

## Natural mass matrices

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We introduce the idea of natural mass matrices, an organizing principle useful in the search for GUT scale quark mass matrix patterns that are consistent with known CKM constraints and quark mass eigenvalues. An application of this idea is made in the context of SUSY GUT's and some potentially "successful" GUT scale mass patterns are found. The CKM predictions of these patterns are presented and some relevant strong *CP* issues are discussed.

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### I. INTRODUCTION

It has been a theoretical quest for nearly 20 years to devise interesting mass matrix patterns which could provide sound predictions for the Cabibbo-Kobayashi-Maskawa (CKM) matrix and quark mass eigenvalues [1]. Recently, these efforts have centered on constructing patterns at the grand unified theory (GUT) scale in supersymmetric (SUSY) theories [2,3]. Most of these attempts, although quite successful, have failed to produce results in complete agreement with precise low energy data (LED) [4]. Part of the difficulty lies in the fact that one has to rely upon "guesses" at the GUT scale which are then extrapolated down to the weak scale, with the hope that the mass matrices so obtained will give rise to acceptable fits to the LED. A somewhat more promising approach, perhaps, would be to reverse the process: constructing LED-consistent mass matrices at some low energy scale and then evolve them upwards to see if interesting GUT patterns would emerge. However, since there exists considerable arbitrariness in this construction, one typically has to be content with studying only certain special cases [5,6] often guided merely by simplicity. Here,

we would like to suggest an organizing principle which may be helpful. This is the idea of natural mass matrices, which severely restricts the aforementioned arbitrariness in the mass pattern construction, thereby allowing a search for viable GUT patterns more systematically and efficiently.

This paper is organized as follows: In Sec. II, we summarize our present knowledge of the CKM parameters and quark masses; in Sec. III, we introduce the idea of natural mass matrices along with our method of mass-matrix parametrization that facilitates its implementation; in Sec. IV, we present some Hermitian GUT scale mass patterns that emerge from this approach and their CKM predictions; in Sec. V, we discuss issues connected with Hermiticity breakdown and associated problems with strong *CP* violation; finally in Sec. VI, we present our conclusions.

### II. CKM AND QUARK MASSES: A SUMMARY OF PRESENT STATUS

#### A. CKM matrix

In its standard form [7], the Cabibbo-Kobayashi-Maskawa (CKM) matrix is

$$[\text{CKM}]_s = \begin{pmatrix} c_1 c_3 & s_1 c_3 & s_3 e^{-i\delta} \\ -s_1 c_2 - c_1 s_2 s_3 e^{i\delta} & c_1 c_2 - s_1 s_2 s_3 e^{i\delta} & s_2 c_3 \\ s_1 s_2 - c_1 c_2 s_3 e^{i\delta} & -c_1 s_2 - s_1 c_2 s_3 e^{i\delta} & c_2 c_3 \end{pmatrix}. \quad (2.1)$$

Taking into account the experimental hierarchy in the mixing angles, one can write

$$s_1 \equiv \sin \theta_1 \equiv \lambda \approx 0.22, \quad s_2 \equiv \sin \theta_2 \equiv A \lambda^2, \quad s_3 \equiv \sin \theta_3 \equiv A \sigma \lambda^3, \quad (2.2)$$

with  $A, \sigma$  being parameters roughly of  $O(1)$ . The CKM matrix then takes the Wolfenstein [8] form<sup>1</sup>

$$[\text{CKM}]_s = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda & A\sigma\lambda^3 e^{-i\delta} \\ -\lambda & 1 - \frac{1}{2}\lambda^2 - \left(\frac{1}{2}A^2 + \frac{1}{8}\right)\lambda^4 & A\lambda^2 \\ A\lambda^3(1 - \sigma e^{i\delta}) & -A\lambda^2 + \frac{1}{2}A\lambda^4 & 1 - \frac{1}{2}A^2\lambda^4 \end{pmatrix} + O(\lambda^5). \quad (2.3)$$

<sup>1</sup>Wolfenstein uses the parameters  $\rho, \eta$  instead of  $\sigma, \delta$ . They are related by  $\sigma e^{-i\delta} \equiv \rho - i\eta$ .

The parameter  $A$  is fixed by  $V_{cb}$  which, in turn, is determined from semileptonic  $B$  decays. The most recent analysis from CLEO data [9] gives

$$|V_{cb}| = 0.0378 \pm 0.0026 \leftrightarrow A = 0.78 \pm 0.05. \quad (2.4)$$

The parameter  $\sigma$  is fixed by the ratio  $|V_{ub}|/|V_{cb}|$ . This in turn can be extracted from a study of semileptonic  $B$  decays near the end point region of the lepton spectrum, where  $b \rightarrow c$  quark transitions are forbidden. The most recent analysis [9] gives

$$|V_{ub}|/|V_{cb}| = 0.08 \pm 0.02 \leftrightarrow \sigma = 0.36 \pm 0.09. \quad (2.5)$$

The phase  $\delta$  (or the parameter  $\eta$ ) can be gotten by combining the measurements of the  $\epsilon$  parameter in  $K-\bar{K}$  mixing and those of  $\Delta m_{B_d}$  in  $B_d-\bar{B}_d$  mixing with the value of  $|V_{ub}|/|V_{cb}|$ . A recent analysis [10] gives an allowed region in the  $\rho-\eta$  plane roughly specified by the ranges

$$\eta \simeq [0.2, 0.5], \quad \rho \simeq [-0.4, 0.3], \quad (2.6)$$

with a corresponding CKM phase

$$\delta \simeq [45^\circ, 158^\circ]. \quad (2.7)$$

### B. Quark masses

For the purpose of calculating quark mass ratios and their renormalization group (RG) scaling, it is convenient to express all quark masses as running masses at some common energy scale. We shall choose this scale here to be the mass of the top quark  $m_t$ . We summarize below what is known about the quark masses and then extrapolate all the results to the scale  $m_t$ . From the recent ‘‘discovery’’ papers on the top quark [11], one infers a value for the physical mass  $m_t^{\text{phys}}$ , which is related to the running mass by

$$m_t(m_t) = \frac{m_t^{\text{phys}}}{1 + \frac{4}{3\pi} \alpha_s(m_t)}.$$

These results [11] suggest that

$$m_t(m_t) \simeq (175 \pm 15) \text{ GeV}.$$

For medium heavy quarks, the analyses of charmonium and bottomonium spectra [12] give

$$m_c(m_c) = (1.27 \pm 0.05) \text{ GeV},$$

$$m_b(m_b) = (4.25 \pm 0.10) \text{ GeV}.$$

Finally, for light quarks, current algebra analyses [12] give the following values for the masses at a scale of 1 GeV:

$$m_u(1 \text{ GeV}) = (5.1 \pm 1.5) \text{ MeV},$$

$$m_d(1 \text{ GeV}) = (8.9 \pm 2.6) \text{ MeV},$$

$$m_s(1 \text{ GeV}) = (175 \pm 55) \text{ MeV}.$$

These mass values are individually uncertain to  $O(30\%)$  but are rather better constrained [13,14] by the current algebra relation

$$\left(\frac{m_u}{m_d}\right)^2 + \frac{1}{Q^2} \left(\frac{m_s}{m_d}\right)^2 = 1 \text{ with } Q = 24 \pm 1.6. \quad (2.8)$$

The RG scaling of the medium heavy and light quark masses to  $m_t$  are calculated to three loops in Ref. [3], with the result sensitive to the precise value of the strong coupling constant  $\alpha_s(M_Z)$ . Using  $\alpha_s(M_Z) = 0.117 \pm 0.05$  and expressing all quark mass ratios in terms of the small parameter  $\lambda \simeq 0.22$ , the above results allow one to write the diagonal quark mass matrices as

$$M_u^{\text{diag}}(m_t) = 175 \text{ GeV} \begin{pmatrix} \xi_{ut} \lambda^7 & 0 & 0 \\ 0 & \xi_{ct} \lambda^4 & 0 \\ 0 & 0 & \xi_{tt} \end{pmatrix} \\ \equiv m_t(m_t) \tilde{M}_u^{\text{diag}}, \quad (2.9)$$

where  $\xi_{ut} = 0.49 \pm 0.15$ ,  $\xi_{ct} = 1.46 \pm 0.13$ ,  $\xi_{tt} = 1 \pm 0.09$ , and

$$M_d^{\text{diag}}(m_t) = 2.78 \text{ GeV} \begin{pmatrix} \xi_{db} \lambda^4 & 0 & 0 \\ 0 & \xi_{sb} \lambda^2 & 0 \\ 0 & 0 & \xi_{bb} \end{pmatrix} \\ \equiv m_b(m_t) \tilde{M}_d^{\text{diag}}, \quad (2.10)$$

where  $\xi_{db} = 0.58 \pm 0.18$ ,  $\xi_{sb} = 0.55 \pm 0.18$ ,  $\xi_{bb} = 1 \pm 0.05$ .

## III. NATURAL MASS MATRICES

### A. Heuristic two-generation example: The notion of naturalness

To introduce the idea of natural mass matrices, we consider first the simple two-quark-generation case, assuming that these mass matrices are Hermitian. A general  $2 \times 2$  Hermitian mass matrix for the first two quark families can always be rewritten, after some trivial phase redefinitions of the quark fields, as some real symmetric matrix. Two such matrices for the  $u$  quarks and  $d$  quarks,  $M_u \equiv m_c \tilde{M}_u$ ,  $M_d \equiv m_s \tilde{M}_d$ , in turn can be diagonalized by some orthogonal matrices  $O_u$  and  $O_d$ , resulting in a Cabibbo-quark-mixing matrix  $C$ . One has<sup>2</sup>

$$O_u^T \tilde{M}_u O_u = \tilde{M}_u^{\text{diag}} \equiv \begin{pmatrix} \xi_{uc} \lambda^4 & 0 \\ 0 & 1 \end{pmatrix}, \\ O_d^T \tilde{M}_d O_d = \tilde{M}_d^{\text{diag}} \equiv \begin{pmatrix} \xi_{ds} \lambda^2 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.1)$$

and

<sup>2</sup>Note that  $\xi_{uc} = \xi_{ut}/(\xi_{ct}\lambda) = (1.53 \pm 0.49)$  while  $\xi_{ds} = \xi_{db}/\xi_{sb} = (1.05 \pm 0.47)$ .

$$C = O_u^T O_d = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}. \quad (3.2)$$

In the above,  $\lambda \equiv \sin\theta_C$  as before and the matrices  $O_u$ ,  $O_d$  have the same form as the matrix  $C$  with angles  $\theta_u$ ,  $\theta_d$  instead of  $\theta_C$ . We notice that it follows from Eq. (3.2) that  $\theta_C = \theta_d - \theta_u$ . Moreover, since the matrix  $C$  is invariant under the changes

$$O_d \rightarrow O O_d, \quad O_u \rightarrow O O_u,$$

where  $O$  is some arbitrary orthogonal matrix, we see that  $\tilde{M}_u$  and  $\tilde{M}_d$  are fixed only up to a common similarity transformation

$$\tilde{M}_u \leftrightarrow O^T \tilde{M}_u O, \quad \tilde{M}_d \leftrightarrow O^T \tilde{M}_d O.$$

Because of this freedom and since  $\theta_C \sim \lambda \ll 1$ , we can always arrange to have both<sup>3</sup>  $\theta_d \ll 1$  and  $\theta_u \ll 1$ . We can now contemplate three different options for the angles  $\theta_u$  and  $\theta_d$ :

$$\begin{aligned} \text{(i)} \quad & \sin\theta_d \sim \lambda, \quad \sin\theta_u \sim \lambda, \\ \text{(ii)} \quad & \sin\theta_d \sim \lambda, \quad \sin\theta_u \leq \lambda^2, \\ \text{(iii)} \quad & \sin\theta_d \leq \lambda^2, \quad \sin\theta_u \sim \lambda. \end{aligned} \quad (3.3)$$

Upon examining the expressions for the mass matrices  $\tilde{M}_u$  and  $\tilde{M}_d$ ,

$$\begin{aligned} \tilde{M}_u &= O_u \tilde{M}_u^{\text{diag}} O_u^T = \begin{pmatrix} \xi_{uc} \lambda^4 + \sin^2 \theta_u & \sin \theta_u \\ \sin \theta_u & 1 \end{pmatrix}, \\ \tilde{M}_d &= O_d \tilde{M}_d^{\text{diag}} O_d^T = \begin{pmatrix} \xi_{ds} \lambda^2 + \sin^2 \theta_d & \sin \theta_d \\ \sin \theta_d & 1 \end{pmatrix}, \end{aligned} \quad (3.4)$$

one sees that if options (i) and (iii) were to hold, one would require a severe fine-tuning of the matrix element  $[\tilde{M}_u]_{11}$ , forcing  $[\tilde{M}_u]_{11} \approx ([\tilde{M}_u]_{12})^2$  to arrive at the large  $m_u/m_c \sim \lambda^4$  experimental hierarchy. Such a fine-tuning appears to be unnatural. For the two-generation case, the natural mass matrices have the form

$$\tilde{M}_u \approx \begin{pmatrix} \alpha'_u \lambda^4 & \alpha_u \lambda^2 \\ \alpha_u \lambda^2 & 1 \end{pmatrix}, \quad \tilde{M}_d \approx \begin{pmatrix} \alpha'_d \lambda^2 & \alpha_d \lambda \\ \alpha_d \lambda & 1 \end{pmatrix}, \quad (3.5)$$

corresponding to option (ii) in which

$$\begin{aligned} \sin\theta_u &= \alpha_u \lambda^2, & \alpha'_u - \alpha_u^2 &= \xi_{uc}, \\ \sin\theta_d &= \alpha_d \lambda, & \alpha'_d - \alpha_d^2 &= \xi_{ds}, \end{aligned}$$

and

$$\alpha_d - \alpha_u \lambda \approx 1.$$

With the parameters  $\alpha$  and  $\alpha'$  of  $O(1)$ , one gets the observed hierarchy [i.e., the  $\xi$ 's being of  $O(1)$ ] without any need for fine-tunings.

### B. Three-generation extension

It is straightforward, though somewhat tedious, to extend the idea of natural mass matrices to the three-generation case. To facilitate its implementation, however, we need to introduce a convenient parametrization of the mass matrices based on a perturbative expansion in  $\lambda$ . The procedure we shall adopt is analogous to, but a slight generalization of, a method employed by Ramond, Roberts, and Ross [6]. The benefits of our generalization, aside from allowing a search for natural mass patterns, also include the flexibility of simultaneous adjustments of matrix elements in  $M_u$  and  $M_d$  (useful in imposing 0's or arranging for equalities among them).

Consider some general  $3 \times 3$  Hermitian mass matrices  $M_u \equiv m_t(m_t) \tilde{M}_u$  and  $M_d \equiv m_b(m_t) \tilde{M}_d$ . These matrices can be diagonalized by some unitary matrices  $U$  and  $D$ , resulting in a CKM-quark-mixing matrix [CKM]

$$\tilde{M}_u = U \tilde{M}_u^{\text{diag}} U^\dagger, \quad (3.6)$$

$$\tilde{M}_d = D \tilde{M}_d^{\text{diag}} D^\dagger, \quad (3.7)$$

$$[\text{CKM}] = U^\dagger D. \quad (3.8)$$

To proceed, it is useful to make two observations.

(1) As in the two-generation case, if we change  $U \rightarrow NU$  and  $D \rightarrow ND$  (where  $N$  is some arbitrary unitary matrix), the matrix [CKM] remains unchanged. The matrices  $\tilde{M}_u, \tilde{M}_d$  are thus unique up to an arbitrary (but common) unitary transformation  $\tilde{M}_u \leftrightarrow N^\dagger(\tilde{M}_u)N$ ,  $\tilde{M}_d \leftrightarrow N^\dagger(\tilde{M}_d)N$ . As a result of [CKM]  $\approx 1$ , this arbitrariness allows us to focus only on "small" transformations, i.e.,  $U \approx 1$ ,  $D \approx 1$ . Furthermore, in the particular case where  $N = \phi_L$ ,  $\phi_L$  being some phase matrix, the induced changes in  $\tilde{M}_u$  and  $\tilde{M}_d$  ( $\tilde{M}_u \rightarrow \phi_L \tilde{M}_u \phi_L^\dagger$ ;  $\tilde{M}_d \rightarrow \phi_L \tilde{M}_d \phi_L^\dagger$ ) can be absorbed by a trivial quark field phase redefinition.

(2) So far as constructions of the mass matrices  $\tilde{M}_u$  and  $\tilde{M}_d$  out of the matrices  $\tilde{M}_u^{\text{diag}}, \tilde{M}_d^{\text{diag}}$ , and [CKM] are concerned, a change of

$$[\text{CKM}] \rightarrow \phi_u^\dagger [\text{CKM}] \phi_d \leftrightarrow D \rightarrow D \phi_d, \quad U \rightarrow U \phi_u,$$

with the  $\phi$ 's some arbitrary phase matrices, is inconsequential due to the fact that  $\phi_u \tilde{M}_u^{\text{diag}} \phi_u^\dagger = \tilde{M}_u^{\text{diag}}$  and  $\phi_d \tilde{M}_d^{\text{diag}} \phi_d^\dagger = \tilde{M}_d^{\text{diag}}$ .

Based on these observations, one is always justified working with a specific form of the CKM matrix. Thus one can write

$$U^\dagger D = [\text{CKM}] \equiv \phi_u^\dagger [\text{CKM}]_s \phi_d,$$

with

$$D \equiv \phi_L^\dagger D_s \phi_d, \quad U \equiv \phi_L^\dagger U_s \phi_u.$$

The phase matrices  $\phi_L, \phi_d$  can be further chosen so as to render  $D_s$  to have the same form as the matrix  $[\text{CKM}]_s$  of Eq. (2.1), and it follows that  $U_s^\dagger D_s = [\text{CKM}]_s$ . (Notice that having chosen  $D_s$  to be of the same form as the matrix

<sup>3</sup>As will become apparent below, this restriction need not to be imposed separately for it is in conformity with our naturalness requirement. It is done here for ease in the discussion that follows.

[CKM]<sub>s</sub>, there is now no more freedom to redefine  $U_s$  and  $U_s$ , in fact, will not have quite the standard CKM form.) The mass matrices  $\tilde{M}_u$ ,  $\tilde{M}_d$  constructed as

$$\begin{aligned}\tilde{M}_u &= U_s \tilde{M}_u^{\text{diag}} U_s^\dagger, \\ \tilde{M}_d &= D_s \tilde{M}_d^{\text{diag}} D_s^\dagger,\end{aligned}\quad (3.9)$$

are still perfectly general for our considerations.<sup>4</sup> Having established their generality, we shall drop the subscript  $s$  in the matrices [CKM]<sub>s</sub>,  $D_s$ , and  $U_s$  hereafter for notational brevity.

With these preliminaries out of the way, it is straightforward to give explicit parametrizations for the matrices  $U$  and  $D$ . For these purposes, it is useful to note that, in its standard form, the matrix [CKM] can be expressed as

$$\begin{aligned}[\text{CKM}] &= C_2 \Delta C_3 \Delta^\dagger C_1 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \Delta \begin{pmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{pmatrix} \\ &\quad \times \Delta^\dagger \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

with

$$\Delta \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}.$$

Because  $D$ , by assumption, also takes the CKM form, we can likewise write

$$D = C_{2d} \Delta_d C_{3d} \Delta_d^\dagger C_{1d}. \quad (3.10)$$

Here the matrices  $C_{id}$  and  $\Delta_d$  are defined analogously to  $C$  and  $\Delta$ , except they involve some new angles  $\theta_{id}$  ( $i=1,2,3$ ) and  $\delta_d$ . It is convenient, in addition, to define three more

<sup>4</sup>The matrices  $\tilde{M}_u$  and  $\tilde{M}_d$  in Eq. (3.9) can still be changed by an overall common unitary transformation. It is possible to remove this remaining freedom by adopting a (slightly) unconventional parametrization of the CKM matrix. Defining  $\tilde{M}_u$  and  $\tilde{M}_d$  as in Eqs. (3.6) and (3.7), one has always the freedom to choose  $D$  to be of the standard CKM form  $D = D_s$ , so that the matrix  $\tilde{M}_d$  is described by six real variables and one phase. However, then there is no further freedom of redefinition; hence,  $U$  being a general  $3 \times 3$  Hermitian matrix must involve six real variables and three phases. It is easy to show that this more general parametrization corresponds to a definition of the CKM matrix (in the standard form), [CKM]<sub>s</sub> =  $U^\dagger D_s \Phi$ , with  $\Phi$  being a diagonal phase matrix containing two arbitrary phases. We find it more convenient to take  $\Phi = 1$  and have the matrices  $\tilde{M}_u$  and  $\tilde{M}_d$  undetermined by an overall common unitary transformation.

orthogonal matrices  $C_{iu}$  (of the same form as the  $C_i$ 's) involving angles  $\theta_{iu}$  ( $i=1,2,3$ ) and satisfying the relations

$$C_{iu}^T C_{id} = C_i \quad (\text{no sum over the index } i).$$

In analogy to the two-generation case, the angles  $\theta_{iu}$ ,  $\theta_{id}$ , and  $\theta_i$  obey

$$\theta_i = \theta_{id} - \theta_{iu} \quad (i=1,2,3). \quad (3.11)$$

Using these definitions, the matrix  $U$  is easily seen to be

$$\begin{aligned}U &= D[\text{CKM}]^\dagger = \{C_{2u}\} \{C_2(\Delta_d C_{3u} \Delta_d^\dagger) C_2^\dagger\} \\ &\quad \times \{C_2(\Delta_d C_3 \Delta_d^\dagger) C_{1u}(\Delta C_3^\dagger \Delta^\dagger) C_2^\dagger\}.\end{aligned}\quad (3.12)$$

This matrix is not quite of the standard CKM form. However, since the matrices  $C_i \sim 1$ , the matrix  $U$  is not that different, apart from the placement of some phase factors.

Once one has explicit forms of the matrices  $U$  and  $D$ , the  $\lambda$  expansion of these matrices and hence of  $\tilde{M}_u$ ,  $\tilde{M}_d$  is accomplished by letting

$$\theta_{1d} \equiv \sum_{n=1} \alpha_n \lambda^n, \theta_{2d} \equiv \sum_{n=2} \beta_n \lambda^n, \theta_{3d} \equiv \sum_{n=4} \gamma_n \lambda^n. \quad (3.13)$$

The expansion of the  $\theta_{iu}$  angles are fixed by Eq. (3.11) in conjunction with the magnitudes of the  $\theta_i$ 's in the CKM matrix, as specified<sup>5</sup> in Eq. (2.2).

Having written out the matrices  $\tilde{M}_u$ ,  $\tilde{M}_d$  according to Eq. (3.9) [with  $\tilde{M}_u^{\text{diag}}$ ,  $\tilde{M}_d^{\text{diag}}$  given by Eqs. (2.9) and (2.10)] and expanded each matrix element in a  $\lambda$  expansion similar to Eq. (3.4), one can follow a procedure analogous to that in the two-generation case and infer which of the mixing angle options give rise to natural mass matrices. The detailed expressions for  $\tilde{M}_u$  and  $\tilde{M}_d$  are quite lengthy and not tremendously illuminating, and so we shall omit them and instead only present our findings regarding the naturalness conditions. We find that for natural mass patterns we must require again that

$$(1) \quad \theta_{1d} \sim \lambda, \quad \theta_{1u} \lesssim \lambda^2,$$

and, for the other angles, that one of the following options from each set holds:

<sup>5</sup>Notice, in particular, that according to Eqs. (2.4) and (2.5),  $s_3 \equiv A \sigma \lambda^3 = O(\lambda^4)$ , if one uses the central values for  $A$  and  $\sigma$ .

$$\begin{aligned}
(2) \quad & \theta_{2u} \sim \theta_{2d} \sim \lambda^2 \text{ or } \theta_{2u} \sim \lambda^2 \gg \theta_{2d} \text{ or } \theta_{2d} \sim \lambda^2 \gg \theta_{2u}, \\
(3) \quad & \theta_{3u} \sim \theta_{3d} \sim \lambda^4 \text{ or } \theta_{3u} \sim \lambda^4 \gg \theta_{3d} \text{ or } \theta_{3d} \sim \lambda^4 \gg \theta_{3u}.
\end{aligned} \tag{3.14}$$

The above conditions severely restrict the form of the mass matrices to which they apply. As a result, the general expressions for these matrices are readily obtained. (The detailed results and their discussion are relegated to Appendix A.) Here, as an example, we give a mass pattern with  $\theta_{1d} \sim \lambda$ ,  $\theta_{1u} \sim \theta_{2d} \sim \lambda^2$ ,  $\theta_{2u} \sim \theta_{3u} \sim \lambda^4$ , and  $\theta_{3d} \sim \lambda^5$ . This pattern corresponds to

$$\begin{aligned}
\tilde{M}_u &\approx \begin{pmatrix} u_{11}\lambda^7 & u_{12}\lambda^6 & e^{-i\delta_u}u_{13}\lambda^4 \\ u_{12}\lambda^6 & u_{22}\lambda^4 & u_{23}\lambda^4 \\ e^{i\delta_u}u_{13}\lambda^4 & u_{23}\lambda^4 & 1 \end{pmatrix}, \\
\tilde{M}_d &\approx \begin{pmatrix} d_{11}\lambda^4 & d_{12}\lambda^3 & e^{-i\delta_d}d_{13}\lambda^5 \\ d_{12}\lambda^3 & d_{22}\lambda^2 & d_{23}\lambda^2 \\ e^{i\delta_d}d_{13}\lambda^5 & d_{23}\lambda^2 & 1 \end{pmatrix}, \tag{3.15}
\end{aligned}$$

where the real coefficients  $u_{ij}$ 's,  $d_{ij}$ 's are functions of the following  $O(1)$  parameters: the CKM parameters  $A$ ,  $\sigma$ ; the quark mass ratios  $\xi$ 's; and the  $\lambda$  expansion coefficients  $\{\alpha_1, \beta_2, \gamma_5\}$ . The coefficient  $u_{13}$  and the phase  $\delta_u$  in addition also depend on the arbitrary phase parameter  $\delta_d$  from the matrix  $\Delta_d$ [Eq. (3.10)] as well as on the CKM phase  $\delta$ .

The principal goal of our construction is to allow us to extrapolate LED-consistent natural mass patterns to some GUT scale where we can look for hints of ‘‘new’’ physics. Nevertheless, low energy (defined here as  $\sim m_t$ ) mass patterns such as the one discussed above are also interesting in their own right. For instance, for the pattern given in Eq. (3.15), by appropriately choosing the signs of the quark masses one can arrange to have

$$\begin{aligned}
\tilde{M}_u &\approx \begin{pmatrix} \xi_{ut}\lambda^7 + (A\sigma)^2\lambda^6 & \alpha\xi_{ct}\lambda^6 & -A\sigma e^{-i\delta}\lambda^3 \\ \alpha\xi_{ct}\lambda^6 & \xi_{ct}\lambda^4 & 0 \\ -A\sigma e^{i\delta}\lambda^3 & 0 & 1 \end{pmatrix}, \\
\tilde{M}_d &\approx \begin{pmatrix} 0 & \sqrt{\xi_{db}\xi_{sb}}\lambda^3 & 0 \\ \sqrt{\xi_{db}\xi_{sb}}\lambda^3 & \xi_{sb}\lambda^2 & A\lambda^2 \\ 0 & A\lambda^2 & 1 \end{pmatrix}, \tag{3.16}
\end{aligned}$$

with  $\alpha = \{\sqrt{\xi_{db}/\xi_{sb}} - 1\}/\lambda \approx 0.12$ . This new pattern now has a large number of the much sought-after ‘‘texture zeros’’: It has three exact ones to begin with, i.e.,  $[\tilde{M}_u]_{23}, [\tilde{M}_d]_{11}$ , and  $[\tilde{M}_d]_{13}$ , and two more to the accuracy level  $O(\lambda^4)$  of the CKM matrix, i.e.,  $[\tilde{M}_u]_{11}, [\tilde{M}_u]_{12} = O(\lambda^7)$ . Moreover, this new pattern exhibits useful features commonly exploited in the study of mass matrix patterns: Sensible CKM and other ‘‘predictions’’ can come about when one imposes equalities among matrix elements of approximately the same order (e.g., demanding that  $|[\tilde{M}_u]_{13}| = [\tilde{M}_u]_{22}$  result in the predic-

tion  $|V_{ub}| \approx m_c/m_t$ ), or when one assigns specific values (usually 0) to certain (usually small) matrix elements (e.g., setting  $[\tilde{M}_u]_{12} = 0$  results in the prediction  $\sin\theta_C = \sqrt{m_d/m_s}$ ).

#### IV. POTENTIALLY SUCCESSFUL GUT SCALE MASS PATTERNS

##### A. A pattern

Having constructed certain low energy natural mass patterns, one can then apply RG equations (RGE's) to evolve these patterns to some high mass scales where global symmetries originating from some GUT texture should become manifest, hopefully gaining some useful insights. As a study case, in this paper we examine the evolution of our natural mass patterns in the minimal supersymmetric standard model (MSSM) theory. For simplicity, we consider only the scenario where the vacuum expectation values (VEV's) of the Higgs bosons coupled to  $u$  quarks and  $d$  quarks are approximately equal, i.e.,  $\tan\beta \sim 1$ . The relevant one-loop RGE's [2,15] are

$$\begin{aligned}
\frac{d[h_U]_{ij}}{dt} &\approx \frac{1}{(4\pi)^2} \{ (3[h_U]_{33}^2 - c_k g_k^2)[h_U]_{ij} \\
&\quad + 3[h_U]_{i3}[h_U]_{33}[h_U]_{3j} \}, \tag{4.1}
\end{aligned}$$

$$\begin{aligned}
\frac{d[h_D]_{ij}}{dt} &\approx \frac{1}{(4\pi)^2} \{ (3[h_D]_{33}^2 + \text{tr}\{h_E^2\} - c'_k g_k^2)[h_D]_{ij} \\
&\quad + [h_U]_{i3}[h_U]_{33}[h_D]_{3j} \}, \tag{4.2}
\end{aligned}$$

where the  $g_k$ 's are the three gauge couplings,  $c_k = (13/15, 3, 16/3)$ ,  $c'_k = (7/15, 3, 16/3)$ , and  $h_U, h_D$ , and  $h_E$  are the Yukawa coupling matrices for  $u$  quarks,  $d$  quarks, and leptons,<sup>6</sup> respectively. To pursue our analysis further, we need to solve the above equations to find the mass matrices at the GUT scale  $m_G \approx 10^{16}$  (GeV) [2]. An input is then necessary at the energy scale  $m_t$ . For definiteness, we assume as a boundary condition  $[h_U(m_t)]_{33} = 1$ , although the general pattern of our result is largely independent of this choice. For the concrete example of our mass pattern of Eq. (3.15), the solution of Eqs. (4.1) and (4.2) gives

<sup>6</sup>Although our analysis so far does not concern the lepton mass matrix, one can still argue that it must be of the form which reflects the lepton mass hierarchy, especially in light of the fact that one wishes to implement the successful Georgi-Jarlskog GUT mass relation [16]  $m_b/m_\tau = 3m_s/m_\mu = m_d/3m_e = 1$  at some point. It then follows that  $\text{tr}\{h_E^2\} \sim [h_E]_{33}^2 \sim [h_D]_{33}^2$  in which case, the contribution of the term  $\text{tr}\{h_E^2\}$  to the solution of Eq. (4.2) is very minimal.

$$\begin{aligned} \tilde{M}_u(m_G) &\simeq \{0.82\} \begin{pmatrix} (\{0.61\}u_{11} + \{0.08\}u_{13}^2)\lambda^7 & \{0.61\}u_{12}\lambda^6 & e^{-i\delta_u}u_{13}\lambda^4 \\ \{0.61\}u_{12}\lambda^6 & \{0.61\}u_{22}\lambda^4 & u_{23}\lambda^4 \\ e^{i\delta_u}u_{13}\lambda^4 & u_{23}\lambda^4 & 1 \end{pmatrix}, \\ \tilde{M}_d(m_G) &\simeq \{0.38\} \begin{pmatrix} \{0.85\}d_{11}\lambda^4 & \{0.85\}d_{12}\lambda^3 & (\{0.85\}e^{-i\delta_d}d_{13} + \Delta_{13})\lambda^5 \\ \{0.85\}d_{12}\lambda^3 & \{0.85\}d_{22}\lambda^2 & \{0.85\}d_{23}\lambda^2 + \Delta_{23}\lambda^4 \\ e^{i\delta_d}d_{13}\lambda^5 & d_{23}\lambda^2 & 1 \end{pmatrix}, \end{aligned} \quad (4.3)$$

with  $\Delta_{13} = \{0.68\}e^{-i\delta_u}u_{13}$  and  $\Delta_{23} = \{0.15\}u_{23}$ . The above result illustrates several general features of the RG runnings of natural Hermitian mass patterns.

(1) As is obvious from the form of  $\tilde{M}_d(m_G)$ , the Hermiticity of the mass matrices is not strictly preserved by the RG evolution. However, the extent to which Hermiticity is broken is relatively minor.

(2) Because of the hierarchy in the mass matrices at  $m_t$ , the RG runnings of various mass matrix elements are quite different.

(3) This notwithstanding, the likely mass-matrix-element candidates for “texture zeros” at  $m_G$  are the same ones which are present at  $m_t$ . These are the matrix elements of  $O(\lambda^4)$  or smaller.

These observations suggest a strategy on how to proceed in the search for GUT patterns. First of all, since in practice it makes more sense to imagine the mass matrices at the GUT scale to be Hermitian (or symmetric), one should really reverse the procedure. Clearly, if one chooses the GUT pattern to be Hermitian [e.g., by manipulating Eq. (4.3) into its nearest Hermitian form], one should expect only modest deviation in the low energy mass matrices from being perfectly LED consistent. This is even less of a real problem since, as a matter of fact, one can obtain LED-consistent mass matrices which are non-Hermitian (see the more detailed discussion in Sec. V). Second, one can exploit the differences in the RG runnings of mass matrix elements to arrange for possible equalities among them at the GUT scale. Finally, since “texture zeros” track between high and low energy scales, one can look for possible “texture zeros” of a GUT pattern in the “texture zeros” or “near-texture zeros” of its corresponding low energy pattern.<sup>7</sup>

For concreteness, it is useful to demonstrate our ideas with a specific example. Let us consider again the mass pattern of Eq. (4.3). By choosing the signs of the quark masses appropriately, one finds among other possibilities, a potentially “successful” GUT mass pattern in which

$$\tilde{M}_u(m_G) = \begin{pmatrix} 0 & C & B e^{-i\phi} \\ C & B & B \\ B e^{i\phi} & B & A \end{pmatrix}, \quad (4.4)$$

<sup>7</sup>Equalities among matrix elements and “texture zeros” are always desirable in GUT mass patterns in that they reduce the number of input parameters and, as a result, enhance the predictive power of the patterns.

$$\tilde{M}_d(m_G) = \begin{pmatrix} 0 & F & 0 \\ F & E & E \\ 0 & E & D \end{pmatrix}, \quad (4.5)$$

where the magnitudes of the parameters are  $A = O(1)$ ,  $B = O(\lambda^4)$ ,  $C = O(\lambda^6)$ ;  $D = O(1)$ ,  $E = O(\lambda^2)$ , and  $F = O(\lambda^3)$ . Given this ansatz, one can then try to find out whether it fits the LED. As we shall see below, some straightforward computation shows that indeed it does.<sup>8</sup>

## B. CKM predictions

Running the RGE’s backwards, at the energy scale  $m_t$  the matrices of our mass pattern become

$$\begin{aligned} \tilde{M}_u &\simeq \begin{pmatrix} -0.6B'^2 & C' & B' e^{-i\phi} \\ C' & 1.6B' & B' \\ B' e^{i\phi} & B' & 1 \end{pmatrix}, \\ \tilde{M}_d &\simeq \begin{pmatrix} 0 & F' & -0.18B' e^{-i\phi} \\ F' & 1.2E' & 1.2E' - 0.18B' \\ 0 & E' & 1 \end{pmatrix}. \end{aligned} \quad (4.6)$$

Notice that to our accuracy only  $\tilde{M}_d$  is slightly non-Hermitian. For ease in the computations that follow, it is convenient to specify the approximate magnitudes of the various parameters above by defining  $B' \equiv b\lambda^4$ ,  $C' \equiv c\lambda^6$ ,  $E' \equiv e\lambda^2$ , and  $F' \equiv f\lambda^3$ . With these choices, one can relate the parameters  $b, c, e, f$  to the quark mass eigenvalues (with the signs of  $u$ ,  $c$ , and  $d$  quark masses chosen to be negative) by solving the corresponding eigenequations of the mass matrices: i.e.,

$$\det\{\tilde{M}_u + \xi_{ct}\lambda^4\} = 0,$$

$$\det\{\tilde{M}_d \tilde{M}_d^\dagger - (\xi_{sb}\lambda^2)^2\} = 0, \dots$$

These computations give

<sup>8</sup>Alternatively, and in fact more efficiently, we could take as our starting point the Hermitian mass pattern in Eq. (3.15) to be our GUT pattern with all the quark mass ratios and CKM parameters therein evaluated at the energy scale  $m_G$  (Appendix B) and directly arrange for “texture zeros” and equalities among its matrix elements.

$$\begin{aligned}
b &\simeq -0.6\xi_{ct}, \\
c &\simeq \pm \sqrt{\xi_{ct}(-4.5\xi_{ut} + 0.6\xi_{ct}^2)}, \\
e &\simeq 0.8\xi_{sb}, \\
f &\simeq \sqrt{\xi_{sb}\xi_{db}}.
\end{aligned}$$

With these values, one can further calculate the diagonalizing unitary matrices  $U_L$ ,  $D_L$  from the equations

$$\begin{aligned}
U_L^\dagger(\tilde{M}_u)U_L &= \tilde{M}_u^{\text{diag}}, \\
D_L^\dagger(\tilde{M}_d\tilde{M}_d^\dagger)D_L &= \{\tilde{M}_d^{\text{diag}}\}^2.
\end{aligned}$$

The final result is

$$[\text{CKM}] = U_L^\dagger D_L \simeq \begin{pmatrix} 1 & \Delta_{12} & \Delta_{13} \\ -\Delta_{12} & 1 & \Delta_{23} \\ \Delta_{31} & -\Delta_{23} & 1 \end{pmatrix},$$

where

$$\begin{aligned}
\Delta_{12} &= \sqrt{\xi_{db}/\xi_{sb}}\lambda \pm \sqrt{-4.5\xi_{ut}/\xi_{ct} + 0.6\xi_{ct}^2}\lambda^2, \\
\Delta_{13} &= 0.7\xi_{ct}e^{-i\phi}\lambda^4 \pm \xi_{sb}\sqrt{-4.5\xi_{ut}/\xi_{ct} + 0.6\xi_{ct}^2}\lambda^4, \\
\Delta_{23} &= \xi_{sb}\lambda^2 + 0.7\xi_{ct}\lambda^4, \\
\Delta_{31} &= \sqrt{\xi_{db}\xi_{sb}}\lambda^3 - 0.7\xi_{ct}e^{i\phi}\lambda^4.
\end{aligned}$$

Comparing this matrix with the CKM matrix in the Wolfenstein parametrization and denoting the absolute values of the quark masses as  $m_q$ 's, one arrives at the ‘‘predictions’’<sup>9</sup>

$$\begin{aligned}
\sin\theta_C &\simeq \sqrt{\frac{m_d}{m_s}} \pm \sqrt{-\frac{m_u}{m_c} + 0.6\frac{m_c}{m_t}}, \\
V_{cb} &\simeq \frac{m_s}{m_b} + 0.7\frac{m_c}{m_t}, \\
V_{ub} &\simeq 0.7\frac{m_c}{m_t}e^{-i\phi} \pm \frac{m_s}{m_b}\sqrt{-\frac{m_u}{m_c} + 0.6\frac{m_c}{m_t}}. \quad (4.7)
\end{aligned}$$

To check the soundness of these results, we choose the + sign in the above expressions and input various quark mass ratios. Although these are not the only possible choices, we find that, for

$$\xi_{ct} \simeq 1.55, \quad \xi_{ut} \simeq 0.30, \quad \xi_{sb} \simeq 0.73, \quad \xi_{db} \simeq 0.66,$$

we have a decent fit corresponding to the central values of  $\sin\theta_C \simeq 0.22$ ,  $A \simeq 0.78$ , and  $\sigma e^{-i\delta} \simeq 0.31e^{-i\phi} + 0.05$ . Choosing  $\phi = 90^\circ$  in the last equation, for example, gives the point (0.05, 0.31) in the  $\rho$ - $\eta$  plane, which is well within the known constraints [Eq. (2.6)]. We note also that the mass ratios

above are in reasonable agreement with the much more restrictive light-quark-mass constraint relation of Eq. (2.8).

### C. Other GUT patterns

By examining the general expressions for the GUT scale Hermitian matrices  $\tilde{M}_u$  and  $\tilde{M}_d$  (Appendixes A and B), it is not difficult to find other potentially interesting mass patterns. We list here four more such patterns which are slight variations of the one we discussed in detail above. The first two have the same form for the  $\tilde{M}_d$  matrices as the one in Eq. (4.5): i.e.,

$$(1) \text{ and } (2) \quad \tilde{M}_d(m_G) = \begin{pmatrix} 0 & F & 0 \\ F & E & E \\ 0 & E & D \end{pmatrix},$$

with  $D \sim 1$ ,  $E = O(\lambda^2)$ , and  $F = O(\lambda^3)$ , but have somewhat different  $\tilde{M}_u$ 's:

$$(1) \quad \tilde{M}_u(m_G) = \begin{pmatrix} C & 0 & Be^{-i\phi} \\ 0 & B & B \\ Be^{i\phi} & B & A \end{pmatrix},$$

with  $A \sim 1$ ,  $B = O(\lambda^4)$ , and  $C = O(\lambda^7)$ ,

$$(2) \quad \tilde{M}_u(m_G) = \begin{pmatrix} C & C & Be^{-i\phi} \\ C & B & B \\ Be^{i\phi} & B & A \end{pmatrix},$$

again with  $A \sim 1$ ,  $B = O(\lambda^4)$ , and  $C = O(\lambda^7)$ . The remaining two patterns have a different  $\tilde{M}_d$  matrix which takes the form

$$(3) \text{ and } (4) \quad \tilde{M}_d(m_G) = \begin{pmatrix} 0 & F & Fe^{-i\phi} \\ F & E & E \\ Fe^{i\phi} & E & D \end{pmatrix},$$

with  $D \sim 1$ ,  $E = O(\lambda^2)$ , and  $F = O(\lambda^3)$ , and the  $\tilde{M}_u$  matrices

$$(3) \quad \tilde{M}_u(m_G) = \begin{pmatrix} C & 0 & 0 \\ 0 & B & B \\ 0 & B & A \end{pmatrix},$$

with  $A \sim 1$ ,  $B = O(\lambda^4)$ , and  $C = O(\lambda^7)$ ,

$$(4) \quad \tilde{M}_u(m_G) = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & B \\ 0 & B & A \end{pmatrix},$$

with  $A \sim 1$ ,  $B = O(\lambda^2)$ , and  $C = O(\lambda^7)$ . The CKM ‘‘predictions’’ of the above GUT patterns are most readily obtained in terms of various quark mass ratios and the parameter  $r$  defined in Appendix B, by comparing the matrices of these patterns with the general results of Appendix A applied at  $m_G$  (see the example and comments in Appendix B for details). The relevant predictions for these GUT patterns are tabulated in Table I.

Several comments are in order at this point.

<sup>9</sup>In terms of the scaling parameter  $r$  defined in Appendix B, the numerical factors in these expressions correspond to  $r \simeq 0.85$  (and hence  $r^2 \simeq 0.7, r^3 \simeq 0.6$ ).

TABLE I. GUT pattern predictions for CKM parameters.

	(1)	(2)	(3)	(4)
$\sin\theta_C$	$\sqrt{\frac{m_d}{m_s}}$	$\sqrt{\frac{m_d}{m_s} - \frac{m_u}{m_c} + r^3 \frac{m_c}{m_t}}$	$\sqrt{\frac{m_d}{m_s}}$	$\sqrt{\frac{m_d}{m_s}}$
$V_{cb}$	$\frac{m_s}{m_b} + r^2 \frac{m_c}{m_t}$	$\frac{m_s}{m_b} + r^2 \frac{m_c}{m_t}$	$\frac{m_s}{m_b} + r^2 \frac{m_c}{m_t}$	$\sqrt{r \frac{m_c}{m_t} - \frac{m_s}{m_b}}$
$V_{ub}$	$r^2 \frac{m_c}{m_t} e^{-i\phi}$	$r^2 \frac{m_c}{m_t} e^{-i\phi} - \frac{m_s}{m_b} \left\{ \frac{m_u}{m_c} - r^3 \frac{m_c}{m_t} \right\}$	$\sqrt{\frac{m_d m_s}{m_b^2}} e^{-i\phi}$	$-\sqrt{\frac{m_d m_s}{m_b^2}} e^{-i\phi}$

(1) Although the mass patterns listed here all contain “texture zeros,” we have not tried to impose “texture zeros” in all possible places. For example, one could have in the matrix  $\tilde{M}_u(m_G)$  of pattern (1) an extra “texture zero” by taking  $C=0$ . The resultant new pattern would, in addition to its CKM “predictions,” generate a GUT scale quark mass relation corresponding to  $m_u m_t \approx r^3 m_c^2$  which, according to Eq. (2.9), is actually allowed. A systematic search for LED-consistent patterns with the maximum number of “texture zeros” has already been thoroughly carried out in Ref. [6] where, specifically, a total of five patterns with five “texture zeros” were found and discussed in substantial detail.

(2) Because of the specific mass matrix parametrization scheme we have chosen, certain frequently encountered Hermitian mass patterns in the literature may not seem transparent from the constructions of our natural mass matrices. Still, in general, these patterns can be related to our easily derivable patterns by some simple unitary transformations. For example, consider the following pattern which can easily be arranged from our general results in Appendix A:

$$\tilde{M}_u \approx \begin{pmatrix} C & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{pmatrix}, \quad \tilde{M}_d \approx \begin{pmatrix} 0 & F & G e^{-i\phi} \\ F & E & E' \\ G e^{i\phi} & E' & D \end{pmatrix},$$

where  $A, D \sim 1$ ;  $E, E' = O(\lambda^2)$ ;  $F = O(\lambda^3)$ ;  $B, G = O(\lambda^4)$ , and  $C = O(\lambda^7)$ . With the various parameters carefully chosen, this pattern can be transformed into a much more familiar-looking form [6]

$$\tilde{M}'_u \approx \begin{pmatrix} 0 & C' & 0 \\ C' & B & 0 \\ 0 & 0 & A \end{pmatrix}, \quad \tilde{M}'_d \approx \begin{pmatrix} 0 & F' e^{-i\phi'} & 0 \\ F' e^{i\phi'} & E & E' \\ 0 & E' & D \end{pmatrix}$$

by a unitary matrix  $T$  (i.e.,  $T \tilde{M}_{u,d} T^\dagger \approx \tilde{M}'_{u,d}$ ) with

$$T \approx \begin{pmatrix} e^{i\phi} & -\omega & 0 \\ \omega e^{i\phi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $\omega \equiv G/E' = O(\lambda^2)$ .

(3) If one wishes to incorporate the Georgi-Jarlskog mass relation [16], the corresponding lepton mass matrices of the GUT patterns listed in this section can be chosen in a straightforward manner. For instance, following Ref. [2], one can simply let, for patterns (1) and (2),

$$(1) \text{ and } (2) \quad \tilde{M}_l(m_G) = \begin{pmatrix} 0 & F & 0 \\ F & -3E & E \\ 0 & E & D \end{pmatrix},$$

and similarly for patterns (3) and (4), one can let

$$(3) \text{ and } (4) \quad \tilde{M}_l(m_G) = \begin{pmatrix} 0 & F & F e^{-i\phi} \\ F & -3E & E \\ F e^{i\phi} & E & D \end{pmatrix}.$$

With these matrices, it is easy to see that the Georgi-Jarlskog relation results directly.

## V. HERMITICITY BREAKDOWN AND STRONG CP COMPLICATIONS

In our introductory discussion we described how to construct Hermitian mass matrices from LED information. We must, however, face the fact that imagining the mass matrices are Hermitian at the weak scale is not a very compelling assumption. Indeed, as we have argued in the preceding sections it is much more sensible to imagine that quark mass matrices are Hermitian (or symmetric) at the GUT scale. When this is the case, the RG evolution definitely introduces some non-Hermitian (or nonsymmetric) components at the weak scale. This was illustrated in explicit detail in the example based on the mass matrices given in Eqs. (4.4) and (4.5). Thus, to be realistic, we should instead display a set of natural non-Hermitian weak scale mass matrices constructed from the LED and then evolve these matrices to the GUT scale.

If one attempts this kind of a general construction from the LED without any further constraints, one is immediately faced with considerable arbitrariness and little progress appears possible. However, if one assumes that the resulting weak scale matrices are only “slightly” non-Hermitian, because they are Hermitian at the GUT scale, then a general construction becomes feasible. In fact, such a construction is

really not necessary in the case of SUSY GUT's with  $\tan\beta\sim 1$ . In this latter case one can simply detail how the CKM parameters and the quark mass ratios evolve to the GUT scale. With these parameters in hand one can directly construct natural Hermitian mass patterns at the GUT scale. The resulting non-Hermitian mass matrices, by construction, will be natural and reproduce the LED. This is basically the technique used to deduce Table I. The details of this procedure is further illustrated through an example in Appendix B.

The presence of non-Hermitian mass matrices at the weak scale, incidently, raises the issue of strong  $CP$  violation. Because of the nontrivial nature of the QCD vacuum [17], the standard model is augmented by an extra  $CP$ -violating term involving the gluon field strength and its dual

$$\mathcal{L}_{\text{strong}CP} = \frac{\alpha_s}{8\pi} \bar{\theta} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}.$$

The parameter  $\bar{\theta}$  is a linear combination of a phase angle  $\theta$  connected with the QCD vacuum and another connected with the quark mass matrices [18]:

$$\bar{\theta} = \theta + \sum_{i=u,d} \text{Arg}\{\det M_i\}.$$

One knows, however, that this parameter must be extremely small ( $\bar{\theta}\leq 10^{-9}$ ) [19], so as to avoid being in conflict with the present bound on the neutron electric dipole moment. Why should the QCD vacuum angle be so precisely aligned as to cancel (or very nearly cancel)  $\text{Arg}\{\det M_{u,d}\}$  is not known and constitutes the strong  $CP$  problem.

For the quark mass matrices we have been discussing, if we assume that at the GUT scale these matrices are Hermitian, then obviously

$$\text{Arg}\{\det M_{u,d}(m_G)\} = 0.$$

However, as we have seen from our analysis, RG evolution induces non-Hermiticity. Thus, starting with some Hermitian mass matrices  $M_{u,d}$  at the GUT scale, in general, these matrices become slightly non-Hermitian at the scale of  $m_t$ . This is a direct consequence of the RGE's not being Hermitian-conjugation invariant. In the one-loop RGE [Eq. (4.2)], for example, the term  $[h_U][h_U^\dagger][h_D]$  is responsible for this non-invariance. Nevertheless, such a term is found to be insufficient to generate

$$\text{Arg}\{\det M_{u,d}(m_t)\} \neq 0.$$

In general, however, one expects eventually that at sufficiently high order such a term will ensue from the RG evolution.

The actual order at which a nonzero value for  $\text{Arg}\{\det M_{u,d}\}$  at the top scale appears depends on the underlying theory. For instance, in a globally supersymmetric theory this mass matrix phase is never generated since it is not renormalized [20], while with the standard model it may first appear at six loops in the Higgs sector, with an additional gauge boson loop [21]. In supersymmetric theories where SUSY is broken softly the actual contribution depends on the breaking. In certain instances no mass matrix phase appears [22] but, in general, if there are nonvanishing el-

ementary or induced gluino masses one expects a phase to appear [23]. For instance, with an explicit gluino mass, one induces a nonvanishing  $\text{Arg}\{\det M_{u,d}(m_t)\}$  at two loops [23].

It is quite possible that with the right underlying theory, imposing  $\text{Arg}\{\det M_{u,d}(m_G)\} = 0$  at the GUT scale, suffices to guarantee that  $\text{Arg}\{\det M_{u,d}(m_t)\}$  is much below  $10^{-9}$ . However, this still does not solve the strong  $CP$  problem unless, somehow,  $\bar{\theta}$  vanishes at  $m_G$  [which certainly is not sufficiently guaranteed by just having  $\text{Arg}\{\det M_{u,d}(m_G)\} = 0$ ]. These additional observations indicate, perhaps compellingly, the necessity of having some dynamical strong- $CP$ -removal mechanism, conceivably by imposing a  $U(1)_{PQ}$  symmetry [24].

## VI. CONCLUSIONS

Interesting patterns of quark masses are surely signals of “new” physics. The task of searching for them therefore can be very rewarding. In order to conduct these searches more effectively, we have suggested in this paper the idea of natural mass matrices as an organizing principle. This idea, along with the efficient mass-matrix-parametrization scheme we have described, allows a procedure whereby one can systematically input low energy data to construct viable GUT patterns. Encouragingly, this procedure has produced a rather small set of “working” mass patterns and our preliminary work in extrapolating these patterns to GUT scales has generated some interesting possibilities. We have discussed, specifically, one such application in the context of SUSY GUT's and some potentially successful GUT mass patterns were readily found. Although we do not particularly wish to assign too much significance to these mass patterns and their predictions, such examples do indicate the usefulness of our approach. An important future task is to perform a more systematic and complete investigation, with different RGE boundary conditions and perhaps different matter contents.

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## APPENDIX A: GENERAL EXPRESSIONS FOR NATURAL MASS MATRICES

Adopting the parametrization scheme we have developed in Sec. III B, we arrive at the following general expressions for natural mass matrices, incorporating all naturalness requirements [Eq. (3.14)]:

$$\tilde{M}_u \simeq \begin{pmatrix} u_{11}\lambda^7 & u_{12}\lambda^6 & u_{13}\lambda^4 \\ u_{12}^*\lambda^6 & u_{22}\lambda^4 & u_{23}\lambda^2 \\ u_{13}^*\lambda^4 & u_{23}\lambda^2 & 1 \end{pmatrix} + \begin{pmatrix} O(\lambda^9) & O(\lambda^8) & O(\lambda^6) \\ O(\lambda^8) & O(\lambda^6) & O(\lambda^6) \\ Q(\lambda^6) & Q(\lambda^6) & Q(\lambda^4) \end{pmatrix}, \quad (\text{A1})$$

$$\tilde{M}_d \simeq \begin{pmatrix} d_{11}\lambda^3 & d_{12}\lambda^3 & d_{13}\lambda^3 \\ d_{12}\lambda^3 & d_{22}\lambda^2 & d_{23}\lambda^2 \\ d_{13}^*\lambda^4 & d_{23}\lambda^2 & 1 \end{pmatrix} + \begin{pmatrix} O(\lambda^6) & O(\lambda^5) & O(\lambda^6) \\ O(\lambda^5) & O(\lambda^4) & O(\lambda^4) \\ O(\lambda^6) & O(\lambda^4) & O(\lambda^4) \end{pmatrix}. \quad (\text{A2})$$

Here,

$$\begin{aligned} u_{11} &= \xi_{ut} + \{\alpha_u^2 \xi_{ct} + |\gamma_u e^{-i\delta_d} - \alpha_u A + \Lambda(e^{-i\delta_d} - e^{-i\delta})|^2\} \lambda, \\ u_{12} &= \alpha_u \xi_{ct} + \beta_u [\gamma_u e^{-i\delta_d} - \alpha_u A + \Lambda(e^{-i\delta_d} - e^{-i\delta})], \\ u_{13} &= \gamma_u e^{-i\delta_d} - \alpha_u A + \Lambda(e^{-i\delta_d} - e^{-i\delta}), \\ u_{22} &= \xi_{ct} + \beta_u^2, \\ u_{23} &= \beta_u, \\ d_{11} &= \xi_{db} + \alpha_d^2 \xi_{sb}, \\ d_{12} &= \alpha_d \xi_{sb}, \\ d_{13} &= \gamma_d e^{-i\delta_d} - \alpha_d \beta_d \xi_{sb} \lambda, \\ d_{22} &= \xi_{sb}, \\ d_{23} &= \beta_d. \end{aligned} \quad (\text{A3})$$

The various parameters in the above equations are as follows:  $\lambda, A, \Lambda (\equiv \sigma A \lambda^{-1})$  and  $\delta$  are the CKM parameters [Eq. (2.3)];  $\xi$ 's are the quark mass ratios [Eqs. (2.9) and (2.10)];  $\delta_d$  is a free input phase parameter [Eq. (3.10)]; and finally  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's are input parameters defined from the  $\lambda$  expansion coefficients of the  $\theta_{u,d}$ 's (Sec. III B):

$$\begin{aligned} \sin\theta_{1d} &\equiv \alpha_d \lambda, & \sin\theta_{1u} &\equiv \alpha_u \lambda^2, \\ \sin\theta_{2d} &\equiv \beta_d \lambda^2, & \sin\theta_{2u} &\equiv \beta_u \lambda^2, \\ \sin\theta_{3d} &\equiv \gamma_d \lambda^4, & \sin\theta_{3u} &\equiv \gamma_u \lambda^4. \end{aligned}$$

The magnitudes of these last parameters are specified below in accordance with our naturalness condition [Eq. (3.14)]:

$$\begin{aligned} \alpha_d &\sim 1, \alpha_u \lesssim 1, \text{ with } \alpha_d - \alpha_u \lambda = 1 + O(\lambda^3), \\ \beta_d &\lesssim 1, \beta_u \lesssim 1, \text{ with } \beta_d - \beta_u = A + O(\lambda^4), \\ \gamma_d &\lesssim 1, \gamma_u \lesssim 1, \text{ with } \gamma_d - \gamma_u = \Lambda + O(\lambda^8). \end{aligned} \quad (\text{A4})$$

The general expressions summarized here are particularly useful for the purpose of arranging interesting mass patterns.

Specifically, ‘‘texture zeros’’ and equalities among mass matrix elements can, whenever possible, be rather conveniently imposed by adjusting the parameters  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's, subject to the constraints in Eq. (1.4). Furthermore, CKM and other ‘‘predictions’’ then ensue when the aforementioned constraints overspecify these parameters.

## APPENDIX B: MASS RATIOS, CKM PARAMETERS AND CONSTRUCTION OF HERMITIAN MASS PATTERNS AT THE GUT SCALE

The relevant formulas for calculating the RG scaling of mass ratios and CKM parameters are derived in Refs. [15,25]. Here, we give only a brief summary of the results. For the SUSY GUT case we are considering (Sec. IV A) where  $\tan\beta \sim 1$ , one finds the following simple RG scaling relations.

Mass ratios:

$$\begin{aligned} \xi_{ct}(m_G) &\simeq r^3 \xi_{ct}, \xi_{ut}(m_G) \simeq r^3 \xi_{ut}, \\ \xi_{sb}(m_G) &\simeq r \xi_{sb}, \xi_{db}(m_G) \simeq r \xi_{db}. \end{aligned} \quad (\text{B1})$$

CKM parameters:

$$\begin{aligned} \lambda(m_G) &\simeq \lambda, \\ A(m_G) &\simeq rA, \\ \sigma(m_G) &\simeq \sigma. \end{aligned} \quad (\text{B2})$$

The scaling parameter  $r$  in these relations is defined by

$$r = \exp\left(-\frac{1}{(4\pi)^2} \int_0^{\ln(m_G/m_t)} [h_U(\mu)]_{33}^2 dt\right) \quad (t \equiv \ln\{\mu/m_t\}) \quad (\text{B3})$$

which, based on Eq. (4.1) and the one-loop RGE's for the gauge couplings,<sup>10</sup> can also be expressed as

$$r = \left\{ \frac{[h_U(m_G)]_{33}}{[h_U(m_t)]_{33}} \right\}^{-1/6} \{\eta(m_G)\}^{1/12}, \quad (\text{B4})$$

with

$$\begin{aligned} &\left\{ \frac{[h_U(m_G)]_{33}}{[h_U(m_t)]_{33}} \right\} \\ &\simeq \{\eta(m_G)\}^{1/2} \left\{ 1 - \frac{3}{4\pi^2} [h_U(m_t)]_{33}^2 I(m_G) \right\}^{-1/2}, \end{aligned} \quad (\text{B5})$$

$$\eta(\mu) \equiv \prod_i \{g_i(m_i)/g_i(\mu)\}^{2c_i/b_i},$$

and

$$I(\mu) \equiv \int_0^{\ln(\mu/m_t)} \eta(\mu) dt.$$

<sup>10</sup>These are  $dg_i/dt = b_i g_i^3/16\pi^2$  ( $i=1,2,3$ ) with  $b_i = (33/5, 1, -3)$ .

To obtain a numerical value for  $r$ , we input  $g_i^2(m_t)/4\pi \approx (0.017, 0.033, 0.100)$  as values for the gauge couplings<sup>11</sup> (at  $m_t$ ) along with the boundary condition  $[h_U(m_t)]_{33} = 1$  into the above results, and we find<sup>12</sup>  $r \approx 0.85$ .

The general expressions for natural Hermitian mass matrices at the GUT scale can be gotten by substituting the mass ratios and CKM parameters evaluated at  $m_G$  in Eqs. (B1) and (B2) into the expressions given in Appendix A. For illustrative purposes, we “derive” a somewhat generic Hermitian GUT pattern and its CKM “predictions” below.

Choosing in Eqs. (1.1)–(1.3) the parameters  $\delta_d = \delta$ ,  $\beta_u = O(\lambda^2)$ , and demanding  $u_{11}, d_{11} \leq O(\lambda^2)$ , one arrives at a mass pattern which can be written as

$$\tilde{M}_u \approx \begin{pmatrix} 0 & C & y_u B e^{-i\phi_u} \\ C & B & x_u B \\ y_u B e^{i\phi_u} & x_u B & A \end{pmatrix},$$

$$\tilde{M}_d \approx \begin{pmatrix} 0 & F & y_d F e^{-i\phi_d} \\ F & E & x_d E \\ y_d F e^{i\phi_d} & x_d E & D \end{pmatrix}, \quad (\text{B6})$$

where  $A, D \sim 1$ ,  $E = O(\lambda^2)$ ,  $F = O(\lambda^3)$ ,  $B = O(\lambda^4)$ ,  $C = O(\lambda^6)$ , and  $x$ 's,  $y$ 's are adjustable parameters which are constrained only by the naturalness requirement.

Mapping the matrix elements in Eq. (2.6) onto those in Eq. (1.3), one can immediately establish the following:

<sup>11</sup>These numbers were also used to produce Eq. (4.3). They correspond to a set of values for the gauge couplings (at  $m_Z$ ) used as inputs in Ref. [2] where, solving the one-loop RGE's with these inputs, the three gauge couplings were found to merge at  $m_G \approx 10^{16}$  (GeV).

<sup>12</sup>Notice from Eq. (2.5) that the solution for  $[h_U(m_G)]_{33}$  depends rather sensitively on the choice of the boundary condition. However, while still important, this dependence is, comparatively speaking, much milder for  $r$ .

$$\beta_u \approx x_u \xi_{ct} \lambda^2, \quad \beta_d \approx x_d \xi_{sb},$$

$$-\alpha_u A + \gamma_u e^{-i\delta} \approx y_u \xi_{ct} e^{-i\phi_u},$$

$$-\alpha_d \beta_d \xi_{sb} \lambda + \gamma_d e^{-i\delta} \approx y_d \alpha_d \xi_{sb} \lambda^{-1} e^{-i\phi_d},$$

$$\xi_{ut} + \alpha_u^2 \xi_{ct} \lambda + (y_u \xi_{ct})^2 \lambda \approx 0,$$

$$\xi_{db} + \alpha_d^2 \xi_{sb} \approx 0.$$

Next, solving for the parameters  $\alpha$ 's,  $\beta$ 's, and  $\gamma$ 's from the above equations and subsequently applying the CKM constraint relations given in Eq. (1.4), one has

$$1 \approx \sqrt{-\xi_{db}/\xi_{sb}} \pm \sqrt{-\xi_{ut}/(\xi_{ct}\lambda) - y_u^2 \xi_{ct} \lambda},$$

$$A \approx x_d \xi_{sb} - x_u \xi_{ct} \lambda^2,$$

$$\Lambda e^{-i\delta} \approx y_d \xi_{sb} \sqrt{-\xi_{db}/\xi_{sb}} e^{-i\phi_d} \lambda^{-1} - y_u \xi_{ct} e^{-i\phi_u}$$

$$\pm A \sqrt{-\xi_{ut}/(\xi_{ct}\lambda) - y_u^2 \xi_{ct} \lambda} + x_d \sqrt{-\xi_{db} \xi_{sb}^3} \lambda.$$

Finally, in the above equations, if one keeps only the significant terms and takes into account the RG scaling of mass ratios and CKM parameters [i.e., Eqs. (2.1) and (2.2)], one sees that the GUT pattern of Eq. (2.6) has the following CKM “predictions”:

$$\sin \theta_C \approx \sqrt{-\frac{m_d}{m_s}} \pm \sqrt{-\frac{m_u}{m_c} - y_u^2 r^3 \frac{m_c}{m_t}},$$

$$V_{cb} \approx x_d \frac{m_s}{m_b} - x_u r^2 \frac{m_c}{m_t},$$

$$V_{ub} \approx y_d \frac{m_s}{m_b} \sqrt{-\frac{m_d}{m_s}} e^{-i\phi_d} - y_u^2 r^2 \frac{m_c}{m_t} e^{-i\phi_u}$$

$$\pm x_d \frac{m_s}{m_b} \sqrt{-\frac{m_u}{m_c} - y_u^2 r^3 \frac{m_c}{m_t}}.$$

The signs of the quark masses above have yet to be chosen as either + or −, depending on which choice is more sensible and gives better agreement with experimental measurements of the CKM parameters.

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