PHYSICAL REVIEW D VOLUME 53, NUMBER 6 15 MARCH 1996

Nonlinear wave equations for relativity

Maurice H. P. M. van Putten*
Center for Radiophysics and Space Research and Cornell Theory Center, Cornell University, Ithaca, New York 14853-6801

Douglas M. Eardley†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

(Received 15 May 1995)

Gravitational wave motion is described by nonlinear wave equations using the tetrad and its connection as field variables. The wave equations result from a Lorentz gauge on the connections. This description separates the physics of wave motion from the causal structure, which is evolved in the tangent space. The initial data constraints are derived in this approach using Yang-Mills scalar and vector potentials, resulting in Lie constraints associated with the additional Poincaré gauge invariance. The analogy of the constraint equations with those in Ashtekar's variables is emphasized.

PACS number(s): 04.20.Jb, 04.30.Nk, 04.60.Ds

I. INTRODUCTION

The prediction of gravitational wave structure from the coalescence of astrophysical black holes or neutron stars is presently being pursued in connection with the construction of the Laser Interferometric Gravitational Wave Observatory (LIGO) and VIRGO [1]. The asymptotic wave structure at a distant observer is in quantitative relation to the system parameters of compact binaries, which makes gravitational radiation a new spectrum for astronomical observations. The prediction of gravitational wave forms is advanced through second-post-Newtonian order approximations [2,3] and large scale numerical simulation (e.g., [4]). The current numerical approaches are based on the 3+1 Hamiltonian formulation by Arnowitt, Deser, and Misner [5], strategies employing null coordinates (e.g., [6]) or systems of conservation laws [7].

The significance of gravity waves in the process of coalescence of compact binaries suggests to focus on a description of relativity by nonlinear wave equations. In doing so, new avenues towards numerical relativity are explored, in regard to the computational challenges with horizon boundary conditions [8] and the extraction of gravitational wave forms at finite boundaries [9,10]. Wave equations ensure that the physical quantities and their derived quantities propagate along light cones (hyperbolicity) also after discretization, enabling rigorous implementation of the physical boundary conditions. Wave equations further provide a framework for the physics of nonlinear wave motion, by way of establishing close connection with electromagnetics and Yang-Mills theory. In this paper, we take a first step in these directions.

We shall derive wave equations in the tetrad approach, taking the tetrad and their connections as field variables. Our starting point is motivated by Pirani's arguments concerning the role of the Riemann tensor in gravitational wave motion [11,12], and by formulations of general relativity as a Yang-Mills theory, going back to Utiyama and developed by many authors since, most notably Ashtekar and co-workers [13]. The Riemann tensor then obeys a divergence equation with matter (if it is present) appearing as a source term. In the tetrad approach, this equation is of Yang-Mills-type for the spacetime connection, with the gauge group SO(3,1,R), the proper Lorentz group. The gauge group acts on the internal degrees of freedom of the tetrad. To fix the gauge we impose a Lorentz gauge condition, i.e., a four-divergence condition on the connection. Thus, unique evolution equations are provided for the internal gauge, resulting in nonlinear wave equations for the connections. Complemented by the equations of structure for the evolution of the tetrad legs, a complete and manifestly hyperbolic system of evolution equations is obtained, which we propose as new evolution equations for numerical relativity.

The interwoveness of wave motion and causal structure distinguishes gravity from the other field theories. In the present description, this twofold nature of gravity is made explicit in the wave equations for the connections on the one hand and the equations of structure for the tetrad (the "square root" of the metric) on the other hand. Essentially, gravitational waves are now propagated by wave equations on the (curved) manifold, while the metric is evolved in the (flat) tangent bundle by the equations of structure. Of course, such interpretation is only meaningful for wave motion with wave lengths above the Planck scale, below which the causal structure is not well defined and wave motion cannot be distinguished from quantum fluctuations.

The tetrad approach taken here has much in common with the Ashtekar program on nonperturbative quantum gravity [13–17]; however there are also important differences. The original Ashtekar's variables are the SU(2,C) soldering form and an associated complex connection in which the constraint equations become polynomial, while our basic variable is the real SO(3,1,R) connection (a comparison study can be found in [17]). In Ashtekar's variables, a real spacetime is recovered from the complex one by reality constraints. Still another approach is that of Barbero [18,19], who recently carried through Ashtekar's procedure in the SO(3,R) phase space with real connections. SL(2,C) being the universal covering group (see [20] for a treatment of

*Electronic address: vanputte@spacenet.tn.cornell.edu
†Electronic address: doug@itp.ucsb.edu

Ashtekar's program with SL(2,C) soldering form) of the Poincaré group used as the gauge group in the tetrad approach, it is therefore not surprising to find that the constraints associated with our wave equations contain the analogs of those in Ashtekar's variables. The main innovation in our work is the incorporation of the Lorentz gauge condition to obtain new evolution equations. It would also be of interest to explore ways of doing the same in the Ashtekar's variables with forementioned soldering forms, something which lies beyond the scope of this work.

After this work was submitted, Choquet-Bruhat and York independently announced the results of some very interesting, related work, on the treatment of the Einstein equations as nonlinear wave equations [21,22]. Their work does not depend on a gauge-theory approach, but they do use derivatives of the metric as field variables, and their results are in many respects similar to the present ours.

II. EQUATIONS FOR R_{abcd}

We will work on a four-dimensional manifold, M, with hyperbolic metric g_{ab}. In discussions on the problem of initial data and constraints, M is considered to be the product of a smooth initial hypersurface, Σ, and a timelike coordinate. In a given coordinate system $\{x^b\}$ the line element on M is given by

$$ds^2=g_{ab}dx^a dx^b. \tag{1}$$

The natural volume element on M is $\epsilon_{abcd}=\sqrt{-g}\Delta_{abcd}$, where g denotes the determinant of the metric in the given coordinate system, and Δ_{abcd} denotes the completely antisymmetric symbol. Coordinate differentiation will be denoted by ∂_a and covariant differentiation associated with g_{ab} will be referred to by ∇_a. Following Pirani [11], our presentation starts from the view that all gravitational wave motion in M is contained in the Riemann tensor, R_{abcd}. The Riemann tensor satisfies the Bianchi identity

$$3\nabla_{[e}R_{ab]cd}=\nabla_e R_{abcd}+\nabla_a R_{bacd}+\nabla_b R_{eacd}=0. \tag{2}$$

Using the volume element ϵ_{abcd}, the dual $*R$ is defined as $(1/2)\epsilon_{ab}^{\cdot\cdot\cdot ef}R_{efcd}$. The Bianchi identity then takes the form

$$\nabla^a *R_{abcd}=0. \tag{3}$$

The interaction of matter, described by an energy-momentum tensor T_{ab}, with gravity is described by Einstein's equations, in which the Ricci tensor $R_{ab}=R_{acb}{}^c$ and scalar curvature $R=R_c^{\cdot c}$ satisfy

$$G_{ab}\equiv R_{ab}-\frac{1}{2}g_{ab}R=8\pi T_{ab}. \tag{4}$$

The dynamics of the geometry on M can be described in terms of the Riemann tensor by (2) (see, e.g., [23])

$$\nabla^d R_{abcd}=2\nabla_{[b}R_{a]c}. \tag{5}$$

In the presence of Einsteins equations (4), $R_{ab}=8\pi[T_{ab}-(1/2)g_{ab}T]\equiv 8\pi\tilde{T}_{ab}$. Therefore, we have (e.g., [24])

$$E:\nabla^a R_{abcd}=16\pi\left(\nabla_{[c}T_{d]b}-\frac{1}{2}g_{b[d}\nabla_{c]}T\right). \tag{6}$$

The quantity on the right-hand side shall be referred to as $16\pi\tau_{bcd}$. In vacuo, E has been discussed by Klainerman [25], who refers to E (with $\tau_{bcd}=0$) together with (2) as the spin-two equations. The tensor τ_{bcd} is divergence-free:

$$\nabla^b\tau_{bcd}=0, \tag{7}$$

in consequence of the conservation laws $\nabla^a T_{ab}=0$, and in agreement with $\nabla^b\nabla^a R_{abcd}=0$.

Taken as evolution equations, it is of interest to note that a cosmological constant does not enter E, as the presence of a term λg_{ab} drops out of the right-hand side of (5). A cosmological constant does enter initial data constraints, of course, and thus be a conserved scalar by E.

A. Yang-Mills equations

We proceed in the language of tetrads, using a tetrad and its connections as field variables. In doing so, additional invariances arise due to the liberty of choosing the tetrad position at each space-time point. This invariance is described by the proper Poincaré group (at each point of M). Details can be found, e.g., in [26–28]. We let $\{(e_\mu)^b\}$ denote a tetrad, satisfying

$$(e_\mu)^c(e_\nu)_c=\eta_{\mu\nu},$$
$$\eta^{\mu\nu}(e_\mu)_a(e_\nu)^b=\delta_a^b,$$
$$\eta_{\mu\nu}=\text{diag}(-1,1,1,1). \tag{8}$$

Here, δ_a^b denotes the Kronecker symbol. Neighboring tetrads are related by their connection one-forms $\omega_{a\mu\nu}$:

$$\omega_{a\mu\nu}:=(e_\mu)^c\nabla_a(e_\nu)_c. \tag{9}$$

The connection one-forms $\omega_{a\mu\nu}$ serve as Yang-Mills connections in the gauge covariant derivative

$$\hat{\nabla}_a=\nabla_a+[\omega_a,\cdot], \tag{10}$$

satisfying $\hat{\nabla}_a(e_\mu)^b=0$. Here, the commutator is defined by its action on tensors $\phi_{a_1\cdots a_k\alpha_1\cdots\alpha_l}$ as

$$[\omega_a,\phi_{a_1\cdots a_k}]_{\alpha_1\cdots\alpha_l}=\sum_i\ \omega_{a\alpha_i}^{\cdot\cdot\cdot\alpha_j}\omega_{a_1\cdots a_k\alpha_1\cdots\alpha_j\cdots\alpha_l}. \tag{11}$$

In particular, we have

$$[\omega_a,\omega_b]_{\mu\nu}=\omega_{a\mu}^{\cdot\cdot\cdot\alpha}\omega_{b\alpha\nu}-\omega_{a\nu}^{\cdot\cdot\cdot\alpha}\omega_{b\alpha\mu}. \tag{12}$$

In what follows, Greek indices stand for contractions with tetrad elements: if v^b is a vector field, then $v_\mu=v^b(e_\mu)_b$, and $v^\mu=\eta^{\mu\nu}v_\nu$.

The equations for the Riemann tensor can be stated in the language of tetrads. The Bianchi identity (2) becomes

$$\hat{\nabla}^a *R_{ab\mu\nu}=0, \tag{13}$$

and the equivalent of E is

$$E':\hat{\nabla}^a R_{ab\mu\nu}=16\pi\tau_{b\mu\nu}. \tag{14}$$

Throughout our presentation, the representation of the Riemann tensor as follows from the Bianchi identity (13) plays a central role. In terms of the connections $\omega_{a\mu\nu}$, (13) implies (cf. [23]):

$$R_{ab\mu\nu}=\nabla_a\omega_{b\mu\nu}-\nabla_b\omega_{a\mu\nu}+[\omega_a,\omega_b]_{\mu\nu}. \tag{15}$$

The antisymmetries in the Riemann tensor introduce conditions on initial data on an initial hypersurface, Σ. If ν^b denotes the normal to Σ, and

$$\nabla_a=-\nu_a(\nu^c\nabla_c)+D_a \text{ on } \Sigma, \tag{16}$$

we obtain the Gauss-Riemann relations

$$\nu^b D^a R_{abcd}=16\pi\rho_{cd},$$

$$\nu^b D^a * R_{abcd}=0, \tag{17}$$

where $\rho_{cd}=\nu^b\tau_{bcd}$. Conditions (17) find their equivalents in the tetrad formulation. To this end, we write, analogous to (16), on Σ the derivative as

$$\hat{\nabla}_a=-\nu_a(\nu^c\hat{\nabla}_c)+\hat{D}_a. \tag{18}$$

The Gauss-Riemann relations (17) thus become

$$\nu^b\hat{D}^a R_{ab\mu\nu}=16\pi\rho_{\mu\nu},$$

$$\nu^b\hat{D}^a * R_{ab\mu\nu}=0. \tag{19}$$

B. Equations for the tetrads

The relationships between the tetrad elements on neighboring hypersurfaces in M as given by the connections $\omega_{a\mu\nu}$ provide for evolution equations for the tetrad. These equations are usually referred to as the equations of structure [23]

$$\partial_{[a}(e_\mu)_{b]}=(e^\nu)_{[b}\omega_{a]\nu\mu}. \tag{20}$$

In (20), $\partial_t(e_\mu)_t$ is left undefined. Defining $\xi^b=(\partial_t)^b$, the four time components

$$N_\mu:=(e_\mu)_a\xi^a \tag{21}$$

become freely specifiable functions. The evolution equations for the tetrad legs thus become

$$\partial_t(e_\mu)_b+\omega_{t\mu}{}^\nu(e_\nu)_b=\partial_b N_\mu+\omega_{b\mu}{}^\nu N_\nu. \tag{22}$$

The tetrad lapse functions, N_μ, are algebraically related to the familiar lapse, N, and shift functions, N_ν, in the Hamiltonian formalism through

$$g_{at}=N_\alpha(e^\alpha)_a=(N_qN^q-N^2,N_p). \tag{23}$$

Note, therefore, that the N_μ's are not parameters related to the internal gauge freedom in the tetrad legs. Internal gauge freedom is associated with Lorentz transformations of the tetrad legs by a Lorentz transformation applied to all four legs. The term $\omega_{b\mu\nu}N^\nu$ on the right-hand side of (22) shows that the tetrad lapse functions introduce different transformations of each of the legs. In contrast, the term $\omega_{t\mu}{}^\nu(e_\nu)_b$ on the left-hand side introduces a transformation which applies to all four legs simultaneously. It follows that it is the infinitesimal Lorentz transformations $\omega_{t\mu}{}^\nu$ which provide the internal gauge transformations.

We will now turn to evolution equations for the connection one-forms.

III. EQUATIONS FOR $\omega_{a\mu\nu}$

Evolution equations for the connections follow from E', upon using the connection representation (15) for the Riemann tensor. Clearly, this defines a partial evolution of the connections, leaving the evolution of those connected with the internal gauge freedom ($\omega_{t\mu\nu}$) undefined. In establishing a complete system of evolution equations, we define a Lorentzian cross section of the tangent bundle of the spacetime manifold by [12]

$$c_{\mu\nu}:=\nabla^d\omega_{d\mu\nu}=0. \tag{24}$$

In a different context (in the presence of a compact gauge group and metric with Euclidean signature) (24) has been given a geometrical interpretation by Lewandowski *et al.*[29]. The Lorentz gauge (24) provides[1] a complete, sixfold connection between neighboring tetrads. The six constraints $c_{\mu\nu}=0$ are incorporated in E' by application of the divergence technique [30,31]:

$$E'':\hat{\nabla}^a\{R_{ab\mu\nu}+g_{ab}c_{\mu\nu}\}=16\pi\tau_{b\mu\nu}. \tag{25}$$

To see the equivalence of E'' and E, it suffices to show that E'' preserves the Lorentz gauge (24) in the future domain of dependence $D^+(\Sigma)$ of an initial hypersurface Σ. To this end, first note that the inhomogeneous Gauss-Riemann relation (17) is implied by antisymmetry of the Riemann tensor in its coordinate indices, giving

$$\begin{aligned}0&=\nu^b\{\hat{\nabla}^a(R_{ab\mu\nu}+g_{ab}c_{\mu\nu})-\tau_{b\mu\nu}\}\\&=\nu^b(\hat{D}^aR_{ab\mu\nu}-\tau_{b\mu\nu})+(\nu^b\hat{\nabla}_b)c_{\mu\nu}\\&=(\nu^b\hat{\nabla}_b)c_{\mu\nu}.\end{aligned} \tag{26}$$

The inhomogeneous Gauss-Riemann relations are gauge covariant, so that we are at liberty to consider initial data satisfying both (26) and the gauge choice (38), whence

$$c_{\mu\nu}=(\nu^c\hat{\nabla}_c)c_{\mu\nu}=0 \text{ on } \Sigma. \tag{27}$$

Second, $c_{\mu\nu}$ satisfies a homogeneous wave equation in (25) (cf. [30,31]):

$$\hat{\nabla}^c\hat{\nabla}_c c_{\mu\nu}=0. \tag{28}$$

[1]Conceptually, $c_{\mu\nu}=f(\omega_{a\mu\nu},g_{ab})$ will also serve its purpose, where $f(\cdot,\cdot)$ depends analytically on its arguments.

Summarizing, we see that the scalars $c_{\mu\nu}$ satisfy an initial value problem for a homogeneous wave equation (28) with trivial Cauchy data (27), so that

$$c_{\mu\nu}=0 \ \text{ in } D^+(\Sigma). \tag{29}$$

This establishes that solutions to E'' are solutions to E'.

We now elaborate on the existence of the Lorentz gauge. Recall the transformation rule for the connection (see, e.g., [28]):

$$\omega_{a\bar{\mu}\bar{\nu}}=\Lambda_{\bar{\mu}}{}^{\alpha}\Lambda_{\bar{\nu}}{}^{\beta}\omega_{a\alpha\beta}+\Lambda_{\bar{\mu}}{}^{\alpha}\partial_a\Lambda_{\bar{\nu}\alpha}. \tag{30}$$

In the present tetrad language, this gauge transformation is readily established by consideration of two tetrads, $\{(e_\mu)^b\}$ and $\{(e_{\bar{\mu}})^b\}$. The construction $\Lambda_{\bar{\mu}}{}^{\nu}:=(e_{\bar{\mu}})_c(e^\nu)^c$ provides a finite transformation $v_{\bar{\mu}}=\Lambda_{\bar{\mu}}{}^{\alpha}v_\alpha$ of a field $v_\alpha=(e_\alpha)^b v_b$ in the $\{(e_\mu)^b\}$-tetrad representation into $v_{\bar{\alpha}}$ in the $\{(e_{\bar{\mu}})^b\}$-tetrad representation. This applied to (9) produces (30). By suitable choice of $\Lambda_{\bar{\mu}}{}^{\nu}$, therefore, (24) can be made to satisfy. Reversing this argument, we see that (24) introduces specific evolution of the internal gauge.

A. Existence of Lorentz gauge

To proceed, we consider in an open neighborhood $\mathcal{N}(\Sigma)$ of an initial hypersurface Σ with Gaussian normal coordinates $\{\tau,x^p\}$ $(\nu^c\nabla_c\tau=1)$ the infinitesimal Lorentz transformation (dropping the $\bar{\mu}$ notation)

$$\Lambda_\mu{}^{\nu}=\delta_\mu{}^{\nu}+\frac{1}{2}\tau^2\sigma_\mu{}^{\nu} \ \text{ in } \mathcal{N}(\Sigma). \tag{31}$$

It is convenient to employ scalar and vector potentials, $\Phi_{\mu\nu}$ and $A_{a\mu\nu}$, respectively, as defined in the decomposition [cf. Eq. (15ab) in [20]]

$$\omega_{a\mu\nu}=\nu_a\Phi_{\mu\nu}+A_{a\mu\nu} \ \text{ in } \mathcal{N}(\Sigma). \tag{32}$$

Upon transforming $\omega_{a\mu\nu}$, and denoting the effect of (31) via (30) by a superscript (r), we have

$$\omega^{(r)}_{a\mu\nu}=\omega_{a\mu\nu}+\tau\delta^\tau_a\sigma_{\mu\nu} \ \text{ in } \mathcal{N}(\Sigma), \tag{33}$$

so that

$$\Phi^{(r)}_{\mu\nu}=\Phi_{\mu\nu}-\tau\sigma_{\mu\nu} \ \text{ in } \mathcal{N}(\Sigma). \tag{34}$$

Using the geodesic extension of the normal ν^b off Σ, the Lorentz condition (24) becomes

$$c_{\mu\nu}=\nabla_c\omega_{\mu\nu}{}^{c}\equiv\dot{\Phi}_{\mu\nu}+D_c\omega^c \ \text{ on } \Sigma. \tag{35}$$

Expressed in $A_{a\mu\nu}$, this obtains

$$\begin{aligned}c_{\mu\nu}&=\dot{\Phi}_{\mu\nu}+D_c(\nu^c\Phi_{\mu\nu}+A^c{}_{\cdot\mu\nu})\\&=\dot{\Phi}_{\mu\nu}+K\Phi_{\mu\nu}+D_cA^c{}_{\cdot\mu\nu}\end{aligned} \tag{36}$$

on Σ, where $K_{ab}=D_a\nu_b$ is the extrinsic curvature of Σ and $K=K^c_c$. Consequently, (31) effectuates the transformations

$$\begin{aligned}A^{(r)}_{a\mu\nu}&=A_{a\mu\nu},\\ \Phi^{(r)}_{\mu\nu}&=\Phi_{\mu\nu},\\ \dot{\Phi}^{(r)}_{\mu\nu}&=\dot{\Phi}_{\mu\nu}-\sigma_{\mu\nu}\end{aligned} \tag{37}$$

on Σ. By choice of $\sigma_{\mu\nu}=c_{\mu\nu}$, the constraints (24) are therefore transformed into

$$c^{(r)}_{\mu\nu}=c_{\mu\nu}-\sigma_{\mu\nu}=0. \tag{38}$$

Notice that bringing the tetrad in Lorentz gauge (38) by (31) is achieved by adjustment of the second time derivatives $\partial^2_\tau(e_\mu)^b_{|\tau=0}$, which does not involve the tetrad position nor the velocities of the legs at $\tau=0$.

We now elaborate further on E''.

B. Nonlinear wave equations

In Lorentz gauge (24), the divergence equation E'' obtains wave equations for the connection 1-forms through the representation of the Riemann tensor (15). Indeed, by explicit calculation, we have

$$\hat{\Box}\omega_{a\mu\nu}-R^c_a\omega_{c\mu\nu}-[\omega^c,\nabla_a\omega_c]_{\mu\nu}=16\pi\tau_{a\mu\nu}. \tag{39}$$

Here, we have used $c_{\mu\nu}=0$, so that $\hat{\nabla}_a c_{\mu\nu}=\nabla_a c_{\mu\nu}$. $\hat{\Box}$ is used to denote the Yang-Mills wave operator $\hat{\nabla}^c\hat{\nabla}_c$. The Ricci tensor R_{ab} in (39) is understood in terms of T_{ab} using Einstein's equations (4). Similarly, we can obtain a system of scalar equations for the Ricci rotation coefficients, $\omega_{\alpha\mu\nu}=(e_\alpha)^a\omega_{a\mu\nu}$ by multiplication of both sides in (39) by $(e_\alpha)^a$.

We interpret these wave equations as the following.

B.1. Separation theorem. Gravity waves propagate through the curved space-time manifold by wave equations (39). In response to the wave motion, the causal structure of the manifold evolves in the tangent bundle by the equations of structure (22). The Arnowitt-Deser-Misner (ADM) lapse and shift functions find their counterparts in the tetrad lapse functions N_μ (23).

IV. INITIAL VALUE PROBLEM

An initial value problem for the wave equations in the field variables $\{(e_\mu)^b,\omega_{a\mu\nu}\}$ contains the values of $(e_\mu)^b$, $\omega_{a\mu\nu}$, and $\mathcal{L}_n\omega_{a\mu\nu}$ on Σ. These data must satisfy certain constraints on Σ, commensurate with the initial distribution of energy-momentum and the Gauss-Riemann equations (19). We shall express these equations in terms of scalar and vector potentials (32). The projection tensor h_{ab} onto Σ is

$$h_{ab}=g_{ab}+\nu_a\nu_b. \tag{40}$$

The covariant derivative associated with h_{ab} shall be denoted by $\bar{D}_a$, i.e., $\bar{D}_ah_{cd}=0$. The unit normal is extended geodesically [so that $\nu^c\nabla_c\nu^b=0$, $(\nu^c\nabla_c)h_{ab}=h_{ab}(\nu^c\nabla_c)$]. We further define

$$\begin{aligned}\Phi'&:=\dot{\Phi}_{\mu\nu}+K\Phi_{\mu\nu},\\ A'_{a\mu\nu}&:=\mathcal{L}_nA_{a\mu\nu}=\dot{A}_{a\mu\nu}+K_a{}^{\cdot c}A_{c\mu\nu}.\end{aligned} \tag{41}$$

The derivation of the initial data constraints involve a number of steps, which are streamlined by suitable reduction formulae for the Riemann tensor $R_{ab\mu\nu}$. Of course, these are most conveniently derived using a tetrad which contains a triad in the initial hypersurface. To this end, consider a normal tetrad, $\{(E_\mu)^b\}$, in which one of the legs is (initially) everywhere normal to the initial hypersurface:

$$(E_{\mu=n})^b=\nu^b \text{ on } \Sigma. \tag{42}$$

Then $\{(E_\mu)^c|_{\mu\neq n}\}$ is a triad in Σ. We are now in the position to obtain the reduction formulas in terms of quantities intrinsic to Σ and the extrinsic curvature $K_{ab}=\bar{D}_a\nu_b$ of Σ.

A. Expressions for $A_{a\mu\nu}$

Helicity can be associated pairwise with the legs of a tetrad as the twist in a strip swept out by the integral curves of a leg $(E_\nu)^b$ passing through an integral curve of leg $(E_\mu)^b$:

$$H_{\mu\nu}:=\nu_b(E_\mu)^c\nabla_c(E_\nu)^b \text{ on } \Sigma. \tag{43}$$

For a normal tetrad, $H_{\mu\nu}$, therefore, describes twist in Σ when $\mu\neq\nu$ and curvature of Σ when $\mu=\nu$. It follows that

$$H_{\alpha\beta}h^\alpha_\mu h^\beta_\nu \text{ on } \Sigma=-K_{\mu\nu} \tag{44}$$

in view of (42) [so that $\nu_a(E_\alpha)^a=-\delta^\nu_\alpha$ on Σ]. The normal $(\mu=n)$ and tangent $(\mu\neq n)$ helicities are, respectively,

$$H_{\mu\nu}=\nu_b(E_\mu)^c\{\omega_{c\gamma\nu}(E^\gamma)^b\}=\begin{cases}-\Phi_{n\nu},\\ A_{\mu n\nu}.\end{cases} \tag{45}$$

Note that by (44), the symmetry of K_{ab} and $A_{\mu n\nu}=-A_{\mu\nu n}$, we have for $\mu,\nu\neq n$ the symmetry $A_{[\mu\nu]n}=0$. If $\mu\neq n$, $(E_\mu)_a=h_{ab}(E_\mu)^b$, and so

$$\begin{aligned}A_{a\mu\nu}&=h^b_a(E_\mu)_c\nabla_b(E_\nu)^c\\&=(E_\mu)_c[h^c_d h^b_a\nabla_b](E_\nu)^d\\&\equiv(E_\mu)_c\bar{D}_a(E_\nu)^c.\end{aligned} \tag{46}$$

If $\nu\neq n$, $\bar{D}_a$ acts in (46) on tangent legs $(E_\nu)^b$, which is determined by h_{pq} (p,q coordinatizing Σ). In the normal tetrad, therefore, $A_{a\mu\nu}$ falls into two groups: (i) the symmetric, extrinsic part $A_{\mu\nu n}=K_{\mu\nu}$, and the (ii) intrinsic part $A_{a\mu\nu}$ with $\mu,\nu\neq n$. We shall write $\bar{A}_{a\mu\nu}=A_{a\alpha\beta}h^\alpha_\mu h^\beta_\nu$, so that on Σ

$$\begin{aligned}A_{\alpha\mu\nu}&=\bar{A}_{\alpha\mu\nu}+A_{\alpha n\nu}\delta^n_\mu+A_{\alpha\mu n}\delta^n_\nu\\&=\bar{A}_{\alpha\mu\nu}+2K_{\alpha[\mu}\delta^n_{\nu]}.\end{aligned} \tag{47}$$

B. Expressions for $R_{ab\mu\nu}$

Reductions of $R_{ab\mu\nu}$ on Σ can be obtained by combining the decompositions (16), (18), and (32):

$$\begin{aligned}R_{ab\mu\nu}&=\nabla_a\omega_{b\mu\nu}-\nabla_b\omega_{a\mu\nu}+[\omega_a,\omega_b]_{\mu\nu}\\&=2\nu_{[b}\hat{D}_{a]}\Phi_{\mu\nu}+2\nu_{[b}\dot{A}_{a]\mu\nu}\\&\quad+2D_{[a}A_{b]\mu\nu}+[A_a,A_b]_{\mu\nu}.\end{aligned} \tag{48}$$

In deriving this equation, use has been made of the symmetry of the extrinsic curvature tensor. Using (47), the latter may be further reduced to

$$\begin{aligned}R_{ab\mu\nu}=&2\nu_{[b}\{\hat{D}_{a]}\Phi_{\mu\nu}+\dot{A}_{a]\mu\nu}\}+2D_{[a}\bar{A}_{b]\mu\nu}+[\bar{A}_a,\bar{A}_b]_{\mu\nu}\\&+4\hat{D}_{[a}K_{b]\,[\mu}\delta^n_{\nu]}+2K_{a[\mu}K_{\nu]b},\end{aligned} \tag{49}$$

where $\hat{D}_aK_{\mu b}=D_aK_{\mu b}+A_{a\mu}{}^\gamma K_{\gamma b}$. Notice that the three-curvature tensor, ${}^3R_{abcd}$, associated with h_{pq} is given by ${}^3R_{ab\mu\nu}=\bar{D}_a\bar{A}_{b\mu\nu}-\bar{D}_b\bar{A}_{a\mu\nu}+[\bar{A}_a,\bar{A}_b]_{\mu\nu}$, so that

$$\begin{aligned}h^c_ah^d_bR_{cd\mu\nu}&=2\bar{D}_{[a}\bar{A}_{b]\mu\nu}+[\bar{A}_a,\bar{A}_b]_{\mu\nu}+2K_{a[\mu}K_{\nu]b}\\&\quad+4\bar{D}_{[a}K_{b][\mu}\delta^n_{\nu]}\\&={}^3R_{ab\mu\nu}+4\bar{\hat{D}}_{[a}K_{b][\mu}\delta^n_{\nu]}+2K_{a[\mu}K_{\nu]b}.\end{aligned} \tag{50}$$

This last equation slightly extends a similar expression obtained in [32], where only the triad components $\mu,\nu\neq n$ are considered.

C. Constraint equations

The constraints as they arise in the initial value problem for our wave equations (39) comprise the familiar vector (momentum) and scalar (energy) equations from the ADM Hamiltonian formalism, together with equations which are specific to the tetrad approach. These constraints fall into two gauge invariant groups, namely the scalar and vector constraints and Gauss-Riemann relations, supplemented with the Lorentz gauge conditions (24).

Vector and scalar constraints. The vector and scalar equations are readily recovered from the second reduction formula (50), giving

$$h^\alpha_\mu h^\beta_\nu h^c_a h^d_b R_{ab\alpha\beta}={}^3R_{abcd}+K_{a\mu}K_{\nu b}-K_{a\nu}K_{\mu b}, \tag{51}$$

whereby the scalar Gauss-Codacci relation

$$\begin{aligned}2G_{ab}\nu^a\nu^b&=h^{ac}h^{bd}R_{abcd}=h^{a\mu}h^{b\beta}R_{ab\alpha\beta}\\&={}^3R+K^2-K_{a\mu}K^{a\mu}\\&={}^3R+K^2-K_{ab}K^{ab}\end{aligned} \tag{52}$$

is obtained. The second reduction formula (50) with $\nu=n$ yields $h^c_ah^d_bR_{cd\mu n}=2\bar{\hat{D}}_{[a}K_{b]\mu}$, thereby obtaining the vector Gauss-Codacci relation

$$h^d_bR_{dn}=2h^d_bh^{a\mu}\bar{\hat{D}}_{[a}K_{d]\mu}=\bar{D}^aK_{ab}-\bar{D}_bK. \tag{53}$$

By Einsteins equations, therefore, the usual ADM constraints follow:

$$\begin{aligned}{}^3R+K^2-K_{ab}K^{ab}&=16\pi T_{nn},\\ \bar{D}^aK_{ab}-\bar{D}_bK&=8\pi h^a_bT_{an}.\end{aligned} \tag{54}$$

However, the additional gauge freedom in working with the field variables $\{(e_\mu)^b,\omega_{a\mu\nu}\}$ results in an extended number of independent degrees of freedom in the Riemann tensor (compared with the case when the Riemann tensor is generated by a metric). For example, arbitrary connections do not satisfy $R_{ab\mu\nu}=R_{\mu\nu ab}$. Consequently, the Riemann tensor is constraint, in order to be consistent with an underlying metric. This is reflected in the alternative derivation of the vector constraints mentioned above:

$$h_a^c \nu^b R_{cb\mu\nu} = -\hat{D}_a \Phi_{\mu\nu} - A'_{a\mu\nu}, \tag{55}$$

using reduction formula (50). A second set of vector constraints is thus obtained:

$$\hat{D}^\mu \Phi_{\mu\nu} + A_{\mu\nu}{}'^{\mu} = -8\pi \tilde{T}_{n\nu}. \tag{56}$$

Note that these new constraints involve the Lie derivative $\mathscr{L}_n$, so that they can be understood arising from working with a second-order formulation; in going from a first-order Hamiltonian description to a second-order description, the ADM vector constraints must also be satisfied on a neighboring future hypersurface, thereby constraining $\mathscr{L}_n \omega_{a\mu\nu}$. These constraints can also be viewed as arising from the four symmetry conditions $R_{[n\nu]}=0$ (and hence $T_{[n\nu]}=0$). In the language of constraint Hamiltonian systems [33], symmetry conditions are primary constraints, giving rise to (56) as a secondary constraint.

Gauss-Riemann relations. Gauss-Riemann relations (19) expressed in terms of $\Phi_{\mu\nu}$ and $A_{a\mu\nu}$ obtain divergence constraints analogous to those in electromagnetics. To see this, note that the antisymmetry of the Riemann tensor in its coordinate indices implies

$$\nu^b D^a R_{ab\mu\nu} = D^a(\nu^b R_{ab\mu\nu}), \tag{57}$$

where use has been made of $R_{ab\mu\nu}K^{ab}\equiv 0$. By the first reduction formula (49) we have

$$\nu^b R_{ab\mu\nu} = -\hat{D}_a \Phi_{\mu\nu} - A'_{a\mu\nu}. \tag{58}$$

Therefore, the first, inhomogeneous Gauss-Riemann constraint in (19) reads

$$\begin{aligned} \nu^b \hat{D}^a R_{ab\mu\nu} &= \nu^b D^a R_{ab\mu\nu} + [A^a, \nu^b R_{ab}]_{\mu\nu} \\ &= \hat{D}^a(\nu^b R_{ab\mu\nu}) \\ &= -\hat{D}^a \hat{D}_a \Phi_{\mu\nu} - \hat{D}^a A'_{a\mu\nu} = 16\pi \rho_{\mu\nu}. \end{aligned} \tag{59}$$

The Gauss-Riemann constraints can be regarded as secondary constraints, with the associated antisymmetries in $R_{ab\mu\nu}$ as primary constraints in the nomenclature of constraint Hamiltonian systems [33]. Note that these constraints involve again a Lie derivative of the vector element of the connections. These are, therefore, again to be regarded as a result of working in a second-order formalism. These constraints amount to six constraints associated with the six degrees of freedom in internal gauge. In Sec. III, we saw this structure exploited to incorporate Lorentz gauge (24) in E''.

To summarize, the constraints on the initial data on Σ in terms of the potentials are the Gauss-Codacci constraints

$$\begin{aligned} {}^{(3)}R + K^2 - K_{ab}K^{ab} &= 16\pi T_{nn}, \\ \bar{D}^a K_{ab} - \bar{D}_b K &= 8\pi h_b^a T_{an}, \end{aligned} \tag{60}$$

the Lie constraints

$$\begin{aligned} D^\mu \Phi_{\mu\nu} + A_{\mu\nu}{}'^{\mu} &= -8\pi \tilde{T}_{n\nu}, \\ \hat{D}^a \hat{D}_a \Phi_{\mu\nu} + \hat{D}^a A'_{a\mu\nu} &= -16\pi \rho_{\mu\nu}, \end{aligned} \tag{61}$$

and the Lorentz gauge

$$\Phi'_{\mu\nu} + \hat{D}^a A_{a\mu\nu} = 0. \tag{62}$$

These constraints are given proviso the normality (42) of the tetrad. The presence of a cosmological constant can also be taken into account. The cosmological constant would appear in the initial data constraints, but clearly would be absent from the wave equations, as alluded to before in Sec. II.

D. Restricted Lorentz gauge

The discussion on the existence on the Lorentz gauge shows that (24) is satisfied by proper acceleration of the tetrad legs. That is, the positions and velocities remain as free parameters in the initial data. As a result, we are at liberty to adjust the velocities in exploring ways to simplify some of the constraint equations, at least on Σ. Thus, assume that (24) is satisfied,

$$c_{\mu\nu} = \dot{\Phi}'_{\mu\nu} + D_c A_{\mu\nu}{}^c = 0, \tag{63}$$

and consider restricted gauge transformations by further inclusion of the velocities using Lorentz transformations

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \tau \lambda_\mu{}^\nu + \frac{1}{2}\tau^2 \sigma_\mu{}^\nu \quad \text{in } \mathscr{N}(\Sigma), \tag{64}$$

where τ is a Gaussian normal coordinate of Σ such that $\nu^c \nabla_c \tau = 1$, and $\partial_\tau \lambda_\mu{}^\nu = 0$, as before. Notice that these conditions respect the normality (42). By (30), we have

$$\begin{aligned} \Phi^{(r)}_{\mu\nu} &= \Phi_{\mu\nu} - \lambda_{\mu\nu}, \\ \dot{\Phi}^{(r)}_{\mu\nu} &= \dot{\Phi}_{\mu\nu} - \sigma_{\mu\nu}, \\ A^{(r)}_{a\mu\nu} &= A_{a\mu\nu} \end{aligned} \tag{65}$$

on Σ. Now choose $\lambda_{\mu\nu}$ and $\sigma_{\mu\nu}$ to satisfy

$$\begin{aligned} \Phi^{(r)}_{\mu\nu} &= \Phi_{\mu\nu} - \lambda_{\mu\nu} = 0, \\ (\Phi^{(r)})'_{\mu\nu} &\equiv \dot{\Phi}^{(r)}_{\mu\nu} + K\Phi^{(r)}_{\mu\nu}, \\ &= \dot{\Phi}_{\mu\nu} - \sigma_{\mu\nu} = \Phi'_{\mu\nu} \end{aligned} \tag{66}$$

on Σ. With $A^{(r)}_{a\mu\nu} = A_{a\mu\nu}$, $\sigma_\mu{}^\nu = -K\Phi_{\mu\nu}$ ensures that we remain in Lorentz gauge (24), while $\lambda_{\mu\nu} = \Phi_{\mu\nu}$ gives

$$\Phi^{(r)} = 0 \quad \text{on } \Sigma. \tag{67}$$

With this restricted Lorentz gauge, the Lie constraints reduce to

$$\begin{aligned} A_{\mu\nu}{}'^{\mu} &= -8\pi \tilde{T}_{n\nu}, \\ \hat{D}^a A'_{a\mu\nu} &= -16\pi \rho_{\mu\nu}. \\ &\text{on } \Sigma. \end{aligned} \tag{68}$$

It appears not to be feasible to ensure $\Phi_{\mu\nu}=0$ throughout $D^{+}(\Sigma)$ by suitable choice of restricted Lorentz gauge.

V. DISCUSSION

The physics of gravitational interaction in a compact binary has been described in terms of wave equations (39) by using the field variables $\{(e_\mu)^b,\omega_{a\mu\nu}\}$ in Lorentz gauge. These equations are proposed as new evolution equations for numerical relativity. The wave equations and their accompanying equations of structure are strictly hyperbolic, thereby facilitating the ''grand challenges'' in the present NSF grand challenge in gravitational radiation form the coalescence of compact binaries, namely those associated with rigorous ingoing horizon boundary conditions, extraction of wave forms at the outer boundary with outgoing boundary conditions, together with a clearcut separation between the physics of wave motion and the evolution of causal structure (Separation theorem B.1). This formulation applies both to the problem of coalescence of a binary black hole system, and that of a binary neutron star system by inclusion of matter as source terms.

Avenues for numerical implementation of the wave equations are given by (39), their scalar form in the Ricci rotation coefficients, or E''. In either case, the choice of gauge (slicing and shift in ADM language) is completely free to be adapted to specifics of the problem, while enabling existing numerical techniques to be exploited for accurate and stable numerical implementation. In the case of implementation through E'' the computational techniques can draw directly from computational fluid dynamics, where in the case of compressible fluids the equations are usually cast in divergence form. The hyperbolicity of the equations ensures the variables to be well behaved, and suitable for discretization. A recent numerical implementation [34] demonstrates the soundness of the approach, by obtaining proper second order convergence through implementation of E'' and (22) in a test computation against an analytic solution of a nonlinear, polarized Gowdy wave [35,36].

The constraint equations as implied by E'' show additional Lie constraints, including the (sixfold) analogue of the (threefold) Gauss law in Ashtekar's variables. These Lie constraints can be understood to be arising from working in a second-order formulation, or as secondary constraints resulting from symmetries in the Riemann tensor as primary constraints. The Lorentz gauge condition is specific to our formulation; we expect this to have its equivalence in Ashtekar's variables through specific conditions on the SU(2) factor ${N_A}^B$ of the Gauss law in the total Hamiltonian, but have not sought to make this explicit.

ACKNOWLEDGMENTS

M.V.P. gratefully acknowledges the warm hospitality at ITP, and very stimulating discussions with S. Chaudhuri, C. Cutler, D. Chernoff, G.B. Cook, S.L. Shapiro, and S.A. Teukolsky. D.M.E. is grateful for the hospitality of the Aspen Center for Physics. This work has been supported in part by NSF Grant Nos. PHY 94-08378, PHY 90-08502, and PHY-94-07194, and the Grand Challenge Grant No. NSF PHY 93-18152/ASC 93-18152 (ARPA supplemented). The Cornell Theory Center is supported by NSF, NY State, ARPA, NIH, IBM, and others.

[1] A. Abramovici, W.E. Althouse, R.W.P. Drever, Y. Gursel, S. Kawamura, F.J. Raab, D. Shoemakes, L. Siewers, R.E. Spero, K.S. Thorne, R.E. Vogt R.E., R. Weis, S.E. Whitcomb, and M.E. Zucker, Science **256**, 325 (1992); C. Bradaschia, E. Calloni, M. Cobal, R. Del Fasbro, A. Di Virgilio, A. Giazotta, L. E. Holloway, H. Kautzky, B. Michelozzi, V. Montelatici, D. Pascuello, and W. Velloso, in *Gravitation 1990*, Proceedings of the Banff Summer Institute, Banff, Alberta, edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991).

[2] L. Blanchet, T. Damour, and B. Iyer, Phys. Rev. D **51**, 5360 (1995).

[3] L. Blanchet, T. Damour, B. Iyer, C.M. Will, and A.G. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995).

[4] P. Anninos, K. Camarda, J. Masso, E. Seidel, W. Suen, and J. Towns, Phys. Rev. D **52**, 2059 (1995).

[5] W.D. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[6] C.J.S. Clarke and R.A. d'Inverno, Class. Quantum Grav. **11**, 1463 (1994).

[7] C. Bona and J. Masso, Phys. Rev. Lett. **68**, 1097 (1992).

[8] E. Seidel and W. M. Suen, Phys. Rev. Lett. **69**, 1845 (1992).

[9] A.M. Abrahams and C.R. Evans, Phys. Rev. D **37**, 318 (1990).

[10] A.M. Abrahams, S.L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **51**, 4295 (1995).

[11] F.A.E. Pirani, Phys. Rev. **105**, 1089 (1957).

[12] M.H.P.M. van Putten, in *Fully Covariant and Constraint-Free Divergence System for Numerical Astrophysics*, Proceedings of the November 6-8 Meeting of the Grand Challenge Alliance on Black Hole Collisions, edited by E. Seidel (NCSA, Urbana-Champaign, 1994).

[13] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).

[14] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986).

[15] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).

[16] A. Ashtekar, J.D. Romano, and R.S. Tate, Phys. Rev. D **40**, 2572 (1989).

[17] S. Gotzes, Acta Phys. Pol. B **23**, 433 (1992).

[18] G.J.F. Barbero, Phys. Rev. D **49**, 6935 (1994).

[19] G.J.F. Barbero, Class. Quantum Grav. **12**, 5 (1995).

[20] T. Jacobson, Class. Quantum Grav. **5**, L143 (1988).

[21] Y. Choquet-Bruhat and J.W. York, Jr., ''Geometrical Well Posed Systems for the Einstein Equations,'' Report No. gr-qc/9506071, IFP-UNC-509, TAR-UNC-047, 1995 (unpublished).

[22] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, Jr., Phys. Rev. Lett. **75**, 3377 (1995).

[23] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[24] E.E. Fairchild, Phys. Rev. D **16**, 2438 (1977).
[25] S. Klainerman, Cont. Math. **71**, 125 (1986).
[26] A. Trautman, *Yang-Mills Theory and Gravitation: A comparison*, Lecture Notes in Mathematics: Geometric Techniques in Gauge Theories (Springer-Verlag, Berlin, 1981).
[27] M. Muller, *Consistent Classical Supergravity Theories*, Lecture notes in Physics Vol. 336 (Springer-Verlag, New York, 1989).
[28] F. Gieres, *Geometry of Supersymmetric Gauge Theories*, Lecture Notes in Physics Vol. 302 (Springer-Verlag, New York, 1988).
[29] J. Lewandowksi, J. Tafel, and A. Trautman, Lett. Math. Phys. **7**, 347 (1983).
[30] M.H.P.M. van Putten, Commun. Math. Phys. **141**, 63 (1991).
[31] M.H.P.M. van Putten, Phys. Rev. D **50**, 6640 (1994).
[32] R. Schoen and S.-T. Yau, Commun. Math. Phys. **79**, 231 (1981).
[33] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964).
[34] M.H.P.M. van Putten, in *Nonlinear Wave Equations for Relativity*, Proceedings of the May Meeting of the Grand Challenge Alliance on Black Hole Collisions, edited by G.B. Cook (Cornell University, Ithaca 1995).
[35] B.K. Berger and V. Moncrief, Phys. Rev. D **48**, 4676 (1993).
[36] B.K. Berger, D. Garfinkle, and S. Swamy, J. Gen. Relativ. Gravit. **27**, 511 (1995).