Strong energy condition in $R+R^2$ gravity

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In this paper we study Raychaudhuri's equation in the background of $R+\beta R^2$ gravity with phenomenological matter $[\rho \propto a(t)^{-n}]$. We conclude that even though the strong energy condition (SEC) for Einstein's gravity (which guarantees singularity) is $n \geqslant 2$ for $\rho \propto a(t)^{-n}$, a perturbative analysis of Raychaudhuri's equation reveals that the big bang singularity may not be guaranteed in $R+\beta R^2$ gravity for $n>4$. We derive the following strong energy conditions for $R+\beta R^2$ gravity $(\beta \neq 0)$: (1) For the $k<0$ FRW metric, the SEC is $0 \leqslant n \leqslant 4$, i.e., $-\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$; (2) for the $k=0$ FRW metric, the SEC is $1 \leqslant n \leqslant 4$, i.e., $-\frac{2}{3}\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$; (3) for the $k>0$ FRW metric, the SEC is $2 \leqslant n \leqslant 4$, i.e., $-\frac{1}{3}\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$.

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I. INTRODUCTION

An essential mathematical criterion in the singularity theorem is the strong energy condition (SEC). The theorem predicts, modulo plausible assumptions, that classical matter in the background of classical Einstein's gravity will eventually form black holes and a cosmological singularity [1,2].

The singularity theorem is limited by its classical content. The question of the validity of this theorem or how the theorem should be altered when the quantum effects are incorporated is a fascinating question.

A straightforward method of incorporating the quantum effects is to start from the classical Einstein's equations and quantize both the gravity and the matter sectors via the correspondence principle, i.e., $\{x,p\}_{\text{Poisson}} \rightarrow [x,p]_{\text{QM}}$, specifically, utilizing the Schrödinger representation for the canonical momenta result in the Wheeler-DeWitt equation [3,4]. Recently, an additional method of incorporating the quantum effects has come into focus. Advances in the theory of quantum gravity have revealed that, even if one starts with Einstein's gravity coupled to a matter, the quantum loop effects of the gravity + matter and the renormalization procedures result in a quadratic gravity [5,6]. Therefore even a classical analysis of the quadratic gravity is inherently a semiclassical analysis. (The two methods of incorporating the quantum effects, which differ by the choice of the canonical or the covariant quantization procedures, are not necessarily equivalent. This is because the formal equivalence of the canonical and the covariant quantization procedures is only true for renormalizable theories.)

Several authors have studied classical solutions of $R+\beta R^2$ gravity without matter and have concluded that the big bang singularity may be avoided [7–9]. There are several problems with higher derivative theories [10,11], e.g., the need for additional initial conditions in the formulation of a cauchy problem, the existence of runaway solutions, and the question of whether solutions obtained from the $R+\beta R^2$ gravity reduce to the solutions of Einstein's gravity as $\beta \rightarrow 0$.

In order to resolve these issues, the author has recently proposed an alternate approach to $R+\beta R^2$ gravity in which the βR^2 term is treated as a back reaction on Einstein's gravity [12]. In essence, in this approach the gravitational degrees of freedom are not altered from those of Einstein's gravity. Using such an interpretation of $R+\beta R^2$ gravity, the author investigated the classical and the Wheeler-DeWitt evolutions of $R+\beta R^2$ gravity for the particular sign of β corresponding to the nontachyon case. The matter sector was described by a phenomenological $\rho \propto a(t)^{-n}$. It was concluded that both the Friedmann potential $U(a)$ $[\dot{a}^2+2U(a)=0]$ and the Wheeler-DeWitt potential $W(a)$ $\{[-\partial^2/\partial a^2 + 2W(a)]\psi(a)=0\}$ develop repulsive barriers near $a \approx 0$ for $n>4$ $(p>\frac{1}{3}\rho)$. The interpretations were clear. The repulsive barrier in $U(a)$ implies that a contracting Friedmann Robertson Walker (FRW) universe $(k>0, k=0, k<0)$ will bounce to an expansion phase without a total gravitational collapse. The repulsive barrier in $W(a)$ means that $a \approx 0$ is a classically forbidden region. Therefore the probability of finding a universe with the big bang singularity $(a=0)$ is exponentially suppressed.

Superficially, this prediction of no cosmological singularity for $n>4$ $(p>\frac{1}{3}\rho)$ seems to be in violation of the singularity theorem for Einstein's gravity, which predicts the eventual formation of a singularity for a matter satisfying the SEC $[p \geqslant -\frac{1}{3}\rho$ and $p \geqslant -\rho$ or equivalently $n \geqslant 2$ for $\rho \propto a(t)^{-n}]$ [13].

In this paper we study Raychaudhuri's equation in the background of $R+\beta R^2$ gravity coupled to a matter $[\rho \propto a(t)^{-n}]$. We conclude that the appropriate SEC for $R+\beta R^2$ gravity is different from that of Einstein's gravity. The SEC for $R+\beta R^2$ gravity is derived and is shown to be in agreement with the author's previous work, which demonstrated that both the classical and the Wheeler-DeWitt solutions of $R+\beta R^2$ gravity were free of the cosmological singularity for $n>4$.

The sign conventions used in this paper are $g=(-,+,+,+)$, $R_{ab}-\frac{1}{2}g_{ab}R=(+)8\pi G T_{ab}$, $(+)R(U,V) = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]}$.

II. REVIEW OF THE STRONG ENERGY CONDITION FOR EINSTEIN'S GRAVITY

Even though the goal of this paper is to study the strong energy condition (SEC) for quadratic gravity, much of the

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technique will be borrowed from the derivation of the SEC for Einstein's gravity. Therefore we shall briefly review this derivation for Einstein's gravity. For a pedagogical review please see [13].

Let ξ^a be a tangent vector to a congruence of timelike geodesics. Raychaudhuri's equation is

$$\frac{d\theta}{d\tau}=-\frac{1}{3}\theta^2-\sigma_{ab}\sigma^{ab}+R_{ab}\xi^a\xi^b. \tag{2.1}$$

θ and σ^{ab} are the expansion and the sheer of two nearby tangent vectors, respectively. In Einstein's gravity the SEC and the ensuing singularity theorem follow from requiring that

$$-R_{ab}\xi^a\xi^b=8\pi G[T_{ab}-\tfrac{1}{2}Tg_{ab}]\xi^a\xi^b\geqslant 0 \tag{2.2}$$

for all timelike vectors ξ^a. In such a case $d\theta/d\tau<0$ and the pair of nearby timelike geodesic vectors converges and will eventually intersect.

We will be interested in the cosmological singularity. In a FRW metric the matter sector is described by

$$T^a_b=\begin{pmatrix}-\rho & & & \\ & p & & \\ & & p & \\ & & & p\end{pmatrix}=\rho t^a t_b+px^a x_b+py^a y_b+px^a z_b. \tag{2.3}$$

$\{t^a,x^a,y^a,z^a\}$ are the eigenvectors of T^a_b. They are normalized as $-t^a t_a=x^a x_a=y^a y_a=z^a z_a=1$. Because of the isotropy one can always rotate the coordinates such that the most general forward timelike vector is

$$\xi^a=At^a+(A^2-1)^{1/2}x^a,\quad [A\in(1,\infty),\ \ \xi^a\xi_a=-1]. \tag{2.4}$$

Then

$$\begin{aligned}-R_{ab}\xi^a\xi^b&=8\pi G[T_{ab}-\tfrac{1}{2}Tg_{ab}]\xi^a\xi^b\\ &=8\pi G[\rho(A^2-\tfrac{1}{2})+p(A^2+\tfrac{1}{2})].\end{aligned} \tag{2.5}$$

The SEC is the requirement on an equation of state (p,ρ) for which $(-R_{ab}\xi^a\xi^b\geqslant 0)$ is true for all timelike vectors ξ^a $[A\in(1,\infty)]$.

The expression (2.5) is a monotonic function of A^2. Therefore if we can find an equation of state (p,ρ) for which $(-R_{ab}\xi^a\xi^b\geqslant 0)$ is satisfied by two extreme timelike vectors ($A=1$ and $A\to\infty$), then this (p,ρ) is guaranteed to satisfy $(-R_{ab}\xi^a\xi^b\geqslant 0)$ for all timelike vectors $A\in(1,\infty)$. Requiring $(-R_{ab}\xi^a\xi^b\geqslant 0)$ for $A=1$ gives $\rho+3p\geqslant 0$. Requiring $(-R_{ab}\xi^a\xi^b\geqslant 0)$ for $A\to\infty$ gives $\rho+p\geqslant 0$. These are the familiar SEC [13].

III. DERIVATION OF THE STRONG ENERGY CONDITION FOR $R+R^2$ GRAVITY

Raychaudhuri's equation describes how a congruence of timelike geodesics deviate from one another. Therefore Raychaudhuri's equation (2.1) is valid even in a background of a quadratic gravity. As in Einstein's gravity one can be assured of the convergence of a congruence of timelike geodesics by requiring $-R_{ab}\xi^a\xi^b\geqslant 0$. In order to proceed we need the relevant "Einstein's" equations for a quadratic gravity (a pedagogical derivation is given in the Appendix).

$$\begin{aligned}&\tfrac{1}{2}Rg_{ab}-R_{ab}+16\pi G\beta(\tfrac{1}{2}R^2g_{ab}-2RR_{ab}+2R^{;n}_{;n}g_{ab}-2R_{;a;b})\\ &\quad=8\pi GT_{ab}.\end{aligned} \tag{3.1}$$

The trace of this equation is

$$6\times 16\pi G\beta R^{;n}_{;n}+R=8\pi GT. \tag{3.2}$$

Some comments are in order. First, by dimensional consideration, β is dimensionless. Second, as noted by various authors [14,15], (3.2) resembles the scalar field equation with $m^2=-(6\times 16\pi G\beta)^{-1}$. Therefore $\beta<0$ is needed to eliminate tachyons. Third, an order of magnitude estimate reveals that the contributions from the quadratic terms are smaller than those of Einstein's gravity by $\beta GR\approx\beta G^2\rho\approx\beta\rho/\rho_{\text{Planck}}$. Therefore if we make a reasonable assumption that $\beta\approx 1$, then we will be justified in treating the quadratic terms as perturbations until the very early universe. For a more elaborate discussion on the justification for treating βR^2 as a perturbation, please see [12]. The subsequent analysis should be viewed as a perturbative analysis of a possible nonlinear phenomenon. On the other hand, it is worth noting that Mijic *et al.*, [16] Starobinsky *et al.*, [17] and Berkin [18] have studied the large β range and have concluded that even the pure $R+\beta R^2$ gravity ($T_{\mu\nu}=0$) can generate an inflationary phase.

For use in Raychaudhuri's equation we need an expression for the Ricci tensor (R_{ab}) in terms of T_{ab}. Combining (3.1) and (3.2) we get

$$\begin{aligned}&-R_{ab}+16\pi G\beta(\tfrac{1}{2}R^2g_{ab}-2RR_{ab}-R^{;n}_{;n}g_{ab}-R_{;a;b})\\ &\quad=8\pi G(T_{ab}-\tfrac{1}{2}Tg_{ab}).\end{aligned} \tag{3.3}$$

We are interested in the first-order contributions from the βR^2 term to Raychaudhuri's equation. Therefore for the terms already multiplied by β in (3.1) we may substituting $R_{ab}=-8\pi G(T_{ab}-\tfrac{1}{2}Tg_{ab})+O(\beta)$ and $R=8\pi GT+O(\beta)$ to get

$$\begin{aligned}&(\tfrac{1}{2}R^2g_{ab}-2RR_{ab}-2R^{;n}_{;n}g_{ab}-2R_{;a;b})\\ &\quad=-\tilde{G}(\tfrac{1}{2}T^2g_{ab}\tilde{G}-2TT_{ab}\tilde{G}+T^{;n}_{;n}g_{ab}+2T_{;a;b})\\ &\qquad+O(\beta).\end{aligned} \tag{3.4}$$

We have introduced the notation $\tilde{G}\equiv 8\pi G$. And, finally, for the use in Raychaudhuri's equation, we get

$$\begin{aligned}-R_{ab}\xi^a\xi^b=&[\tilde{G}(T_{ab}-\tfrac{1}{2}Tg_{ab})+2\beta\tilde{G}^2(\tfrac{1}{2}T^2g_{ab}\tilde{G}\\ &-2TT_{ab}\tilde{G}+T^{;n}_{;n}g_{ab}+2T_{;a;b})]\xi^a\xi^b+O(\beta^2).\end{aligned} \tag{3.5}$$

From (3.5) it is clear that $R+R^2$ gravity with $T_{\mu\nu}$ have been replaced by Einstein's gravity with an effective $T^{\text{eff}}_{\mu\nu}$. (For a fuller discussion, please see [12].) The rest of the procedure more of less mimicks Einstein's case. The expression for a general timelike vector ξ^a is (2.4). The T_{ab} that is appropriate for a FRW metric is (2.2).

We will assume that during any epoch in the evolution of the universe, the universe is dominated by a matter with the characteristic dependence on the scale factor ($\rho=\rho_0/a^n$). A local conservation of T_{ab} gives $p=(n-3)/3\rho_0/a^n$.

A purist may argue that the proper way of including a matter in the early universe is by incorporating a quantum field (e.g., scalar field). We feel that in the study of Raychaudhuri's equation, describing the matter by a phenomenological $\rho=\rho_0/a^n$ is adequate. This is because Raychaudhuri's equation studies the geometrical optics limit of a field in a background of metric and matter. If we had used a scalar field, then we would have had a difficult task of splitting the field $\phi(x,t)=\phi_0(x,t)+\delta\phi(x,t)$, where the $\phi_0(x,t)$ and the $\delta\phi(x,t)$ are the background low frequency and the fluctuating high frequency components, respectively. The background component $\phi_0(x,t)$ would again result in an effective $\rho=\langle T_0^0[\phi_0(x,t)]\rangle_{\text{space}}\propto a^{-n}$ for some (n). The value of (n) would depend on the detail properties of the field [e.g., mass, self-coupling $V(\phi)$], and Raychaudhuri's equation would correspond to the geometrical optics limit of the $\delta\phi(x,t)$.

Continuing, it is a laborious exercise to show that (3.5) becomes

$$-R_{ab}\xi^a\xi^b=\frac{n-2}{2}\tilde{G}\rho_n+\beta\tilde{G}^2[3n(n-1)(n-4)\tilde{G}\rho_n^2 -6n^2(n-4)ka^{-2}\rho_n] +(A^2-1)\left\{\frac{n}{3}\tilde{G}\rho_n+\beta\tilde{G}^2[2n^2(n-4)\tilde{G}\rho_n^2 -4n(n+2)(n-4)ka^{-2}\rho_n]\right\}. \tag{3.6}$$

A reader who may want to derive (3.6) from (3.5) will find the following helpful.

$$T\equiv T_{ab}g^{ab}=3p-\rho=(n-4)\rho_n, \tag{3.7}$$

$$T_{ab}\xi^a\xi^b=A^2(p+\rho)-p, \tag{3.8}$$

$$T_{;a;b}\xi^a\xi^b=A^2\left(\partial_t^2T-\frac{\dot{a}}{a}\partial_tT\right)+\frac{\dot{a}}{a}\partial_tT, \tag{3.9}$$

$$T_{;a}^{;a}=-a^{-3}\partial_t(a^3\partial_tT)=-(n-4)a^{-3}\partial_t(a^3\partial_t\rho_n). \tag{3.10}$$

Again, the strong energy condition (SEC) is the requirement on an equation of state for which $-R_{ab}\xi^a\xi^b\geqslant 0$ for all forward timelike vectors ξ^a. There is a subtlety to keep in mind. From (2.1) we can conclude that even though $(-R_{ab}\xi^a\xi^b\geqslant 0)$ implies that a pair of timelike geodesics converges, the opposite $(-R_{ab}\xi^a\xi^b\leqslant 0)$ does not imply that a pair of timelike geodesics diverges. This is because of the other negative negative terms in (2.1).

The expression (3.6) is a monotonic function of A^2. Similar to Einstein's case, if we can find an equation of state (p,ρ) for which $(-R_{ab}\xi^a\xi^b\geqslant 0)$ is satisfied by the two extreme timelike vectors ($A=1$ and $A\to\infty$), then this (p,ρ) is guaranteed to satisfy $(-R_{ab}\xi^a\xi^b\geqslant 0)$ for all timelike vectors $A\in(1,\infty)$.

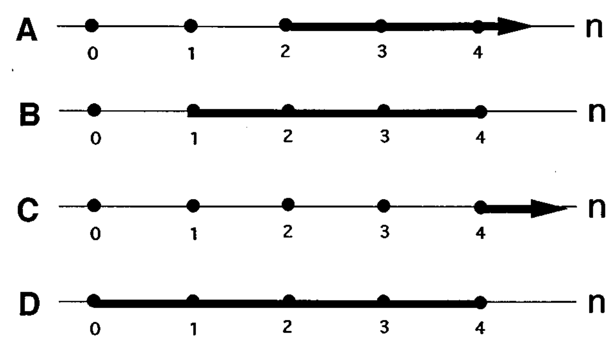

FIG. 1. (a) The strong energy condition for Einstein's gravity ($\beta=0$). (b) $-R_{ab}\xi^a\xi^b\geqslant 0$ determined under the assumption of ρ_n^2 domination. (c) $-R_{ab}\xi^a\xi^b\geqslant 0$ determined under the assumption of $ka^{-2}\rho_n$ domination ($k>0$). (d) $-R_{ab}\xi^a\xi^b\geqslant 0$ determined under the assumption of $ka^{-2}\rho_n$ domination ($k<0$).

Requiring $-R_{ab}\xi^a\xi^b\geqslant 0$ for $A=1$ gives

$$\frac{n-2}{2}\tilde{G}\rho_n+\beta\tilde{G}^2[3n(n-1)(n-4)\tilde{G}\rho_n^2 -6n^2(n-4)ka^{-2}\rho_n]\geqslant 0. \tag{3.11}$$

Requiring $-R_{ab}\xi^a\xi^b\geqslant 0$ for $A\to\infty$ gives

$$\frac{n}{3}\tilde{G}\rho_n+\beta\tilde{G}^2[2n^2(n-4)\tilde{G}\rho_n^2-4n(n+2)(n-4)ka^{-2}\rho_n] \geqslant 0. \tag{3.12}$$

The problem of finding the SEC has been reduced to solving for (n) that will satisfy (3.11) and (3.12) as $a\to 0$.

As a partial check in the algebra, consider the case of $\beta=0$. (3.11) gives $n\geqslant 2$, which is equivalent to $p_n\geqslant -\frac{1}{3}\rho_n$. And (3.12) gives $n\geqslant 0$, which is equivalent to $p_n\geqslant -\rho_n$. As expected, these are the SEC for Einstein's gravity.

Now for the $\beta\neq 0$ case ($\beta<0$ for a tachyon free system) the argument is significantly more subtle. We shall first extract various collection of results from (3.11) and (3.12). We note that as one goes further back in time, depending on the value of (n), one of the $(\rho_n,\rho_n^2,a^{-2}\rho_n)$ terms in (3.11) and (3.12) will grow fastest and hence will dominate the expressions. We already know the result if the ρ_n term dominates. Because this is equivalent to Einstein's case ($\beta=0$), we can conclude that $n\geqslant 2$ is necessary to satisfy (3.11) and (3.12) [Fig. 1(a)]. Now if the ρ_n^2 term dominates, then (3.12) reduces to $\beta n(n-1)(n-4)\geqslant 0$ ($1\leqslant n\leqslant 4$ since $\beta<0$), and (3.12) reduces to $\beta n^2(n-4)\geqslant 0$ ($0\leqslant n\leqslant 4$). The two are simultaneously satisfied by $1\leqslant n\leqslant 4$ [Fig. 1(b)]. And, finally, if the curvature term ($ka^{-2}\rho_n$) dominates, then (3.11) reduces to $-k\beta n^2(n-4)\geqslant 0$ and (3.12) reduces to $-k\beta n(n+2)(n-4)\geqslant 0$. The two reduced equations are simultaneously satisfied for the $k>0$ FRW metric by $n\geqslant 4$

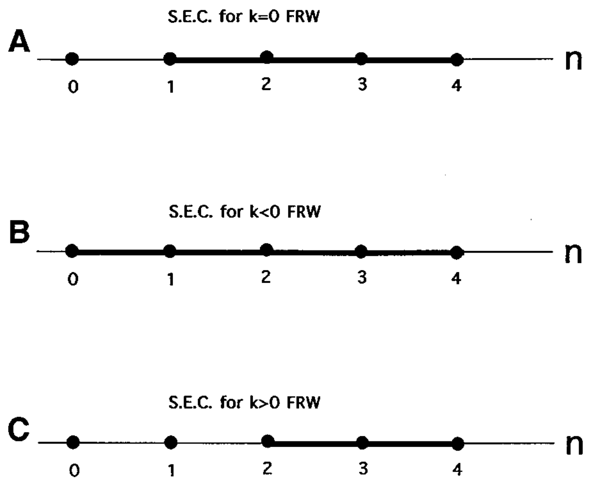

FIG. 2. (a) The strong energy condition for the $k=0$ FRW metric ($1\leqslant n \leqslant 4$, i.e., $-\frac{2}{3}\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$). The universe ends in ρ_n^2 domination. (b) The strong energy condition for the $k<0$ FRW metric ($0\leqslant n \leqslant 4$, i.e., $-\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$). For $0\leqslant n \leqslant 2$ the universe ends in $ka^{-2}\rho_n$ domination. For $2\leqslant n\leqslant 4$ the universe ends in ρ_n^2 domination. (c) The strong energy condition for the $k>0$ FRW metric ($2\leqslant n \leqslant 4$, i.e., $-\frac{1}{3}\rho_n \leqslant p_n \leqslant \frac{1}{3}\rho_n$). The universe ends in ρ_n^2 domination.

[Fig. 1(c)], and for the ($k<0$) FRW metric by $0\leqslant n \leqslant 4$ [Fig. 1(d)]. We have tactfully assumed that $n\geqslant 0$, i.e., energy density should not get less dense when squeezed. (Note that only when $0\leqslant n\leqslant 2$ can the curvature term $ka^{-2}\rho_n$ ever grow to dominate over the ρ_n^2 term as $a\rightarrow 0$.)

We have obtained a various collection of results. A slightly more useful conclusion would be whether a particular set of (ρ_n, k, β) has the big bang singularity. We will proceed to address this question in two steps. The strategy will be as follows. First, for a given (ρ_n, k, β) we will have to determine which one of the $(\rho_n, \rho_n^2, a^{-2}\rho_n)$ terms will end up dominating (3.11) and (3.12) as $a\rightarrow 0$. Then from Figs. 1(b)–1(d) we will be able to read off whether such a set of (ρ_n, k, β) has the big bang singularity.

Let us first consider the $k=0$ FRW metric, with $\rho \propto a(t)^{-n}$. As $a\rightarrow 0$, the ρ_n^2 term should dominate over the ρ_n term. Therefore from Fig. 2(a) we can deduce that the appropriate SEC is $(1\leqslant n \leqslant 4)$.

A careful combination of the previous results reveals that the SEC for the $k<0$ FRW metric is $0\leqslant n\leqslant 4$ [Fig. 2(b)]. The reasoning is as follows. As $a\rightarrow 0$, either the $ka^{-2}\rho_n$ term or the ρ_n^2 term could end up dominating (3.11) and (3.12). For $0\leqslant n \leqslant 2$, the $ka^{-2}\rho_n$ term will dominate, and from Fig. 1(d) we can deduce that all of this region $(0\leqslant n\leqslant 2)$ result in the convergence of a pair of timelike vectors. On the other hand for $n\geqslant 2$, the ρ_n^2 term will end up dominating. From Fig. 1(b), of this region, only $2\leqslant n\leqslant 4$ result in the convergence of a pair of timelike vectors. Q.E.D.

Similarly, an analysis reveals that the SEC for $k>0$ FRW universe is $2\leqslant n\leqslant 4$ [Fig. 2(b)]. Again for $0\leqslant n \leqslant 2$, the $ka^{-2}\rho_n$ term will dominate. But from Fig. 1(d) none of this region results in the convergence of a pair of timelike vectors. On the other hand for $n\geqslant 2$, the ρ_n^2 term will again end up dominating. And from Fig. 1(b), of this region, only $2\leqslant n \leqslant 4$ result in the convergence of a pair of timelike vectors. Q.E.D.

These results, of the cosmological strong energy condition for $R+R^2$ gravity, are summarized in Figs. 2(a)–2(c).

IV. DISCUSSIONS AND CONCLUSION

We were primarily interested in how the big bang singularity would be affected by the quantum effects. Granted that even a classical analysis of $R+\beta R^2$ gravity can be interpreted as a semiclassical analysis, there are several limitations to our analysis, and we would like to briefly raise these points.

(1) Near the Planck epoch, large quantum fluctuations will undoubtedly result in an inhomogeneous universe, yet, to make the problem analytically tractable, we have assumed a homogeneous and isotropic metric in (2.3) and (2.4). A future work will have to address how the present conclusion is affected by an anisotropy and inhomogeneity.

(2) Raychaudhuri's equation is only a classical geometric optics limit which may become invalid near the Planck epoch.

(3) In Raychaudhuri's equation we have taken the limit as $a\rightarrow 0$. A more physical limit might be to cut off the limit at $\rho_n \rightarrow \rho_{\text{Planck}}, ka^{-2} \rightarrow (\text{Planck length})^{-2}$. The various sets of conclusions obtained by this limiting process turn out to be sensitive to the fine-tuning of the parameters and will not be discussed here. An interested reader is invited to explore the various possibilities.

(4) In retrospect, in deriving the SEC for $R+\beta R^2$ gravity we have pushed the analysis until the $k\beta a^{-2}\rho_n$ term or the $\beta\rho_n^2$ term dominated over the ρ_n term. Yet if these these terms, which are linear in β, ever grew larger than the terms from Einstein's gravity, then this would be beyond the validity of the perturbation analysis. This is a serious objection. But as it is common practice in perturbation theory, we are inclined to interpret these results as a possible glimpse of a nonperturbative effect. With this in mind we must settle for the following weaker conclusion.

"The cosmological SEC for Einstein's gravity is $n\geqslant 2$. But for $R+\beta R^2$ gravity, the first-order perturbation analysis indicates the deformation of this SEC from Einstein's case $(n\geqslant 2)$ such that (1) for the $k<0$ FRW metric, $n\geqslant 4$ may have to be excluded and $0\leqslant n\leqslant 2$ may have to be included, (2) for the $k=0$ FRW metric, $n\geqslant 4$ may have to be excluded and $1\leqslant n\leqslant 2$ may have to be included, and (3) for the $k>0$ FRW metric, $n\geqslant 4$ may have to be excluded."

Finally, let us address the interesting question of whether the pure $R+R^2$ gravity $(T_{\mu\nu}=0)$ can avoid the big bang singularity. From Figs. 2(a)–2(c) we can conclude that $\rho_n = p_n = 0$ satisfy the SEC (one gets $0\geqslant 0\geqslant 0$). Therefore there must be pure $R+R^2$ gravity models with the big bang singularity.

Superficially, this is in conflict with works of [7–9]. The difference originates from two different views of the quadratic gravity. We are interested in interpreting the quadratic terms as semiclassical corrections induced by one-loop effects. But for $T_{\mu\nu}=0$ the induced corrections vanish (3.4). On the other hand, these authors have interpreted quadratic gravity as a fundamental theory. Within such a context, they were then able to demonstrate that the classical evolutions of

$R+R^2$ gravity with $T_{\mu\nu}=0$ may avoid the big bang singularity.

A direct support of our result can be found in the work of [19]. 't Hooft and Veltman have shown, using the background field method, that the one-loop corrections to pure Einstein's gravity must vanish. The proof is as follows. The one-loop corrections to the gravitational action for the vacuum is $\triangle L_{\text{grav}}=1/\epsilon[\frac{1}{120}R^2+\frac{7}{20}R_{\mu\nu}R^{\mu,\nu}]$. In the background field method, one must impose the background field equations, $R_{\mu\nu}=R=0$. Q.E.D.

In closing, the derivation of the SEC for $R+R^2$ gravity complements the authors recent work, which demonstrated that both the classical and the Wheeler-DeWitt solutions of $R+\beta R^2$ gravity were free of the big bang singular for $n>4$.

APPENDIX

The action for the quadratic gravity is

$$I=-\frac{1}{16\pi G}\int d^4x\sqrt{-g}R-\int d^4x[\beta_1R^2+\beta_2R_{ab}R^{ab}+\beta_3R_{abcd}R^{abcd}]+I_{\text{matter}}+\text{surface term}. \tag{A1}$$

We have formally included a surface term to cancel any boundary term that would result in applying the variational principle. We will be interested in applying the formalism to a homogeneous and isotropic metric, i.e., the Weyl tensor vanishes $C_{abcd}=0$ [20]. By definition of the Weyl tensor, $C_{abcd}C^{abcd}=R_{abcd}R^{abcd}-2R_{ab}R^{ab}+\frac{1}{3}R^2$. This gives one relationship among the possible quadratic terms. The second relationship is from the four-dimensional generalization of the Gauss-Bonnet formula [5]

$$R^2-4R_{ab}R^{ab}+R_{abcd}R^{abcd}=\text{exact derivative}. \tag{A2}$$

The two relationships, combined with the fact that Euler Lagrange equations are unchanged by addition of an exact differential, allow any two of β_1, β_2, and β_3 to be set equal to zero in the action (A.1). We choose to set $\beta_3=\beta_2=0$.

The resulting Euler Lagrange equations are

$$\frac{1}{2}Rg_{ab}-R_{ab}+16\pi G\beta(\tfrac{1}{2}R^2g_{ab}-2RR_{ab}+2R_{;\sigma}^{;\sigma}g_{ab}-2R_{;a;b})=8\pi GT_{ab}. \tag{A3}$$

The trace of this equation is

$$6\times16\pi G\beta R_{;\sigma}^{;\sigma}+R=8\pi GT. \tag{A4}$$

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