Divergence problem in the black hole brick-wall model

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(Received 24 March 1995)

In this work we review, in the framework of the so-called brick-wall model, the divergence problem arising in the one-loop calculations of various thermodynamical quantities, such as entropy, internal energy, and heat capacity. Particularly we find that, if one imposes that entanglement entropy is equal to the Bekenstein-Hawking one, the model gives problematic results. Then a proposal of a solution to the divergence problem is made following the zeroth law of black hole mechanics.

PACS number(s): 04.70.Dy, 04.60.Kz, 04.62.+v, 98.80.Hw

I. INTRODUCTION

Recently it was proposed to explain the dynamical origin of black hole entropy by identifying it with entanglement entropy [1].

One of the puzzles related with this kind of interpretation is the so-called divergence problem. Entanglement entropy is not finite at the black hole horizon [1,2]; in order to compute that, it is necessary to introduce a kind of regularization by imposing an ultraviolet cutoff [1,2] or a renormalization of gravitational coupling constant [3] and of constants related to second-order curvature terms [4]. Indeed the first idea of entanglement entropy was implicitly proposed by 't Hooft in 1985 [5] applied to his "brick-wall model".[1] In a certain sense cutoff-dependent models [1,2] are up to date versions of the former. One of the problems 't Hooft proposed in this seminal work was the divergence of not only entropy but also of quantum matter contribution to the internal energy of the black hole, which has to be regularized by using the same cutoff one has to introduce for entropy. He found that, fixing the cutoff in order to obtain $S_{\text{ent}} = S_{\text{Bek-Haw}} = A/4$, one obtains $U = \frac{3}{8}M$. So the matter contribution to the internal energy appeared to be a very consistent fraction of the black hole mass M. As 't Hooft underlined, this is a signal for a strong back reaction effect, not a good aim for a model based on the semiclassical (negligible back reaction) approximation.

We shall show that the same problem is present in the Barvinsky-Frolov-Zelnikov (BFZ) model [2] and that one also finds a surprising behavior of heat capacity.

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[1] An extensive study of the divergence problem for generic value of the space-time dimension is made in [6].

II. ENTANGLEMENT ENTROPY AND BFZ MODEL

In the BFZ model entropy is computed from the global vacuum density matrix by tracing over the degrees of freedom of matter outside the black hole. In so doing one obtains a mixed state density matrix for matter inside the black hole.

BFZ define the global wave function of the black hole as

$$\Psi = \exp(\Gamma/2)\langle\phi_-|\exp(-\beta\hat{H}/2)|\phi_+\rangle \tag{2.1}$$

and

$$\hat{\rho} = |\Psi\rangle\langle\Psi| \tag{2.2}$$

the related density matrix. Here $|\phi_+\rangle$ ($|\phi_-\rangle$) are the external (internal) states of matter (a massless scalar field for simplicity) on the black hole fixed background. Tracing over $|\phi_+\rangle$ gives the internal density matrix:

$$\begin{aligned}\rho_{\text{int}}(\phi'_-,\phi_-) &= \langle\phi'_-|\hat{\rho}|\phi_-\rangle \\ &= \int \mathcal{D}\phi_+ \Psi^*(\phi'_-,\phi_+)\Psi(\phi_-,\phi_+) \\ &= \exp(\Gamma_\beta)\langle\phi'_-|\exp(-\beta\hat{H})|\phi_-\rangle\ .\end{aligned} \tag{2.3}$$

The entanglement entropy associated with this reduced density matrix is

$$S_{\text{ent}} = -\text{Tr}_{\text{int}}(\rho_{\text{int}}\ln\rho_{\text{int}})\ .$$

Here Γ_β is a normalization factor fixed in order to obtain trρ=1, but it also corresponds to the one-loop effective action:

$$\begin{aligned}\Gamma_\beta &= -\ln\left[\int \mathcal{D}\phi_-\langle\phi_-|\exp(-\beta\hat{H})|\phi_-\rangle\right] \\ &= -\frac{1}{2}\ln\det\left[\frac{\delta(x-y)}{2(\cosh\beta\omega-1)}\right]\ ,\end{aligned} \tag{2.4}$$

where $\hat{\omega}$ is the operator associated with the frequency of field's modes. BFZ calculate the entanglement entropy as the trace over the internal modes of $-\rho_{\rm int}\ln\rho_{\rm int}$, so their calculation is relative to the internal degrees of freedom. Instead, in the common definition of entanglement entropy, one usually refers to the trace over the external degrees of freedom of $-\rho_{\rm ext}\ln\rho_{\rm ext}$. In the following, we shall show that, given the symmetry existing in BFZ study between internal and external variables, the two definitions of entanglement entropy coincide. Besides, given the relation existing between thermofield dynamics and BFZ,[2] we can use the "external" quantities, as defined below, to calculate free energy and internal energy relative to the external field.

We recall that BFZ calculation is made in WKB approximation.

From the definitions we have

$$\begin{aligned}
\rho_{\rm ext} &= {\rm Tr}_-|\Psi\rangle\langle\Psi| = \int \mathcal{D}\phi_- \Psi^*(\phi'_+,\phi_-)\Psi(\phi_-,\phi_+)\\
&= \int \mathcal{D}\phi_- \langle\phi'_+|\exp\left(-\frac{\beta\hat{H}}{2}\right)|\phi_-\rangle\langle\phi_-|\exp\left(-\frac{\beta\hat{H}}{2}\right)|\phi_+\rangle \exp(\Gamma_{\rm ext}) \ ,\\
\rho_{\rm int} &= {\rm Tr}_+|\Psi\rangle\langle\Psi| = \int \mathcal{D}\phi_+ \Psi^*(\phi'_-,\phi_+)\Psi(\phi_+,\phi_-)\\
&= \int \mathcal{D}\phi_+ \langle\phi'_-|\exp\left(-\frac{\beta\hat{H}}{2}\right)|\phi_+\rangle\langle\phi_+|\exp\left(-\frac{\beta\hat{H}}{2}\right)|\phi_-\rangle \exp(\Gamma_{\rm int}) \ .
\end{aligned} \tag{2.5}$$

A. Symmetry theorem for the BFZ model

We show that for the BFZ model it holds an extended version of the symmetry theorem [8]. Following Bekenstein path [9] we put

$$\mathcal{C}_{\phi_-,\phi_+} \equiv \left\langle \phi_- \left| \exp\left(-\frac{\beta\hat{H}}{2}\right)\right| \phi_+ \right\rangle .$$

So doing the formulas (2.5) become

$$\begin{aligned}
\rho_{\rm ext} &= (\mathcal{C}^\dagger\mathcal{C})_{\phi'_+,\phi_+}\exp(\Gamma_{\rm ext}) \ ,\\
\rho_{\rm int} &= (\mathcal{C}^*\mathcal{C}^T)_{\phi'_-,\phi_-}\exp(\Gamma_{\rm int}) \ .
\end{aligned} \tag{2.6}$$

From unitarity request for density matrices we obtain

$$\begin{aligned}
\exp(\Gamma_{\rm ext}) &= \frac{1}{\int \mathcal{D}\phi_+ (\mathcal{C}^\dagger\mathcal{C})_{\phi'_+,\phi_+}|_{\phi'_+=\phi_+}} \ ,\\
\exp(\Gamma_{\rm int}) &= \frac{1}{\int \mathcal{D}\phi_- (\mathcal{C}^*\mathcal{C}^T)_{\phi'_-,\phi_-}|_{\phi'_-=\phi_-}}
\end{aligned} \tag{2.7}$$

so

$$\begin{aligned}
\exp(\Gamma_{\rm ext}) &= \frac{1}{{\rm Tr}(\mathcal{C}^\dagger\mathcal{C})} \ ,\\
\exp(\Gamma_{\rm int}) &= \frac{1}{{\rm Tr}(\mathcal{C}^*\mathcal{C}^T)} \ .
\end{aligned} \tag{2.8}$$

By using invariance of the trace under transposition and permutation, it is easy to see that

$$\Gamma_{\rm int} = \Gamma_{\rm ext} \ . \tag{2.9}$$

Now $\Gamma_{\rm ext}$ can be identified with the product βF relative to the field degrees of freedom external to the horizon.

The well-known relations between entropy, internal energy, heat capacity, and free energy

$$\begin{aligned}
S &= (\beta\partial_\beta - 1)\Gamma \ ,\\
U &= \partial_\beta\Gamma \ ,\\
c &= -\beta^2\partial_\beta^2\Gamma \ ,
\end{aligned} \tag{2.10}$$

and (2.9) imply that not only is there no clash between BFZ and Bombelli, Koul, Lee, and Sorkin [10] definition of entanglement entropy, but moreover that "symmetry theorem" is generalizable to other thermodynamical quantities, such as internal energy and heat capacity.

As a concluding remark of this section, we note that while the interpretation of the "internal" entropy such as BFZ is clear, it does not appear obvious which meaning to attribute to the "internal" free energy and internal energy. For example, the internal energy for the external field is

$$E^{\rm ext} = {\rm Tr}(\rho_{\rm ext}H) \tag{2.11}$$

and the equality (2.9) implies

$$E^{\text{``int''}} = {\rm Tr}(\rho_{\rm int}H) = E^{\rm ext} \ . \tag{2.12}$$

The Hamiltonian in (2.12) is relative to the external de-

[2] See for example Jacobson [7].

grees of freedom: the one relative to the internal ones is $-H$. We think that the high symmetry in the variables internal-external is the reason for this "extended symmetry theorem." In other words, the property Frolov [11] calls *duality* is related to this very special feature of BFZ's model, that in general is not implemented. In fact in this case we are defining "external" and "internal" thermodynamical quantities on the asymptotically flat regions of the Einstein-Rosen bridge. Symmetry theorem is just the statement of the impossibility, for an observer which "lives" in one of these regions to discriminate in which of the two he is. In what follows, we refer always to the "external" quantities.

B. Effective action

Let us calculate Γ_β, from (2.4) we obtain

$$\Gamma_\beta = \int dx \left[\ln\left(2\sinh\frac{\beta\hat{\omega}}{2}\right)\delta(x-y)\right]_{y=x} , \tag{2.13}$$

where we used the property $\ln \det A = \mathrm{Tr}\ln A$. As BFZ we calculate the expression below by expanding all the functions $\phi(x)$ in terms of eigenfunctions $R_\lambda(x)$ of the operator $\hat{\omega}$:

$$\begin{aligned}\phi(x) &= \sum_\lambda \phi_\lambda R_\lambda(x) ,\\ \hat{\omega}^2 R_\lambda(x) &= \omega_\lambda^2 R_\lambda(x) ,\\ \delta(x-y) &= \sum_\lambda g^{00} g^{1/2} R_\lambda(x) R_\lambda(y) ,\end{aligned} \tag{2.14}$$

where $\sum_\lambda$ denotes the sum over all quantum numbers, g^{00} is the timelike component of the matrix tensor and $g = \det g_{\mu\nu} = g^{00}\det g^{ab}$ $(a, b, \ldots = 1, 2, 3)$. So we obtain

$$\Gamma_\beta = \int_{2M}^{R_{\rm bos}} dr \frac{r^2}{(r-2M)} \int_0^\infty \sum_{l=0}^\infty d\omega (2l+1) R_{\lambda\omega}^2(r)\gamma(\beta\omega), \tag{2.15}$$

where

$$\gamma(\beta\omega_\lambda) = \frac{\beta}{2}\omega_\lambda + \ln(1 - e^{-\beta\omega_\lambda}) \tag{2.16}$$

and where $R_{\lambda\omega}(r)$ are the radial eigenfunctions. We are interested in the behavior of Γ_β near the horizon; using the BFZ result

$$\sum_{l=0}^\infty (2l+1) R_{\lambda\omega}^2(r) \sim \frac{4}{\pi}\omega^2 \frac{M}{r-2M} \tag{2.17}$$

we get

$$\Gamma_\beta \sim \frac{4M}{\pi}\int_{2M}^{r_{\rm box}} dr \frac{r^3}{(r-2M)^2}\int_0^{+\infty} d\omega\omega^2\gamma(\beta\omega) , \tag{2.18}$$

where $r_{\rm box}$ is the radius of the box in which we have to put the black hole to regularize infrared divergences. To compute the second integral we have to subtract the zero-point term from (2.16). We find the following leading term near the horizon:

$$\Gamma_\beta = \beta F(\beta) \sim -\frac{32\pi^3 M^4}{45}\frac{1}{\beta^3}\frac{1}{h} , \tag{2.19}$$

where the cutoff is defined as $h \equiv Inf(r - 2M)$.

From the free energy (2.19), it is possible to find the other thermodynamical quantities by using (2.10) relations. We obtain

$$\begin{aligned} S &\sim \frac{128\pi^3 M^4}{45}\frac{1}{\beta^3}\frac{1}{h} ,\\ U &\sim \frac{32\pi^3 M^4}{15}\frac{1}{\beta^4}\frac{1}{h} ,\\ c &\sim \frac{128\pi^3 M^4}{15}\frac{1}{\beta^3}\frac{1}{h} .\end{aligned} \tag{2.20}$$

Rewriting the above formulas in terms of a proper distance cutoff,

$$\epsilon \sim 2\sqrt{r_{\rm BH} h} \Leftrightarrow h \sim \frac{\epsilon^2}{4r_{\rm BH}} , \tag{2.21}$$

we find, for F, U, S, and c at the Hawking temperature,

$$\begin{aligned} F(\beta h) &\simeq -\frac{M}{720\pi}\frac{1}{\epsilon^2} ,\\ S(\beta_h) &\simeq \frac{2M^2}{45}\frac{1}{\epsilon^2} ,\\ U(\beta_h) &\simeq \frac{M}{240\pi}\frac{1}{\epsilon^2} ,\\ c(\beta_h) &\simeq \frac{4M^2}{30\epsilon^2} .\end{aligned} \tag{2.22}$$

The entropy in (2.22) is exactly the same as that in BFZ.

Note that (2.20) depends on the implicit standard assumption [2,3,5,12–16] of β independence of the cutoff h appearing in the regularized effective action (2.19). We will call in the following "standard" brick-wall model a model in which it holds the above assumption. Further discussion is found in the conclusions.

C. Interpretive problems

The divergences appearing in (2.22) for the entropy and the other thermodynamical quantities requires a renormalization scheme or a brick-wall cutoff. The standard position consists in identifying the black hole entropy with the leading divergent regularized term:

$$S_{\rm BH} \equiv S_{\rm radiation,\ leading} . \tag{2.23}$$

Our line is to follow most of the papers on the same problem [2,3,5,12–16] and to check which results one can obtain from (2.23) calculating the regularized terms for the other thermodynamical quantities. We will work in the framework of the brick-wall regularization of the divergences [2,5,16]; the cutoff in (2.22) is the same for all the thermodynamical quantities: we have to use the same value of the cutoff, fixed by Eq. (2.23), for all of them.

The identification of Bekenstein-Hawking entropy with the entanglement entropy of course generates a problem of interpreting the classical (tree level) entropy due to gravity in the path-integral approach. The first aim of entanglement approach is to explain as dynamical matter entropy all black hole entropy. The matter leading term is not a new one-loop contribution to be added to the tree level one. So it appears as a necessary complement of this program a clear explanation for ignoring the presence of the tree level contribution of gravity. As a matter of fact in literature this problem appears to be often ignored or gone around. We can quote in this sense only a work by Jacobson [17] and an alternative proposal by Frolov [16].

The interpretative problem[3] is worse for the other thermodynamical quantities. For the internal energy, we will find that it seems impossible the identification of the brick-wall value with the tree level one; on the other hand, it does not seem possible to understand the radiation term as a perturbative contribution to the black hole tree level one. In the second case, the underlying idea is that geometry (black hole internal energy M) is not induced by linearized matter fields in thermal equilibrium with the black hole. A similar situation is found for heat capacity. We stress that a self-consistency check of (2.23) imposes to compare tree level gravitational values with the one-loop matter ones.

D. Free energy and internal energy

The cutoff fixing necessary to obtain the required value $S_{\text{ent}} = S_{\text{Bek-Haw}} = A/4$ is

$$\epsilon^2 = \frac{1}{90\pi}, \tag{2.24}$$

leading to the following values for free energy and internal energy:

$$\begin{aligned} F &= -\tfrac{1}{8}M \ , \\ U &= \tfrac{3}{8}M \ . \end{aligned} \tag{2.25}$$

The results in (2.25) are the same obtained by 't Hooft in his pioneering paper [5] and exactly the same are found if one calculates U and F with heat kernel expansion truncated to the first De Witt coefficient in the optical metric.[4]

E. Heat capacity

We now want to show the behavior of black hole heat capacity in the brick-wall model. For heat capacity, imposing again the cutoff value (2.42), one obtains, from[5] (2.22),

$$c = +12\pi M^2 \ . \tag{2.26}$$

It is important to note that this is a positive value and in module bigger than the classical well-known result

$$c_{\text{class}} = -8\pi M^2 \ . \tag{2.27}$$

So if we accept the brick-wall model plus entanglement entropy frame as dynamical explanation of black hole entropy we find, in the most naive interpretation of (2.26), that black holes are stabilized by one-loop contribution of matter. On the other side (2.26) is different from its classical counterparts not only for a numerical difference but also because they describe completely different thermodynamical objects.

The same value (2.26) can also be found by using well-known results [19] for thermodynamical quantities for scalar field confined in spatial cavity in a static space-time at finite temperature. In [19] a high temperature expansion, in terms of the De Witt coefficients, is performed. From [19] the main contribution to entropy is given by

$$S = \frac{2\pi^2}{45}\frac{1}{\beta^3}c_0 \ .$$

Here $\bar{c}_0$ is the first De Witt coefficient in optical metric $g^{\text{opt}}_{\mu\nu} \equiv \bar{g}_{\mu\nu} = g_{\mu\nu}/g_{00}$ which in our case is

$$\begin{aligned} \bar{c}_0 &= \int d^3x\sqrt{\bar{g}} \\ &= 4\pi \int dr r^2 \frac{1}{g_{00}^2(r)} \\ &= 4\pi \int_{2M}^{R} dr \frac{r^4}{(r-2M)^2} \\ &\approx 4\pi \frac{(2M)^4}{h} \ . \end{aligned} \tag{2.28}$$

So at the Hawking temperature the entropy for the Schwarzschild black hole is

$$S = \frac{M}{180}\frac{1}{h} \ . \tag{2.29}$$

To get $S = A/4 = 4\pi M^2$ we have to fix (as 't Hooft) $h \equiv \frac{1}{720\pi M}$. We can now calculate heat capacity. We find

[3] 't Hooft, as a matter of fact seems aware of the interpretative problem; his insight is that also the mass of the black hole should be entirely due to the radiation [18]. Anyway, the problem of how to implement an identification of black hole quantities and radiation ones, is left open.

[4] See the following section.

[5] Also in this case, it is possible to obtain the same value from 't Hooft results [5] with a simple computation using his internal energy.

$$c = \frac{2\pi^2}{15}\frac{1}{\beta^3}\bar{c}_0 = \frac{M}{60}\frac{1}{h} \; . \quad (2.30)$$

Here we have used (2.28).

Introducing in (2.30) the former cutoff fixing, we finally obtain

$$c = \frac{M}{60} \times 720\pi M = 12\pi M^2 \; . \quad (2.31)$$

III. CONCLUSIONS AND PERSPECTIVES

Our analysis suggests that brick-wall model interpretation of black hole entropy brings problematic results for internal energy and heat capacity. The internal energy, as 't Hooft remarked [5], is of the same order of magnitude of black hole mass: at this point one must question the applicability of the assumption of the negligible back reaction. Even if one passes over this problem, we still find that the one-loop contribution of matter to black hole heat capacity is positive and so it would stabilize the black hole. We believe it is an inconsistent result because quantum correction is for the heat capacity bigger than its background counterpart; it could be more plausible if we would have in the gravitational action quadratic terms in curvature tensor, but this is not our case.

Our results can be interpreted as a "warning bell" of a structural problem embedded in the standard brick-wall approach to black hole thermodynamics. Perhaps it is due to the fact that one ignores the back reaction of matter field on the gravitational background. Maybe that relaxing the standard assumption of β independence of the cutoff h appearing in the regularized effective action (2.19) one eventually gets a consistent of all thermodynamical quantities.

Our proposal is to review critically [20] the key idea underlying the usual approach, according to which, in order to take the β derivatives necessary to compute from the partition function the various thermodynamical quantities we need to displace slightly from β_h, introducing a conical singularity in the manifold. The above approach understands that, as it is made in "common" manifolds not characterized by the Hawking effect, it is possible to give to the parameter β the meaning of an allowed physical equilibrium temperature for the quantum fields living on the manifold. So all the physical quantities are calculable from the partition function by means of β derivatives.

The case of a manifold characterized by the Hawking effect is in the conical approach treated on the same foot: in order to study the equilibrium thermodynamics at the Hawking temperature, we have to take the β derivatives and then to put $\beta = \beta_h$. This is equivalent to introducing a displacement in the manifold from its natural period.

Our ansatz [20] is that equilibrium thermodynamics in a black hole manifold requires a different approach that takes its stand essentially from a literal interpretation of the zeroth law of black hole mechanics [21].

Indeed, we know that for manifolds with a bifurcate Killing horizon the equivalent of the temperature concept is given by the surface gravity k; they are related by the formula

$$T = \frac{k}{2\pi} \; . \quad (3.1)$$

So, if we allow the temperature, or equivalently, its inverse β, to vary freely (as it is true in the canonical ensemble, in which all the above calculations are made), then we have to think the geometrical parameters entering in the surface gravity as functions of the temperature by means of (3.1). To be more clear, the equation defining the proper period of the manifold

$$\beta = \frac{2\pi}{k} \quad (3.2)$$

has to be considered a constraint equation for the geometrical parameters appearing in (3.2). In the case of the Schwarzschild black hole β_h is the Euclidean time period to be naturally selected with the aim of avoiding the conical singularity. The equilibrium thermodynamics of the manifold is such that one has to adjust the relevant geometric parameters in order to match the generic β and the surface gravity k associated with the horizon: given the relation

$$\beta = \frac{2\pi}{k} = 8\pi M \quad (3.3)$$

we have to substitute in the metric

$$M = \frac{\beta}{8\pi} \; . \quad (3.4)$$

We note that, to make the above match, the metric becomes β dependent.

The ansatz is coherent with the perturbative expansion of the path integral of the finite temperature quantum field theory: the starting point in the path-integral calculation of the partition function is a tree level approximation for the gravitational part; the classical solution we are interested in is the Schwarzschild solution. This solution is smooth and gives the relation (3.4) between its period in Euclidean time and the black hole radius.

All the matter field contributions in the linearized theory are perturbations of the tree level ones. Besides, as far as the matter field back reaction is not included, matter fields are not able to modify the link between geometry and thermodynamics given by (3.1). We stress that, in this way, all the diseases due to introduction of the conical defect are eliminated. In support to this approach we cite analogous statements in [22,16]. We think that it incorporates the tree level back reaction phenomenon, and on the other hand, it should make the one-loop contributions to be a small correction to the tree level ones; particularly, the radiation entropy should be only a small perturbation with respect to the big Bekenstein-Hawking entropy.

We intend to return to these topics in a future publication [23].

ACKNOWLEDGMENTS

The authors wish to thank M. Martellini and K. Yoshida for extensive discussions and some constructive advice. S. Liberati thanks V. Ferrari for illuminating remarks.

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