Nontrivial directions for scalar fields

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We study the eigenvectors of the renormalization-group matrix for scalar fields at the Gaussian fixed point, and find that there exist "relevant" directions in parameter space. They correspond to theories with exponential potentials that are nontrivial and asymptotically free. All other potentials, including polynomial potentials, are "irrelevant," and lead to trivial theories. Away from the Gaussian fixed point, renormalization does not induce derivative couplings, but it generates nonlocal interactions.

PACS number(s): 11.10.Hi, 11.10.Kk, 11.10.Lm

I. INTRODUCTION AND SUMMARY

In a previous work [1], we discussed the renormalization group (RG) for scalar field theories, and reported RG trajectories near the Gaussian fixed point along which the scalar theory is nontrivial and asymptotically free. In this paper, we give the details, including a critical analysis of the calculations. In particular, we address the question of whether renormalization generates interactions not originally present in the Lagrangian.

To address the question of closure under RG, we start with the most general action conceivable for a real scalar field $\phi(x)$ in d space-time dimensions. Eventually, we focus our attention on a theory with local nonderivative couplings, whose Euclidean action is given by

$$A[\phi] = \int d^d x \left[\frac{1}{2} (\partial \phi)^2 + U(\phi^2) \right],$$

$$U(\phi^2) = g_2 \phi^2 + g_4 \phi^4 + \cdots. \tag{1}$$

The potential $U(\phi^2)$ is arbitrary, and not necessarily polynomial. For simplicity we work with a one-component real field here; but extension to a multicomponent field with $O(N)$ symmetry is straightforward, and we shall quote results for that case. There is a high-momentum cutoff Λ. To make calculations feasible, we use a sharp cutoff, which also proves to be a limitation, for it leads to ambiguous nonlocal interactions. We only report results that are believed to be independent of the cutoff function.

Scalar fields are used in the Higgs sector of the standard model, where it is customary to assume that $U(\phi^2)$ is quartic in ϕ. It turns out that such a choice leads to "triviality," in that the renormalized value of g_4 vanishes in the limit $\Lambda \to \infty$, and one is left with a free field. This startling result was implicit in the work of Larkin and Khumel'nitskii [2], and demonstrated by Wilson [3]. It has been verified in a number of independent Monte Carlo simulations [4–8]. There are proposals on how to deal with this awkward situation.

(a) Physical quantities are insensitive to the value of the cutoff, because the approach to the free-field limit proceeds with logarithmic slowness [2]. Thus, one can keep the cutoff finite as a parameter of the model. Considerations of self-consistency [9] impose an upper bound, estimated to be 600 GeV [8], on the Higgs boson mass.

(b) Even in the free-field limit, the theory is not entirely trivial. The field can have a nonvanishing vacuum expectation, as shown in Monte Carlo simulations [6]. Thus it can still be used as a phenomenological method to generate particle masses.

These alternatives are not completely satisfactory, for they do not take the field theory seriously. The purpose of this paper is to show that escape routes do exist in the framework of renormalized quantum field theory. In the rest of this section, we describe our approach to the problem, and summarize the results.

Common belief holds that only ϕ^4 theories are renormalizable, in the sense that higher powers in the potential will give Feynman graphs requiring an infinite number of subtraction constants. This is true if the higher coupling constants, which generally have dimensions, set independent scales. From a physical standpoint, however, these scales contain information about the system at momenta higher than Λ, of which nothing is supposedly known. Accordingly, we shall assume that Λ is the only intrinsic scale in the problem. This means that all coupling constants should be scaled by appropriate powers of the cutoff:

$$g_\alpha = u_\alpha \Lambda^{\alpha + d - \alpha d/2}, \tag{2}$$

where the u_α are dimensionless parameters. These factors of Λ supply extra convergence to Feynman graphs, rendering them renormalizable in the usual sense [10]. It can be shown that the S matrix of the theory in $d=4$ is the same as that of an effective ϕ^4 theory, whose effective coupling is a function of the u_α [1]. However, the RG behavior of the effective coupling is not the same as that of a ϕ^4 theory, for it depends on the RG flow of the u_α, which can only be obtained from the original theory.

Renormalization relates the coupling constants at different momentum scales. In Wilson's formulation [3], the relation is found through a RG transformation that represents a coarse-graining process, eliminating the degrees of freedom with momenta between Λ and Λ/b, and effectively lowering the cutoff by a factor b. The new action should have the same form as the old, except that the "bare" couplings u_α are replaced by the "renormalized" ones u'_α. Making an in-

finitesimal RG transformation in the neighborhood of $b=1$ yields differential equations for u_α, the RG equations. They generate RG trajectories in the parameter space spanned by the u_α. The flow along a trajectory always proceeds in the coarse-graining direction, i.e., direction of increasing length scale. If A and B are two points on a trajectory, with the sense of flow from A to B, then A corresponds to a bare system, and B a renormalized system.

It should be emphasized that the cutoff Λ does not appear in $A[\phi]$ explicitly, for we can set $\Lambda=1$ by choosing appropriate units. Its value is reflected solely in the values of the coupling constants u_α. Thus, the RG equations give the tangent vector to a trajectory at an arbitrary point.

The actual value of the cutoff can be deduced only by computing some physical quantity, such as the correlation length. Thus, the only way to approach the limit $\Lambda \to \infty$ is to go to some point in the parameter space at which the correlation length is infinite. Since the length scale increases under a RG transformation, such a point must be a fixed point, where the system is invariant under RG transformations.

If a trajectory flows into a fixed point (in the coarse-graining direction) then, to systems lying on that trajectory, the fixed point is infrared (IR), representing the low-energy limit of the theory. If a trajectory flows out of a fixed point, then to systems on this trajectory the fixed point is ultraviolet (UV), corresponding to the high-energy limit of the theory.

Although we are free to choose a bare action, the renormalized action is determined by the RG transformation, and is not under our control. For example, if we start with a ϕ^4 theory at some value of the cutoff, a RG transformation may generate ϕ^6 and other couplings. Only at a fixed point are the couplings determined. When we approach a fixed point along a trajectory, in the coarse-graining sense, some couplings not destined to be in the fixed-point action will tend to zero, and these are called "irrelevant" couplings. Conversely, when we go away from a fixed point in a coarse-graining sense, some couplings that were infinitesimally small will grow, and these are termed "relevant." Triviality comes from the existence of an IR fixed point at zero couplings, the Gaussian fixed point. By examining all possible trajectories in the neighborhood of the Gaussian fixed point, we find that, although the fixed point is IR in theories with polynomial potentials, it is UV to a class of potentials with exponential asymptotic behavior.

To insure that the parameter space is closed under RG transformations, we have to consider an arbitrary action, which should include derivative couplings as well as nonlocal interactions. A derivative coupling refers to terms containing a derivative of the field not of the form of the kinetic term $\int d^d x (\partial \phi)^2$, as, for example,

$$\int d^d x (\partial^2 \phi)^2. \tag{3}$$

A nonlocal term involves fields or derivatives at different space-time points, as, for example,

$$\int d^d x d^d y\, \phi(x) K(x-y) \phi(y). \tag{4}$$

Actually, the action with a momentum cutoff is nonlocal within a spatial distance of order Λ^{-1}. By "nonlocal terms,"

we specifically refer to those for which the range of nonlocality is large compared to Λ^{-1}.

The exact RG equations for the most general case have been obtained by Wegner and Houghton [11], and we shall review the derivation. This remarkable calculation is made possible by the simplicity of the sharp cutoff. The equations show that RG transformations do not induce derivative couplings if none were present from the start. On the other hand, nonlocal terms are always generated. Some of these have infinite range, being of the form $V^{-1}[\int d^d x\, \phi(x)]^2$, where V is the space-time volume. Though consistent with the fact that the action is $O(V)$, such a term is indeterminate in the limit $V \to \infty$. The ambiguity can be ascribe to the infinitesimal RG transformation made with a sharp momentum cutoff. It would disappear if gentle cutoff functions were used, or if the momentum-shell integration had extended over a finite instead of an infinitesimal shell. Both of these alternatives, however, make the problem intractable.

Fortunately, the ambiguous nonlocal terms are second order in the bare couplings. We can therefore neglect them in a linear approximation about the Gaussian fixed point, and the action (1) becomes closed under RG in this approximation. We study the eigenvalue problem based on the RG matrix, which should be insensitive to the form of the cutoff. It tell us about the characteristics of various "principal axes" in parameter space at the origin. Our main results are as follows.

(a) There exist trajectories flowing into the Gaussian fixed point, as well as flowing out of it. That is, the Gaussian fixed point is IR with respect to some trajectories, and UV with respect to others.

(b) For all theories with polynomial potentials, the Gaussian fixed point is IR. These theories are consequently trivial. A similar result was obtained earlier by Hasenfratz and Hasenfratz [12].

(c) For a class of nonpolynomial potentials, the Gaussian fixed point is UV. For $d > 2$, potentials in this class behave like $U(\phi) \sim \exp[c(d-2)\phi^2]$ for large ϕ, where c is a constant. Theories with such potentials are nontrivial and asymptotically free. Some of the potentials exhibit spontaneous symmetry breaking.

In summary, we can say that in a sufficiently small neighborhood of the Gaussian fixed point, conventional scalar theories with polynomial interactions are trivial, and that certain models with exponential potentials are nontrivial. For conventional potentials, the road to oblivion is clear and inescapable, because with each RG step we are closer to the fixed point, and the linear approximation improves. For the nontrivial models, on the other hand, the escape route is clouded, since RG steps tend to take us out of the linear region into unknown territory.

II. RENORMALIZATION PROCEDURE

We shall begin with the most general scalar field theory, with arbitrary derivative and nonlocal couplings, and choose units such that the cutoff momentum is unity:

$$\Lambda = 1. \tag{5}$$

We enclose the system in a periodic hypercube of volume V, and define the Fourier transform of the field by

$$\phi_k = V^{-1/2} \int d^d x\, e^{-ik\cdot x} \phi(x) \qquad (6)$$

with $\phi_k^* = \phi_{-k}$. Eventually, we take the limit $V \to \infty$, in which the Fourier component is replaced by the continuum version $\phi(k) = V^{-1/2}\phi_k$. For illustration, the action (1) can be written as

$$A[\phi] = \frac{1}{2}\sum_{|k|<1}(k^2+r)\phi_k\phi_{-k} + \frac{u_4}{V}\sum_{|k_i|<1}\delta(k_1+\cdots$$
$$+ k_4)\phi_{k_1}\cdots\phi_{k_4} + \cdots, \qquad (7)$$

where $\delta(k)$ is the Kronecker delta δ_{k0}.

To generalize the action, all we have to do is to replace u_α by an arbitrary function $u_\alpha(k_1,\ldots,k_\alpha)$, which we abbreviate as $u_\alpha(k)$. Thus, our starting point is the action

$$A[\phi] = \sum_{\alpha=2}^{\infty} V^{1-\alpha/2}\sum_{|k_i|<1}\delta(k)u_\alpha(k)\phi_{k_1}\cdots\phi_{k_\alpha}, \qquad (8)$$

where $\delta(k)$ is an abbreviation for $\delta(k_1+\cdots+k_\alpha)$. Without loss of generality, we may assume that $u_\alpha(k)$ is a symmetric function of its arguments. To fix the normalization of the field, we normalize $u_2(k_1,k_2)$ as follows:

$$v(k) \equiv 2u_2(k,-k) = k^2 + r + c_4 k^4 + c_6 k^6 + \cdots . \qquad (9)$$

The generalized kinetic term is

$$A_2[\phi] \equiv \frac{1}{2}\sum_{|k|<1} v(k)\phi_k\phi_{-k}$$
$$= \frac{1}{2}\int_{|k|<1}\frac{d^d k}{(2\pi)^d}v(k)\phi(k)\phi(-k) \qquad (10)$$

from which we can see that $v(k) = v(-k)$ is the inverse propagator for Feynman graphs.

Wilson's RG transformation [3,13] is defined in terms of the partition function

$$Z = \int D\phi\, e^{-A[\phi]}. \qquad (11)$$

The object is to eliminate the Fourier components with momentum magnitudes between 1 and $1/b$, without changing the partition function. We decompose the field into a "slow" part S_k and a "fast" part f_k:

$$\phi_k = S_k + f_k, \qquad (12)$$

where

$$S_k = 0 \quad \text{unless } |k| < 1/b,$$
$$f_k = 0 \quad \text{unless } 1/b \leq |k| \leq 1. \qquad (13)$$

Let us split off the kinetic term in the action by writing

$$A[\phi] = A_2[\phi] + A_I[\phi], \qquad (14)$$

where A_I is the "interaction" part. Since $S_k f_{-k} = 0$, as their domains do not overlap, $A_2[S+f]$ is additive:

$$A_2[S+f] = A_2[S] + A_2[f]. \qquad (15)$$

We now write

$$Z = \int DS\int Df\, e^{-A_2[S]-A_2[f]-A_I[S+f]}$$
$$= N\int DS\, e^{-A_2[S]}\langle e^{-A_I[S+f]}\rangle_f$$
$$\equiv N\int DS\, e^{-\tilde{A}[S]}, \qquad (16)$$

where N is a constant, and $\langle O\rangle_f$ denotes averaging over f with weight $\exp\{-A_2[f]\}$. The new action

$$\tilde{A}[S] \equiv A_2[S] - \ln\langle e^{-A_I[S+f]}\rangle_f \qquad (17)$$

contains only the slow fields, with the cutoff lowered to $1/b$. Writing out the first few terms, we have

$$\tilde{A}[S] = \frac{1}{2}\sum_{|k|<1/b}[zk^2+r_1+\cdots]S_k S_{-k} + \cdots. \qquad (18)$$

The parameters z, r_1, etc., are proportional b^{-y}, where y is a characteristic index.

To make comparison with the original action, we must restore the cutoff to 1, and normalize the field according to the convention (9). The cutoff can be restored by changing the momentum integration variable to

$$k' = bk. \qquad (19)$$

To restore the normalization, we transform the field to

$$\phi'_{k'} \equiv S_{k'/b}b^{-1-d/2-\eta/2}, \qquad (20)$$

where η is the index of z in (18), i.e., $z = b^{-\eta}$. The partition function can now be put in the form

$$Z = N\int D\phi'\, e^{-A'[\phi']}, \qquad (21)$$

where

$$A'[\phi'] \equiv \tilde{A}[S]. \qquad (22)$$

The action $A'[\phi']$ should have the same form as $A[\phi]$ in (8), except that the bare coupling function $u_\alpha(k)$ is replaced by the renormalized coupling function $u'_\alpha(k')$, which is of course a function of b.

The RG transformation can be formulated in terms of Feynman graphs. By expanding $\exp\{-A_I[S+f]\}$ in powers of f, we can obtain $\tilde{A}[S]$ as a sum of connected Feynman graphs, in which all external momenta are "slow," while all internal momenta are "fast." That is, an external line is associated with S_k; an internal line is associated with f_k, and gives the propagator $1/v(k)$ after functional integration weighted with $A_2[f]$. A vertex represents a momentum-dependent factor $u_\alpha(k)$.

III. RENORMALIZATION-GROUP EQUATIONS

We shall carry out an infinitesimal RG transformation at the cutoff momentum. The fast momenta are contained in a shell σ in momentum space:

$$\sigma = \{k \mid e^{-t} < |k| < 1\}, \qquad (23)$$

where we have put $b = e^t$. Calculating to first order in t will yield equations for du_α/dt, which are the RG equations. To this order, all internal momenta in Feynman graphs are integrated over a shell of infinitesimal thickness t, just below the surface of the unit sphere. Each independent integration therefore yields $O(t)$. This circumstance leads to the following simplifications: (a) To first order, we need to keep only tree and one-hoop graphs; (b) a one-loop graph with two or more vertices must have two or more propagators, and is superficially $O(t^2)$. But it is $O(t)$ when the total momentum
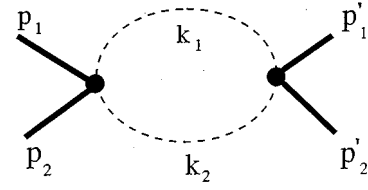

FIG. 1. A one-loop graph. The internal lines correspond to high-momentum components to be eliminated in the RG transformation. The external lines represent low-momentum components left untouched.

of the external lines emerging from any one vertex is zero. An equivalent statement is that all the internal lines should carry exactly the same loop momentum.

To show (b), consider the simple one-loop graph in Fig. 1, which is proportional to

$$\int_{k_1 \in \sigma} d^d k_1 \int_{k_2 \in \sigma} d^d k_2 \, \delta^d(p_1 + p_2 - k_1 - k_2) \delta^d(p_1' + p_2' - k_1 - k_2) \frac{u_4(p_1, p_2, k_1, k_2) u_4(k_1, k_2, p_1', p_2')}{v(k_1) v(k_2)}. \qquad (24)$$

This is $O(t^2)$ in general; but an exception occurs when $p_1 + p_2 = 0$. The integrations are then constraint by $\delta(k_1 + k_2)$, and the graph becomes $O(t)$. This argument applies to any vertex of a graph, even if it is a subgraph. Thus, in order for a one-loop graph to be $O(t)$ instead of $O(t^2)$, the total external momentum emerging from any one vertex must be zero.

Wegner and Houghton [11] sum the tree and one-loop graphs by means of a functional method, as follows. First, expand the action in powers of f:

$$A[S + f] = A[S] + \sum_{k \in \sigma} P_k f_k + \frac{1}{2} \sum_{k \in \sigma} Q_k f_k f_{-k} + \cdots, \qquad (25)$$

where

$$P_k = \left[\frac{\partial A[\phi]}{\partial f_k} \right]_{f=0},$$

$$Q_k = \left[\frac{\partial^2 A[\phi]}{\partial f_k \partial f_{-k}} \right]_{f=0}. \qquad (26)$$

The terms represented by the dots in (25) may be omitted because they do not contribute to $O(t)$. In the second term in (25), we have a single k sum instead of a sum over two independent k's, because of the restriction to a single-loop momentum. This circumstance makes it possible to calculate the functional integral over f to obtain

$$Z = N \int DS e^{-\tilde{A}[S]}, \qquad (27)$$

where

$$\tilde{A}[S] = A[S] + t B[S],$$

$$B[S] = \frac{1}{2t} \sum_{k \in \sigma} \left[\ln Q_k - \frac{|P_k|^2}{Q_k} \right]. \qquad (28)$$

The quantity Q_k arises from one-loop graphs, while $|P(k)|^2$ arises from tree graphs.

We now transform to the rescaled variables (19) and (20). To first order in t it is only necessary to do so in the first term of $\tilde{A}[S]$, since the second term is $O(t)$. We obtain, after a straightforward calculation,

$$Z = N \int D\phi' e^{-A'[\phi']}, \qquad (29)$$

where

$$A'[\phi'] = A[\phi'] + t\{B[\phi'] + C[\phi']\},$$

$$C[\phi] = \frac{1}{t} \sum_{\alpha=2}^{\infty} \sum_{|k_i| < 1} \delta(k) [\phi_{k_1} \cdots \phi_{k_\alpha}]$$

$$\times \left[d + \frac{\alpha}{2}(2 - \eta - d) - \sum_i k_i \frac{\partial}{\partial k_i} \right] u_\alpha(k). \qquad (30)$$

This is the result of Wegner and Houghton [11].

By expanding $B[\phi]$ and $C[\phi]$ in powers of ϕ we can express the new action $A'[\phi]$ in the form (8), and read off the new coupling functions $u'_\alpha(k)$. The first-order change of the action can be written in the form

$$A'[\phi] - A[\phi] = t \sum_{\alpha=2}^{\infty} \sum_{|k_i| < 1} \delta(k) \beta_\alpha(k) \phi(k_1) \cdots \phi(k_\alpha), \qquad (31)$$

where

$$\beta_\alpha(k) \equiv u'_\alpha(k) - u_\alpha(k). \qquad (32)$$

Note that, to RG transformations, $u_\alpha(k)$ is a function of t only, with α and k acting as labels for the type of coupling. Thus we can write

$$\frac{du_\alpha(k)}{dt} = \beta_\alpha(k), \qquad (33)$$

which is an exact RG equation. The function $\beta_\alpha(k)$ depends on the $u_\alpha(k)$, but not on t explicitly. This equation therefore gives the tangent vector to the trajectory at an arbitrary point in parameter space. Although this point is identified as $t=0$ in the derivation, we can shift the origin of t at will, because the equation is invariant under a translation in t.

Since the coupling function u_α obeys a differential equation in t, we can trace its evolution both forward and backward in t. This might seem puzzling, since the RG transformation as defined appears to be irreversible. What renders it reversible is the fact that one and only one trajectory passes through any given point in the parameter space, except at a fixed point.

At this point, we can easily see that no derivative couplings are induced if none were present initially. Terms involving derivatives are generated by the momentum-dependent terms in $B[\phi]+C[\phi]$. As we can see from (28) and (30), such terms can occur only in $C[\phi]$, through the expression

$$\sum_i k_i \frac{\partial}{\partial k_i} u_\alpha(k). \qquad (34)$$

If only nonderivative local couplings were present at the start, then the above vanishes except for $\alpha=2$, for which it gives a term proportional to k^2. Therefore no derivative couplings are generated. This also shows that a massless free field, which corresponds to the origin of the parameter space, is invariant under RG. The origin is therefore a fixed point—
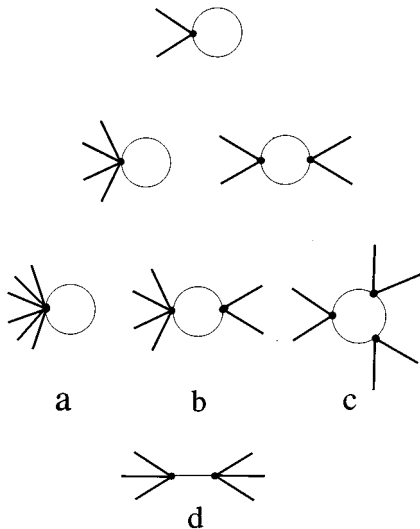

FIG. 2. Contributions to renormalized n-field couplings for $n=2,4,6$.

the Gaussian fixed point. It can be seen that if there were no odd powers of the field initially, then none will be generated. The reason is that Q_k in (28) is even in the field.

Graphs with n external lines contribute to u'_n, and are shown in Fig. 2 for $n=2,4,6$. In any one-loop graph, the j external lines emerging from any vertex give rise to a factor $\int d^d x \phi^j(x)$, since they have total momentum zero. Thus, a one-loop graph is generally proportional to a product of such factors. For example, Figs. 2(a)–2(c) lead to the following contributions to the action $A'[\phi]$:

$$G_a = u_8 \int d^d x \phi^6(x),$$

$$G_b = \frac{u_4 u_6}{V} \int d^d x \phi^4(x) \int d^d y \phi^2(y),$$

$$G_c = \frac{u_4^3}{V^2} \left[\int d^d x \phi^2(x) \right]^3. \qquad (35)$$

The first contribution, coming from the "diamond ring" graph with only one vertex, gives a local interaction. All others give uncorrelated products of the fields, which correspond to nonlocal interactions of infinite range. The powers of the space-time volume V in front of these expressions arise from the fact that the action should be $O(V)$. All these uncorrelated nonlocal contributions are indeterminate in the infinite-volume limit. The ambiguity clearly arises from the infinitesimal RG step implemented with a sharp momentum cutoff. The products of field would have been correlated, if a gentle cutoff function had been used, or if the internal lines were integrated over a finite instead of infinitesimal shell. However, the nonlocal terms are second order in the bare couplings, and can be neglected in a linear approximation about the Gaussian fixed point.

The tree graph d in Fig. 2 contributes to $A'[\phi]$ a term of the form

$$G_d = u_4^2 \sum_{|k_i|<1} \delta(k_1 + \cdots + k_6)$$

$$\times \delta(|k_1+k_2+k_3|-1)\phi_{k_1} \cdots \phi_{k_6} \qquad (36)$$

which gives rise to a correlated nonlocal interaction. As shown in [11] this term gives rise to the "nontrivial fixed point" in $d=4-\epsilon$ ($\epsilon \to 0$). But, since it is second order in the couplings, we shall ignore it here.

In view of the critical examination above, those results in [1,12] pertaining to nonlinear terms in the RG equation must be taken with reservation.

IV. LINEARIZED RG EQUATIONS

In the linear approximation, the action (1) is closed under RG, and we have a well-defined system. To obtain the linearized RG equations, we need $B[\phi]$ defined in (28), in which the term $|P_k|^2$ can be neglected. A straightforward calculation gives

$$Q_k = 1 + r + \tilde{Q},$$

$$\tilde{Q} = \sum_{\alpha=2}^{\infty} \alpha(\alpha+1) V^{1-\alpha/2} u_{\alpha+2}$$

$$\times \sum_{|k_i|<1} \delta(k_1 + \cdots + k_\alpha) \phi_{k_1} \cdots \phi_{k_\alpha}, \qquad (37)$$

which is a sum over "diamond rings," and is independent of k. To first order in the u_α, we have

$$B[\phi] = \frac{1}{2t} V_\sigma \tilde{Q}, \qquad (38)$$

where V_σ is the volume of the thin momentum shell σ.

We quote the linearized RG equations generalized to an N-component field $\phi_i(x)$ ($i=1,\ldots,N$) with $O(N)$ internal symmetry:

$$\frac{du_{2n}}{dt} = (2n+d-nd)u_{2n} + S_d(n+1)(2n+N)u_{2n+2}$$

$$(n=1,2,\ldots,\infty), \quad (39)$$

where S_d is the surface area of a unit d sphere divided by $(2\pi)^d$:

$$S_d = \frac{2^{1-d}\pi^{-d/2}}{\Gamma(d/2)},$$

$$S_4 = \frac{1}{8\pi^2}. \qquad (40)$$

Let ψ be the column matrix whose elements are u_{2n}. We can write (39) in the form

$$\frac{d\psi}{dt} = M\psi, \qquad (41)$$

where M is a matrix. Consider now the eigenvalue problem

$$M\psi = \lambda\psi. \qquad (42)$$

The eigenvectors ψ correspond to "principal axes" in the parameter space, along which we have the behavior $d\psi/dt = \lambda\psi$, or

$$\psi(t) = \psi(t_0)e^{\lambda(t-t_0)}. \qquad (43)$$

The origin t_0 is arbitrary, except that it should be such that ψ is small; but it should not correspond to the Gaussian fixed point, where $\psi \equiv 0$.

The eigenvalue λ characterizes the trajectory tangent to the corresponding principal axis at the Gaussian fixed point.

(a) If $\lambda < 0$, then $\psi \to 0$ as $t \to \infty$. The coupling constants are said to be "irrelevant." Under coarse graining, they tend to the Gaussian fixed point, or triviality. On such a trajectory, the Gaussian fixed point is IR.

(b) If $\lambda > 0$, then ψ grows with t. The coupling constants are said to be "relevant." Under coarse graining, they tend to go away from Gaussian fixed point. On such a trajectory the Gaussian fixed point is UV, and the theory is nontrivial. The

trajectory is specified by some initial condition at an arbitrary point $t=t_0$, and it flows away from the Gaussian fixed point. The latter can be reached by letting $t \to -\infty$, in which limit the couplings vanish. This is asymptotic freedom.

(c) The case $\lambda=0$ corresponds to "marginal" coupling constants. In this case, we have to go beyond the linear approximation in order to determine the true behavior.

Using (39), we can put the eigenvalue equation (42) in the form

$$u_{2n+2} = \frac{n(d-2)-d+\lambda}{S_d(n+1)(2n+N)}u_{2n} \quad (n=1,2,\ldots,\infty) \quad (44)$$

which is a recursion relation starting with $u_2 = r/2$. To solve it in terms of known functions, it is convenient to introduce a parameter a by writing the eigenvalue in the form

$$\lambda = 2 + (d-2)a. \qquad (45)$$

The recursion relation can then be put in the form

$$u_{2n+2} = \frac{(d-2)(a+n-1)}{2S_d(n+1)(n+N/2)}u_{2n} \qquad (46)$$

whose solution is

$$u_{2n} = \frac{r}{2}\left(\frac{d-2}{2S_d}\right)^{n-1}$$

$$\times \frac{a(a+1)\cdots(a+n-2)}{n!(n-1+N/2)(n-2+N/2)\cdots(1+N/2)}. \qquad (47)$$

The potential with these coupling constants is referred to as the "eigenpotential." Using the abbreviation

$$z = \frac{(d-2)\phi^2(x)}{2S_d}, \qquad (48)$$

where $\phi^2 = \Sigma_i \phi_i^2$, we have

$$U_a[\phi^2(x)] \equiv \sum_{n=1}^{\infty} u_{2n}\phi^{2n}(x)$$

$$= r\frac{NS_d}{2(a-1)(d-2)}[M(a-1,N/2,z)-1], \qquad (49)$$

where $M(a,b,z)$ is the Kummer function [14]:

$$M(a,b,z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots$$

$$= \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 dt e^{zt}t^{a-1}(1-t)^{b-a-1}. \qquad (50)$$

If a is a negative integer, the power series breaks off to become a polynomial of degree $|a|$. Otherwise, its asymptotic behavior for large z is given by

$$M(a,b,z) \approx \frac{\Gamma(b)z^{a-b}e^z}{\Gamma(a)}[1+O(z^{-1})]. \qquad (51)$$

The eigenpotential $U_a(\phi^2)$ describes a field theory lying on a trajectory tangent to a particular principal axis with respect to the Gaussian fixed point. The principal axis is identified only through the eigenvalue parameter a.

For a polynomial potential of even degree $2K$, then, we have $a = -2K$. The corresponding eigenvalues are

$$\lambda = 2[1-(d-2)K] \quad (K=1,2,\ldots), \qquad (52)$$

which is negative for $d=4$. In $d=3$ it is negative except for the marginal case of $K=1$; but that corresponds to a free theory. Therefore, in $d>2$, all polynomial even potentials lead to triviality.

For $d=2$, the linear approximation breaks down completely. The reason is undoubtedly the formation of vortices that lead to the Kosterlitz-Thouless phase transition [15]. It would be very interesting to discover vortices within the present framework, for in the existing literature they are simply put in by hand. We shall not pursue this topic here, and will assume $d>2$ from now on.

V. NONTRIVIALITY AND ASYMPTOTIC FREEDOM

Nontrivial theories correspond to positive eigenvalues $\lambda>0$, which means that

$$a > -\frac{2}{d-2}. \qquad (53)$$

They correspond to nonpolynomial potentials with the following asymptotic behavior for large ϕ:

$$U(\phi^2) \sim \exp\left[\frac{(d-2)\phi^2}{2S_d}\right]. \qquad (54)$$

Nothing in canonical field theory rules out such a potential.

Sufficiently close to the Gaussian fixed point, the potential is proportional to r, which evolves in t according to

$$r(t) = r(t_0)e^{\lambda(t-t_0)} = Ce^{\lambda t} \qquad (55)$$

with $C = r(t_0)\exp(-t_0)$. This is a running coupling constant, with a given renormalized value $r(t_0)$ at the reference point t_0. The theory is nontrivial because the potential does not tend to zero in the low-momentum limit. Instead, we have asymptotic freedom, corresponding to the fact that the potential vanishes in the limit $t \to -\infty$, which corresponds to infinite momentum.

In order to have spontaneous symmetry breaking on the semiclassical level, the eigenpotential must have at least one minimum in ϕ. The power-series expansion for the eigenpotential reads

$$U_a(\phi^2) = \frac{rS_d}{(d-2)}\left[z + \frac{az^2}{(1+N/2)2!} + \frac{a(a+1)z^3}{(1+N/2)(2+N/2)3!}\right.$$

$$\left. + \cdots\right]. \qquad (56)$$

A sufficient condition is that $U'(0)<0$, and $U>0$ for large z. The first is satisfied by choosing $r<0$. Asymptotically U is proportional to $r[(a-1)\Gamma(a)]^{-1}$, the rest of the factors being positive. Thus we must have $(a-1)\Gamma(a)<0$, which is equivalent to $\Gamma(a-1)<0$. Using the formula $\Gamma(a)\Gamma(-a) = \pi/\sin(\pi a)$, and the fact that $\Gamma(a)$ is positive for $a>0$, we find that a must be in one of the open intervals $(0,-1),(-2,-3)$, etc. For a nontrivial theory, we have $\lambda>0$, or $2+(d-2)a>0$. Combining these requirements, we obtain the sufficient condition

$$-1 < a < 0. \qquad (57)$$

A family of eigenpotentials for this range of a, and $d=N=4$, is plotted in Fig. 3.

The eigenpotential U_a corresponds to a theory that lies on a trajectory tangent to a principal axis. Generally, we can consider a theory on an arbitrary trajectory, which is represented near the Gaussian fixed point by a linear superposition of the eigenpotentials. This gives us considerable freedom in choosing potentials.

The asymptotically free theory may be useful for models of the inflationary universe [16], for it offers a nontrivial quantum field theory with spontaneous symmetry breaking. From a philosophical point of view, it seems more sensible to have a cosmological potential that was zero at the Big Bang and grow at decreasing energies, rather than the conventional polynomial potential, which would have the opposite behavior if taken seriously. For such applications, one needs a potential whose ϕ^2 term is very small, of order 10^{-12} [17]. This turns out to be very natural in terms of our eigenpotentials $U_a(\phi^2)$. As we can see from the power-series expansion (56), the ϕ^2 term is independent of a. Therefore the difference of any two eigenpotentials
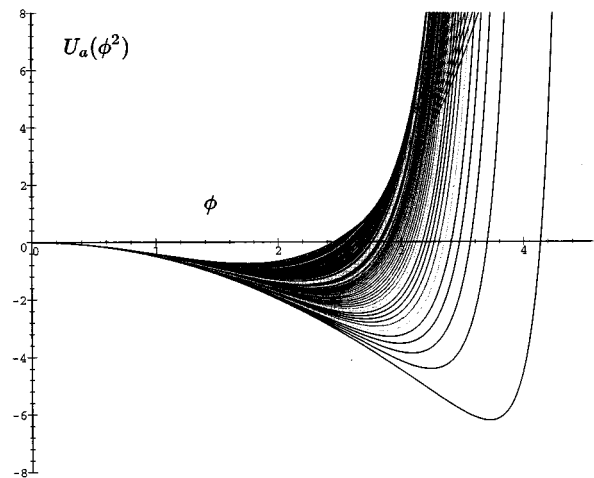

FIG. 3. Eigenpotentials $U_a(\phi^2)$ as functions of $\phi \equiv \sqrt{\sum_{i=1}^{N}\phi_i^2}$, for $d=N=4$, in units in which the momentum cutoff is unity. The ordinate is in arbitrary units. From top to bottom, they correspond, respectively, to values of the eigenvalue parameter a uniformly spaced from -0.999 to -0.001. All of the potentials behave like e^{ϕ^2} for large ϕ, and lead to theories with asymptotic freedom. The limiting case $a=-1$ represents a ϕ^4 potential, which gives a trivial theory.

$$V(\phi^2) = U_a(\phi^2) - U_{a'}(\phi^2)$$

$$= \frac{r(a-a')S_d}{(1+N/2)(d-2)}$$

$$\times \left[\frac{z^2}{2!} + \frac{(a-a'+1)z^3}{(2+N/2)3!} + \cdots \right] \quad (58)$$

has no ϕ^2 term. Since this is a linear approximation, it means that the ϕ^2 term is $O(r^2)$. By taking $r < 0$ and $a > a'$, we make the potential go negative for small ϕ^2. At large ϕ^2 it must turn positive, because the curves of the U_a with different a's intersect, as we can see in Fig. 3. Therefore the potential has a negative minimum.

VI. CONCLUSION AND OUTLOOK

We have shown that, near the Gaussian fixed point, all scalar theories are trivial free fields in the low-energy limit, except for a specific class with exponentially rising potentials, which are nontrivial at low energies, but become free in the high-energy limit.

The renormalized coupling constants used in this paper are not the same as the conventional ones in particle physics; the latter are defined in terms of physical scattering amplitudes, which contain extra momentum scales. The conventional renormalized coupling constants may be calculated by integrating the RG equations along a trajectory. We plan to address this topic in a separate paper.

The low-energy behavior of the asymptotically free theories lies beyond the capability of the present formulation, because the sharp momentum cutoff used here introduces ambiguities. It is an important problem to implement Wilson's renormalization program with a gentle cutoff function, and extract results independent of the cutoff function.

An interesting extension of the present work would be to make similar analyses of gauge fields and spinor fields. We hope the present paper will stimulate interest in this direction.

Note added in proof. G. Immirzi (private communication) has pointed out that Eq. (44) for $d = 2$ leads to an eigenpotential corresponding to the sine-Gordon equation. V. Periwal [Princeton University report (unpublished)] has derived our results using Polchinski's form of the RG.

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