

Polaron variational methods in the particle representation of field theory: II. Numerical results for the propagator

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(Received 27 March 1995)

For the scalar Wick-Cutkosky model in the particle representation we perform a similar variational calculation for the two-point function as was done by Feynman for the polaron problem. We employ a quadratic nonlocal trial action with a retardation function for which several *Ansätze* are used. The variational parameters are determined by minimizing the variational function and in the most general case the nonlinear variational equations are solved numerically. We obtain the residue at the pole, study analytically and numerically the instability of the model at larger coupling constants, and calculate the width of the dressed particle.

PACS number(s): 11.80.Fv, 11.10.St, 11.15.Tk

I. INTRODUCTION

The need of nonperturbative methods in quantum physics is obvious considering the many problems where strong coupling and/or binding effects render perturbation theory inadequate. In nonrelativistic many-body physics variational methods are widely used under these circumstances while this is *not* the case in relativistic field theory. However, Feynman's successful treatment of the polaron problem [1] shows that variational methods may also be used for a nonrelativistic field theory provided that the fast degrees of freedom can be integrated out and their effects properly taken into account in the trial action. In a previous paper [2] [henceforth referred to as (I)], we have extended the polaron variational method to the simplest scalar field theory which describes heavy particles ("nucleons") interacting by the exchange of light particles ("mesons"). In Euclidean space-time the Lagrangian of the Wick-Cutkosky model is given by

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} M_0^2 \Phi^2 + \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 - g \Phi^2 \varphi. \quad (1)$$

Here, M_0 is the bare mass of the nucleon, m is the mass of the meson, and g is the (dimensionful) coupling constant of the Yukawa interaction between them. In the quenched approximation the meson field can be integrated out and one ends up with an effective action for the nucleons only

$$S_{\text{eff}}[x(\tau)] = \int_0^\beta d\tau \frac{1}{2} \dot{x}^2 - \frac{g^2}{2} \int_0^\beta d\tau_1 \times \int_0^\beta d\tau_2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2} e^{iq \cdot [x(\tau_1) - x(\tau_2)]}. \quad (2)$$

Note that this is formulated in terms of trajectories $x(\tau)$ of the heavy particle ("particle representation") which

are parametrized by the proper time τ and obey the boundary conditions $x(0) = 0$ and $x(\beta) = x$. To obtain the two-point function, one has to perform the path integral over all trajectories and to integrate over β from zero to infinity with a certain weight. It is, of course, impossible to perform this path integral exactly and, again following Feynman, we have approximated it variationally by a retarded quadratic two-time action. In (I) we have proposed various parametrizations for the retardation function which enters this trial action and derived variational equations for the most general case when its form was left free.

The purpose of the present paper is to investigate numerically these parametrizations as well as to solve the variational equations. This fixes the variational parameters which will be used to calculate physical observables in forthcoming applications. One quantity which we evaluate in the present paper is the residue on the pole of the propagator. Another one is related to the well-known instability [3] of the Wick-Cutkosky model: although the effective action (2) is very similar to the one in the polaron model the ground state of the system is only metastable. This does not show up in any order of perturbation theory but, as we have demonstrated in (I), the variational approach knows about it. Indeed an approximate solution of the variational equations has revealed that there are no real solutions beyond a certain critical coupling. In the present paper we will find exact numerical values for this critical coupling and calculate the width of the unstable particle for couplings beyond it.

This paper is organized as follows. The essential points of the polaron variational approach are collected in Sec. II, while Sec. III is devoted to the numerical methods and results. In Sec. IV we investigate the instability of the Wick-Cutkosky model in our variational method and determine analytically and numerically the width of

the dressed particle. The variational principle can also be applied away from the pole, which is explored in Sec. V and used to calculate the residue at the nucleon pole. The main results of this work are summarized in the Sec. VI.

II. POLARON VARIATIONAL APPROACH

Following Feynman's treatment of the polaron problem we have performed in (I), a variational calculation of the two-point function with the quadratic trial action

$$S_t[x] = \int_0^\beta d\tau \frac{1}{2} \dot{x}^2 + \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2 f(\tau_1 - \tau_2) \times [x(\tau_1) - x(\tau_2)]^2. \quad (3)$$

Here, $f(\tau_1 - \tau_2)$ is an undetermined "retardation function" which takes into account the time lapse occurs when mesons are emitted and reabsorbed on the nucleon. In actual calculations we rather have used the Fourier space form

$$S_t = \sum_{k=0}^{\infty} A_k b_k^2, \quad (4)$$

where the b_k are the Fourier components of the path $x(\tau)$ and the Fourier coefficients A_k are considered as variational parameters. The variational treatment is based on the decomposition of the action S_{eff} into $S_{\text{eff}} = S_t + \Delta S$ and on Jensen's inequality

$$\langle e^{-\Delta S} \rangle \geq e^{-\langle \Delta S \rangle}. \quad (5)$$

Near $p^2 = -M_{\text{phys}}^2$, the two-point function should behave like

$$G_2(p^2) \rightarrow \frac{Z}{p^2 + M_{\text{phys}}^2}, \quad (6)$$

where $0 \leq Z \leq 1$ is the residue. As was shown in (I) this requires the proper time β to tend to infinity. One then obtains the inequality

$$M_{\text{phys}}^2 \leq \frac{M_1^2}{2\lambda} + \frac{\lambda}{2} M_{\text{phys}}^2 + \frac{1}{\lambda} (\Omega + V), \quad (7)$$

where

$$M_1^2 = M_0^2 - \frac{g^2}{4\pi^2} \ln \frac{\Lambda^2}{m^2} \quad (8)$$

is a finite mass into which the divergence of the self-energy has been absorbed and λ a variational parameter. For $\beta \rightarrow \infty$, all discrete sums over Fourier modes A_k turn into integrals over the "profile function" $A(E = k\pi/\beta)$ and one finds

$$\Omega = \frac{2}{\pi} \int_0^\infty dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right], \quad (9)$$

as well as

$$V = -\frac{g^2}{8\pi^2} \int_0^\infty d\sigma \int_0^1 du \times \left[\frac{1}{\mu^2(\sigma)} e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}}\sigma}{\mu(\sigma)}, u \right) - \frac{1}{\sigma} e(m\sqrt{\sigma}, 0, u) \right]. \quad (10)$$

Here, we use the abbreviations

$$e(s, t, u) = \exp \left(-\frac{s^2}{2} \frac{1-u}{u} - \frac{t^2}{2} u \right) \quad (11)$$

and

$$\mu^2(\sigma) = \frac{4}{\pi} \int_0^\infty dE \frac{1}{A(E)} \frac{\sin^2(E\sigma/2)}{E^2}. \quad (12)$$

Because $\mu^2(\sigma)$ behaves like σ and σ/A_0 for small and large σ , respectively, we have called it a "pseudotime." Note that in Eq. (10) the particular renormalization point $\mu_0 = 0$ has been used to regularize the small- σ behavior of the integrand. As we have shown in (I) the total result is, of course, independent of μ_0 .

The profile function $A(E)$ is linked to the retardation function $f(\sigma)$ by

$$A(E) = 1 + \frac{8}{E^2} \int_0^\infty d\sigma f(\sigma) \sin^2 \frac{E\sigma}{2}. \quad (13)$$

In (I) we have studied the following parametrizations. "Feynman" parametrization:

$$f_F(\sigma) = w \frac{v^2 - w^2}{4} e^{-w\sigma}, \quad (14)$$

which leads to

$$A_F(E) = \frac{v^2 + E^2}{w^2 + E^2}. \quad (15)$$

"Improved" parametrization:

$$f_I(\sigma) = \frac{v^2 - w^2}{2w} \frac{1}{\sigma^2} e^{-w\sigma}, \quad (16)$$

which entails

$$A_I(E) = 1 + 2 \frac{v^2 - w^2}{wE} \left[\arctan \frac{E}{w} - \frac{w}{2E} \ln \left(1 + \frac{E^2}{w^2} \right) \right]. \quad (17)$$

In both cases v, w are variational parameters whose values have to be determined by minimizing Eq. (7).

As well as the above parametrizations, it was possible not to impose any specific form for the retardation function but to determine it from varying equation (7) with respect to λ and $A(E)$. This gave the relations

$$\frac{1}{\lambda} = 1 + \frac{g^2}{8\pi^2} \int_0^\infty d\sigma \frac{\sigma^2}{\mu^4(\sigma)} \times \int_0^1 du u e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}}\sigma}{\mu(\sigma)}, u \right) \quad (18)$$

$$\begin{aligned}
A(E) = & 1 + \frac{g^2}{4\pi^2} \frac{1}{E^2} \int_0^\infty d\sigma \frac{\sin^2(E\sigma/2)}{\mu^4(\sigma)} \int_0^1 du \\
& \times \left[1 + \frac{m^2}{2} \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2 M_{\text{phys}}^2 \sigma^2}{2\mu^2(\sigma)} u \right] \\
& \times e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}} \sigma}{\mu(\sigma)}, u \right). \quad (19)
\end{aligned}$$

Together with Eq. (12), they constitute a system of coupled variational equations which have to be solved. Assuming $\mu^2(\sigma) \simeq \sigma$ and $m \simeq 0$ we have found in (I) an approximate solution which had the same form of the retardation function as the “improved” parametrization and exhibited the instability of the system beyond a critical coupling constant. In the general case we can read off the variational retardation function from Eq. (19):

$$f_{\text{var}}(\sigma) = \frac{g^2}{32\pi^2} \frac{1}{\mu^4(\sigma)} \int_0^1 du \left[1 + \frac{m^2}{2} \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2 M_{\text{phys}}^2 \sigma^2}{2\mu^2(\sigma)} u \right] e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}} \sigma}{\mu(\sigma)}, u \right). \quad (20)$$

Obviously, it has the same $1/\sigma^2$ behavior for small relative times as the “improved” parametrization (16). Finally, we mention that by means of the variational equation (19), one can find an expression for the “kinetic term” Ω defined in Eq. (9) as

$$\Omega_{\text{var}} = \frac{g^2}{8\pi^2} \int_0^\infty d\sigma \int_0^1 du \left[1 + \frac{m^2}{2} \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2 M_{\text{phys}}^2 \sigma^2}{2\mu^2(\sigma)} u \right] e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}} \sigma}{\mu(\sigma)}, u \right) \frac{\partial}{\partial \sigma} \left(\frac{\sigma}{\mu^2(\sigma)} \right). \quad (21)$$

This is demonstrated in the Appendix and will be used in Sec. IV. That the kinetic term Ω can be combined with the “potential term” V is a consequence of the virial theorem for a two-time action [4] which the variational approximation fulfills.

III. NUMERICAL RESULTS

In this section we will compare numerically the various parametrizations for the retardation function. Because we are primarily interested in an eventual application in pion-nucleon physics, we have chosen the masses and coupling constants appropriately. Of course, the model does not really give a realistic description of the pion-nucleon interaction as spin and isospin degrees of freedom as well as chiral symmetry are missing.

In short, we use

$$m = 140 \text{ MeV}, \quad (22)$$

$$M_{\text{phys}} = 939 \text{ MeV}, \quad (23)$$

and the results are presented as functions of the dimensionless coupling constant

$$\alpha = \frac{g^2}{4\pi} \frac{1}{M_{\text{phys}}^2}. \quad (24)$$

The relevant quantity for the physical situation is the

strength of the Yukawa potential between two nucleons because of one pion exchange [5], which is approximately given by (depending on the spin-isospin channel)

$$f^2 = \frac{g'^2}{4\pi} \left(\frac{m}{2M_{\text{phys}}} \right)^2 \cong 0.08, \quad (25)$$

where $g'^2/4\pi \cong 14$ is the pion-nucleon coupling. In the Wick-Cutkosky scalar model the corresponding strength is just the dimensionless coupling constant α that we have defined in Eq. (24). It should also be remembered that a Yukawa potential only supports a bound state [6] if

$$\alpha > 1.680 \frac{m}{M} = 0.2505. \quad (26)$$

We have minimized [cf. Eq. (7)]

$$-M_1^2 \leq (\lambda^2 - 2\lambda) M_{\text{phys}}^2 + 2(\Omega + V) \quad (27)$$

with the “Feynman” ansatz (15) and the “improved” ansatz (17). This minimization was performed numerically with respect to the parameters λ, v, w by using the standard CERN program MINUIT. The numerical integrations were done with typically 2×72 Gauss-Legendre points after mapping the infinite-range integrals to finite range. For the “improved” retardation function we had to calculate $\mu^2(\sigma)$ and Ω numerically. Tables I and II

TABLE I. Variational calculation for the nucleon self-energy in the Wick-Cutkosky model using the “Feynman” parametrization (15) for the profile function. The parameters v, w obtained from minimizing Eq. (27) are given as well as λ and the intermediate renormalized mass M_1 [see Eq. (8)]. The last column lists $A(0) = v^2/w^2$.

α	\sqrt{v} [MeV]	\sqrt{w} [MeV]	λ	M_1 [MeV]	$A(0)$
0.1	1850	1845	0.97300	890.23	1.0120
0.2	1805	1794	0.94400	839.73	1.0257
0.3	1756	1739	0.91250	787.29	1.0417
0.4	1702	1678	0.87773	732.69	1.0606
0.5	1641	1608	0.83843	675.70	1.0838
0.6	1569	1527	0.79223	616.09	1.1142
0.7	1477	1424	0.73355	553.93	1.1582
0.8	1325	1254	0.63714	490.60	1.2485

TABLE II. Same as in Table I but using the “improved” parametrization (17) for the profile function.

α	\sqrt{v} [MeV]	\sqrt{w} [MeV]	λ	M_1 [MeV]	$A(0)$
0.1	677.2	674.6	0.97297	890.25	1.0158
0.2	661.4	656.0	0.94390	839.78	1.0338
0.3	640.8	632.3	0.91223	787.43	1.0548
0.4	613.7	601.9	0.87715	732.97	1.0808
0.5	596.7	581.2	0.83741	676.20	1.1109
0.6	570.4	550.7	0.79040	616.97	1.1514
0.7	534.3	509.2	0.72996	555.45	1.2118
0.8	468.2	434.5	0.62429	493.44	1.3482

give the results of these calculations. We also include the value of M_1 although it does not have a physical meaning: finite terms (which, for example, arise when a different renormalization point is chosen) can be either grouped with M_1 or with V . However, from the variational inequality (27) we see that M_1 is a measure of the quality of the variational approximation: the larger the M_1 , the better the approximation.

Although the value of the parameters v and w are rather different for the Feynman and the “improved” parametrization, the parameter λ and the value of the profile function at $E = 0$ are very close. This is also reasonable when we study the behavior of these quantities under a *reparametrization* of the particle path: it can be shown that a rescaling of the proper time $\beta \rightarrow \beta/\kappa$ leaves the variational functional invariant if

$$A^{(\kappa)}\left(\frac{E}{\kappa}\right) = A^{(\kappa=1)}(E). \quad (28)$$

We are working in the “proper time gauge” $\kappa = 1$. In a general “gauge” κ the variational parameters v, w then obviously are different [see Eqs. (15, 17)],

$$v^{(\kappa)} = \kappa v, \quad w^{(\kappa)} = \kappa w, \quad (29)$$

but $A(0) = v^2/w^2$ and λ are gauge invariant.

For both parametrizations, no minimum of Eq. (27) was found beyond

$$\alpha > \alpha_c, \quad (30)$$

where

$$\alpha_c = \begin{cases} 0.824 & \text{ (“Feynman”)}, \\ 0.817 & \text{ (“improved”)}. \end{cases} \quad (31)$$

This value of the critical coupling is surprisingly close to the value $\alpha_c \simeq \pi/4$ which we obtained from the approximate solution of the variational equations in (I). On the other hand, when the parameter λ is fixed to $\lambda = 1$, i.e., a less general trial action for “momentum averaging” [see (I)] is used, then a minimum is found for *all* values of α . This points to the important role played by this parameter. Indeed, in the approximate solution of the variational equations found in (I) the branching of the real solutions into complex ones is most clearly seen in the approximate solution for λ . We can also trace the instability to the inequality (7) for the physical mass: a clear minimum as a function of λ exists only as long as the coefficient of $1/\lambda$, i.e., $M_1^2/2 + \Omega + V$ stays posi-

tive. However, with increasing coupling, M_1 shrinks and V becomes more negative until at the critical coupling the collapse finally occurs.

We have also solved the coupled nonlinear variational equations (18) and (19) together with (12) numerically.¹ This was done by the following *iterative* method: we first mapped variables with infinite range to finite range, e.g.,

$$E = M_{\text{phys}}^2 \tan \theta, \quad (32)$$

$$\sigma = \frac{1}{M_{\text{phys}}^2} \tan \psi, \quad (33)$$

and then discretized the integrals by the standard Gauss-Legendre integration scheme, with typically 72 or 96 Gaussian points per integral. The functions $A(\theta), \mu^2(\psi)$, as given by the variational equations, were then tabulated at the Gaussian points using as input the values of $\lambda, A(\theta), \mu^2(\psi)$ from the previous iteration. We started with the perturbative values

$$\begin{aligned} \lambda^{(0)} &= A(\theta_i)^{(0)} = 1, \\ \mu^2(\psi_i)^{(0)} &= \frac{1}{M_{\text{phys}}^2} \tan \psi_i \end{aligned} \quad (34)$$

and monitored the convergence with the help of the largest relative deviation

$$\Delta_n = \text{Max} \left(\frac{|\lambda^{(n)} - \lambda^{(n-1)}|}{\lambda^{(n)}}, \frac{|A(\theta_i)^{(n)} - A(\theta_i)^{(n-1)}|}{A(\theta_i)^{(n)}}, \frac{|\mu^2(\psi_i)^{(n)} - \mu^2(\psi_i)^{(n-1)}|}{\mu^2(\psi_i)^{(n)}} \right), \quad n = 1, 2, \dots \quad (35)$$

Some numerical results are given in Table III. Comparing with Table II we observe a remarkable agreement with the values from the “improved” parametrization. It is only near $\alpha = 0.8$ that the variational solution is appreciably better as demonstrated by the numerical value of M_1 which measures the quality of the corresponding approximation.

This may also be seen in Figs. 1–3 where the different profile functions and pseudotimes are plotted for $\alpha = 0.2$ and $\alpha = 0.8$. One can also confirm from the

¹Note that the variational solution is also reparametrization invariant: Equations (19) and (12) are consistent with the condition (28).

TABLE III. The variational parameter λ , the renormalized mass M_1 , and the value of the profile function at $E = 0$ from the solution of the variational equations.

α	λ	M_1 [MeV]	$A(0)$
0.1	0.97297	890.25	1.0151
0.2	0.94389	839.78	1.0322
0.3	0.91223	787.43	1.0520
0.4	0.87718	732.97	1.0755
0.5	0.83738	676.20	1.1044
0.6	0.79030	616.98	1.1421
0.7	0.72968	555.47	1.1972
0.8	0.62262	493.55	1.3188

graphs that the numerical results indeed have the limits for σ, E either small or large which we expect from the analytical analysis. Furthermore, it is clear that the “improved” parametrization of the trial action is, in general extremely close to the “variational” one, while the “Feynman” parametrization deviates much more. Finally, it is interesting to note that the profile function of the “variational” calculation has a small dip near $E = 0$ which is a result of the additional terms in the retardation function (20). These rather innocent-looking deviations will become important if an analytic continuation back to Minkowski space is performed in which physical scattering processes take place.

Examples for the convergence of the iterative scheme are shown in Fig. 4. It is seen that for small coupling constant we have rapid convergence which becomes slower and slower as the critical value

$$\alpha_c = 0.815 \quad (\text{“variational”}) \quad (36)$$

is reached. Finally, beyond $\alpha > \alpha_c$, only a minimal relative accuracy can be reached and the deviations increase again with additional iterations.

How the critical coupling depends on the meson mass is shown in Fig. 5. It turns out that the good agreement of the approximate value of $\alpha_c \approx \pi/4$ with the numerical value obtained for $m = 140$ MeV was an accidental one:

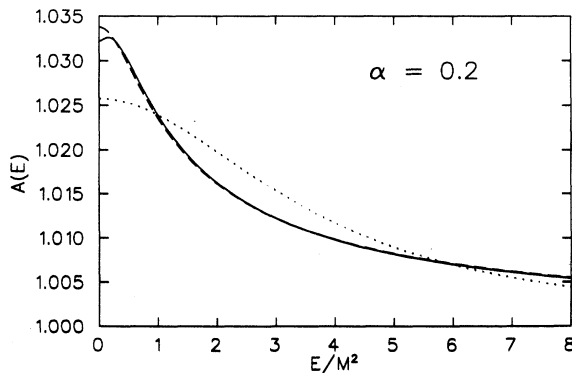


FIG. 1. Profile function $A(E)$ as function of E for the “Feynman” parametrization (15) (dotted line), the “improved” parametrization (17) (dashed line), and the “variational” solution (solid line). The dimensionless coupling constant is $\alpha = 0.2$.

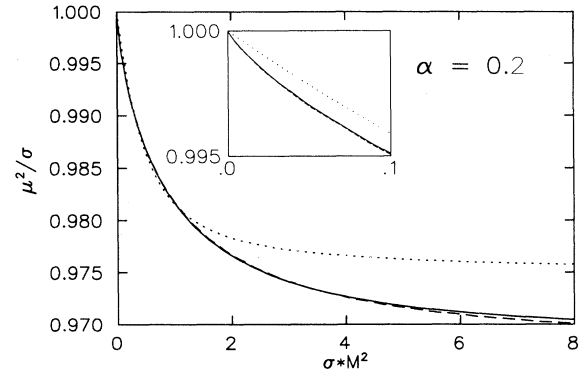


FIG. 2. Ratio of pseudotime $\mu^2(\sigma)$ to proper time σ for $\alpha = 0.2$. The labeling of the curves is as in Fig. 1. An expanded view of the small- σ region is shown in the inset.

at $m = 0$ we have $\alpha_c = 0.641$. There is also a surprisingly strong but nearly linear m dependence which we cannot reproduce from an approximate solution of the variational equations when taking $m \neq 0$ but still assuming $\mu^2(\sigma) \approx \sigma$.

IV. INSTABILITY AND WIDTH OF THE DRESSED PARTICLE

In all parametrizations of the profile function $A(E)$ which we investigated numerically in the previous section, it turned out to be impossible to find a (real) solution of the variational equations or the variational inequality for coupling constants above a critical value α_c . This is a signal of the instability of the model which is already seen in the classical “potential”

$$V^{(0)}(\Phi, \varphi) = \frac{1}{2}M_0^2\Phi^2 + \frac{1}{2}m^2\varphi^2 - g\Phi^2\varphi \quad (37)$$

and tells us that the physical mass of the dressed particle becomes complex

$$M_{\text{phys}} = M - i\frac{\Gamma}{2}. \quad (38)$$

In the following we take the real part of the nucleon mass to be $M = 939$ MeV and try to determine the width Γ .

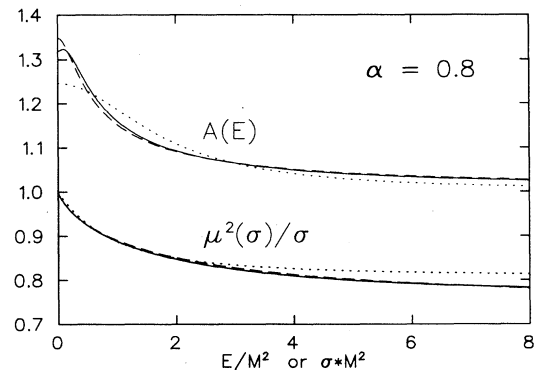


FIG. 3. $A(E)$ and $\mu^2(\sigma)$ for $\alpha = 0.8$. The labeling of the curves is as in Fig. 1.

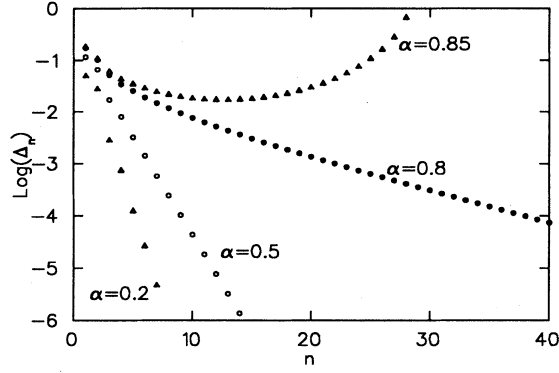


FIG. 4. Convergence of the iterative solutions of the variational equations as functions of the number of iterations n . The convergence measure Δ_n is defined in Eq. (35).

Note that in a perturbative calculation no sign of the instability shows up: the one-loop self-energy

$$\Sigma(p^2) = -\frac{g^2}{4\pi^2} \ln \frac{\Lambda^2}{m^2} + \frac{g^2}{4\pi^2} \int_0^1 du \times \ln \left[1 + \frac{p^2}{m^2} u + \frac{M_0^2}{m^2} \frac{u}{1-u} \right] \quad (39)$$

is perfectly well behaved. Also the one-loop effective potential [7, 8] is not very indicative: in the quenched approximation it is given by

$$V_{\text{eff}}^{(1)}(\Phi, \varphi) = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left[1 - \frac{4g^2 \Phi^2}{p^2 + m^2} \times \frac{1}{p^2 + M_0^2 - 2g\varphi - i\epsilon} \right]. \quad (40)$$

A detailed analysis shows that the quantum corrections lower the barrier which makes the ground state metastable in $V^{(0)}(\Phi, \varphi)$ but do not remove it. In addition, the one-loop effective potential develops an imaginary part but it is easy to see that $\text{Im } V_{\text{eff}}^{(1)}$ vanishes for

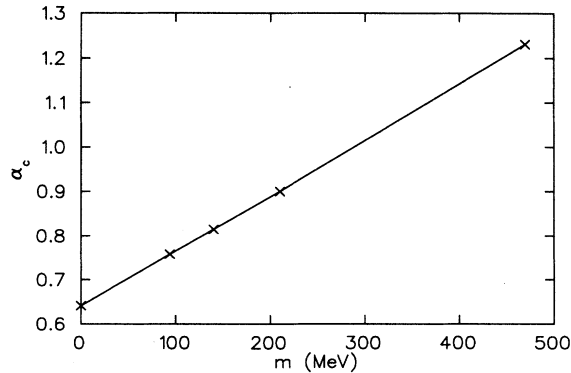


FIG. 5. Critical coupling constant as a function of the meson mass m . The nucleon mass is fixed at $M = 939$ MeV. The crosses indicate the points at which the critical coupling has been calculated, the line through them being drawn to guide the eye.

$M^2 - 2g\Phi > 4g^2\phi^2/m^2$, i.e., it contains a (nonanalytic) step function. Therefore, all proper one-loop Green functions (which are generated by the effective action) carry no sign of the instability.

In contrast, the variational approach for the two-point function *knows* about the instability if we allow the parameter λ in the trial action to vary. Since the approximate solution of the variational equation for λ in (I) clearly showed the impossibility to obtain a real solution beyond α_c , we will first study the width of the state using similar approximative methods before turning to the exact numerical evaluation.

A. Approximate analytical treatment

In order to discuss complex solutions of the variational equations it is useful to introduce the complex quantity

$$\zeta = \lambda M_{\text{phys}} \quad (41)$$

and to write it in the form

$$\zeta = \zeta_0 e^{-ix}. \quad (42)$$

It is a phase $\chi \neq 0$ which will lead to the complex pole (38) of the two-point function. With the same approximation $m = 0$ and $\mu^2(\sigma) \approx \sigma$, which was used before, we now obtain

$$\frac{1}{\lambda} = 1 + \frac{\alpha}{\pi} \frac{M^2}{\zeta^2}. \quad (43)$$

Here, the dimensionless coupling constant is defined in terms of the real part of the physical mass

$$\alpha \equiv \frac{g^2}{4\pi^2 M^2}. \quad (44)$$

Because of Eq. (21) the kinetic term vanishes under the same approximation

$$\Omega_{\text{var}} \approx 0, \quad (45)$$

and the potential term becomes

$$\begin{aligned} V &\approx -\frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty d\sigma \frac{1}{\sigma} \\ &\times \left[\exp \left(-\frac{m^2}{2} \frac{1-u}{u} \sigma - \frac{\zeta^2}{2} u \sigma \right) \right. \\ &\quad \left. - \exp \left(-\frac{m^2}{2} \frac{1-u}{u} \sigma \right) \right] \\ &= \frac{g^2}{8\pi^2} \int_0^1 du \ln \left[1 + \frac{\zeta^2}{m^2} \frac{u^2}{1-u} \right]. \end{aligned} \quad (46)$$

These are rather drastic simplifications but the exact numerical calculations show that the imaginary part of Ω is indeed smaller (by a factor of 5) than $\text{Im } V$. Note that V is not infrared stable, i.e., it diverges if the meson mass m is set to zero. With the above approximations, the stationarity equation (7) then reads

$$M_1^2 = \left(\frac{2}{\lambda} - 1 \right) \zeta^2 - \frac{\alpha}{\pi} M^2 \int_0^1 du \ln \left[1 + \frac{\zeta^2}{m^2} \frac{u^2}{1-u} \right]. \quad (47)$$

Using Eq. (43) this is equivalent to

$$\zeta^2 = M_1^2 - \frac{2\alpha}{\pi} M^2 + \frac{\alpha}{\pi} M^2 \int_0^1 du \ln \left[1 + \frac{\zeta^2}{m^2} \frac{u^2}{1-u} \right]. \quad (48)$$

If we take the *imaginary* part of this equation, it is possible to set $m = 0$ and we obtain

$$\zeta_0^2 \sin 2\chi = \frac{2\alpha}{\pi} M^2 \chi. \quad (49)$$

How do we determine the width of the unstable state? We take the defining equation (41) for ζ , eliminate λ by means of Eq. (43), and use Eq. (38). This gives

$$M - i \frac{\Gamma}{2} = \zeta + \frac{\alpha M^2}{\pi \zeta}. \quad (50)$$

The real and imaginary parts of this equation allow us to express ζ_0 and the width as functions of the phase χ . A simple calculation gives

$$\zeta_0 = \frac{M}{2 \cos \chi} \left[1 + \sqrt{1 - \frac{4\alpha}{\pi} \cos^2 \chi} \right] \quad (51)$$

and the width is

$$\Gamma = 2M \tan \chi \sqrt{1 - \frac{4\alpha}{\pi} \cos^2 \chi}. \quad (52)$$

We have chosen the root which results in a positive width for $0 \leq \chi \leq \pi/2$. Finally, substituting Eq. (51) into Eq. (49) gives the transcendental equation which determines the phase χ . After some algebraic transformations, we obtain it in the form

$$\alpha = \pi \frac{2\chi \sin 2\chi}{(2\chi + \sin 2\chi)^2 \cos^2 \chi} \equiv \frac{\pi}{4} h(\chi). \quad (53)$$

It is easy to see that the function $h(\chi)$ grows monotonically from $h(0) = 1$ to $h(\pi/2) = \infty$. Solutions $\chi(\alpha)$, therefore, only exist for

$$\alpha > \alpha_c = \frac{\pi}{4}, \quad (54)$$

which is the same critical value of the coupling constant at which previously the real (approximate) solutions of the variational equations ceased to exist. It is also easy to find solutions for the transcendental equation (53) for small χ : from $h(\chi) = 1 + \chi^2 + O(\chi^4)$ we find

$$\chi \approx \sqrt{\frac{\alpha - \alpha_c}{\alpha_c}}, \quad (55)$$

where, of course, $\alpha_c = \pi/4$ should be used. Since the expression (52) for the width can be transformed into

$$\Gamma = 2M \tan \chi \frac{2\chi - \sin 2\chi}{2\chi + \sin 2\chi}, \quad (56)$$

we obtain the following *nonanalytic* dependence of the width on the coupling constant

$$\Gamma \approx \frac{2}{3} M \left(\frac{\alpha - \alpha_c}{\alpha_c} \right)^{3/2}. \quad (57)$$

This should be valid near the critical coupling constant.

B. Numerical results

For the numerical solution of the complex variational equations we follow the approximate analytical solution as closely as possible. However, some of the relations used previously do not hold exactly. For example, the quantity

$$L = \frac{\zeta^2}{2} \int_0^\infty d\sigma \frac{\sigma^2}{\mu^4(\sigma)} \int_0^1 du u e \left(m\mu(\sigma), \frac{\zeta\sigma}{\mu(\sigma)}, u \right) \quad (58)$$

would be unity for $m = 0$, $\mu^2(\sigma) = \sigma$ but has some complex value in the exact treatment. Similarly, $\Omega \neq 0$ and V deviate from the approximate value (46). Without invoking the simplifying assumptions, Eq. (48) changes to

$$\zeta^2 = M_1^2 - \frac{2\alpha}{\pi} M^2 L + 2(\Omega + V). \quad (59)$$

Following the same steps as in the approximate treatment, we obtain

$$\zeta_0 = \frac{M}{2 \cos \chi} \left[1 + \sqrt{1 - \frac{4\alpha}{\pi} \cos \chi \operatorname{Re}(Le^{i\chi})} \right] \quad (60)$$

which replaces Eq. (51) and

$$\alpha = \pi \frac{K}{[\operatorname{Re}(Le^{i\chi}) + K \cos \chi]^2} \quad (61)$$

which supersedes Eq. (53). Here,

$$K = \frac{2}{\sin 2\chi} \operatorname{Im} \left[L - \frac{\pi}{\alpha M^2} (\Omega + V) \right]. \quad (62)$$

Instead of Eq. (56), one can show that the width itself has now the exact form

$$\Gamma = 2M \frac{K \sin \chi - \operatorname{Im}(Le^{i\chi})}{K \cos \chi + \operatorname{Re}(Le^{i\chi})}. \quad (63)$$

We have solved the coupled complex equations by specifying a value for the phase χ and determining the corresponding value of the coupling constant α by means of Eq. (61). Of course, this could be done only iteratively by starting with

$$L^{(0)} = 1, \quad K^{(0)} = 2\chi / \sin 2\chi, \\ \mu^2(\sigma)^{(0)} = \sigma, \quad A^{(0)}(E) = 1.$$

Typically, 20–25 iterations were needed to get a relative accuracy of better than 10^{-5} . Table IV gives the results of our calculations. It is seen that the width grows rapidly after the coupling constant exceeds the critical value. In Fig. 6 this is shown together with the approximate (small- χ) behavior predicted by Eq. (57). After the critical coupling constant in this formula has been shifted to the precise value, one observes a satisfactory agreement with the exact result.

Finally, Figs. 7 and 8 depict the complex profile

TABLE IV. The width Γ of the unstable state from the complex solution of the variational equations for $\alpha > \alpha_c = 0.815$. The width is given as a function of the phase χ which determines the corresponding coupling constant α according to Eq. (61). The complex value of the profile function at $E = 0$ is also listed.

χ	α	Γ [MeV]	$A(0)$
0.05	0.818	0.13	1.405 + 0.045 i
0.10	0.827	1.05	1.396 + 0.088 i
0.15	0.843	3.54	1.382 + 0.130 i
0.20	0.865	8.42	1.362 + 0.169 i
0.25	0.893	16.5	1.338 + 0.205 i
0.30	0.929	28.6	1.309 + 0.236 i
0.35	0.972	45.5	1.277 + 0.263 i
0.40	1.024	68.2	1.243 + 0.285 i
0.45	1.084	97.6	1.207 + 0.301 i
0.50	1.153	134	1.171 + 0.313 i
0.55	1.232	180	1.134 + 0.319 i
0.60	1.323	235	1.099 + 0.320 i
0.65	1.425	301	1.065 + 0.316 i
0.70	1.540	379	1.032 + 0.307 i

function $A(E)$ and the complex pseudotime $\mu^2(\sigma)$ for $\chi = 0.5$, i.e., $\alpha = 1.153$. Compared to the real solutions below α_c (cf. Figs. 1–3), one does not notice any qualitative change in the real part of $A(E)$ as one crosses the critical coupling.

V. THE TWO-POINT FUNCTION AWAY FROM THE POLE

Up to now we only have determined the variational parameters on the nucleon pole. However, the variational principle also applies to $p^2 \neq -M_{\text{phys}}^2$. This forces us to consider subasymptotic values of the proper time β . We first deal with the residue at the pole which gives us the probability to find the bare nucleon in the dressed particle.

A. The residue

To calculate the residue it is most convenient to use the “momentum averaging” scheme developed in (I) be-

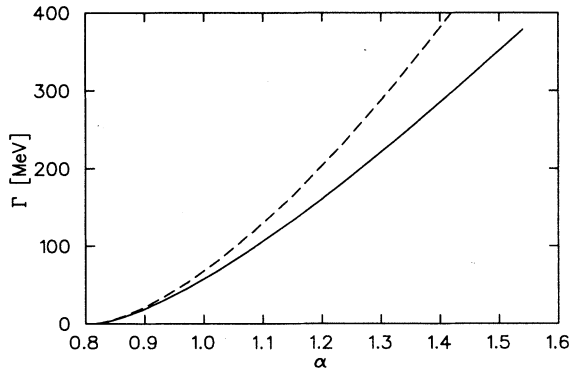


FIG. 6. Width of the unstable state as a function of the coupling constant as obtained from the solution of the complex variational equations (see Table IV). The dashed line shows the approximate solution (57).

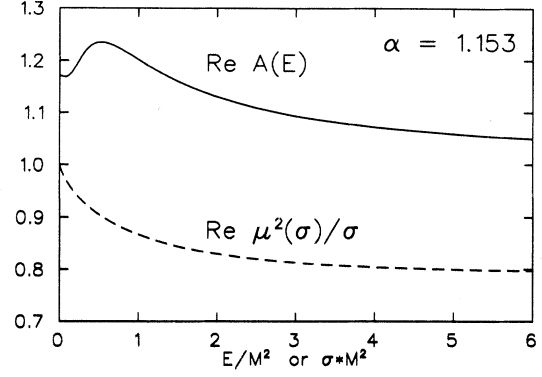


FIG. 7. Real part of the profile function and of the ratio of pseudotime to proper time for $\alpha = 1.153$.

cause in this approach there are only a few subasymptotic terms. To be more specific the quantity $\tilde{\mu}^2(\sigma, T)$ introduced in Eq. (I.98) has an additional term which exactly cancels the $1/\beta$ term which arises from application of the Poisson summation formula. With exponential accuracy we, therefore, have

$$\tilde{\mu}^2(\sigma, T) \simeq \frac{4}{\pi} \int_0^\infty dE \frac{1}{A(E)} \frac{\sin^2(E\sigma/2)}{E^2}. \quad (64)$$

This is a big advantage as we do not have to expand the potential term $\langle\langle S_1 \rangle\rangle$ in Eq. (I.97) in inverse powers of β . The only source of subasymptotic terms in $\langle\langle S_1 \rangle\rangle$ is then from the T integration from $\sigma/2$ to $\beta - \sigma/2$ which simply gives a factor $\beta - \sigma$. Applying the Poisson formula to the kinetic term $\tilde{\Omega}$ defined in Eq. (I.100), we obtain, again with exponential accuracy,

$$\tilde{\Omega}(\beta) = \frac{2}{\pi} \int_0^\infty dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right] + \frac{1}{\beta} \left[\ln A(0) + \frac{1}{A(0)} - 1 \right]. \quad (65)$$

We recall from (I) that the two-point function may be written near the pole as

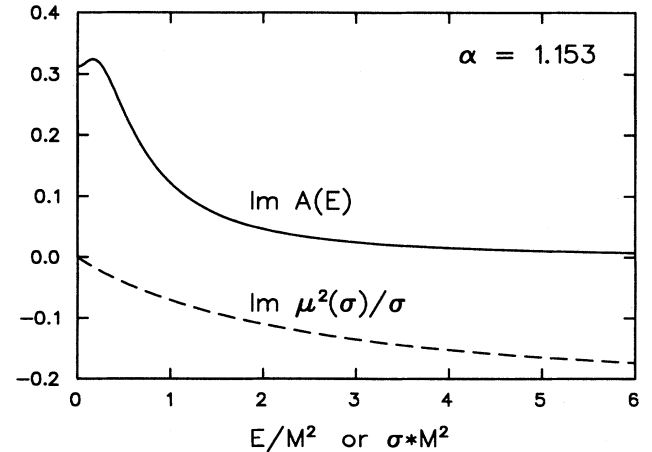


FIG. 8. Imaginary part of the profile function and of the ratio of pseudotime to proper time for $\alpha = 1.153$.

$$G_2(p) \simeq \frac{1}{2} \int_0^\infty d\beta \exp \left[-\frac{\beta}{2} F(\beta, p^2) \right]. \quad (66)$$

Collecting all nonexponential terms, the function $F(\beta, p^2)$ therefore has the large- β expansion

$$F(\beta, p^2) \simeq F_0(p^2) + \frac{2}{\beta} F_1(p^2), \quad (67)$$

where

$$F_0(p^2) = p^2 + M_0^2 - p^2(1-\lambda)^2 + 2\Omega - \frac{g^2}{4\pi^2} \int_0^\infty \frac{d\sigma}{\mu^2(\sigma)} \int_0^1 du e \left(m\mu(\sigma), \frac{-i\lambda p\sigma}{\mu(\sigma)}, u \right) \quad (68)$$

is what we have used before on the nucleon pole ($p = iM_{\text{phys}}$) and

$$F_1(p^2) = \ln A(0) + \frac{1-A(0)}{A(0)} + \frac{g^2}{8\pi^2} \int_0^\infty d\sigma \frac{\sigma}{\mu^2(\sigma)} \times \int_0^1 du e \left(m\mu(\sigma), \frac{-i\lambda p\sigma}{\mu(\sigma)}, u \right). \quad (69)$$

Note that the potential term in $F_0(p^2)$ develops a small- σ singularity which renormalizes the bare mass M_0 but $F_1(p^2)$ is finite.

Neglecting the exponentially suppressed terms and performing the proper time integration, we thus obtain the expression for the two-point function

$$G_2(p^2) \simeq \frac{e^{-F_1(p^2)}}{F_0(p^2)} = \exp \left[-\ln F_0(p^2) - F_1(p^2) \right]. \quad (70)$$

It is now very easy to calculate the residue Z at the pole [see Eq. (6)] by expanding around the point $p^2 = -M_{\text{phys}}^2$, where F_0 vanishes. We obtain

$$Z = \frac{\exp \left[-F_1(-M_{\text{phys}}^2) \right]}{F_0'(-M_{\text{phys}}^2)}, \quad (71)$$

where the prime denotes differentiation with respect to p^2 . Explicitly, we find

$$Z = \frac{N_0 N_1}{D}, \quad (72)$$

where

$$N_0 = \exp \left(-\ln A(0) + 1 - \frac{1}{A(0)} \right), \quad (73)$$

$$N_1 = \exp \left[-\frac{g^2}{8\pi^2} \int_0^\infty d\sigma \frac{\sigma}{\mu^2(\sigma)} \times \int_0^1 du e \left(m\mu(\sigma), \frac{\lambda\sigma M_{\text{phys}}}{\mu(\sigma)}, u \right) \right], \quad (74)$$

$$D = 1 - (1-\lambda)^2 - \frac{g^2}{8\pi^2} \lambda^2 \int_0^\infty d\sigma \times \frac{\sigma^2}{\mu^4(\sigma)} \int_0^1 du u e \left(m\mu(\sigma), \frac{\lambda\sigma M_{\text{phys}}}{\mu(\sigma)}, u \right) = \lambda. \quad (75)$$

In the last line the stationarity equation (18) for λ was used to simplify the denominator D . Note that this also applies to the case where one parametrizes the profile function $A(E)$. This demonstrates that

$$Z = \frac{N_0 N_1}{\lambda} \quad (76)$$

is always positive. It seems to be more difficult to prove in general that $Z \leq 1$, although all numerical calculations clearly give this result. Finally, it is again useful to check the variational result in perturbation theory. With $A(0) = 1 + O(g^2)$, one sees that $N_0 = 1 + O(g^4)$. Similarly, $(1-\lambda)^2 = 1 + O(g^4)$. Expanding N_1 and $1/\lambda$ to order g^2 , we obtain

$$Z = 1 - \frac{g^2}{8\pi^2} \int_0^\infty d\sigma \int_0^1 du (1-u) \times \exp \left(-\frac{\sigma m^2}{2} \frac{1-u}{u} - \frac{\sigma M_{\text{phys}}^2}{2} u \right) + O(g^4) = 1 - \frac{g^2}{8\pi^2} \int_0^1 du \frac{u(1-u)}{M_{\text{phys}}^2 u^2 + m^2(1-u)} + O(g^4). \quad (77)$$

This coincides with what one obtains from the perturbative result for the self-energy (39) in the usual way.

Table V contains the numerical values of the residue obtained with different parametrizations as well as the perturbative result from Eq. (77):

TABLE V. Residue at the pole of the two-point function for different parametrizations of the profile function. The heading "Feynman" gives the result in the Feynman parametrization whereas "Improved" refers to the improved parametrization from Eq. (17). The residue calculated with the solution of the variational equations is denoted by "Variational." For comparison, the perturbative result is also given.

α	"Feynman"	"Improved"	"Variational"	Perturbative
0.1	0.96090	0.96087	0.96087	0.96200
0.2	0.91934	0.91914	0.91918	0.92399
0.3	0.87467	0.87418	0.87428	0.88599
0.4	0.82600	0.82494	0.82521	0.84798
0.5	0.77184	0.76996	0.77036	0.80998
0.6	0.70940	0.70610	0.70672	0.77198
0.7	0.63216	0.62597	0.62697	0.73397
0.8	0.51086	0.49187	0.49284	0.69597

$$Z_{\text{perturb}} = 1 - 0.38004 \alpha. \quad (78)$$

It is seen that for α near the critical value appreciable deviations from the perturbative result occur. For example, at $\alpha = 0.8$ perturbation theory says that there is a probability of nearly 70% to find the bare particle in the dressed one whereas the variational results estimate this probability to be less than 50%. It should be also noted that the residue is *not* an infrared stable quantity, i.e., for $m \rightarrow 0$, Z also vanishes. From the variational equations one can deduce that

$$Z \underset{m \rightarrow 0}{\sim} \text{const.} \times m^\kappa \quad (79)$$

with $\kappa = \alpha/(\pi\lambda^2)$. For massless mesons the residue at the nucleon pole must vanish because it is well known (e.g., from quantum electrodynamics) that in this case the two-point function does not develop a pole but rather a branch point at $p^2 = -M_{\text{phys}}^2$.

B. Variational equations for the off-mass-shell case

It is also possible to apply the variational principle away from the pole of the two-point function by varying equation (70). This gives

$$\delta F_0(p^2) + F_0(p^2) \delta F_1(p^2) = 0. \quad (80)$$

Note that on mass shell, where F_0 vanishes, the previous variational equations follow. We will not elaborate on Eq. (80) further but only point out that the perturbative self-energy (39) is *not* obtained from the off-shell variational equations (80). In the limit $\mu^2(\sigma) \rightarrow \sigma$, $A(E) \rightarrow 1$, $\lambda \rightarrow 1$, one rather finds

$$\begin{aligned} \Sigma_{\text{var}}(p^2) \rightarrow & -\frac{g^2}{4\pi^2} \ln \frac{\Lambda^2}{m^2} \\ & + \frac{g^2}{4\pi^2} \int_0^1 du \ln \left[1 - \frac{p^2}{m^2} \frac{u^2}{1-u} \right] \\ & + \frac{g^2}{4\pi^2} \int_0^1 du u \frac{p^2 + M_1^2}{m^2(1-u) - p^2 u^2} \end{aligned} \quad (81)$$

which is an expansion of Eq. (39) around $p^2 = -M_0^2$ (or M_1^2 which is the same in lowest order perturbation theory). The reason for this somehow unexpected result is the neglect of exponentially suppressed terms in Eq. (70). Indeed, it is easy to see that one obtains the correct perturbative self-energy only if the upper limit of the σ integral is kept at β and not extended to infinity as we have done in deriving Eq. (70). The difference is one of the many exponentially suppressed terms which we have neglected. Thus, the off-shell variational equations (80) only hold in the vicinity of the nucleon pole and in order to investigate variationally the two-point function far away [say near the meson production threshold $p^2 = -(M_{\text{phys}} + m)^2$], one has to include consistently all terms which are exponentially suppressed in β . This is beyond the scope of the present work.

VI. DISCUSSION AND SUMMARY

In the present work we have performed variational calculations for the ‘‘Wick-Cutkosky polaron’’ following the

approach which was developed previously [2]. We have determined different parametrizations as well as the full variational solution for the retardation function which enters the trial action. Since the nucleon mass is fixed on the pole of the two-point function the value of the functional which we minimize is of no physical significance but only a measure of the quality of the corresponding ansatz. This is in contrast with the familiar quantum-mechanical case where an upper limit to the ground-state energy of the system is obtained. However, our calculation fixes the variational parameters with which we then can calculate other observables of physical interest.

One of these quantities was the residue on the pole of the propagator for which we have compared numerically the results of the variational calculations to first-order perturbation theory in Table V. For small couplings all results for the residue agree, since in this case the variational approach necessarily reduces to perturbation theory independent of the value of the variational parameters. What is rather remarkable is that for larger couplings the three parametrizations of the profile function in our variational approach yield rather similar results, which are now, of course, different from the perturbative calculation. As we have seen, the ‘‘improved’’ and ‘‘variational’’ actions have the same singularity behaviors, for small relative times, as the true actions, so here one might expect some similarity in the results. This is, however, not true for the ‘‘Feynman’’ parameterization which has a rather different form, so its agreement with the other two is not preordained. This similarity is also exhibited in Tables I and II for λ and $A(0)$, but of course not for the parameters v and w which enter the respective profile functions and which are ‘‘gauge’’ (i.e., reparametrization)-dependent quantities. Also, the critical coupling at which real solutions ceased to exist was nearly identical in all three parametrizations. The similarity of the results for the different *Ansätze* presumably indicates that these results are not too far away from the exact ones.

We were not only able to determine the critical coupling but also to deduce qualitatively and quantitatively the width which the particle acquires beyond the critical coupling. This was achieved by finding *complex* solutions of the variational equations, first approximately by an analytic approach and then exactly by an iterative method which closely followed the analytic procedure. Although the present approach does not describe tunneling (which we expect to render the system unstable even at small coupling constants but with exponentially small width [9]), the polaron variational method is clearly superior to any perturbative treatment in this respect.

We have concentrated mostly, although not exclusively, on the on-shell two-point function, i.e., the nucleon propagator. This corresponds to the limit where the proper time goes towards infinity. It is possible, however, to go beyond the on-shell limit. This was necessary, for example, for the calculation of the residue of the two-point function in Sec. V A. Nevertheless, the residue is a quantity which is calculated at the pole and thus only requires off-shell information from an infinitesimal region around it. This has the effect that the variational parameters for

the calculation of the residue are the same as those for the on-shell ones. As one moves a finite distance away from the pole, the variational parameters themselves become functions of the off-shellness p^2 (see Sec. VB).

In conclusion, we think that the present variational approach has yielded nonperturbative numerical results which look very reasonable and are encouraging. We, therefore, believe it worthwhile to try to extend it in several ways. First, in a sequel to this work we will generalize the present approach to the case with n external mesons and thereby study physical processes such as meson production or meson scattering from a nucleon. This can be done by employing the quadratic trial function whose parameters have been determined in the present work on the pole of the two-point function. Such a “zeroth order” calculation is similar in spirit to a quantum-mechanical calculation in which wave functions determined from minimizing the energy functional are used to evaluate other observables. More demanding is the consistent “first-order” variational calculation of higher-order Green functions as this requires the amputation of precisely the nonperturbative nucleon propagators which have been determined in the present work. That this is indeed possible will be demonstrated in another paper in this series [13].

Of course, finally we would like to apply these nonperturbative techniques to the theories which are of a more physical nature. Among these one may mention scalar QED, the Walecka model [10, 11], or QED. The latter two will require introduction of Grassmann variables in order to deal with spin in a path integral. As such, this should not pose a fundamental problem. A greater challenge, however, is to extend such an approach beyond the quenched approximation or to non-Abelian theories where the light degrees of freedom cannot be integrated out analytically.

Note added. After completion of this work we became aware of the pioneering work by Mano [12] in which similar methods are applied to the Wick-Cutkosky model with zero meson mass. Mano uses the proper time formulation, the quenched approximation, and the Feynman parametrization for the retardation function to derive a variational function for the self-energy of a scalar nucleon [the expression following his Eq. (6.18)] which is identical with our Eq. (7) after proper identification of quantities is made. However, for minimizing the variational function, Mano sets (in our nomenclature) $v = w(1 + \epsilon)$, expands to second order in ϵ , and finds an instability of the ground state for $g_{\text{Mano}}^2/8\pi M^2 > 0.34$. Note that $g_{\text{Mano}} = \sqrt{\pi}g$ so that this translates into a critical coupling $\alpha_c \approx 0.22$ which is much smaller than the value which we obtain from the exact minimization. In addition, in the present work we consider nonzero meson masses, employ more general retardation functions, and calculate residue and width of the dressed particle.

ACKNOWLEDGMENTS

We would like to thank Dina Alexandrou and Yang Lu for many helpful discussions and Geert Jan van Oldenborgh for encouragement and a careful reading of the manuscript.

APPENDIX: AN ALTERNATIVE EXPRESSION FOR Ω_{var}

Here, we derive Eq. (21) for the kinetic term Ω when the variational equations are satisfied. We first perform an integration by parts in the definition (9) of Ω . The slow fall off of the variational profile function with E ,

$$\begin{aligned} A(E) &\underset{E \rightarrow \infty}{\sim} 1 + \frac{g^2}{4\pi^2} \frac{1}{E^2} \int_0^\infty d\sigma \frac{\sin^2(E\sigma/2)}{\sigma^2} + \dots, \\ &= 1 + \frac{g^2}{16\pi} \frac{1}{E} + \dots, \end{aligned} \quad (\text{A1})$$

leads to a contribution at $E = \infty$:

$$\Omega_{\text{var}} = \frac{g^2}{8\pi^2} + \frac{2}{\pi} \int_0^\infty dE \left[-E \frac{A'(E)}{A(E)} + \frac{1 - A(E)}{A(E)} \right]. \quad (\text{A2})$$

We then write the variational equation (19) for $A(E)$ in the form

$$\frac{1}{A(E)} - 1 = -\frac{g^2}{4\pi^2} \int_0^\infty d\sigma \frac{\sin^2(E\sigma/2)}{E^2 A(E)} \frac{1}{\mu^4(\sigma)} X(\sigma), \quad (\text{A3})$$

where

$$\begin{aligned} X(\sigma) &= \int_0^1 du \left[1 + \frac{m^2}{2} \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2 M_{\text{phys}}^2 \sigma^2}{2\mu^2(\sigma)} u \right] \\ &\times e \left(m\mu(\sigma), \frac{\lambda M_{\text{phys}} \sigma}{\mu(\sigma)}, u \right). \end{aligned} \quad (\text{A4})$$

The integration over E can now be performed giving a factor $\pi\mu^2(\sigma)/4$ because of Eq. (12). Therefore, we have

$$\int_0^\infty dE \left[\frac{1}{A(E)} - 1 \right] = -\frac{g^2}{16\pi} \int_0^\infty d\sigma \frac{1}{\mu^2(\sigma)} X(\sigma) \quad (\text{A5})$$

which is just one term in the expression (A2) for Ω . To get the other one, we differentiate the variational equation for $A(E)$ with respect to E and observe that

$$\frac{\partial}{\partial E} \sin^2 \left(\frac{E\sigma}{2} \right) = \frac{\sigma}{E} \frac{\partial}{\partial \sigma} \sin^2 \left(\frac{E\sigma}{2} \right). \quad (\text{A6})$$

One has to be careful not to interchange the E integration and the σ differentiation. We therefore perform an integration by parts and obtain

$$\begin{aligned} - \int_0^\infty dE E \frac{A'(E)}{A(E)} &= \frac{g^2}{4\pi^2} \int_0^\infty dE \frac{1}{E^2 A(E)} \left[\sigma X(\sigma) \frac{\sin^2(E\sigma/2)}{\mu^4(\sigma)} \right]_0^\infty + \int_0^\infty d\sigma \frac{\sin^2(E\sigma/2)}{\mu^4(\sigma)} \left(2 + \frac{\partial}{\partial \sigma} \sigma \right) X(\sigma) \\ &= \frac{g^2}{16\pi} \left[- \lim_{\sigma \rightarrow 0} \frac{\sigma X(\sigma)}{\mu^2(\sigma)} + \int_0^\infty d\sigma \frac{1}{\mu^2(\sigma)} \left(2 + \frac{\partial}{\partial \sigma} \sigma \right) X(\sigma) \right]. \end{aligned} \quad (\text{A7})$$

Note that the boundary term at $\sigma = 0$ gives a contribution because of $X(0) = 1$. This contribution exactly cancels the term $g^2/8\pi^2$ in Eq. (A2). Combining both terms for Ω [which do not exist separately because of the slow falloff of $A(E)$], we obtain

$$\begin{aligned} \int_0^\infty dE \left[-E \frac{A'(E)}{A(E)} + \frac{1 - A(E)}{A(E)} \right] &= \frac{g^2}{16\pi} \left[-1 + \int_0^\infty d\sigma \frac{1}{\mu^2(\sigma)} \left(1 - \frac{\partial}{\partial \sigma} \sigma \right) X(\sigma) \right] \\ &= \frac{g^2}{16\pi} \left[-1 + \int_0^\infty d\sigma X(\sigma) \left(1 + \sigma \frac{\partial}{\partial \sigma} \right) \frac{1}{\mu^2(\sigma)} \right] \end{aligned} \quad (\text{A8})$$

from which Eq. (21) follows. In the last line, again an integration by parts has been performed but this time there is no contribution from the boundary terms.

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