vacua with enhanced gauge symmetry and orbits of the complexified gauge group which do not contain all their limit points. At such vacua, the moduli space is often singular. These singularities continue to play an important role in the quantum theory, where they may change structure or disappear by being blown up [3]. It seems natural to pursue a further understanding of the classical and quantum moduli spaces of vacua using this geometrical point of view.

APPENDIX: PROOF THAT $\mathscr{M}_0$ IS A VARIETY

In this Appendix we give a proof that for any gauge group G and matter fields ϕ in any representation of G the classical moduli space $\mathscr{F} \| G^c$ can be parametrized by a finite set of gauge-invariant polynomials $P_a(\phi)$ subject to a finite number of relations. Specifically, we show that $\mathscr{F} \| G^c$ is the natural algebraic variety associated with the ring of *all* invariant polynomials in ϕ. The proof is valid when there is a superpotential present, in which case the space $\mathscr{F}$ is the set of values for the fields ϕ at which the superpotential is stationary. The presence of a superpotential simply imposes additional relations on the polynomials P_a, as described in Sec. II A. In fact, the result holds for any theory where $\mathscr{F}$ can be described as a variety in terms of a set of fields transforming linearly under G and satisfying a set of algebraic equations.

The proof we give here is essentially a distillation of results contained in a related proof in Ref. [8]. Our goal in presenting this proof here is to make this result accessible to the physics community by giving a self-contained derivation using fairly elementary methods. We will use the language of algebraic geometry but we will only use a few basic definitions and results from this subject. We begin by reviewing those concepts and results that we will use, all of which can be found on the first few pages of any standard textbook (such as Hartshorne [10]).

The set A of points $(x_1, \ldots x_n)$ in the complex vector space $\mathbf{C}^n$ satisfying a system of polynomial equations $f_\alpha(x_1, \ldots, x_n) = 0$ is called an *algebraic set*. The algebraic sets define a special topology on $\mathbf{C}^n$ called the *Zariski topology*. In the Zariski topology the closed sets are the algebraic sets. Open sets are those sets whose complement is closed. All of the usual statements of topology hold in the Zariski topology; e.g., the intersection of a finite number of closed sets is closed, etc. We will distinguish sets closed in the Zariski topology from sets closed in the usual topology by using the terms *Z-closed* and *closed*, respectively. It is easy to see that every Z-closed set is closed and thus every Z-open set is open. A *constructable* set is a set which can be constructed from Z-closed and Z-open sets with a finite number of operations such as unions or intersections. Constructable sets have the nice property that every point in their Z-closure is also in their closure. [This can be shown, for example, by first proving the assertion for an algebraic curve (one-dimensional variety) and then proceeding by induction, reducing the dimension of the initial variety by one by imposing the constraint that an additional equation vanishes.]

Associated with every algebraic set A there is a ring of polynomials $I(A)$ that consists of all polynomials in the variables x_i that vanish at all points of A. $I(A)$ is an *ideal* (invariant subring) of the ring $\mathbf{C}[x_1, \ldots, x_n]$ of all polynomials in the x_i's. The *Hilbert basis theorem* states that $I(A)$ always has a finite number of generators, so that A can always be described as the set of points on which a finite set of polynomials vanishes.

An algebraic set A is *irreducible* when it cannot be written as a union $A = B \cup C$ of two algebraic sets that are proper subsets of A. An irreducible algebraic set is an *affine variety*. A Z-open subset (with respect to the induced topology) of an affine variety is a *quasiaffine variety*. We refer to both as simply varieties. Every variety A has associated with it a ring $R(A)$ of rational functions without poles on A. It can be shown that for an affine variety A, $R(A)$ is just $\mathbf{C}[x_1, \ldots, x_n]/I(A)$, the polynomials in the x_i's subject to the relations defined by $I(A)$. The essential point of algebraic geometry is that all the geometric information about the variety A is encoded in the algebraic structure of the ring $R(A)$. Thus, in algebraic geometry the fundamental objects are commutative rings rather than geometric objects.

The simplest example of how $R(A)$ encodes geometric information about A is given by the algebraic description of points in A. From the above definitions, it is clear that any Z-closed subset B in A can be associated with an ideal $I(B) \supset I(A)$. Thus, $I(B)$ naturally corresponds to an ideal $I(B)/I(A)$ in $R(A)$. Conversely, every nontrivial ideal I of A (an ideal that is neither $\{0\}$ nor A) can be associated with a closed, nonempty algebraic set, the *zero set* $Z(I)$ of I. Using another theorem due to Hilbert (the *Nullstellensatz*), it can be shown that the points in A are in 1-1 correspondence with the ideals $I \subset R(A)$ that are *maximal* in the sense that there exists no larger ideal $I' \supset I$ other than $I' = R$.

An *algebraic map* (or *morphism*) is a map from a variety $A \subset \{(x_1, \ldots, x_n)\}$ to another variety $B \subset \{(y_1, \ldots, y_m)\}$ that can be described by writing the y_i's as rational functions of the x_i's with denominators that are nonvanishing everywhere on A. Such a map gives rise to a ring homomorphism $R(B) \rightarrow R(A)$. It can be shown that the image of a variety under an algebraic map is always a constructable set.

This concludes our brief review of concepts from algebraic geometry. In terms of this language, the statement that we wish to prove is the following.

Theorem. Given a group G^c acting on a variety A, there is a 1-1 correspondence between $A \| G^c$ and the set of points in the affine variety A^G defined by the ring R_G of G-invariant elements in $R = R(A)$.

We are making the technical assumptions (which are always valid in the relevant physical theories) that A is an affine variety in a complex vector space $\mathbf{C}^n$ on which G^c acts linearly, and that G is the product of a semisimple Lie group with a torus $U(1)^k$. Thus, G^c is itself a variety (a so-called *algebraic group*), and the action of G^c on A is described by an algebraic map $\tau : G \times A \rightarrow A$. The G^c orbits in A are the image under τ of $G \times \{p\}$ where p is a point in A; therefore each orbit is a constructable set. (In fact, it can be shown that each orbit is a variety but we will not need this fact.)

Implicit in the statement of the theorem is the result that A^G is an affine variety. This follows from the fact that R_G is finitely generated, which is a consequence of the Hilbert basis theorem and the fact (used and proven in the proof below) that every ideal $I \subset R_G$ generates an ideal M in R with $M \cap R_G = I$.

It will be convenient for us to think of A as lying in the complex vector space with coordinates $x_1, \ldots x_n$. We can then take a set of generators for R_G to be some set $P_1, \ldots P_l$ of G-invariant polynomials in the x_i's. There is a

natural map $\pi: A \to A^G$ that can be defined by simply evaluating the polynomials P_a at a point $x \in A$. Since the polynomials are invariants, this map is constant on orbits of G^c, so for any point $p \in A^G$ the preimage $\pi^{-1}(p)$ is a union of disjoint orbits. Furthermore, by continuity π must be constant on extended G^c orbits in A, so it induces a well-defined map from $A\|G^c$ to A^G.

It should be noted that the variety A^G is a simple example of a general class of varieties that are the subject of a deep and beautiful area of mathematics called geometric invariant theory [8]. Fortunately, in the specific case we are interested in here we can prove the desired result without using any particularly sophisticated or delicate methods from algebraic geometry.

Proof of theorem. We prove two basic statements, of which the theorem is a consequence: (i) For $p \in A^G$, $\pi^{-1}(p)$ contains at most a single extended G^c orbit; (ii) π is onto.

It will be useful to define a *Reynolds operator* $E: R \to R_G$, which is a projection onto the subring R_G of invariants. Because R is a direct sum of finite dimensional irreducible representations of G, such an operator always exists. Important properties of the Reynolds operator are that it is linear, and that for any $f \in R_G$ and $g \in R$, $E(fg) = fE(g)$.

To prove (i), we begin by noting that every extended orbit is Z-closed. This follows from the fact that every orbit is constructable, which implies that the Z-closure and closure of each orbit are identical. Now, suppose that there were two distinct extended orbits O and O' in $\pi^{-1}(p)$. Since O and O' are disjoint, the ideal $I(O) + I(O')$ in R generated by $I(O)$ and $I(O')$ must be all of R. (To see this, note that the ideal $I(O) + I(O')$ cannot be contained in any maximal ideal of R or the corresponding point would be in both O and O'.) Thus, $1 \in I(O) + I(O')$, and we can write, for some $f \in I(O)$ and $f' \in I(O')$, $1 = f + f'$. But then we have $1 = E(1) = E(f) + E(f')$. We now claim that $E(f) \in I(O) \cap R_G$ and $E(f') \in I(O') \cap R_G$. This follows from the fact that the ideals $I(O)$ and $I(O')$ are invariant under G^c (since the extended orbits are invariant) and therefore can be written as a direct sum of linear spaces on which G^c acts irreducibly. We have thus shown that $E(f)$ is an invariant function that takes the value 0 on O and 1 on O'. Thus, $\pi(O) \neq \pi(O')$, completing the proof of (i).

To show that π is onto, we fix a point $p \in A^G$ and show that there exists a nontrivial ideal M in R with zero set $Z(M) = \pi^{-1}(p)$. We define M to be the ideal in R generated by the maximal ideal $I(p) \subset R_G$. M satisfies $Z(M) = \pi^{-1}(p)$ by construction, but we must prove that M is not all of R, so that it is nontrivial. To do this, note that every $g \in M$ can be written as $g = \Sigma e_i f_i$, where the $\{e_i\}$ generate $I(p)$ and $f_i \in R$. If g is invariant, we have $g = E(g) = \Sigma e_i E(f_i) \in I(p)$, which shows that $R_G \cap M = I(p)$. Thus, M is nontrivial, proving (ii).

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