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Ghost-free spectrum of a quantum string in SL(2, R) curved spacetime

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The unitarity problem in curved spacetime is solved for the string described by the $SL(2,R)$ Wess-Zumino-Witten (WZW) model. The spectrum is computed exactly and demonstrated to be ghost-free. The new features include (i) $SL(2,R)$ left-right symmetry currents that have logarithmic cuts on the world sheet but that satisfy the usual local operator products or commutation rules, (ii) physical states consistent with the monodromy condition of closed strings despite the logarithmic singularity in the currents, and (iii) a new free boson realization for these currents which render the $SL(2,R)$ WZW model completely solvable.

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I. INTRODUCTION

A string propagating in curved spacetime with one time and $(d-1)$ space coordinates is described by an action in the conformal gauge that has the form

$$S = \int d\tau d\sigma \, [\partial_+ X^\mu \partial_- X^\nu \, G_{\mu\nu}(X) + \cdots] \,, \tag{1.1}$$

where $G_{\mu\nu}(X)$ is a background metric in d dimensions with signature $(-1,1,1,\ldots)$. The terms in the action denoted by the ellipses may contain additional background fields such as an antisymmetric tensor $B_{\mu\nu}(X)$, a dilaton $\Phi(X)$, etc. The overall theory must be conformally invariant at the quantum-mechanical level. In general it is difficult to impose the conformal invariance condition nonperturbatively, but some solutions do exist. However, it is even more difficult to solve the system. Thus, up to this point there has not been any solution presented to the quantum-mechanical spectrum or other quantum-mechanical features of a string propagating in curved space and time. There are some solutions of models for two-dimensional (2D) gravity. But the spacetime interpretation remains somewhat obscure in the matrix formulation of such models and furthermore there seems to be no prospect for extending the approach to higher dimensions.

One knows the spectrum and correlation functions for many string models in curved space (without time) in several dimensions; these are the string models for compactified space that represent possible string vacuua. However, when the time coordinate is included as part of the curved space, as it would be during the early universe, little is known about the quantum string theory. The lack of such solutions has prevented the understanding of the role and true impact of string theory on the structure of symmetries and matter as observed in the present universe. If string theory is truly relevant, it must play its major role in the presence of quantum gravity during the early part of the universe when spacetime is curved. Thus, the structure of gauge symmetries, matter content in the form of families of quarks and leptons, and all that, would be determined by the structure of string theory while the universe is at an early stage with its time coordinate being part of curved spacetime. Given this intuitive fact, it would be premature to try to predict from string theory the structure of low energy physics (at accelerator energies) by considering only flat four-dimensional spacetime plus additional curved compactified spaces in extra dimensions, as is usually done by string phenomenologists. The role of quantum gravity and its impact on low energy physics could not be assessed without a better understanding of strings during the early universe while time and space are curved.

Many models that are exactly conformally invariant in one time plus $(d-1)$ space dimensions have been constructed by now [1–3]. But no further understanding of the quantum properties of these models has been obtained. One of the stumbling blocks has been the issue of the unitarity of such theories (negative norm states) [1,4]. Although there have been many attempts to solve the problem [5,6], no real progress has been made until now.

In this paper we solve the unitarity problem that has plagued a certain class of models, and provide new free field methods for the computation of many quantum properties of the models. In particular we give the explicit solution of the exact spectrum of a string propagating in the $SL(2,R)$ curved spacetime of a Wess-Zumino-Witten (WZW) model. This is the simplest of all curved spacetime models. This resolves also the unitarity and spectrum issues for the $SL(2,R)/R$ black hole model [1,7] and provides the methods of spectrum computation for all higher dimensional models that are in a similar class [1,2], including supersymmetric and heterotic versions. The free field methods that are introduced in this paper open the way to the computation of correlation functions as well.

II. UNITARITY PROBLEM

One of the first attempts to solve string propagation in curved spacetime was to consider some current algebra

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models that potentially can be solved through algebraic methods [2]. However, one immediately comes across an unexpected problem with the unitarity of the theory. Because of the presence of a time coordinate, there are negative norm oscillators. These create negative norm states. However, since one must also impose the Virasoro constraints, on the basis of naive counting one may hope to prove that all negative norm states are removed from the theory. A similar situation occurs also in the flat theory. As is well known, in the flat case one can indeed prove the no-ghost theorem [8] which implies that the theory is unitary. However, in the case of curved spacetime current algebra models, one finds that even after imposing the Virasoro constraints there remain negative norm states that render the theory nonunitary. This has been the main stumbling block that has discouraged the application of these ideas to model building for the past five years.

To illustrate the problem consider SL(2,R) currents, and a stress tensor defined in the form of a Laurent series:

$$\begin{aligned} \tilde{J}^i(z) &\equiv \sum_{n=-\infty}^{\infty} J_n^i\, z^{-n-1}, \quad i=0,1,2, \\ \tilde{T}(z) &\equiv \sum_{n=-\infty}^{\infty} \tilde{L}_n\, z^{-n-2}\ , \\ \tilde{L}_n &\equiv \frac{1}{k-2} \sum_{m=-\infty}^{\infty} :J_{-m}^i J_{n+m}^j:\ \eta_{ij}\ , \end{aligned} \tag{2.1}$$

where $\eta_{ij} = \mathrm{diag}(-1,1,1)$ is the Minkowski metric and is proportional to the Killing metric for SL(2,R). The commutation rules are

$$\begin{aligned} \left[J_n^i, J_m^j\right] &= i\epsilon^{ijk}\eta_{kl}J_{n+m}^l + \frac{k}{2} n\delta_{n+m}\eta^{ij}\ , \\ \left[\tilde{L}_n, J_m^i\right] &= -m\, J_{n+m}^i\ , \\ \left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\ \tilde{L}_{n+m} + \frac{c}{12} n\left(n^2-1\right)\ \delta_{n+m}\ , \\ c &= 3k/(k-2). \end{aligned} \tag{2.2}$$

J^0 corresponds to the compact generator, and the other two correspond to the noncompact generators. For positive k these commutation rules imply that there is one time and two space dimensions. One indication of this is the large k limit (semiclassical limit) in which the commutation rules of the currents reduce to those of flat spacetime oscillators, with one timelike and two spacelike oscillators. Therefore, it is useful to think of J^0 as a current in the timelike direction and of $J^{1,2}$ as two currents in two spacelike directions. Since we are interested in the covering group of SL(2,R), the time coordinate can take all values, and it is not restricted to a compact range. For example, one can see this by analyzing the classical solutions of either the particle or the string in the SL(2,R) curved spacetime in the form of a WZW model. In the large $k\to\infty$ limit the model reduces to flat spacetime in 2+1 dimensions with $c=3$. But for finite k it describes curved spacetime with a metric induced by the group, as seen in the WZW model formulation (see Sec. VII), with $c=3k/(k-2)$, which can be made critical $c=26$ if desired. Otherwise, the $c<26$ theory may be considered a piece of a critical theory.

The states of the theory are constructed as usual from the current algebra Verma module (analogue of Fock space of flat spacetime)

$$|\psi\rangle = \prod \left(J_{-n}^i\right)^{p_{in}}\ |jm\rangle\ , \tag{2.3}$$

where $|jm\rangle$ is a representation of the zero mode currents. The physical states $|\phi\rangle$ are those linear combinations of Verma module states that satisfy the Virasoro constraints

$$\begin{aligned} \tilde{L}_n|\phi\rangle &= 0, \quad n\geq 1\ , \\ \tilde{L}_0|\phi\rangle &= \tilde{a}\,|\phi\rangle. \end{aligned} \tag{2.4}$$

In a critical theory one may take $\tilde{a}=1$ and $c=26$ (or $k-2=6/23$). It is also possible to take the SL(2,R) model as a piece of a critical conformal theory. Then $\tilde{a}\leq 1$ and $c\leq 26$ (or $k-2\geq 6/23$).

The eigenvalues of $\tilde{L}_0$ are determined by the Casimir and the level of excitation of the string $l\equiv\sum_{i,n} np_{in}$:

$$\tilde{L}_0 = \frac{-j(j+1)}{k-2} + l = \tilde{a}\ . \tag{2.5}$$

Thus, an excited string at level $l\geq 1$ must have positive values of $j(j+1)$ and $j+1$ that depend on the level of excitation:[1]

$$\begin{aligned} j(j+1) &= (k-2)(l-\tilde{a})\ , \\ j+1 &= \frac{1}{2} + \sqrt{(k-2)(l-\tilde{a})+1/4}. \end{aligned} \tag{2.6}$$

Such values of the Casimir operator can occur in unitary representations of SL(2,R) only for the discrete series $|jm\rangle$, for which the lowest (or highest) state has $|\,m| = j+1 > 0$, and then the values of m are given by $|\,m| = (j+1+n)$, where n is a positive integer.

It is easy to see that there are plenty of excited states that satisfy the Virasoro constraint, but whose norm is negative. An explicit example is [1]

$$\begin{aligned} &|\phi, l\rangle = \left(J_{-1}^1 - iJ_{-1}^2\right)^l\ |j, m=j+1\rangle, \\ &\tilde{L}_n|\phi, l\rangle = 0, \qquad n\geq 1, \\ &\langle\phi, l|\phi, l\rangle = N_{j(l)}\ (l!)\prod_{r=0}^{l-1}\left[k-2j(l)-2+r\right], \end{aligned} \tag{2.7}$$

where $N_{j(l)} = \langle j, m=j+1\,|j, m=j+1\rangle$ is the norm of the state at the base. Evidently, for sufficiently large values of the excitation number l the norm switches between positive and negative values. Hence, despite the Virasoro constraints, this model is not unitary and cannot describe a physical string.

Until now a solution to this problem, and the related SL(2,R)/R black hole problem, has not been found despite many attempts[2] [5,6]. Suggestions included (1) re-

[1] The analogue of this equation in flat spacetime is the mass shell condition $-(p_\mu)^2 = M^2 = l-\tilde{a}$, where $|p_\mu\rangle$ denotes the labeling of the base.

[2] There has been a claim of a proof of the no-ghost theorem for this problem (second reference in [6]). Evidently the "proof" is wrong since we are able to display explicitly negative norm states that satisfy the Virasoro conditions.

stricting (artificially) $j(l)+1<k/2$ so that the norm never becomes negative, (2) allowing large values of $j(l)$ as needed by the excited level l, but also permitting the base state to have negative norm $N_{j(l)}$ in such a way as to make the norm of the excited state $\langle\phi,l|\phi,l\rangle$ positive, and (3) hoping that modular invariants will fix the problem.

All of these suggestions are rejected as follows. (1) If $j+1$ cannot exceed $k/2$, then the string cannot be excited to arbitrary levels, as seen from (2.6). In addition to being an artificial condition not justified by the formalism, this also leads to inconsistent physical results: For example, classical string solutions in which the string is arbitrarily excited exist in curved spacetime; there is no intuitive or physical reason not to expect them in the quantum theory as well. (2) Even if the explicit state in (2.7) is forced to have positive norm by changing the norm of the base, there are other states at the same level that would have opposite norm to $|\phi,l\rangle$. For example, the state $|\phi,l,+\rangle\equiv(J_0^1+iJ_0^2)|\phi,l\rangle$, which also satisfies the Virasoro constraint, has the norm

$$\langle\phi,l,+|\phi,l,+\rangle=2(j+1-l)\langle\phi,l|\phi,l\rangle. \tag{2.8}$$

The two norms have opposite signs when $j(l)+1-l$ is negative at sufficiently high level l. Thus, the second suggestion does not work either. Independent of this argument, it seems unreasonable to have a negative norm base in a unitary theory.

The third possibility is more elusive since modular invariants are not well understood for SL(2,R) or other noncompact groups. For the past five years this possibility remained open. One could have hoped that the problem could be resolved through modular invariants that are needed in order to complete the construction of the physical theory. A modular invariant provides instructions for putting together the left-moving and the right-moving states

$$|\psi\rangle=\sum\gamma_{ab}\,|\phi_L^a\rangle|\phi_R^b\rangle. \tag{2.9}$$

It could happen that the physical modular invariant would choose only those combinations of states that have overall positive norm, even though the left- or right-moving states $|\phi_L^a\rangle,|\phi_R^b\rangle$ contained in it may have negative norm. However, as described below, we recently found an argument that destroys this possibility too.

Thus, consider an open string rather than a closed one. The boundary conditions turn out to relate the left- and right-moving currents, so that only one set of currents and states is sufficient to describe the full open string (see Sec. IV C). The convenient current is neither the left nor the right mover, but it can be related to either one by a transformation with the group element. In any case, the quantum theory for the open string propagating in the SL(2,R) curved spacetime reduces to the same mathematics outlined above, with no further consideration of left-right states since there is a single current. So the mechanism of squaring two minus signs hoped for through Eq. (2.9) cannot help. Therefore, this model is not unitary. Something is wrong with the open as well as the closed string. The problem will be solved in this paper.

On the other hand one may analyze the classical theory underlying such models in the form of a WZW or gauged WZW theory based on a noncompact group, such that there is a single time coordinate [2]. The classical solutions in a class of models were outlined some time ago [9], and a more detailed discussion of the 2D special case has been given more recently [10,11]. The same approach yields classical string solutions for SL(2,R) as well. For a classical string to make sense one must consider only those solutions for which the string time coordinate $X^0(\tau,\sigma)$ (defined through an appropriate coordination of the group element) monotonically increases as a function of τ for the whole string (all σ), just as in flat spacetime. Under such conditions one finds that there are perfectly well-behaved physical string solutions that are the generalizations of string motions in flat spacetime.[3] There must be a corresponding quantum theory formulated in terms of the symmetry currents of the WZW theory. Evidently, it cannot be the theory outlined above. A radical solution is needed.

III. MODIFIED CURRENTS

In this paper a key departure from the usual currents is introduced in the form of a logarithmic cut $\ln z$ in the complex z plane. The new currents satisfy the standard local operator products, with only poles as singularities so that the standard commutation rules are not altered. To compensate for the cut, the physical states are restricted by monodromy conditions.

First we explain in general terms why a cut is possible. The string equations of motion that follow from an action (e.g., WZW or gauged WZW) consist of differential equations and boundary conditions. The differential equations for the left-right currents,

$$\partial_{\bar z}J_L=0,\quad \partial_z J_R=0, \tag{3.1}$$

are required to be satisfied when $z,\bar z$ are on the circle, i.e., $z=\exp[i(\tau+\sigma)]$, $\bar z=\exp[i(\tau-\sigma)]$, in the Minkowski world sheet, and therefore the cut starting at $z=0,\infty$ presents no problem with the physical equations of motion. Thus the left-moving currents J_L are functions of z, including the possibility of $\ln z$ (and similarly right-moving currents J_R are functions of $\bar z$). Another aspect of minimizing the string action is the boundary conditions that require periodicity in the σ variable. The physics should be consistent with the periodicity condition $\sigma\to\sigma+2\pi n$ (or $z\to ze^{i2\pi n}$ in the complex z plane). This is the reason that the currents are usually taken as functions of only the powers z^n and hence are holomorphic for complex z, except for poles at $z=0,\infty$ [as in Eq. (2.1)].

The periodicity requirement will be satisfied in our

[3]These solutions also seem to display new *classical string physics* at singularities, such as penetration to the other side of the black hole spacetime, which is not possible for particle geodesics (see, e.g., [10,11]).

solution in a more subtle way. A cut in the currents presents problems with the monodromy. Instead of taking periodic currents, the periodicity condition will be implemented on the Hilbert space in our theory. So the physical states will be identified as the subset of states that are invariant under the monodromy. In other words, the matrix elements of the currents will be periodic in the physical subspace. Furthermore, we will see that although the new currents have a logarithmic cut, the stress tensor is free of cuts and can be written as a Laurent series in powers of z. This feature permits the simultaneous imposition of the Virasoro constraints as well as the monodromy on the states.

Except for a new definition of the currents we keep the entire formalism the same. That is, the stress tensor will be constructed as the Sugawara form with the new currents, and the Virasoro condition will be implemented with this stress tensor. The structure of the new currents was discovered by trial and error, in the process of building a new free boson realization for SL(2,R) currents, which will be presented below. But later it was understood that the logarithmic structure follows naturally from the WZW model. It turns out that the natural variables for the quantization of the model, and for performing computations, are the free bosons, rather than the currents. However, it is instructive to state the resolution of the unitarity problem directly in terms of currents without considering the details of free boson realizations.

Thus define the following new set of currents constructed from the old ones (in light-cone-type combinations):

$$J^0(z)+J^1(z)=\left[\tilde{J}^0(z)+\tilde{J}^1(z)\right] ,$$

$$\begin{aligned}J^0(z)-J^1(z)=&\left[\tilde{J}^0(z)-\tilde{J}^1(z)\right]\\&-2i\alpha_0^-\ln z\;\tilde{J}^2(z)-\frac{k}{z}\alpha_0^-\\&+\left(-i\alpha_0^-\ln z\right)^2\left[\tilde{J}^0(z)+\tilde{J}^1(z)\right] ,\end{aligned}\qquad(3.2)$$

$$J^2(z)=\tilde{J}^2(z)-i\alpha_0^-\ln z\left[\tilde{J}^0(z)+\tilde{J}^1(z)\right] .$$

The old currents $\tilde{J}^i(z)$ are analytic, and are written in the form of a Laurent series as in (2.1), and the coefficients J_n^i satisfy the commutation rules in (2.2). In addition to these coefficients we have introduced a new zero mode α_0^- which commutes with all the current modes J_n^i. In the limit $\alpha_0^-\to 0$ we get the old currents. One can compute the local operator products and/or the commutators and show that for any α_0^- the *local* commutation rules are the same for both sets of currents:

$$\begin{aligned}\left[\tilde{J}^i(z),\tilde{J}^j(w)\right]&=i\epsilon^{ijk}\eta_{kl}\tilde{J}^l(w)\,\delta(z-w)\\&\quad-\frac{k}{2}\partial_z\delta(z-w)\eta^{ij},\qquad(3.3)\\\left[J^i(z),J^j(w)\right]&=i\epsilon^{ijk}\eta_{kl}J^l(w)\,\delta(z-w)\\&\quad-\frac{k}{2}\partial_z\delta(z-w)\eta^{ij}.\end{aligned}$$

The first line is given and is equivalent to the commutation rules of the current modes J_n^i given in (2.2). The second line is derived from the first line plus the definition of the new currents. The $\ln z$ structure plays a nontrivial role in arriving at this result. In particular the crucial commutator to check is

$$\left[J^2(z),\;J^0(w)-J^1(w)\right]=-i\delta(z-w)\;\left[J^0(w)-J^1(w)\right] ,\qquad(3.4)$$

where the term $-i\;\delta(z-w)\;\left(-\frac{k}{w}\alpha_0^-\right)$ on the right-hand side must be reproduced as part of the new currents. This term is obtained from the combination of the central extensions coming from

$$\left[\tilde{J}^2(z),-2i\alpha_0^-\ln w\;\tilde{J}^2(w)\right]\qquad(3.5)$$

and

$$\left[-i\alpha_0^-\ln z\,[\tilde{J}^0(z)+\tilde{J}^1(z)],\,[\tilde{J}^0(z)-\tilde{J}^1(z)]\right] .$$

Collecting the central extensions of these two terms we have

$$\begin{aligned}-\frac{k}{2}\partial_z\delta(z-w)\;&\left\{-2i\alpha_0^-\ln w+2i\alpha_0^-\ln z\right\}\\&=\frac{ik\alpha_0^-}{w}\delta(z-w) ,\end{aligned}\qquad(3.6)$$

which gives the desired result. Thus, even though the new currents have different global properties on the world sheet, they have the same local singularities as the old currents when their products are considered. The global structure has an effect on the spectrum of the theory through the monodromy, as will be discussed in Sec. VIII.

We now claim that the physical model has the new currents as the symmetry currents, and that the stress tensor is constructed from the new currents

$$T(z)=\frac{1}{k-2}:\left\{-\left[J_0(z)\right]^2+\left[J_1(z)\right]^2+\left[J_2(z)\right]^2\right\}: ,\qquad(3.7)$$

where normal ordering is defined by splitting the points, subracting the singularity, and then sending $z\to w$ in the finite part. This procedure produces the following expression when written in terms of the old currents:

$$\begin{aligned}T(z)=\frac{1}{k-2}&:\left\{-\left[\tilde{J}_0(z)\right]^2+\left[\tilde{J}_1(z)\right]^2+\left[\tilde{J}_2(z)\right]^2\right\}:\\&+\frac{1}{z}\alpha_0^-\;\left[\tilde{J}^0(z)+\tilde{J}^1(z)\right] .\end{aligned}\qquad(3.8)$$

This implies that the Virasoro constraints are modified as[4]

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[4]This form of stress tensor was considered before from a different point of view [12], but apparently without realizing that it comes from a current with a logarithmic singularity, and that it implies monodromy constraints on the states.

$$L_n = \tilde{L}_n + \alpha_0^- \left(J_n^0 + J_n^1\right). \tag{3.9}$$

Using the algebra (2.2) for $\tilde{L}_n, J_n^i$ it is easily shown that the central charge of the new L_n is the one given before (for any α_0^-):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}. \tag{3.10}$$

The new structure (3.9) changes the physical conditions on the Hilbert space in a profound way. Furthermore, instead of using the basis $|jm\rangle$ in which the zero mode J_0^0 is diagonal, it is necessary to use the basis $|j,p^+,p^-\rangle$ in which $J_0^0 + J_0^1 = p^+$ and $\alpha_0^- = p^-$ are diagonal in order to diagonalize L_0. The eigenvalue of the Casimir operator is insensitive to which zero mode generator is diagonalized. Then the eigenvalue of L_0 is

$$L_0 = p^-p^+ - \frac{j(j+1)}{k-2} + l = a\ , \tag{3.11}$$

with $a \leq 1$. The contribution from $\alpha_0^- = p^-$ alters the balance of this equation as compared to (2.5). When p^- was absent $j(j+1)$ had to be positive, which in turn required the discrete series. However, now $p^-p^+ = -M^2/2$ plays the role of a mass^2 in a two-dimensional subspace, and therefore $j(j+1)$ can be negative. In that case the principal as well as the supplementary series become relevant in the description of excited string states. This is basically the way out of the bind with the ghosts. The computations that led to the negative norm states were valid only for the discrete series, but now we have other choices of unitary representations of SL(2,R) at the base. Furthermore, the new Virasoro constraints require physical states that correspond to a different combination of Verma module states. One may compute as in the past "physical" states and then check their norms. However, this procedure is very cumbersome. Furthermore, it may not be very useful to consider the representation space of the old current algebra, since from that point of view the states do not fall into degenerate representations of the central SL(2,R) generated by J_0^i. The stress tensor is covariant under the transformations generated by the new currents, and the old symmetry looks like spontaneously broken [as in (3.9)]. Therefore, in order to solve the new theory we must resort to better methods. This will be done below, where we will show that the theory can be solved completely in terms of free fields. In particular we will show that in the free field representation that corresponds to the WZW model only the principal series is selected, and that there are no ghosts after the Virasoro constraints are satisfied.

IV. SL(2,R) CURRENTS AND FREE FIELDS

A. Definition of the fields

Consider the free fields $X^-(z)$, $P^+(z)$, $S(z)$, and $T'(z)$. They have naive dimensions 0, 1, 1, and 2, respectively, and they are defined as

$$X^-(z) = q^- - i\alpha_0^- \ln z + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^-\, z^{-n},$$

$$\left(\alpha_n^-\right)^\dagger = \alpha_{-n}^-\ , \tag{4.1}$$

$$P^+(z) = \sum_{n=-\infty}^{\infty} \alpha_n^+\, z^{-n-1}, \quad \left(\alpha_n^+\right)^\dagger = \alpha_{-n}^+\ ,$$

$$S(z) = \sum_{n=-\infty}^{\infty} s_n\, z^{-n-1}, \quad (s_n)^\dagger = s_{-n}\ , \tag{4.2}$$

$$T'(z) = \sum_{n=-\infty}^{\infty} L'_n\, z^{-n-2}, \quad (L'_n)^\dagger = L'_{-n}\ .$$

These fields are Hermitian, defined for any complex z by

$$\left[X^-(z)\right]^\dagger = X^-\left(\frac{1}{z}\right), \quad \left[zP^+(z)\right]^\dagger = \frac{1}{z}P^+\left(\frac{1}{z}\right), \quad \text{etc.}$$

That is, because of the Hermiticity of the modes, the fields $X^-(z)$, $z\,P^+(z)$, $zS(z)$, $z^2T'(z)$ are Hermitian if z is on the unit circle, as it would be for describing left-right moving fields $z = \exp[i(\tau \pm \sigma)]$ in the Minkowski world sheet. We emphasize that under Hermitian conjugation the α_n^+ modes do not go to the α_n^- modes. The zero modes $q^-, \alpha_0^\pm = p^\pm, s_0$, L'_0 play an important role, as we will see below. We also assign the commutation rules

$$\begin{aligned}
&[q^-, \alpha_0^+] = i\,,\\
&[\alpha_n^-\,, \alpha_m^+] = n\,\delta_{n+m,0}\ ,\\
&[s_n\,, s_m] = \left(\tfrac{k}{2}-1\right)\, n\,\delta_{n+m,0}\ ,\\
&[L'_n\,, L'_m] = (n-m)L'_{n+m}\ ,
\end{aligned} \tag{4.3}$$

while all other commutators are zero. In particular all α_n^+ commute among themselves, and all α_n^- commute among themselves, just like light-cone coordinates. Indeed the $\alpha_n^\pm$ oscillators may be rewritten in terms of light-cone-type combinations of one timelike α_n^0 and one spacelike α_n^1 oscillator, i.e., $\alpha_n^\pm = (\alpha_n^1 \pm \alpha_n^0)/\sqrt{2}$. In this sense, $q^-, p^\pm$ are interpreted as light-cone-type canonical variables $q = x^-$, $p = p^+$. We have *not* introduced a canonical variable corresponding to x^+; hence, $\alpha_0^- = p^-$ commutes with all the operators and acts like a constant. Similarly the zero mode s_0 also acts like a constant.

The L'_n operators act like Virasoro operators with zero central charge. The minimal construction of SL(2,R) currents does not need these operators. In fact, the WZW model that we discuss in Sec. VII does not have the L'_n. But in this section we would like to present a new more general construction of SL(2,R) currents that include the L'_n since they may find physical applications in more general models. In their absence only the principal series arises, but in their presence all unitary representations of the SL(2,R) current algebra can be constructed, as will be seen below.

The field $X^-(z)$ contains the logarithmic term $-i\alpha_0^- \ln z$. This is the source of the $\ln z$ terms in the currents discussed in the previous section. If this term is left out, all the expressions are analytic, and we obtain the old currents. However, the presence of this term is crucial for the description of excited strings in a unitary theory.

Normal ordering is defined in the usual way for all the oscillators; i.e., positive modes are moved to the right and negative modes to the left. However, for the zero modes q^-, p^+ we define normal ordering to mean the Hermitian combination of any powers of q^-, p^+: for example,

$$:q^-p^+: \equiv \frac{1}{2}(q^-p^+ + p^+q^-), \tag{4.4}$$

$$:q^-q^-p^+: \equiv q^-p^+q^-, \tag{4.5}$$

$$:q^-p^+q^-p^+: \equiv \frac{1}{2}\left(q^-p^+q^-p^+ + p^+q^-p^+q^-\right), \text{ etc.}$$

This is important for constructing Hermitian currents. We define the contractions $\langle q^-p^+\rangle = i/2$ and $\langle p^+q^-\rangle = -i/2$ which arise in rewriting ordinary products in terms of Hermitian products:

$$\begin{aligned} q^-p^+ &= :q^-p^+: + \langle q^-p^+\rangle, \quad \langle q^-p^+\rangle = i/2, \\ p^+q^- &= :q^-p^+: + \langle p^+q^-\rangle, \quad \langle p^+q^-\rangle = -i/2. \end{aligned} \tag{4.6}$$

With this definition of normal ordering the usual Wick's theorem for multiplying normal-ordered products with each other is preserved. Then, one obtains, for example,

$$\begin{aligned} :q^-p^+::q^-p^+: &= :q^-p^+q^-p^+: + \langle p^+q^-\rangle\langle q^-p^+\rangle \\ &\quad + \left(\langle p^+q^-\rangle + \langle q^-p^+\rangle\right) :q^-p^+: \\ &= :q^-p^+q^-p^+: + \tfrac{1}{4}. \end{aligned} \tag{4.7}$$

Products of fields may be rewritten in the normal-ordered form as

$$A(z)\,B(w) =: A(z)\,B(w): + \langle A(z)\,B(w)\rangle\ . \tag{4.8}$$

The following c terms arise from the normal ordering of the various fields:

$$\begin{aligned} \langle X^-(z)\,P^+(w)\rangle &\equiv \frac{i}{2w} + \frac{i}{z-w}\ , \\ \langle P^+(z)X^-(w)\,\rangle &\equiv \frac{i}{2z} + \frac{-i}{z-w}\ , \\ \langle S(z)\,S(w)\rangle &\equiv \frac{k/2-1}{(z-w)^2}\ . \end{aligned} \tag{4.9}$$

The $\frac{i}{2w}$ or $\frac{i}{2z}$ terms occur because of the unusual normal ordering of the zero modes p^+, q^-. These are unimportant in the computation of the singular parts of operator products, but they do play a role in the computation of the finite parts, such as the energy-momentum tensor, as seen below. The presence or absence of the $p^-\ln z$ term in the definition of $X^-(z)$ does not change the contractions in (4.9); therefore, the singularity structure in the operator products is unaffected by the $p^-\ln z$ term. However, the presence of the p^- term does change the finite parts in a desirable way.

B. Construction of the currents

Some time ago we suggested a free field construction of SL(2,R) currents that gives only the principal series at the base [5]. This was similar to the Wakimoto construction but with the important difference that the currents were Hermitian. Here we generalize that construction by including all unitary representations of SL(2,R) at the base. In this construction there are new structures that have not been introduced heretofore. After a few trials and errors we found that the following works:

$$J_0(z) + J_1(z) = P^+(z)\ , \tag{4.10}$$

$$\begin{aligned} J_0(z) - J_1(z) &= :X^-(z)\,P^+(z)\,X^-(z): + 2S(z)\,X^-(z) \\ &\quad - ik\partial_z X^-(z) - \frac{(k-2)T'(z)}{P^+(z)}\ , \end{aligned} \tag{4.11}$$

$$J_2(z) = :X^-(z)\,P^+(z): + S(z)\ .$$

All the currents are Hermitian:

$$[zJ^\mu(z)]^\dagger = \frac{1}{z}J^\mu\left(\frac{1}{z}\right)\ . \tag{4.12}$$

The new aspects include the $\ln z$ terms and the $T'(z)/P^+(z)$ parts. T' is a stress tensor with *zero central charge*. It commutes with all other terms in these currents, but it obeys the following operator product with itself:

$$T'(z)T'(w) = \frac{0}{(z-w)^4} + \frac{2T'(w)}{(z-w)^2} + \frac{\partial_w T'(w)}{z-w} + \cdots\ . \tag{4.13}$$

It is possible to give a construction of $T'(z)$ (or L'_n) in terms of other elementary (free) fields, but this is not necessary for the present paper since it will be used only in the form of $T'(z)$.

Because of the $p^-\ln z$ term in $X^-(z)$, the currents can not be written purely as a Laurent series. Let us define another set of currents $\tilde{J}^i$ as the $\alpha_0^- = 0$ limit of the currents above. The $\tilde{J}^i(z)$ have the Laurent expansion with Hermitian coefficients

$$\tilde{J}^i(z) = \sum J_n^i\, z^{-n-1}, \quad \left(J_n^i\right)^\dagger = J_{-n}^i. \tag{4.14}$$

Then the J_n^i become functions of the oscillators $\alpha_n^\pm, s_n$ and L'_n. The relation between the new nonanalytic currents $J^i(z)$ and the analytic currents $\tilde{J}^i(z)$ has precisely the form given in Eq. (3.2).

Using the rules for normal ordering given above we compute the operator product algebra for the currents, and verify that they satisfy the correct relations in the presence or absence of the $\alpha_0^-\ln z$ term:

$$\tilde{J}^\mu(z)\,\tilde{J}^\nu(w) \to \frac{k/2}{(z-w)^2} + i\epsilon^{\mu\nu\lambda}\,\frac{\tilde{J}_\lambda(w)}{z-w} + \cdots\ , \tag{4.15}$$

$$J^\mu(z)\,J^\nu(w) \to \frac{k/2}{(z-w)^2} + i\epsilon^{\mu\nu\lambda}\,\frac{J_\lambda(w)}{z-w} + \cdots \tag{4.16}$$

(see Sec. V and the Appendix for the details of the calculation).

The energy-momentum tensor is obtained from the normal-ordered product of the currents:

$$T(z)=\frac{1}{k-2}:\left\{-[J_0(z)]^2+[J_1(z)]^2+[J_2(z)]^2\right\}: . \tag{4.17}$$

The result of the computation gives (see Sec. VI)

$$T(z)=:P^+i\partial_z X^-:+T_S(z)+T'(z) . \tag{4.18}$$

If the computation is repeated with the $\tilde{J}(z)$ currents, the only difference is dropping the α_0^- term contained in

$$i\partial_z X^-=\sum_{n=-\infty}^{\infty}\alpha_n^- z^{-n-1}. \tag{4.19}$$

In (4.18), T_S is a Hermitian stress tensor:

$$T_S(z)=\frac{1}{k-2}\left[:[S(z)]^2:-\frac{i}{z}\partial_z[zS(z)]+\frac{1}{4z^2}\right] . \tag{4.20}$$

The structure $\frac{i}{z}\partial_z[zS(z)]$ differs from the usual one $i\partial S$, and thus is Hermitian. The operator products of $T_S(z)$ are

$$T_S(z)T_S(w)=\frac{c_s/2}{(z-w)^2}+\frac{2T_S(w)}{(z-w)^2}+\frac{\partial_w T_S(w)}{z-w}+\cdots , \tag{4.21}$$

with the central charge

$$c_s=1+\frac{6}{k-2}. \tag{4.22}$$

Note that the term $P^+i\partial_z X^-$ is identical to the energy-momentum tensor of flat light-cone coordinates constructed from the oscillators $\alpha_n^{\pm}$. Therefore, that part is mathematically equivalent to a $c=2$ stress tensor constructed from one time and one space coordinate in flat spacetime. Then the total central charge is

$$\begin{aligned} c&=2+c_s+c' \\ &=2+\left(1+\frac{6}{k-2}\right)+0 \\ &=\frac{3k}{k-2}, \end{aligned} \tag{4.23}$$

which is the right central charge for the SL(2, R) WZW model. Finally, as a further consistency check, by using only the operator products of the elementary fields, one finds that $T(z)$ has the correct operator products with the currents.

The zero mode of the stress tensor takes the form $L_0=L_0^{\pm}+L_0^S+L_0'$, where each piece has the eigenvalues

$$\begin{aligned} L_0^{\pm}&=p^+p^-+l_{\pm}, \\ L_0^S&=(s_0^2+1/4)/(k-2)+l_s, \\ L_0'&=h'+l', \end{aligned} \tag{4.24}$$

where $l_{\pm}, l_s, l'$ are positive integers and h' is the eigenvalue of L_0' at the base (whose possible values depend on the model for T'). The mass shell condition $L_0=a$ is

$$p^+p^-+(s_0^2+1/4)/(k-2)+h'+\text{integer}=a , \tag{4.25}$$

where $a\le 1$. The term $p^+p^-=-M^2/2$ is crucial since it takes negative values. In comparison to Eq. (3.11) we see that the Casimir operator of the old currents takes the value

$$j(j+1)=-(s_0^2+1/4)-h'(k-2). \tag{4.26}$$

Therefore, if the T' piece is absent in the construction ($h'=0$), then $j=-1/2\pm is_0$ is only in the principal series. The supplementary series occurs for $-1/4<j(j+1)<0$ and the discrete series occurs for $-1/4<j(j+1)$. We see that the field T' with a positive h' contributes only to the principal series and with a negative h' it leads to the other representations as well. This construction may find various applications in the future. We will see below that T' is absent in the SL(2,R) WZW model; hence, only the special case of our construction ($T'=0$) will find an application in the current paper. Then, for excited string states, since the integer in (4.25) is positive, it would not be possible to satisfy the mass shell condition in the absence of the p^-. So the new logarithmic structure will play a role.

The physical state conditions will be analyzed in Sec. VIII, after we prove the above construction.

V. OPERATOR PRODUCTS OF CURRENTS

We want to verify that the operator products of the currents are correct. Some formulas that are useful for this computation are collected in the Appendix. There are six independent operator products that we need to reproduce. In increasing complexity these are

$$\begin{aligned} &(J_0+J_1)(z)(J_0+J_1)(w)\to 0+\cdots , \\ &(J_0+J_1)(z)J_2(w)\to\frac{-i}{z-w}(J_0+J_1)(w)+\cdots , \\ &(J_0+J_1)(z)(J_0-J_1)(w)\to\frac{-k}{(z-w)^2}-\frac{2i}{z-w}J_2(w)+\cdots , \\ &J_2(z)J_2(w)\to\frac{k/2}{(z-w)^2}+\cdots , \\ &J_2(z)(J_0-J_1)(w)\to\frac{-i}{z-w}(J_0-J_1)(w)+\cdots , \\ &(J_0-J_1)(z)(J_0-J_1)(w)\to 0+\cdots . \end{aligned} \tag{5.1}$$

The first one is easy:

$$[J_0(z)+J_1(z)]\,[J_0(z)+J_1(z)] = P^+(z)\times P^+(w)\to 0\ . \quad (5.2)$$

This is correct since the central extension in J_0J_0 cancels the central extension in J_1J_1 due to the indefinite metric $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1)$. The product

$$\begin{aligned}(J_0+J_1)(z)J_2(w) &= P^+(z)[:X^-P^+:+S](w) && (5.3)\\ &\to -iP^+(w)/(z-w) \\ &\to -i\,[J_0(w)+J_1(w)]\,/(z-w) && (5.4)\end{aligned}$$

follows from Eq. (A1). Next, by using the operator product for P^+ and X^- one obtains

$$\begin{aligned}&(J_0+J_1)(z)\,(J_0-J_1)(w)\\ &\quad= P^+(z)\begin{bmatrix} :X^-P^+X^-:-ik\partial_wX^- \\ +2X^-S-\frac{(k-2)T'}{P^+}\end{bmatrix}(w)\\ &\quad\to \frac{-k}{(z-w)^2}+\frac{-2i}{z-w}[:X^-P^+:+S](w)\\ &\quad\to \frac{-k}{(z-w)^2}+\frac{-2i}{z-w}J_2(w). \qquad (5.5)\end{aligned}$$

The product for J_2J_2 is obtained by using (A3) and (4.9):

$$\begin{aligned}J_2(z)J_2(w) &= [:X^-P^+:+S](z)\,[:X^-P^+:+S]\,(w)\\ &\to \frac{1}{(z-w)^2}+\frac{k/2-1}{(z-w)^2}=\frac{k/2}{(z-w)^2}. \qquad (5.6)\end{aligned}$$

The product $J_2\,(J_0-J_1)$ is computed by using (A4) and (4.9):

$$\begin{aligned}J_2(z)\,(J_0-J_1)\,(w) &= (:X^-P^+:+S)\,(z)\left[:X^-P^+X^-:\,-ik\partial_wX^-+2X^-S-\tfrac{(k-2)T'}{P^+}\right](w)\\ &\to \frac{-i\,(:X^-P^+X^- -(k-2)T'/P^+)\,(w)}{z-w}+\frac{-2iX^-(w)}{z-w}S(w)+\frac{2X^-(w)}{(z-w)^2}\\ &\quad -ikX^-(z)\left(\partial_w\frac{-i}{z-w}\right)+2X^-(w)\frac{k/2-1}{(z-w)^2}\\ &\to \frac{-i}{z-w}\left[:X^-P^+X^-:-ik\partial_wX^-+2X^-S-\tfrac{(k-2)T'}{P^+}\right] && (5.7)\\ &\to \frac{-i}{z-w}\,(J_0-J_1)\,(w)\ , && (5.8)\end{aligned}$$

where we have used

$$X^-(z)=X^-(w)+(z-w)\,\partial_wX^-(w)+\cdots \quad (5.9)$$

in the 4th line. Finally the product that is the most complicated to compute follows from the steps that lead to (A18):

$$(J_0-J_1)\,(z)\,(J_0-J_1)\,(w)\to 0. \quad (5.10)$$

Hence we have correctly constructed the current algebra from the elementary free fields. The closure of the algebra goes through whether or not the $\alpha_0^-\ln z$ term is included in the definition of $X^-(z)$. Its presence is felt through Eq. (5.9) since this is how the $1/z$ terms work out. Thus, for any α_0^- the algebra of the new currents is identical to the algebra of the old ones.

VI. STRESS TENSOR

To construct the stress tensor we need to compute the normal-ordered products of currents $\eta_{ij}:J^i(z)J^j(z):$. We define the normal-ordered product by splitting the points, and taking the ordinary product minus the singularity. That is,

$$\eta_{ij}:J^i(z)J^j(w):\equiv \eta_{ij}\,J^i(z)J^j(w)-\frac{3k/2}{(z-w)^2}. \quad (6.1)$$

Then we compute the right-hand side by substituting the free field form and rearrange it by using Wick's theorem for free fields. The final form is a normal-ordered form for the free fields in which all singularities cancel. Then the limit $z\to w$ is taken to define the local stress tensor. In this process it is important to keep track of the finite parts, and not use the operator products naively. We need to compute the products $\frac{1}{2}(J_0-J_1)(z)(J_0+J_1)(w)$, $\frac{1}{2}(J_0+J_1)(z)(J_0-J_1)(w)$, and $J_2(z)J_2(w)$ including the finite parts. The singular part for the first two products is given in (5.5) and the finite part (as $z\to w$) is obtained from Eqs. (4.9) and (A2). Thus,

$$\begin{aligned}-\frac{1}{2}(J_0+J_1)(z)(J_0-J_1)(w) &= -\frac{1}{2}P^+(z)\left[:X^-P^+X^-:\,-ik\partial_wX^-+2X^-S-(k-2)\frac{T'}{P^+}\right](w)\\ &= -\frac{1}{2}(:P^+X^-P^+X^-:)+\frac{ik}{2}(:P^+\partial_wX^-:)-:P^+X^-:S+\left(\frac{k}{2}-1\right)T'\\ &\quad+\frac{k}{2(z-w)^2}-\left(\frac{-i}{z-w}+\frac{i}{2w}\right)J_2(w)+O(z-w)\ . \qquad (6.2)\end{aligned}$$

Here we were careful to keep the finite $1/z$ part in the contraction $\langle P^+(z)X^-(w)\rangle$. Similarly we have

$$\begin{aligned}-\frac{1}{2}(J_0-J_1)(z)(J_0+J_1)(w) &= -\frac{1}{2}\left[: X^- P^+X^- : -ik\partial_z X^- + 2X^-S - (k-2)\frac{T'}{P^+}\right](z)\,P^+(w)\\ &= -\frac{1}{2}(: X^-P^+X^-P^+ :) + \frac{ik}{2}(: P^+\partial_w X^- :) - : P^+X^- : S\\ &\quad+\left(\frac{k}{2}-1\right)T' + \frac{k}{2(z-w)^2} - \left(\frac{i}{z-w}+\frac{i}{2w}\right)J_2(w) - i\partial_w J_2(w) + O(z-w). \qquad (6.3)\end{aligned}$$

The singular part of the product $J_2(z)J_2(w)$ is given in (5.6) while the finite part is obtained through Eqs. (A3) and (4.9) as

$$\begin{aligned}J_2(z)J_2(w) &= \left(: X^- P^+ : +S\right)(z)\left(: X^- P^+ : +S\right)(w)\\ &= : X^-(z)\,P^+(z)X^-(w)\,P^+(w) : + \left(\frac{-i}{z-w}+\frac{i}{2z}\right) : X^-(z)\,P^+(w) :\\ &\quad+\left(\frac{i}{z-w}+\frac{i}{2w}\right) : X^-(w)\,P^+(z) : +2S : X^-P^+ :\\ &\quad+\left(\frac{i}{z-w}+\frac{i}{2w}\right)\left(\frac{-i}{z-w}+\frac{i}{2z}\right) + : S(z)S(w) : + \frac{k/2-1}{(z-w)^2}\\ &\to \frac{k}{2(z-w)^2} - \frac{1}{4w^2} + \left(: i\partial P^+X^- - P^+ i\partial X^- :\right) + \frac{i}{w} : X^-P^+ : + : X^- P^+X^- P^+ :\\ &\quad+ : SS : +2S : X^-P^+ : +O(z-w)\,, \qquad (6.4)\end{aligned}$$

where we have used

$$-\frac{1}{2z}\frac{1}{z-w} = -\frac{1}{2w}\frac{1}{z-w} + \frac{1}{2w^2} + O(z-w) \qquad (6.5)$$

in arriving at the $1/4w^2$ term. Substituting these expressions in (6.1) we obtain, as $z \to w$,

$$\begin{aligned}\frac{\eta_{ij}}{k-2} &: J^i(w)J^j(w) := \left(: P^+ i\partial_w X^- : +T_S + T'\right),\\ T_S &= \frac{1}{k-2}\left(: S^2 : -\frac{i}{w}\partial(wS) + \frac{1}{4w^2}\right). \qquad (6.6)\end{aligned}$$

Therefore the energy-momentum tensor is

$$T =: P^+ i\partial X^- : +T_S + T'\,, \qquad (6.7)$$

as advertized in Eq. (4.18).

VII. QUANTIZATION OF THE SL(2,R) WZW MODEL

Up to now we have worked in a purely algebraic framework. We now relate these structures to the WZW model for SL(2,R). The group element $g(X)$ may be parametrized in terms of one-time and two-space string coordinates $X^\mu(\tau,\sigma)$, $\mu = 0,1,2$. When the model is explicitly written in terms of these, the action looks like Eq. (1.1) with the metric $G_{\mu\nu}(X)$ induced by the Cartan connection of the group as

$$\begin{aligned}dg(X)\,g^{-1}(X) &= (-it_i)\,dX^\mu\,E^i_\mu(X)\,, \qquad (7.1)\\ G_{\mu\nu}(X) &= E^i_\mu(X)\,\eta_{ij}\,E^j_\nu(X),\end{aligned}$$

where t_i is a basis for the SL(2,R) Lie algebra, and $\eta_{ij} = -2\mathrm{tr}(t_it_j) = \mathrm{diag}(-1,1,1)$. A convenient basis that we will use is given in terms of Pauli matrices $t_0 = \sigma_2/2$, $t_1 = i\sigma_1/2$, $t_2 = -i\sigma_3/2$.

The quantum theory is conveniently formulated in terms of the left- and right-moving currents after writing $g(\tau,\sigma) = g_L(\tau+\sigma)\,g_R^{-1}(\tau-\sigma)$:

$$J_L(z) = ik\partial_z g_L g_L^{-1}, \quad J_R(\bar z) = ik\partial_{\bar z} g_R g_R^{-1}, \qquad (7.2)$$

where $z = e^{i(\tau+\sigma)}$, $\bar z = e^{i(\tau-\sigma)}$. Note that $z, \bar z$ are independent complex variables. After normal ordering, the stress tensor takes the form

$$\begin{aligned}T(z) &= \frac{1}{2(k-2)}\mathrm{tr}[: J_L(z)J_L(z) :], \qquad (7.3)\\ \bar T(\bar z) &= \frac{1}{2(k-2)}\mathrm{tr}[: J_R(\bar z)J_R(\bar z) :].\end{aligned}$$

The quantum rules are most conveniently given in terms of operator products among the currents and the group elements. The left movers $J_L(z)$, $g_L(z)$ or right movers $J_R(\bar z)$, $g_R(\bar z)$ obey similar rules. Therefore, to save space, in the following we denote $J(z)$ and $g(z)$ for either the left-moving or right-moving pair:

$$\begin{aligned}J^i(z)\,J^j(w) &\to \frac{k/2}{(z-w)^2} + i\epsilon^{ijk}\,\frac{J_k(w)}{z-w} + \cdots\,, \qquad (7.4)\\ J^i(z)\,g(w) &\to \frac{-t^i}{z-w}g(w) + \cdots\,,\end{aligned}$$

where the ellipses stand for nonsingular terms in the operator product. These quantum rules reflect the left-right symmetry structure that is of fundamental importance in this model. Below we give a construction in terms of oscillators that satisfies these rules.

To obtain a relation to the free boson realization discussed in the previous sections we coordinate the group element (i.e., g_L or g_R) in terms of triangular and diagonal matrices as

$$g(z) = \begin{pmatrix} 1 & 0 \\ X^- & 1 \end{pmatrix} \begin{pmatrix} :e^{-\frac{u}{k-2}}: & 0 \\ 0 & :e^{\frac{u}{k-2}}: \end{pmatrix} \begin{pmatrix} 1 & X^+ \\ 0 & 1 \end{pmatrix} \tag{7.5}$$

and compute the currents

$$i\,(k-2) : \partial_z g g^{-1} :$$

$$= \begin{pmatrix} -J^2(z) & J^0(z)+J^1(z) \\ -J^0(z)+J^1(z) & J^2(z) \end{pmatrix} . \tag{7.6}$$

As compared to the classical currents (7.2) we have shifted $k \to (k-2)$ in both g, Eq. (7.5), and the definition of the current, Eq. (7.6), and applied normal ordering. This renormalization is necessary for the commutation rules to work out, and is consistent with similar phenomena concerning the quantization of the WZW model [13]. One finds

$$\begin{aligned} J^0(z)+J^1(z) &= :(k-2)\, i\partial_z X^+ e^{-2u/(k-2)} : , \\ J^2(z) &= (k-2) : i\partial_z X^+ X^- e^{-2u/(k-2)} : + i\partial_z u , \\ J^0(z)-J^1(z) &= (k-2) : i\partial_z X^+ \left(X^-\right)^2 e^{-2u/(k-2)} : \\ &\quad + 2X^- i\partial_z u - ik\partial_z X^- . \end{aligned} \tag{7.7}$$

The coefficient of $-ik\partial_z X^-$ is ambiguous because of the normal ordering of the term $: i\partial_z X^+ \left(X^-\right)^2 e^{-2u/(k-2)} :$. Again this has to be fixed by requiring that the commutation rules work out. Therefore, instead of having naively $-i\,(k-2)\,\partial_z X^-$, we actually must have $-ik\partial_z X^-$. These results are established by applying the canonical formalism and identifying these structures with canonical conjugate variables. Velocities must be replaced by canonical momenta. Note that for left-right movers ∂_z can be related to time derivatives ∂_τ or space derivatives ∂_σ. So, at the quantum level, we find that we must identify the canonical pairs (X^-, P^+) and (u, S) as

$$\begin{aligned} P^+(z) &= (k-2)\, i\partial_z X^+ e^{-2u/(k-2)} , \\ S(z) &= i\partial_z u , \end{aligned} \tag{7.8}$$

and then the currents take the form

$$\begin{aligned} J^0(z)+J^1(z) &= P^+(z) , \\ J^2(z) &= : X^- P^+ : + S , \\ J^0(z)-J^1(z) &= : X^- P^+ X^- : + 2SX^- - ik\partial_z X^- . \end{aligned} \tag{7.9}$$

This is the form used in the previous section without the extra field $L'(z)$. Thus, as discussed before, only the principal series will emerge in the WZW model. Using the oscillator form introduced in (4.2) we can express $u(z)$ and $X^+(z)$ in terms of the basic oscillators s_n, α_n^+ by inverting the formulas in (7.8); thus,

$$\begin{aligned} u(z) &= u_0 - is_0 \ln z + i\sum_{n\neq 0} \frac{1}{n} s_n z^{-n} , \\ X^+(z) &= -i\int^z dz' \frac{P^+(z')}{(k-2)} \; : \exp\left[\frac{2u(z')}{k-2}\right] : . \end{aligned} \tag{7.10}$$

Then these structures satisfy the operator products

$$\begin{aligned} \langle u(z)\, S(w)\rangle &= \left(\frac{i}{z-w} + \frac{i}{2w}\right)\left(\frac{k}{2}-1\right) , \\ \left[J^0(z)-J^1(z)\right] X^+(w) &\to \frac{i}{z-w} i : ie^{2u(w)/(k-2)} . \end{aligned} \tag{7.11}$$

Thus, $u(z)$ is just the canonical conjugate to $S(z)$. Another property of X^+ that follows from the fundamental operator product is that it is a singlet under the action of $J_2(z)$:

$$J_2(z) X^+(w) \to 0. \tag{7.12}$$

Actually ∂X^+ is a screening current. Its operator product with all the currents is either zero or a total derivative. Therefore, its zero mode commutes with all the currents.

Inserting these expressions in Eq. (7.10) into (7.5) we obtain the quantum operator version of the group element g. The operator products may now be evaluated. We find the correct quantum products (7.4) with the above construction in terms of oscillators. That is,

$$\begin{aligned} &\left[J^0(w)+J^1(w)\right] g(w) \to \tfrac{-i}{z-w} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g(w) , \\ &J^2(z) g(w) \to \tfrac{i/2}{z-w} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(w) , \\ &\left[J^0(w)-J^1(w)\right] g(w) \to \tfrac{i}{z-w} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g(w) . \end{aligned} \tag{7.13}$$

This result, combined with the current $\times$ current operator products that we have proven earlier, is convincing evidence that the free field formalism that we have introduced corresponds to the quantization of the SL(2,R) WZW model.

VIII. PHYSICAL STATES

A. No ghosts

Since we have rewritten the WZW theory in terms of free fields, the space of states consists of the Fock space for the oscillators $\alpha_n^\pm, s_n$ applied on the base $|p^+, p^-, s_0\rangle$ that diagonalizes the zero mode operators $\alpha_0^\pm, s_0$:

$$\prod_{n=1}^{\infty} \left(\alpha_{-n}^+\right)^{a_n} \prod_{m=1}^{\infty} \left(\alpha_{-m}^-\right)^{b_m} \prod_{k=1}^{\infty} \left(s_{-k}\right)^{c_k} \, |p^+, p^-, s_0\rangle , \tag{8.1}$$

where the powers a_n, b_m, c_k are positive integers or zero.

This is the space of states that provide a representation basis for the SL(2,R) currents with only the principal series. The physical states are identified as those linear combinations that are annihilated by the total Virasoro generators:

$$L_n|\psi\rangle = 0, \quad n \geq 1. \tag{8.2}$$

In the present case the total Virasoro generators include the terms

$$L_n = L_n^{\pm} + L_n^S , \tag{8.3}$$

where

$$L_n^{\pm} = \sum_m : \alpha_{-m}^- \alpha_{n+m}^+ : , \tag{8.4}$$

$$L_n^S = \frac{1}{k-2}\left(\sum_m : s_{-m}s_{n+m} : + ins_n + \frac{1}{4}\delta_{n,0}\right) .$$

Note that the $L_n^{\pm}$ is equivalent to the $c = 2$ Virasoro operator in 2D flat spacetime. The full central charge is

$$c = \frac{3k}{k-2}. \tag{8.5}$$

The eigenvalue of the total L_0 is

$$L_0 = p^+p^- + \frac{1}{k-2}\left[s_0^2 + 1/4\right] + \text{integer} . \tag{8.6}$$

Thus, the theory has been reduced to a 2D light cone in flat spacetime plus a Liouville-type spacelike free field that has positive norm. A small but important difference as compared to the standard Liouville formalism is that the linear term in L_n^S is Hermitian in our case, and does not contribute to L_0^S.

The only negative norm states are the ones produced by the timelike oscillator $\alpha_n^0 = (\alpha_n^+ - \alpha_n^-)/\sqrt{2}$. However, this is no worse than the usual flat spacetime case.[5] The space of physical states is defined by

$$(L_n - a\delta_{n,0})\,|\phi\rangle = 0 ,$$

with $a \leq 1$ fixed. A proof of no ghosts can now be given by following step by step the same arguments that prove the no-ghost theorem in flat spacetime [8]. There is no need to repeat it here. We only recall that there are no ghosts as long as $a \leq 1$ and $c \leq 26$.

It is straightforward to construct a few low-lying states that satisfy the Virasoro constraints and check explicitly that they have positive norm. For example, at the level $l = 1$ the mass shell condition is

$$p^+p^- + \frac{1}{k-2}\left[s_0^2 + 1/4\right] + 1 = a . \tag{8.7}$$

One finds the two orthogonal states that satisfy the Virasoro constraints

$$\begin{aligned}
|\phi_1\rangle &= (p^+\alpha_{-1}^- - p^-\alpha_{-1}^+)|p^+,p^-,s_0\rangle , \\
|\phi_2\rangle &= \left[(p^+\alpha_{-1}^- + p^-\alpha_{-1}^+)\left(s_0 + \frac{i}{2}\right)\right. \\
&\qquad \left. -2p^+p^-s_{-1}\right]|p^+,p^-,s_0\rangle, \\
\langle\phi_1|\phi_1\rangle &= -2p^+p^-, \quad \langle\phi_1|\phi_2\rangle = 0, \\
\langle\phi_2|\phi_2\rangle &= -4p^+p^-(1-a).
\end{aligned} \tag{8.8}$$

After taking into account (8.7) one sees that the norms are positive as long as $a < 1$. The second state becomes a zero norm state if the theory is critical, with $a = 1$. The first state survives even at the critical point, and is interpreted as a string state in the 2D subspace. At higher levels one finds that the number of states corresponds to the same counting as if there are two free bosons. This should be expected since we started with three free bosons and basically eliminated one of them through the Virasoro conditions. Some of the norms, but not all, are proportional to $(1-a)$. If the theory is critical with $a = 1$, a subset of these states becomes zero norm states. The same phenomena can be observed in 3D flat string theory when it is considered as a piece of the $D = 26$ critical theory. In fact, as $k \to \infty$ the states described above become the states of the flat 3D noncritical theory, satisfying the 3D flat string Virasoro constraints. This is seen by rescaling $s_n \to \alpha_n^3\sqrt{k/2-1}$, where $\left[\alpha_n^3, \alpha_m^3\right] = n\delta_{n+m}$ become the oscillators of the string in the third dimension, $\alpha_0^3 = s_0/\sqrt{k/2-1}$ becomes the momentum in the third dimension, and the L_n of the present theory tend to the L_n of the 3D flat string theory. This observation is additional confirmation that the SL(2,R) curved space theory behaves in the correct intuitive way as $k \to \infty$ by becoming a flat space theory.

A more efficient approach to construct the physical states is to use the "spectrum generating algebra" as in flat space [14]. This will be presented in a separate paper.

B. Monodromy

So far we have not taken into account the physical effects of the $\ln z$ cut in the currents. As we argued in the beginning of Sec. III, a physical string theory must satisfy the monodromy condition in the physical sector:

$$\langle \text{phy}|J^i(ze^{i2\pi n})|\text{phy}'\rangle = \langle \text{phy}|J^i(z)|\text{phy}'\rangle. \tag{8.9}$$

This condition would have been satisfied automatically if there were no cuts. One of the new features of our theory is to require that the monodromy be satisfied only on a

[5]We have left out the L' sector since this is not part of the WZW model. In a model that includes L' the base state would have the additional label h' and products of Virasoro operators L'_{-n} applied on it. If the L' sector is present, there could be additional negative norm states arising in sectors of negative h', *if such values are permitted* by the model describing L'. For example, note that the norm of $L'_{-n}|h'\rangle$ is $2nh'$. Recall that negative values of h' lead to the discrete series when $j(j+1)\rangle - 1/4$. This observation is, in fact, an explanation of the origin of negative norm states that arose in the beginning.

subset of states that are the physical states. Quantum mechanically it is possible to impose this condition simultaneously with the Virasoro constraints since the latter commute with the monodromy operator as seen below.

To implement the monodromy let us first consider its effect on the currents. From the modified currents in (3.2) we see that under the monodromy the currents undergo a linear transformation

$$\left[J^0+J^1\right](ze^{i2\pi n}) = \left[J^0+J^1\right](z) ,$$

$$\begin{aligned}\left[J^0-J^1\right](ze^{i2\pi n}) &= \left[J^0-J^1\right](z) + 4\pi n\alpha_0^- \, J^2(z) \\ &\quad + \left(2\pi n\alpha_0^-\right)^2 \left[J^0+J^1\right](z) ,\end{aligned} \tag{8.10}$$

$$J^2(ze^{i2\pi n}) = J^2(z) + 2\pi n\alpha_0^- \, \left[J^0+J^1\right](z) .$$

Therefore we expect that the right-hand side can be rewritten as the adjoint action with a *global* SL(2,R) transformation. Since the current $J^0(z)+J^1(z)$ remains unchanged, the generator of this transformation must be the zero mode of this current. Indeed, since α_0^- acts like a number, we can rewrite the monodromy in the form

$$J^i(ze^{i2\pi n}) = e^{-2i\pi n\alpha_0^-\left(J_0^0+J_0^1\right)} J^i(z) \, e^{2i\pi n\alpha_0^-\left(J_0^0+J_0^1\right)} . \tag{8.11}$$

Therefore physical states that satisfy (8.9) are the subset of states that are invariant under the monodromy

$$e^{2i\pi n\alpha_0^-\left(J_0^0+J_0^1\right)}|\text{phys}\rangle = |\text{phys}\rangle . \tag{8.12}$$

In the free boson representation this is easy to implement. Using $\left(J_0^0+J_0^1\right) = \alpha_0^+$ this condition is applied on the Fock space of the free bosons in the form

$$e^{2i\pi n\alpha_0^-\alpha_0^+} \prod_{n,m,k=1}^{\infty} \left(\alpha_{-n}^+\right)^{a_n} \left(\alpha_{-m}^-\right)^{b_m} \left(s_{-k}\right)^{c_k} \, |p^+p^-s_0\rangle . \tag{8.13}$$

Therefore it only requires that the momenta that describe the ground state be quantized in terms of negative integers:

$$e^{2i\pi n\alpha_0^-\alpha_0^+}|p^+,p^-,s_0\rangle = |p^+,p^-,s_0\rangle ,$$

$$\alpha_0^-\alpha_0^+ = p^-p^+ = -r, \quad r=0,1,2,\ldots \; . \tag{8.14}$$

We must take negative integers because according to the mass shell condition p^-p^+ is negative. So the mass shell condition on physical states at the excitation level l takes the form

$$-r + \frac{1}{k-2}\left[s_0^2+1/4\right] + l = a. \tag{8.15}$$

It is always possible to satisfy this condition with some value of s_0 which is quantized in terms of the positive integers r, l. In terms of the original Casimir operator $j(j+1)$ this corresponds to a principal series representation of SL(2,R) with quantized values of j given by

$$\begin{aligned} j &= -\frac{1}{2} + is_0 \\ &= -\frac{1}{2} \pm i\sqrt{(k-2)\,(r-l+a) - 1/4} ,\end{aligned} \tag{8.16}$$

where r must be chosen so that the square root is real.

C. Open and closed strings

An open string action $S=\int d\tau \int_0^\pi d\sigma \, L(\tau,\sigma)$ is minimized by allowing free variation of the end points. For the WZW model for any group G this produces the boundary terms

$$\delta S = \int d\tau \times \begin{cases} \operatorname{Tr}\left[\left(\delta g g^{-1}\right)\left(\partial_\sigma g g^{-1}\right)\right]\big|_\pi , \\ -\operatorname{Tr}\left[\left(\delta g g^{-1}\right)\left(\partial_\sigma g g^{-1}\right)\right]\big|_0 . \end{cases} \tag{8.17}$$

In addition to the equations of motion, these terms must also vanish at each end of the string. That is,

$$\partial_\sigma g g^{-1}\big|_{\sigma=0} = 0 = \partial_\sigma g g^{-1}\big|_{\sigma=\pi} . \tag{8.18}$$

At the conformal critical point the equations of motion are satisfied by the general form $g(\tau,\sigma) = g_L(\tau+\sigma)\, g_R^{-1}(\tau-\sigma)$. Then the boundary conditions require that g_L and g_R be related to each other by the constraint

$$g_L^{-1}(\tau)\,\partial_\tau g_L(\tau) + g_R^{-1}(\tau)\,\partial_\tau g_R(\tau) = 0. \tag{8.19}$$

Furthermore, each term in this equation is required to be periodic. As discussed in the rest of this paper, we impose periodicity on the physical states. The relation (8.19) between $g_L(\tau)$ and $g_R(\tau)$ is not easy to solve explicitly. However, we may carry out the quantum theory in terms of the current

$$J(z) = g_L^{-1}(z)\,\partial_z g_L(z) = -g_R^{-1}(z)\,\partial_z g_R(z).$$

This is neither the left-moving current $J_L = \partial g_L g_L^{-1}$ nor the right-moving one $J_R = \partial g_R g_R^{-1}$, but is related to them by transformations involving g_L or g_R. This current generates transformations on the *right side* of g_L and the left side of g_R^{-1}, and the meaning of (8.19) is that the total current on both g_L and g_R vanishes at the end points. The canonical commutation rules for this current are identical to the ones we have already discussed in the rest of the paper. The stress tensor constructed from it is equal to the stress tensor constructed from either the left movers or the right movers:

$$\operatorname{Tr}(J^2) = \operatorname{Tr}(J_L^2) = \operatorname{Tr}(J_R^2).$$

The quantum spectrum is obtained from the properties of J, whose mathematical structure is the same as either left movers or right movers as discussed in the previous sections. Thus, the quantum spectrum of the open string in the SL(2,R) curved spacetime becomes identical to the spectrum discussed above.

For a closed string we have independent left- and right-moving sectors. The full group element is $g = g_L(z)g_R^{-1}(\bar{z})$ and there are left- and right-moving currents. Therefore we now need two sets of oscillators, the left movers $\alpha_n^{\pm}, s_n$ and the right movers $\tilde{\alpha}_n^{\pm}, \tilde{s}_n$. So the direct product Hilbert space has a base labeled by $|p^-, p^+, s_0; \tilde{p}^-, \tilde{p}^+, \tilde{s}_0\rangle$ with $p^-p^+ = -r$ and $\tilde{p}^-\tilde{p}^+ = -\tilde{r}$ to ensure that the currents obey the monodromy conditions in the physical sector. We now need to figure out if these are all independent labels or if they must be constrained by physical considerations.

For this purpose we recall that a possible modular invariant is the so-called "diagonal invariant" that requires the same unitary representation labeled by the same j for both left and right movers. This may be understood as being related to the representation of the full group element $D^j(g) = D^j(g_L(z))D^j(g_R^{-1}(\bar{z}))$ which requires the same j for both left and right movers. Therefore, we must demand $s_0 = \tilde{s}_0$.

In addition, we examine $g(z,\bar{z})$ in more detail. Keeping the order of operators, it may be written in the form

$$g = g_L(z)g_R^{-1}(\bar{z}) = \begin{pmatrix} u & a \\ -b & v \end{pmatrix} , \tag{8.20}$$

with

$$u = e^{\frac{-u_L+u_R}{k-2}} - e^{\frac{-u_L}{k-2}} \left(X_L^+ - X_R^+\right) X_R^- \, e^{\frac{-u_R}{k-2}} , \tag{8.21}$$

$$v = e^{\frac{u_L-u_R}{k-2}} + e^{\frac{-u_L}{k-2}} X_L^- \left(X_L^+ - X_R^+\right) e^{\frac{-u_R}{k-2}} ,$$

$$a = e^{\frac{-u_L}{k-2}} \left(X_L^+ - X_R^+\right) e^{\frac{-u_R}{k-2}} , \tag{8.22}$$

$$b = -\left(X_L^- - X_R^-\right) e^{\frac{-u_L+u_R}{k-2}} + e^{\frac{-u_L}{k-2}} X_L^- \left(X_L^+ - X_R^+\right) X_R^- \, e^{\frac{-u_R}{k-2}} .$$

We see that g is not periodic under $\sigma \to \sigma + 2\pi n$ since there are logarithms in the expressions for every $X_{L,R}$,$^{\pm}$ $u_{L,R}$. However, provided we impose $p^+ = -\tilde{p}^+$ on physical states (to cancel the nonperiodic behavior in $X_L^+ - X_R^+$) , we find that we can rewrite this monodromy in the form

$$\begin{gathered} g(ze^{i2\pi n}, \bar{z}e^{-i2\pi n}) = U\tilde{U}g(z,\bar{z})\tilde{U}^{-1}U^{-1} , \\ U\tilde{U} = e^{-ip^+p^-2\pi n}e^{-is_0^2 2\pi n}\, e^{i\tilde{p}^+\tilde{p}^-2\pi n}e^{i\tilde{s}_0^2 2\pi n} , \end{gathered} \tag{8.23}$$

where p^+, s_0 are operators which do not commute with q^-, u_0, and similarly for right movers (note that we have never introduced a canonical conjugate to p^- [or $\tilde{p}^-$)]. To ensure that the matrix elements of the overall g are consistent with monodromy in the physical sector it is sufficient to impose the conditions

$$2p^+p^- + 2s_0^2 - 2\tilde{p}^+\tilde{p}^- - 2\tilde{s}_0^2 = 2m , \tag{8.24}$$

where m is an integer. Since we have already seen that $s_0 = \tilde{s}_0$, we find that this condition reduces to $r - \tilde{r} = m$, and does not impose any additional constraints on $r, \tilde{r}$.

Furthermore, for a closed string we should also have $L_0 - \tilde{L}_0 = 0$ on the physical states. According to the mass shell condition (8.15) this requires $r - l = \tilde{r} - \tilde{l}$. So modular invariant physical closed string states must be labeled at the base as

$$\left| -\frac{r}{p^+},\ p^+, s \right\rangle \left| \frac{\tilde{r}}{p^+}, -p^+, s \right\rangle , \tag{8.25}$$

where the restrictions are

$$\begin{gathered} \tilde{p}^+ = -p^+, \qquad \tilde{s}_0 = s_0, \\ p^-p^+ = -r, \quad \tilde{p}^-\tilde{p}^+ = -\tilde{r} , \end{gathered} \tag{8.26}$$

and the excitation numbers for left-right movers must be restricted by

$$r - l = \tilde{r} - \tilde{l}. \tag{8.27}$$

IX. COMMENTS

Two novel features were introduced in this paper. The first is that currents are allowed to contain logarithmic cuts provided monodromy conditions are applied on the physical states. The second is a new representation of the currents in terms of free bosons that render the theory completely solvable. Both of these ideas have generalizations that would allow the construction of a large number of new string models that are especially useful in curved spacetime.

We have shown that a unitary string theory in SL(2,R) curved spacetime can be constructed and its spectrum solved exactly. In a separate publication we will give the spectrum generating algebra which characterizes the physical states more efficiently. Correlation functions can also be computed by using free boson methods.

Using the SL(2,R) solution given here, it is not difficult to figure out the spectrum of the 2D black hole SL(2,R)/R gauged WZW model [1,7] . This requires imposing $J_n^2 = \tilde{J}_n^2 = 0$ for $n \geq 1$ and $J_0^2 + \tilde{J}_0^2 = 0$ on the SL(2,R) states described above. This task is more easily carried out once the spectrum generating algebra is constructed. This will be described elsewhere.

The new methods seem appropriate for understanding quantum string gravity beyond the so-called $c \geq 1$ barrier. In the present SL(2,R) case we have solved a 3D model with $c = 3k/(k-2)$ that can take values between 3 and 26.

We have also shown that the free boson methods permit a more general representation of SL(2,R) current algebra when the extra degrees of freedom L'_n are introduced. These were absent in the WZW model, but they may be present in more general models.

As emphasized in the Introduction, the main purpose for the present exercise is to develop the appropriate methods to study string theory during the early universe and to understand the impact of string theory on the symmetries and matter content observed at accelerator energies. For this purpose the current methods must be generalized to heterotic strings such as those described in [15]. Methods used for other special models of curved spacetimes may also be helpful [16].

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APPENDIX

1. Operator products computation

Here we compute the operator products of certain structures that will be used as building blocks for the operator products of the currents given in Sec. V. The method of computation is to use Wick's theorem for free fields to rearrange the oscillators into normal-ordered form, then expand the result in powers of the singularities, and finally drop the nonsingular terms (noted as ellipses below). The following are needed in the computation:

$$\begin{aligned} P^+(z)\ :X^-(w)\,P^+(w):\ &= \,:P^+(z)\,X^-(w)\,P^+(w):+<P^+(z)\,X^-(w)>P^+(w)\\ &\to \frac{-i}{z-w}P^+(w)+\cdots\ , \end{aligned} \tag{A1}$$

$$\begin{aligned} P^+(z)\ :X^-(w)\,P^+(w)X^-(w):\ &= \ :P^+(z)\,X^-(w)\,P^+(w)X^-(w):+2\langle P^+(z)\,X^-(w)\rangle\ :X^-(w)\,P^+(w):\\ &\to \frac{-2\,i}{z-w}[:X^-(w)\,P^+(w):]+\cdots\ , \end{aligned} \tag{A2}$$

$$\begin{aligned} \big(:X^-\,P^+:\big)(z)\ \big(:X^-\,P^+:\big)(w)\ &= \ :X^-(z)\,P^+(z)X^-(w)\,P^+(w):+\ :X^-(z)\ P^+(w):\ \langle P^+(z)X^-(w)\rangle\\ &\quad+\ :P^+(z)\,X^-(w):\ \langle X^-(z)P^+(w)\rangle\langle P^+(z)X^-(w)\rangle\,\langle X^-(z)P^+(w)\rangle\\ &\to \frac{1}{(z-w)^2}+\cdots\ . \end{aligned} \tag{A3}$$

Similarly,

$$\begin{aligned} &\big(:X^-\,P^+:\big)\ \ (z)\ \big(:X^-\,P^+X^-:\big)(w)\\ &\quad= \,:X^-(z)\,P^+(z)X^-(w)\,P^+(w)X^-(w):+2\langle P^+(z)X^-(w)\rangle\ :X^-(z)\ P^+(w)X^-(w):\\ &\quad\quad+\langle X^-(z)\,P^+(w)\rangle\ :\ X^-(w)\,P^+(z)X^-(w):+2\langle P^+(z)X^-(w)\rangle\,\langle X^-(z)\,P^+(w)\rangle X^-(w)\\ &\quad\to \frac{2X^-(w)}{(z-w)^2}+\frac{-i\,[:X^-(w)\ P^+(w)X^-(w)]}{z-w}+\cdots\ . \end{aligned} \tag{A4}$$

Furthermore,

$$\begin{aligned} &\big(:X^-PX^-:\big)\ \ (z)\ \big(:X^-\,PX^-:\big)(w)\\ &\quad= \ :\big(X^-\,PX^-\big)(z)\,\big(X^-\,P^+X^-\big)(w):+2\langle X^-(z)P^+(w)\rangle:\ P^+(z)X^-(z)X^-(w)\,X^-(w):\\ &\quad\quad+2<P^+(z)X^-(w)\rangle:\ X^-(z)X^-(z)X^-(w)\,P^+(w):\\ &\quad\quad+4\langle X^-(z)P^+(w)\rangle\langle P^+(z)X^-(w)\rangle:\ X^-(z)X^-(w):\\ &\quad\to \frac{4:\ X^-(z)X^-(w):}{(z-w)^2}+\cdots\\ &\quad\to \frac{4:\ X^-(w)X^-(w):}{(z-w)^2}+\frac{4:X^-(w)\partial_wX^-(w):}{z-w}+\cdots \end{aligned} \tag{A5}$$

and

$$\begin{aligned} &[:X^-(z)\,P^+(z)X^-(z):]\ \ \big[-ik\partial_wX^-(w)\big]\\ &\quad= -ik\ \ :X^-(z)\,P^+(z)X^-(z)\partial_wX^-(w):-ik\,\partial_w\langle P^+(z)X^-(w)\rangle\ :X^-(z)X^-(z):\\ &\quad\to -k\frac{:X^-(z)X^-(z):}{(z-w)^2}+\cdots\\ &\quad\to -k\frac{:X^-(w)X^-(w):}{(z-w)^2}-2k\frac{:X^-(w)\partial_wX^-(w):}{z-w}+\cdots\ . \end{aligned} \tag{A6}$$

Similarly,

$$\big[-ik\partial_zX^-(z)\big]\ \big(:X^-\,P^+X^-:\big)(w)\to -k\frac{:X^-(w)X^-(w):}{(z-w)^2}+\cdots\ . \tag{A7}$$

Combining the last three equations we see that

$$[: X^- P^+ X^- : -ik\partial_z X^-](z)\ [: X^- P^+ X^- : -ik\partial_w X^-](w) \to 2(2-k)\left[\frac{:X^- X^-(w):}{(z-w)^2} + \frac{:X^- \partial_w X^-:}{z-w}\right] + \cdots \ . \tag{A8}$$

If k had the value 2, this operator product would not be singular. However, for any k the singularity cancels by including the current S in a modified operator as follows. Consider

$$\begin{aligned}[2X^-(z)S(z)]\ [2X^-(w)S(w)] &= 4\ [: X^-(z)X^-(w) :]\ [: S(z)S(w) :] + 4\frac{k/2-1}{(z-w)^2}[: X^-(z)X^-(w) :] \\ &\to 2(k-2)\left[\frac{:X^-(w)X^-(w):}{(z-w)^2} + \frac{:X^-(w)\partial_w X^-(w):}{z-w}\right] + \cdots \end{aligned} \tag{A9}$$

and the combination

$$\left.[: X^- P^+ X^- : -ik\partial_z X^-](z) \times [2X^- S](w) + [2X^- S](z) \times [: X^- P^+ X^- : -ik\partial_w X^-](w)\right\} \to 0\ , \tag{A10}$$

which is not singular (although each term by itself is). Combining the last three equations we see that the following operator product is not singular:

$$\begin{aligned}&[: X^- P^+ X^- \div ik\partial_z X^- + 2X^- S](z) \\ &\quad \times [: X^- P^+ X^- : -ik\partial_w X^- + 2X^- S](w) \\ &\quad \to 0 + \cdots \ . \end{aligned} \tag{A11}$$

2. Operator products with $1/P^+$

If the L'_n are included in the construction, then we need to compute the operator products with $1/P^+(z)$. This operator may be treated as a series in the oscillators, with the zeroth order term $1/p^+$. Successive terms in the series contain higher powers of $1/p^+$. The series is well defined provided one acts on states for which $1/p^+$ is a well-defined operator. In momentum space $|p^+\rangle$ this simply requires a nonzero eigenvalue $p^+ \neq 0$. In the space $|\,j,m\rangle$ labeled by the eigenvalues of J_0 one has to be careful since the behavior of the wave function $\langle p^+|j,m\rangle$ near the origin is $(p^+)^{j+1/2+is}$ (discrete, supplementary, principal series) or $(p^+)^{-j-1/2-is}$ (principal, supplementary series). Sufficiently high powers of $1/p^+$ may map a given state $|jm\rangle$ out of the normalizable Hilbert space. This would have to be interpreted properly. In the following we assume that the operators are well defined on an appropriate set of states, such as the states $|p^+ \neq 0\rangle$. Then

$$\begin{aligned} X^-(z)\frac{1}{P^+(w)} &= \ : X^-(z)\frac{1}{P^+(w)} : \\ &\quad - \left(\frac{1}{P^+(w)}\right)^2 \langle X^-(z)P^+(w)\rangle \\ &\to \frac{-i}{(z-w)}\left(\frac{1}{P^+(w)}\right)^2 + \cdots \ . \end{aligned} \tag{A12}$$

This leads to

$$\begin{aligned} &: X^-(z)\,P^+(z) : \frac{1}{P^+(w)} \to \frac{-i}{(z-w)}\frac{1}{P^+(w)} + \cdots\ , \\ &\frac{1}{P^+(z)} : X^-(w)\,P^+(w) :\to \frac{i}{(z-w)}\frac{1}{P^+(w)} + \cdots\ , \\ &: X^- P^+ X^- : (z)\,\frac{1}{P^+(w)} \to \frac{-2i}{(z-w)} : X^-(w)\frac{1}{P^+(w)} : \\ &\qquad + \frac{-2}{(z-w)^2}\frac{1}{[P^+(w)]^2} + \cdots\ , \end{aligned} \tag{A13}$$

and

$$\begin{aligned} &[: X^- P^+ X^- : \ -ik\partial_z X^- + 2X^- S](z)\,\frac{T'(w)}{P^+(w)} \\ &\quad \to \frac{-2iT'(w)}{(z-w)}(: X^- \frac{1}{P^+} : +S)(w) \\ &\quad + \frac{k-2}{(z-w)^2}\frac{T'(w)}{[P^+(w)]^2} + \cdots \ . \end{aligned} \tag{A14}$$

Similarly,

$$\begin{aligned} &\frac{T'(z)}{P^+(z)}\ [: X^- P^+ X^- : -ik\partial_w X^- + 2X^- S](w) \\ &\quad \to \frac{2iT'(w)}{(z-w)}\left(: X^- \frac{1}{P^+} : +S\right)(w) \\ &\quad + \frac{k-2}{(z-w)^2}\frac{T'(w)}{[P^+(w)]^2} \\ &\quad + \frac{k-2}{(z-w)}\partial_w\left(\frac{T'(w)}{[P^+(w)]^2}\right) + \cdots \ . \end{aligned} \tag{A15}$$

We also have

$$\frac{T'(z)}{P^+(z)}\frac{T'(w)}{P^+(w)} \to \frac{1}{P^+(z)P^+(w)}\left[\frac{2T'(w)}{(z-w)^2}+\frac{\partial_w T'(w)}{(z-w)}\right] \to \frac{1}{(z-w)^2}\frac{T'(w)}{[P^+(w)]^2} + \frac{1}{(z-w)}\partial_w\left(\frac{T'(w)}{[P^+(w)]^2}\right)+\cdots \quad . \tag{A16}$$

Combining the last three equations together with (A11) shows that the current

$$J_0(z)-J_1(z) = :X^-(z)\,P^+(z)X^-(z): -ik\partial_z X^-(z)+2X^-(z)S(z) - \frac{(k-2)T'(z)}{P^+(z)} \tag{A17}$$

has a nonsingular operator product with itself:

$$[J_0(z)-J_1(z)]\,[J_0(z)-J_1(z)] \to 0+\cdots \quad . \tag{A18}$$

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