Chiral Lagrangian for baryons in the $1/N_c$ expansion

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A $1/N_c$ expansion of the chiral Lagrangian for baryons is formulated and used to study the low-energy dynamics of baryons interacting with the pion nonet π, K, η, and η' in a combined expansion in chiral symmetry breaking and $1/N_c$. Strong CP violation is included. The chiral Lagrangian correctly implements nonet symmetry and contracted spin-flavor symmetry for baryons in the large N_c limit. The implications of nonet symmetry for low-energy baryon-pion interactions are described in detail. The procedure for calculating nonanalytic pion-loop corrections to baryon amplitudes in the $1/N_c$ expansion for finite N_c is explained. Flavor-**27** baryon mass splittings are calculated at leading order in chiral perturbation theory as an example.

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I. INTRODUCTION

Although it is by now well established that the theory of the strong interactions is quantum chromodynamics, first principles calculations of the spectrum and properties of hadrons are not possible because the theory is strongly coupled at low energies. A number of different methods have been used to extract low-energy consequences of QCD. One of the oldest methods is chiral perturbation theory [1] which exploits the symmetry of the QCD Lagrangian under $SU(3)_L \times SU(3)_R \times U(1)_V$ transformations on the three flavors of light quarks u, d, and s in the limit that the quark masses m_u, m_d, and m_s vanish. Chiral symmetry is spontaneously broken to the vector subgroup $SU(3) \times U(1)_V$ by the QCD vacuum, resulting in an octet of pseudoscalar Goldstone bosons, the pions. A perturbative expansion in the pion momenta and the explicit chiral symmetry-breaking parameters m_i over the scale parameter of chiral symmetry breaking Λ_χ leads to flavor symmetry relations among hadronic amplitudes which are valid to a given order in chiral symmetry breaking.

A second method which has been important in the understanding of low-energy QCD hadron dynamics is the $1/N_c$ expansion [2]. This method promotes QCD to a $SU(N_c)$ non-Abelian gauge theory, where N_c is the number of colors. The $1/N_c$ expansion has been used primarily to derive $1/N_c$ power counting rules for hadronic amplitudes [2–4]. For finite and large N_c, planar diagrams dominate the dynamics. Each quark loop is suppressed by one factor of $1/N_c$ and nonplanar gluon exchange is suppressed by two factors of $1/N_c$. The suppression of quark loops in the $1/N_c$ expansion is particularly important for processes involving hadrons, since it implies that diagrams of leading order in the $1/N_c$ expansion contain no quark-antiquark pair creation and annihilation. Thus, planar QCD has a flavor symmetry [5]

$$U(1)_{q_i} \times U(1)_{\bar{q}_i} \tag{1.1}$$

which allows independent rotations on each quark flavor and antiquark flavor and implies the separate conservation of the number of each quark flavor and of each antiquark flavor (light or heavy). The planar QCD flavor symmetry (1.1) is broken at first subleading order due to a single quark loop of order $1/N_c$. It is important to emphasize that (1.1) is a symmetry of the planar approximation of QCD dynamics only and *not* of the QCD Lagrangian itself. Consequences of planar flavor symmetry include Zweig's rule and the formation of ideally mixed meson nonets [in the SU(3) flavor limit] at leading order in $1/N_c$ [6]. Planar flavor symmetry is often called "nonet symmetry" in the literature for this reason.

The combined use of chiral perturbation theory and the $1/N_c$ expansion can constrain the low-energy interactions of hadrons with the pion nonet π, K, η, and η' more effectively than either method alone. An effective Lagrangian describing the spectrum and self-interactions of the pion nonet was constructed some time ago [7–9]. The derivation of this $1/N_c$ chiral Lagrangian led to a number of important theoretical results concerning the QCD vacuum angle θ, and to a consistent picture for phenomenology [7,10] associated with the resolution of the $U(1)_A$ problem [11,12]. Two results deserve special mention here. First, the phenomenological analysis proved that the θ parameter is close to zero in QCD [7,8]. Second, the analysis showed that an η' which is primarily an SU(3) flavor singlet (in violation of ideal mixing) and which has a mass much larger than the pion octet is accommodated for reasonable values of parameters [7]. Understanding these features of the η' is nontrivial because the phenomenology involves an interplay between effects suppressed by m_i/Λ_χ and $1/N_c$.

In this paper, a $1/N_c$ chiral Lagrangian for the lowest-lying baryons is constructed. The Lagrangian describes the interactions of the spin-$\frac{1}{2}$ baryon octet and the spin-$\frac{3}{2}$ baryon decuplet with the pion nonet. The formulation of the $1/N_c$ baryon chiral Lagrangian relies upon recent developments in the study of the spin-flavor structure of baryons in the $1/N_c$ expansion [13–29]. In the large-N_c limit, the baryon sector of QCD possesses an exact contracted spin-flavor symmetry algebra [13,30]. For finite N_c, corrections to the large-N_c limit are parametrized by $1/N_c$-suppressed operators [13,14]. Consistency conditions determine which operators are allowed at any given order in the $1/N_c$ expansion. The (spin $\otimes$ flavor) structure of the $1/N_c$ expansion for baryons is manifest in the baryon chiral Lagrangian presented here. In addition, planar QCD flavor symmetry is implemented at leading order in $1/N_c$, and violated at first subleading order.

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Planar QCD flavor symmetry leads to a number of new results, such as the formation of flavor nonets among baryon flavor octet and singlet amplitudes at leading order in $1/N_c$. The consequences of planar QCD flavor symmetry are examined in detail and are entirely new to this work. Planar QCD flavor symmetry follows from the $1/N_c$ expansion alone, so the results of this symmetry for baryons do not depend on the chiral Lagrangian framework and are valid in general. Strong CP violation enters the baryon chiral Lagrangian in the same manner as earlier treatments [31–34]. Finally, the issue of nonanalytic meson-loop corrections to baryons amplitudes is examined. A number of subtleties arise in the calculation of loop corrections at finite N_c. The procedure for computing loop calculations using operators at finite N_c is explained. The group theoretic and $1/N_c$ structure of these corrections is explicit in this method.

The paper is organized as follows. A presentation of the pion nonet chiral Lagrangian is given in Sec. II to set notation. Readers familiar with the $1/N_c$ chiral Lagrangian for the pseudo Goldstone bosons may skip directly to Sec. III and refer to Sec. II only for definitions of the meson nonet field $\mathbf{\Phi}$ and strong-CP parameters. Section III presents the $1/N_c$ baryon chiral Lagrangian. The baryon chiral Lagrangian is formulated for arbitrary finite N_c in terms of the $1/N_c$ operator expansion for baryons. Planar QCD flavor symmetry is imposed on the Lagrangian at leading order in $1/N_c$. The $1/N_c$ baryon chiral Lagrangian for $N_c=3$ is compared to the chiral Lagrangian for the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet baryons with no $1/N_c$ expansion. Section IV addresses the computation of nonanalytic corrections using the $1/N_c$ baryon chiral Lagrangian. The flavor-**27** nonanalytic contribution to baryon masses is computed to illustrate the method. An understanding of the accuracy of the Gell-Mann–Okubo formula for baryon octet masses is gained from this computation. Section V considers the implications of U(2) planar QCD flavor symmetry for SU(3) breaking of the baryon $1/N_c$ expansion. Conclusions are presented in Sec. VI.

II. PION CHIRAL LAGRANGIAN

The $1/N_c$ chiral Lagrangian describing the interactions of baryons and low-momentum pions has the form

$$\mathscr{L}=\mathscr{L}_{\rm pion}+\mathscr{L}_{\rm baryon}, \tag{2.1}$$

where the pion Lagrangian describes the self-interactions of the pseudo Goldstone boson nonet. In order to calculate in chiral perturbation theory to nontrivial orders in the $1/N_c$ expansion for baryons, it is necessary to understand the $1/N_c$ chiral Lagrangian of the pion sector as well. This section contains a presentation of the pion nonet $1/N_c$ chiral Lagrangian, as originally derived by di Vecchia and Veneziano [7] and Witten [8]. The inclusion of strong-CP violation in the baryon sector involves making the same transformations on the baryon Lagrangian as on the pion Lagrangian, so it is useful to present a self-contained derivation. Readers already familiar with the pion $1/N_c$ chiral Lagrangian may proceed directly to Sec. III.

It is well known that CP violation enters the QCD Lagrangian through the vacuum angle parameter θ, which is a physical observable of the theory. $\mathrm{U}(1)_A$ transformations on the QCD Lagrangian can rotate part or all of this angular dependence among the $G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$ term and the phase of the quark mass matrix. Let us adopt the convention in which all θ dependence initially resides in the quark mass matrix.[1] With this convention, the pion nonet chiral Lagrangian is given by

$$\begin{aligned}\mathscr{L}_{\rm pion}=\frac{f_\pi^2}{4}\bigg[&\mathrm{Tr}\,\partial_\mu\overline{\Sigma}\partial^\mu\overline{\Sigma}^\dagger-\frac{a}{N_c}\bigg[\frac{i}{2}\mathrm{Tr}(\ln\overline{\Sigma}-\ln\overline{\Sigma}^\dagger)\bigg]^2\\&+b\,\mathrm{Tr}(\mathscr{M}_q\overline{\Sigma}+\mathscr{M}_q^\dagger\overline{\Sigma}^\dagger)+\frac{c}{N_c}(\mathrm{Tr}\,\overline{\Sigma}^\dagger\partial_\mu\overline{\Sigma})^2+\cdots\bigg],\end{aligned} \tag{2.2}$$

where $\mathscr{M}_q$ is the quark mass matrix and $\overline{\Sigma}=e^{2i\mathbf{\Phi}/f_\pi}$ depends nonlinearly on the pion nonet field $\mathbf{\Phi}=\pi^a\lambda^a/2+\eta' I/\sqrt{6}$ divided by $f_\pi=93$ MeV. The λ^a are the eight Gell-Mann matrices and I is the 3×3 unit matrix. Thus, the octet component of $\mathbf{\Phi}$ is given by

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & \pi^+ & K^+\\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0+\frac{1}{\sqrt{6}}\eta & K^0\\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta\end{pmatrix}. \tag{2.3}$$

Under $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$ transformations, $\overline{\Sigma}\to L\overline{\Sigma}R^\dagger$. Equation (2.2) is the most general Lagrangian consistent with chiral and planar QCD flavor symmetry and violation, to second order in the derivative expansion and to lowest nontrivial order in $\mathscr{M}_q$ and $1/N_c$. The term proportional to the parameter b is the usual quark mass term of the pion Lagrangian which explicitly breaks $\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R\to\mathrm{SU}(3)$. The a term, the anomaly term, breaks $\mathrm{U}(1)_A$ and is explicitly order $1/N_c$ since the anomaly involves a single quark loop. The a term violates planar QCD flavor symmetry. The c term splits $f_{\eta'}$ from f_π. Thus, the c term violates planar QCD nonet symmetry, and is explicitly order $1/N_c$. Both the a and c terms preserve the $\mathrm{U}(1)_V$ subgroup of planar QCD flavor symmetry. The parameters a and b are dimensionful: a is

[1]For intermediate situations [32], θ dependence appears in the term

$$\frac{\theta_0}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu},\quad \tilde{G}^a_{\mu\nu}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{a\rho\sigma},$$

in the QCD Lagrangian as well as the phase of the quark mass matrix, such that $\theta=\theta_0+\arg(\det\mathscr{M}_q)$. Under $\mathrm{U}(1)_A$ transformations $R=L^\dagger=e^{i\alpha/2}I$ for F light quark flavors,

$$\theta_0\to\theta_0+F\alpha,\quad \mathscr{M}_q\to e^{-i\alpha}\mathscr{M}_q,$$

leaving θ invariant.

$O(\Lambda^2)$ and b is $O(\Lambda)$, where Λ is a hadronic scale. The parameter c is dimensionless. Finally note that the pion Lagrangian (2.2) is of the form

$$N_c \mathscr{L}\left(\frac{\Phi}{\sqrt{N_c}}\right), \tag{2.4}$$

as required by large-N_c power counting rules for mesons, since $f_\pi \sim \sqrt{N_c}$. It will often be convenient to perform the rescaling $f_\pi \to \sqrt{N_c} f$ to keep all N_c-dependence manifest.

Recall that all θ dependence of the theory presently resides in the quark mass matrix $\mathscr{M}_q$, so $\theta = \arg(\det \mathscr{M}_q)$. By performing $SU(3)_L \times SU(3)_R$ transformations, the mass matrix can be written in the form

$$\mathscr{M}_q = \mathscr{M} e^{i\theta/3}, \tag{2.5}$$

where $\mathscr{M} = \mathrm{diag}(m_u, m_d, m_s)$ is real, since all terms in Lagrangian (2.2) are invariant under $SU(3)_L \times SU(3)_R$ transformations except for the quark mass terms which violate chiral symmetry explicitly. Now remove the phase in $\mathscr{M}_q$ by performing a $U(1)_A$ transformation:

$$\overline{\Sigma} \to e^{-i\theta/3} \overline{\Sigma}. \tag{2.6}$$

All terms are invariant under this transformation except for the a term and terms containing the quark mass matrix, so the Lagrangian becomes

$$\mathscr{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4}\Bigg[\mathrm{Tr}\,\partial_\mu \overline{\Sigma} \partial^\mu \overline{\Sigma}^\dagger - \frac{a}{N_c}\left[\theta + \frac{i}{2}\mathrm{Tr}(\ln\overline{\Sigma} - \ln\overline{\Sigma}^\dagger)\right]^2 + b\,\mathrm{Tr}[\mathscr{M}(\overline{\Sigma} + \overline{\Sigma}^\dagger)] + \frac{c}{N_c}(\mathrm{Tr}\overline{\Sigma}^\dagger \partial_\mu \overline{\Sigma})^2 + \cdots \Bigg], \tag{2.7}$$

where the θ dependence of the Lagrangian is now manifest. Lagrangian (2.7) is the chiral Lagrangian with the convention that no θ dependence resides in the quark mass matrix.

The $\overline{\Sigma}$ field spontaneously breaks the SU(3) chiral symmetry down to its diagonal subgroup. The vacuum expectation value of $\overline{\Sigma}$ is determined by minimization of the potential of the pion Lagrangian. The potential $V(\overline{\Sigma})$ is given by minus the nonderivative terms in the Lagrangian. Since the real mass matrix $\mathscr{M}$ is diagonal, the minimum of $\overline{\Sigma}$ is also diagonal, so one looks for a solution of the form

$$\langle \overline{\Sigma} \rangle = \begin{pmatrix} e^{i\phi_u} & 0 & 0 \\ 0 & e^{i\phi_d} & 0 \\ 0 & 0 & e^{i\phi_s} \end{pmatrix}. \tag{2.8}$$

The potential as a function of the ϕ_i is

$$V(\phi_i) = \frac{f_\pi^2}{4}\left[-\sum_i 2bm_i \cos\phi_i + \frac{a}{N_c}\left(\theta - \sum_i \phi_i\right)^2 \right]. \tag{2.9}$$

Minimization of the potential leads to the equations

$$2bm_i \sin\phi_i = \frac{a}{N_c}\left(\theta - \sum_j \phi_j\right). \tag{2.10}$$

The solution of Eq. (2.10) for a given θ determines the angles ϕ_i as a function of θ. Solutions of Eq. (2.10) for different values of θ are discussed in detail in Refs. [7,8].

It is more physical to reexpress the Lagrangian in terms of a Σ field with vacuum expectation value $\langle \Sigma \rangle = I$; this vacuum realignment is performed by making the substitution

$$\overline{\Sigma} = \langle \overline{\Sigma} \rangle \Sigma \tag{2.11}$$

in Eq. (2.7), so that

$$\mathscr{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4}\Bigg[\mathrm{Tr}\,\partial_\mu \Sigma \partial^\mu \Sigma^\dagger - \frac{a}{N_c}\left(\overline{\theta} + \frac{i}{2}\mathrm{Tr}(\ln\Sigma - \ln\Sigma^\dagger)\right)^2 + b\,\mathrm{Tr}(\overline{\mathscr{M}}\Sigma + \overline{\mathscr{M}}^\dagger \Sigma^\dagger) + \frac{c}{N_c}(\mathrm{Tr}\Sigma^\dagger \partial_\mu \Sigma)^2 + \cdots \Bigg], \tag{2.12}$$

where

$$\overline{\mathscr{M}} = \mathscr{M}\langle \overline{\Sigma} \rangle = \mathrm{diag}(m_i e^{\phi_i}) \tag{2.13}$$

and

$$\overline{\theta} = \left(\theta - \sum_i \phi_i\right). \tag{2.14}$$

Using the minimization equation (2.10), the mass can be rewritten as

$$\overline{\mathscr{M}} = \mathscr{M}(\theta) + i\frac{a\overline{\theta}}{2b}\frac{1}{N_c} I, \tag{2.15}$$

where

$$\mathscr{M}(\theta) = \mathrm{diag}(m_i \cos\phi_i). \tag{2.16}$$

Using Eq. (2.15) one obtains the final version of the pion Lagrangian:

$$\mathscr{L}_{\mathrm{pion}} = \frac{f_\pi^2}{4}\Bigg[\mathrm{Tr}\,\partial_\mu \Sigma \partial^\mu \Sigma^\dagger - \frac{a}{N_c}\left(\frac{i}{2}\mathrm{Tr}(\ln\Sigma - \ln\Sigma^\dagger)\right)^2 + b\,\mathrm{Tr}[\mathscr{M}(\theta)(\Sigma + \Sigma^\dagger - 2)] + \frac{c}{N_c}(\mathrm{Tr}\Sigma^\dagger \partial_\mu \Sigma)^2 + i\frac{a\overline{\theta}}{N_c}\left[\tfrac{1}{2}\mathrm{Tr}(\Sigma - \Sigma^\dagger) - \mathrm{Tr}(\ln\Sigma - \ln\Sigma^\dagger)\right] + \cdots \Bigg], \tag{2.17}$$

where a constant term has been dropped relative to Eq. (2.12).

The observed spectrum and mixing of the pion nonet can be understood using Lagrangian (2.17) if the parameters satisfy [7]

$$bm_u, bm_d \ll bm_s < \frac{a}{N_c}. \tag{2.18}$$

FIG. 1. SU$(2F)$ spin-flavor representation for ground-state baryons. The Young tableau has N_c boxes.

In the $\bar{u}u$, $\bar{d}d$, and $\bar{s}s$ basis, the neutral meson mass matrix is given by

$$b\begin{pmatrix} m_u\cos\phi_u & 0 & 0 \\ 0 & m_d\cos\phi_d & 0 \\ 0 & 0 & m_s\cos\phi_s \end{pmatrix} + \frac{a}{N_c}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{2.19}$$

to leading order in explicit chiral symmetry breaking and[2] $1/N_c$. In the chiral limit $m_i \to 0$, the η' is a massive SU(3) singlet with mass $m^2_{\eta'} = Fa/N_c$, and the octet mesons are exact massless Goldstone bosons [7].

III. BARYON CHIRAL LAGRANGIAN

This section formulates a $1/N_c$ baryon chiral Lagrangian for N_c large, finite, and odd. The Lagrangian is first presented in the flavor symmetry limit. Explicit flavor symmetry breaking terms involving the quark mass matrix are then added to the Lagrangian. Strong-CP violation enters the baryon chiral Lagrangian through these terms. The baryon chiral Lagrangian is written in terms of the $1/N_c$ (spin $\otimes$ flavor) operator expansion for baryons. The structure of this operator expansion is reviewed below.

The (spin $\otimes$ flavor) $1/N_c$ expansion for baryons organizes the lowest-lying baryon states into the completely symmetric SU$(2F)$ representation shown in Fig. 1. Under SU(2) $\otimes$ SU(F) symmetry, this representation decomposes into a tower of baryon states with spins $\frac{1}{2}, \ldots, N_c/2$ in the flavor representations displayed in Fig. 2. The weight diagrams of the flavor representations of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ baryons for $F=3$ are given in Figs. 3 and 4, respectively. For $N_c=3$, these flavor multiplets reduce to the baryon octet and decuplet, but for $N_c>3$, the multiplets contain additional baryon states which do not exist for $N_c=3$. Because of the complexity of the flavor representations for $F>2$, it is easier to focus on the operators than the states.

Any QCD operator transforming according to a given SU(2) $\times$ SU(F) representation has an expansion in terms of n-body operators of the form

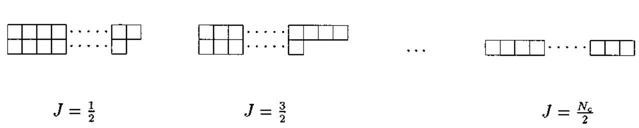

FIG. 2. SU(F) flavor representations for the tower of baryon states with $J=\frac{1}{2}, \frac{3}{2}, \ldots, N_c/2$. Each Young tableau has N_c boxes.

[2]The effects of higher order terms on a leading order bound on the mass ratio η/η' [35] have been considered recently in Ref. [36].

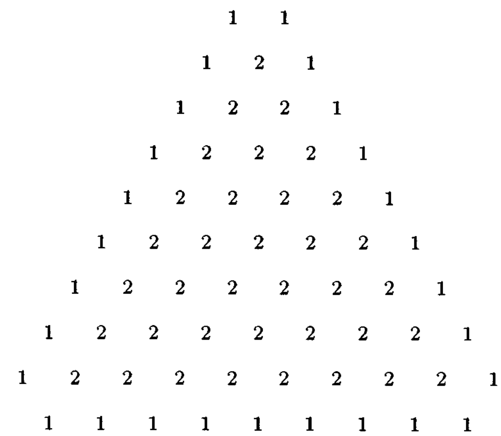

FIG. 3. Weight diagram for the SU(3) flavor representation of the spin-$\frac{1}{2}$ baryons. The long side of the weight diagram contains $\frac{1}{2}(N_c+1)$ weights. The numbers denote the multiplicity of the weights.

$$\mathcal{O}_{\rm QCD} = \sum_n c_{(n)} \frac{1}{N_c^{n-1}} \mathcal{O}_n, \tag{3.1}$$

where the operator basis $\mathcal{O}_n$ consists of polynomials in the spin-flavor generators J^i, T^a, and G^{ia}. The operator coefficients $c_{(n)}(1/N_c)$ have power series expansions in $1/N_c$ beginning at order unity.

The problem of finding a complete and independent set of operators for any spin-flavor representation was solved in Ref. [17]. The basic building blocks of the expansion are the zero-body SU$(2F)$ identity operator $\mathbb{1}$ and the one-body operators J^i, T^a, and G^{ia} which satisfy the SU$(2F)$ commutation relations. Because antisymmetric products of these operators can be reduced using the commutation relations, one only needs to consider operator products which are completely symmetric in noncommuting operators. In addition, it suffices to keep polynomials through order N_c for the lowest-

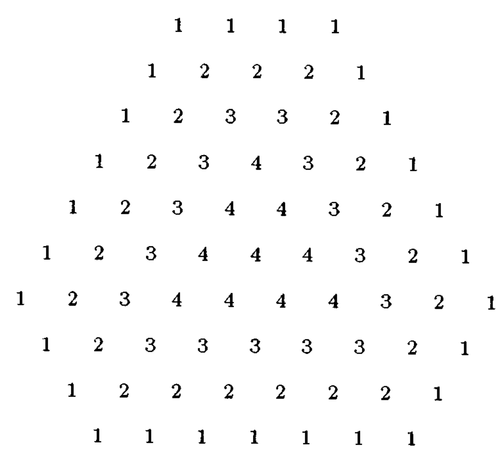

FIG. 4. Weight diagram for the SU(3) flavor representation of the spin-$\frac{3}{2}$ baryons. The long side of the weight diagram contains $\frac{1}{2}(N_c-1)$ weights. The numbers denote the multiplicity of the weights.

lying baryons. There are a number of identities among the polynomials of order less than or equal to N_c which further reduce the operator basis. The complete set of identities were derived in Ref. [17] using quark operators

$$J^i = q^\dagger\left(\frac{\sigma^i}{2}\otimes I\right)q \quad (1,1),$$
$$T^a = q^\dagger\left(I\otimes\frac{\lambda^a}{2}\right)q \quad (0,8), \tag{3.2}$$
$$G^{ia} = q^\dagger\left(\frac{\sigma^i}{2}\otimes\frac{\lambda^a}{2}\right)q \quad (1,8).$$

This paper also uses the quark representation of the $1/N_c$ operator expansion for baryons. Equivalent results can be obtained in the Skyrme representation.

A. Lagrangian in the flavor symmetry limit

In the large-N_c limit, baryons have masses of order N_c and become very heavy relative to mesons with masses of order 1. The $1/N_c$ baryon chiral Lagrangian is formulated treating baryons as heavy static fields with fixed velocity v^μ [37–39]. The $1/N_c$ expansion provides a systematic expansion parameter for this procedure. The following $1/N_c$ chiral Lagrangian is written in the rest frame of the baryon, which is the natural frame for the (spin $\otimes$ flavor) operator expansion. The generalization to an arbitrary velocity frame is straightforward.

The $1/N_c$ baryon chiral Lagrangian for arbitrary N_c is of the form

$$\mathcal{L}_{\text{baryon}} = i\mathcal{D}^0 - M_{\text{hyperfine}} + \text{Tr}(\mathcal{A}^i\lambda^a)A^{ia} + \text{Tr}\left(\mathcal{A}^i\frac{2I}{\sqrt{6}}\right)A^i + \cdots, \tag{3.3}$$

with

$$\mathcal{D}^0 = \partial^0\mathbb{1} + \text{Tr}(\mathcal{V}^0\lambda^a)T^a + \tfrac{1}{3}\text{Tr}(\mathcal{V}^0 I)N_c\mathbb{1}. \tag{3.4}$$

The notation of Eqs. (3.3) and (3.4) is very compact: each term involves a baryon operator. The baryon kinetic energy term is proportional to the spin-flavor identity element $\mathbb{1}$. The hyperfine baryon mass operator describes the spin splittings of the baryon tower. Pion fields appear in the chiral Lagrangian through the vector and axial vector combinations

$$\mathcal{V}^0 = \tfrac{1}{2}(\xi\partial^0\xi^\dagger + \xi^\dagger\partial^0\xi),$$
$$\mathcal{A}^i = \frac{i}{2}(\xi\nabla^i\xi^\dagger - \xi^\dagger\nabla^i\xi), \tag{3.5}$$

which depend nonlinearly on the field $\xi = e^{i\Phi/f_\pi}$. The vector pion combinations couple to baryon vector charges: the flavor octet pion combination couples to the flavor octet baryon charge[3]

$$V^{0a} = \left\langle \mathcal{B}'\left|\left(\bar{q}\gamma^0\frac{\lambda^a}{2}q\right)_{\text{QCD}}\right|\mathcal{B}\right\rangle, \tag{3.6}$$

while the flavor singlet pion combination

$$\text{Tr}\left(\mathcal{V}^0\frac{2I}{\sqrt{6}}\right) \tag{3.7}$$

couples to the flavor singlet baryon charge

$$V^0 = \left\langle \mathcal{B}'\left|\left(\bar{q}\gamma^0\frac{I}{\sqrt{6}}q\right)_{\text{QCD}}\right|\mathcal{B}\right\rangle. \tag{3.8}$$

The baryon vector charges equal

$$V^{0a} = v^0 T^a = T^a,$$
$$V^0 = v^0\frac{1}{\sqrt{6}}N_c\mathbb{1} = \frac{1}{\sqrt{6}}N_c\mathbb{1}, \tag{3.9}$$

to all orders in the $1/N_c$ expansion. The $\ell = 1$ flavor octet axial vector pion combination couples to the flavor octet baryon axial vector current

$$A^{ia} = \left\langle \mathcal{B}'\left|\left(\bar{q}\gamma^i\gamma_5\frac{\lambda^a}{2}q\right)_{\text{QCD}}\right|\mathcal{B}\right\rangle, \tag{3.10}$$

whereas the flavor singlet axial pion combination couples to the flavor singlet baryon axial vector current

$$A^i = \left\langle \mathcal{B}'\left|\left(\bar{q}\gamma^i\gamma_5\frac{I}{\sqrt{6}}q\right)_{\text{QCD}}\right|\mathcal{B}\right\rangle. \tag{3.11}$$

The ellipses in Eq. (3.3) denotes higher partial wave pion couplings which occur at subleading orders in the $1/N_c$ expansion for $N_c > 3$. At leading order in the $1/N_c$ expansion, the pion couplings of baryons are purely p wave for any N_c [17].

The baryon chiral Lagrangian describes the interactions of the pions and baryons in terms of QCD baryon operators. Each of these operators has an expansion in $1/N_c$ of the form Eq. (3.1).

In the limit of exact SU(3) flavor symmetry, the baryon mass operator is defined by

$$M = \langle\mathcal{B}'|\mathcal{H}_{\text{QCD}}|\mathcal{B}\rangle, \tag{3.12}$$

where $\mathcal{H}_{\text{QCD}}$ is the QCD Hamiltonian in the chiral limit $m_i \to 0$. The baryon mass operator transforms as a (0,1) under SU(2) $\times$ SU(3) symmetry. The $1/N_c$ expansion for a (0,1) QCD operator is of the form [14,15,20,22]

$$M = m^{0,1}_{(0)}N_c\mathbb{1} + \sum_{n=2,4}^{N_c-1} m^{0,1}_{(n)}\frac{1}{N_c^{n-1}}J^n. \tag{3.13}$$

The coefficients $m^{0,1}_{(n)}$ are dimensionful parameters of $O(\Lambda)$. The first term in expansion (3.13), the overall spin-independent mass of the baryon multiplet, is removed from the chiral Lagrangian by the heavy baryon field redefinition

[3]The subscript QCD is used to emphasize that the quark fields are QCD quark fields, not the quark creation and annihilation operators of the quark representation.

[37]. The spin-dependent terms in Eq. (3.13) define $M_{\rm hyperfine}$, which appears explicitly in the Lagrangian. The hyperfine mass expansion reduces to a single operator [14]

$$M=m_{(2)}^{0,1}\frac{1}{N_c}J^2, \tag{3.14}$$

for $N_c=3$.

The $1/N_c$ expansions for the baryon flavor octet and singlet axial currents were derived in Ref. [17]. The $1/N_c$ expansion for the (1,8) baryon axial current is given by

$$A^{ia}=a_{(1)}^{1,8}G^{ia}+\sum_{n=2,3}^{N_c} b_{(n)}^{1,8}\frac{1}{N_c^{n-1}}\mathcal{D}_n^{ia}+\sum_{n=3,5}^{N_c} c_{(n)}^{1,8}\frac{1}{N_c^{n-1}}\mathcal{O}_n^{ia}, \tag{3.15}$$

where the $\mathcal{D}_n^{ia}$ are diagonal operators, with nonzero matrix elements only between states with the same spin, and the $\mathcal{O}_n^{ia}$ are purely off-diagonal operators, with nonzero matrix elements only between states with different spin. The operators $\mathcal{D}_n^{ia}$ and $\mathcal{O}_n^{ia}$ are defined in Ref. [17]. Equation (3.15) reduces to

$$A^{ia}=a_{(1)}^{1,8}G^{ia}+b_{(2)}^{1,8}\frac{1}{N_c}J^iT^a+b_{(3)}^{1,8}\frac{1}{N_c^2}\{J^i,\{J^j,G^{ja}\}\}$$
$$+c_{(3)}^{1,8}\frac{1}{N_c^2}\left(\{J^2,G^{ia}\}-\frac{1}{2}\{J^i,\{J^j,G^{ja}\}\}\right) \tag{3.16}$$

for $N_c=3$. The $1/N_c$ expansion for the (1,1) baryon axial current is given by

$$A^{i}=\sum_{n=1,3}^{N_c} b_{(n)}^{1,1}\frac{1}{N_c^{n-1}}\mathcal{D}_n^{i}, \tag{3.17}$$

where $\mathcal{D}_1^i=J^i$ and $\mathcal{D}_{n+2}^i=\{J^2,\mathcal{D}_n^i\}$. Equation (3.17) reduces to

$$A^i=b_{(1)}^{1,1}J^i+b_{(3)}^{1,1}\frac{1}{N_c^2}\{J^2,J^i\}, \tag{3.18}$$

for $N_c=3$.

Lagrangian $\mathcal{L}_{\rm baryon}$ is the most general Lagrangian invariant under $SU(3)_L\times SU(3)_R\times U(1)_V\times U(1)_A$ chiral symmetry and contracted spin-flavor symmetry. The form of the Lagrangian factorizes baryon invariants from pion invariants explicitly, which is necessary because baryons transform under a larger symmetry than mesons in the large-N_c limit. The Lagrangian correctly relates baryon-multipion vertices using chiral symmetry. Under chiral transformations,

$$\xi\rightarrow L\xi U^\dagger=U\xi R^\dagger, \tag{3.19}$$

where U is a vector $SU(3)\times U(1)$ transformation defined by Eq. (3.19). The flavor representations of the baryon spin tower transform as N_c-index tensor representations under U,

$$B^{\alpha_1\cdots\alpha_n\cdots\alpha_{N_c}}\rightarrow\sum_n U^{\alpha_n}_{\alpha_n'}B^{\alpha_1\cdots\alpha_n'\cdots\alpha_{N_c}}, \tag{3.20}$$

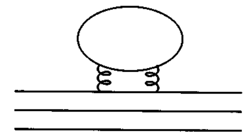

FIG. 5. Planar QCD flavor breaking at order $1/N_c$ due to a single quark loop.

where the symmetry of the baryon flavor tensors is dictated by the Young tableaux of Fig. 2. The pion combinations (3.5) are unaffected by vacuum realignment (2.11), so the SU(3)-symmetric baryon chiral Lagrangian contains no strong CP violation.

Planar QCD flavor symmetry further constrains the parameters of the $1/N_c$ baryon chiral Lagrangian. In the next subsection, planar QCD flavor symmetry is imposed on the baryon chiral Lagrangian. In the following subsection, the chiral Lagrangian for the baryon octet and decuplet is compared with the $1/N_c$ baryon chiral Lagrangian at $N_c=3$.

1. Planar QCD flavor symmetry

Planar QCD flavor symmetry implies that the baryon $1/N_c$ chiral Lagrangian possesses a $SU(2)\otimes U(3)$ (spin $\otimes$ flavor) symmetry at leading order in the $1/N_c$ expansion. The symmetry is broken at first subleading order by diagrams with a single quark loop, as shown in Fig. 5.

Planar QCD flavor symmetry constrains $\mathcal{L}_{\rm baryon}$ by forming a nonet baryon axial vector current out of the singlet and octet baryon axial vector currents at leading order in the $1/N_c$ expansion:[4]

$$A^i=A^{i9}+O(1/N_c). \tag{3.21}$$

This constraint relates the coefficients of the A^i expansion to those of the A^{ia} expansion in the limit $N_c\rightarrow\infty$. The easiest way to impose Eq. (3.21) is to replace the operator coefficients of the singlet axial vector expansion Eq. (3.17) by

$$b_{(n)}^{1,1}\rightarrow\bar{b}_{(n)}^{1,1}+\frac{1}{N_c}b_{(n)}^{1,1}, \tag{3.22}$$

where the coefficients with an overscore are determined by exact nonet symmetry, and the remainders are unconstrained and violate nonet symmetry at first subleading order $1/N_c$. For arbitrary N_c, nonet symmetry implies

$$\bar{b}_{(1)}^{1,1}=\frac{1}{\sqrt{6}}(a_{(1)}^{1,8}+b_{(2)}^{1,8}), \tag{3.23}$$

$$\bar{b}_{(3)}^{1,1}=\frac{1}{\sqrt{6}}(2b_{(3)}^{1,8}+b_{(4)}^{1,8}),$$

[4]The baryon vector currents V^{0a} and V^0 form a flavor nonet to all orders in the $1/N_c$ expansion.

$$\bar{b}^{1,1}_{(5)}=\frac{1}{\sqrt{6}}(2b^{1,8}_{(5)}+b^{1,8}_{(6)}),$$

$$\ldots,$$

where the relative factor of $1/\sqrt{6}$ occurs because the ninth flavor components of G^{ia} and T^a are related to J^i and $N_c\mathbb{1}$ by

$$G^{i9}=q^\dagger\left(\frac{\sigma^i}{2}\otimes\frac{I}{\sqrt{6}}\right)q=\frac{1}{\sqrt{6}}J^i \quad (1,1),$$

$$T^9=q^\dagger\left(I\otimes\frac{I}{\sqrt{6}}\right)q=\frac{1}{\sqrt{6}}N_c\mathbb{1} \quad (0,1). \tag{3.24}$$

Notice that the coefficients of the diagonal operators $\mathscr{D}^i_n$ in the singlet expansion do not depend on the coefficients $c^{1,8}_{(n)}$ of the off-diagonal operators $\mathscr{O}^{ia}_n$ in the octet expansion. For $N_c=3$, the nonet symmetry conditions reduce to

$$\bar{b}^{1,1}_{(1)}=\frac{1}{\sqrt{6}}(a^{1,8}_{(1)}+b^{1,8}_{(2)}),$$

$$\bar{b}^{1,1}_{(3)}=\frac{1}{\sqrt{6}}(2b^{1,8}_{(3)}), \tag{3.25}$$

where the second condition is modified because the four-body operator corresponding to $b^{1,8}_{(4)}$ does not occur in the operator basis for $N_c=3$.

It is important to stress that the nonet symmetry constraint Eq. (3.21) leads to a condition for each operator coefficient in the singlet expansion since this constraint must be satisfied for *all* spin states of the baryon tower (not just the states with spins of order unity). The fact that Eq. (3.21) is satisfied operator by operator in the baryon (spin $\otimes$ flavor) operator expansion is consistent with the violation of planar QCD flavor symmetry by single quark loop diagrams Fig. 5, since this breaking is decoupled from the baryon $1/N_c$ operator expansion.

The final version of the $1/N_c$ baryon chiral Lagrangian can be obtained by rewriting $\mathscr{L}_{\rm baryon}$ in a form which implements the constraints of planar QCD symmetry explicitly:

$$\mathscr{L}_{\rm baryon}=i\mathscr{D}^0-M_{\rm hyperfine}+{\rm Tr}(\mathscr{A}^i\lambda^a)A^{ia}$$
$$+\frac{1}{N_c}{\rm Tr}\left(\mathscr{A}^i\frac{2I}{\sqrt{6}}\right)A^i+\cdots, \tag{3.26}$$

with

$$\mathscr{D}^0=\partial^0\mathbb{1}+{\rm Tr}(\mathscr{V}^0\lambda^a)T^a, \tag{3.27}$$

where $a=1,\ldots,9$, $\lambda^9\equiv 2I/\sqrt{6}$, and the baryon one-body operators T^9 and G^{i9} are defined in Eq. (3.24). Nonet flavor symmetry of the baryon-pion axial vector couplings is broken by the last term, which gives a nonet symmetry-breaking contribution to the singlet current at relative order $1/N_c$.

2. Comparison with octet and decuplet chiral Lagrangian

It is instructive to compare the $1/N_c$ chiral Lagrangian at $N_c=3$ with the chiral Lagrangian for the baryon octet and decuplet without a $1/N_c$ expansion. The flavor octet pion couplings of the octet and decuplet are described by the chiral Lagrangian [37]:

$$\mathscr{L}_{\rm baryon}=i{\rm Tr}\bar{B}_v(v\cdot\mathscr{D})B_v-i\bar{T}^\mu_v(v\cdot\mathscr{D})T_{v\mu}+\Delta\bar{T}^\mu_vT_{v\mu}$$
$$+2D{\rm Tr}\bar{B}_vS^\mu_v\{\mathscr{A}_\mu,B_v\}+2F{\rm Tr}\bar{B}_vS^\mu_v[\mathscr{A}_\mu,B_v]$$
$$+\mathscr{C}(\bar{T}^\mu_v\mathscr{A}_\mu B_v+\bar{B}_v\mathscr{A}_\mu T^\mu_v)+2\mathscr{H}\bar{T}^\mu_vS^\nu_v\mathscr{A}_\nu T_{v\mu}, \tag{3.28}$$

where D, F, $\mathscr{C}$, and $\mathscr{H}$ are the baryon-pion couplings and $\Delta=m_T-m_B$ is the decuplet-octet mass difference. The octet mass m_B has been removed from the Lagrangian by the heavy baryon field redefinition. Flavor singlet baryon-η' couplings can be incorporated into the chiral Lagrangian by adding two terms:

$$2S_B{\rm Tr}\mathscr{A}_\mu{\rm Tr}\bar{B}_vS^\mu_vB_v-2S_T{\rm Tr}\mathscr{A}_\nu\bar{T}^\mu_vS^\nu_vT_{v\mu}, \tag{3.29}$$

where S_B and S_T are the singlet axial vector coupling constants of the octet and decuplet, respectively.

There is a one-to-one correspondence between the parameters of the octet and decuplet chiral Lagrangian and the coefficients of the $1/N_c$ baryon chiral Lagrangian at $N_c=3$. The mass parameters are related to the $1/N_c$ mass coefficients by

$$m_B=3m^{0,1}_{(0)}+\tfrac{1}{4}m^{0,1}_{(2)},$$

$$m_T=3m^{0,1}_{(0)}+\tfrac{5}{4}m^{0,1}_{(2)}, \tag{3.30}$$

so that

$$\Delta=m^{0,1}_{(2)}. \tag{3.31}$$

The flavor octet baryon-pion couplings are related to the coefficients of the $1/N_c$ expansion at $N_c=3$ by

$$D=\tfrac{1}{2}a^{1,8}_{(1)}+\tfrac{1}{6}b^{1,8}_{(3)},$$

$$F=\tfrac{1}{3}a^{1,8}_{(1)}+\tfrac{1}{6}b^{1,8}_{(2)}+\tfrac{1}{9}b^{1,8}_{(3)},$$

$$\mathscr{C}=-a^{1,8}_{(1)}-\tfrac{1}{2}c^{1,8}_{(3)}, \tag{3.32}$$

$$\mathscr{H}=-\tfrac{3}{2}a^{1,8}_{(1)}-\tfrac{3}{2}b^{1,8}_{(2)}-\tfrac{5}{2}b^{1,8}_{(3)}.$$

Notice that the purely off-diagonal operator coefficient $c^{1,8}_{(3)}$ contributes only to the octet-decuplet-pion coupling constant $\mathscr{C}$, and that the diagonal operator coefficients $b^{1,8}_{(n)}$ contribute only to the diagonal couplings D, F, and $\mathscr{H}$. In addition, $b^{1,8}_{(2)}$ is pure F, and does not contribute to D. The flavor singlet baryon-pion couplings are related to the coefficients of the $1/N_c$ expansion at $N_c=3$ by

$$S_B=\frac{1}{\sqrt{6}}(b^{1,1}_{(1)}+\tfrac{1}{6}b^{1,1}_{(3)}),$$

$$S_T=\frac{3}{\sqrt{6}}(b^{1,1}_{(1)}+\tfrac{5}{6}b^{1,1}_{(3)}). \tag{3.33}$$

The factor of 3 in the second relation occurs because the decuplet spin operator in Eq. (3.29) acts only on the spinor portion of the spin-$\frac{3}{2}$ Rarita-Schwinger field [37]. The metric of the spin-one portion of the spin-$\frac{3}{2}$ field T^μ_v cancels the minus sign of the decuplet term in Eq. (3.29).

Relations (3.30)–(3.33) are valid for N_c set equal to three. For arbitrary N_c, the $1/N_c$ expansions for baryons with spins of order unity can be truncated:

$$M=m^{0,1}_{(0)}N_c\mathbb{1},$$

$$A^{ia}=a^{1,8}_{(1)}G^{ia}+b^{1,8}_{(2)}\frac{1}{N_c}J^iT^a, \tag{3.34}$$

$$A^i=b^{1,1}_{(1)}J^i,$$

where Eqs. (3.34) are valid up to terms of relative order $O(1/N_c^2)$ *everywhere* in the flavor weight diagrams. The parameter $b^{1,8}_{(2)}$ produces deviations from SU(6) symmetry. In the limit $N_c\rightarrow3$, Eqs. (3.34) lead to the parameter relations

$$m_B=m_T,$$

$$\mathscr{C}=-2D,\quad \mathscr{H}=3D-9F, \tag{3.35}$$

$$S_B=\tfrac{1}{3}S_T.$$

The implementation of flavor nonet symmetry on the axial vector baryon-pion couplings raises an interesting subtlety. The spin-$\frac{1}{2}$ baryon SU(3) field with mixed symmetry is written as a tensor with an upper index and a lower index by using the flavor SU(3) ϵ-tensor to replace two antisymmetric upper indices by a single lower index:

$$B^\alpha_\beta=\epsilon_{\beta\gamma\delta}B^{\alpha[\gamma\delta]}. \tag{3.36}$$

The octet tensor B^α_β transforms in the same manner as the three-index tensor $B^{\alpha[\gamma\delta]}$ under SU(3) transformations, since the ϵ-tensor is an invariant tensor under SU(3) transformations. The ϵ-tensor, however, is not invariant under U(1) transformations, so replacing $B^{\alpha[\gamma\delta]}$ by B^α_β is not legitimate when U(3) flavor symmetry is present. Nonet flavor symmetry cannot be imposed on the baryon chiral Lagrangian (3.28) by simply promoting $\mathscr{V}_\mu$ and $\mathscr{A}_\mu$ to nonet matrices since the Lagrangian is written in terms of B^α_β. It is not difficult to work out the condition of nonet symmetry for the baryon axial vector couplings using $B^{\alpha[\gamma\delta]}$:

$$S_B\rightarrow\tfrac{1}{3}(3F-D),$$

$$S_T\rightarrow-\frac{1}{3}\mathscr{H}. \tag{3.37}$$

The consistency of Eq. (3.37) with Eq. (3.25) can be checked using Eqs. (3.32) and (3.33).

The generalization of Eq. (3.37) to arbitrary N_c for the spin-$\frac{1}{2}$ baryons also is of interest. Reference [15] defined the pion octet couplings $\mathscr{M}$ and $\mathscr{N}$ of the spin-$\frac{1}{2}$ baryons in terms of the large-N_c baryon tensor with one upper index and $\nu=(N_c-1)/2$ lower indices. The lesson of U(3) symmetry is that the use of baryon tensors with antisymmetric indices lowered by the flavor ϵ symbol is to be avoided in the $1/N_c$ expansion; for general N_c, one should work exclusively with N_c-index baryon flavor tensors. It is straightforward to rewrite these invariants using the N_c-index flavor tensor

$$B^{\alpha_1[\alpha_2\alpha_3]\cdots[\alpha_{N_c-1}\alpha_{N_c}]} \tag{3.38}$$

for the spin-$\frac{1}{2}$ baryons. Planar QCD flavor symmetry relates the singlet invariant of the spin-$\frac{1}{2}$ baryons to the octet invariants:

$$\mathcal{S}_{1/2}=\frac{1}{3}(\mathscr{M}-2\mathscr{N})+O\left(\frac{1}{N_c}\right), \tag{3.39}$$

where $\mathcal{S}_{1/2}$ is the generalization of S_B for large-N_c flavor representations. The $O(1/N_c)$ correction to Eq. (3.39) is due to violation of planar QCD flavor symmetry.[5] Reference [15] proved that the ratio

$$\frac{\mathscr{N}}{\mathscr{M}}=\frac{1}{2}+\frac{\alpha}{N_c}+O\left(\frac{1}{N_c^2}\right), \tag{3.40}$$

where $\mathscr{M}$ and $\mathscr{N}$ are both $O(N_c)$. Substitution into Eq. (3.39) shows that the leading term cancels, so that

$$\frac{\mathcal{S}_{1/2}}{\mathscr{M}}=-\frac{2}{3}\frac{\alpha}{N_c}+O\left(\frac{1}{N_c^2}\right), \tag{3.41}$$

where the $O(1/N_c^2)$ correction depends on nonet symmetry violation and on the $O(1/N_c^2)$ contribution to $\mathscr{N}/\mathscr{M}$. Reference [17] showed that

$$\alpha=-\frac{3}{2}\left(1+\frac{b^{1,8}_{(2)}}{a^{1,8}_{(1)}}\right), \tag{3.42}$$

so Eq. (3.41) implies that the singlet axial vector current is order $1/N_c$ relative to the octet current and that the normalization depends on the ratio of $b^{1,8}_{(2)}$ to $a^{1,8}_{(1)}$ at leading order.

B. Lagrangian with quark mass flavor breaking

Explicit flavor symmetry breaking enters the baryon chiral Lagrangian through terms containing powers of the quark mass matrix. The leading Lagrangian with a single insertion of the quark mass matrix is presented in this subsection. The singlet and octet components of these linear terms form a nonet at leading order in the $1/N_c$ expansion due to planar QCD flavor symmetry. Vacuum realignment generates strong-CP violating terms, which also form a nonet at leading order in the $1/N_c$ expansion. Section III B 1 compares the

[5] The invariant $\mathcal{S}_{1/2}$ is $O(1)$ even though $\mathscr{M}$ and $\mathscr{N}$ are both $O(N_c)$, so the correction to Eq. (3.39) is both of relative and absolute order $1/N_c$.

$1/N_c$ Lagrangian terms with one insertion of the quark mass matrix to the octet and decuplet Lagrangian with no $1/N_c$ expansion. Section III B 2 discusses the implications of nonet symmetry for the proton matrix element $\langle p|m_s\bar{s}s|p\rangle$.

The leading Lagrangian with one power of the quark mass matrix is given by

$$\begin{aligned}\mathcal{L}^{\mathcal{M}}_{\text{baryon}} = {} & \text{Tr}\left((\mathcal{M}\bar{\Sigma} + \mathcal{M}^\dagger \bar{\Sigma}^\dagger)\frac{I}{\sqrt{6}}\right)\mathcal{H}^0 \\ & + \text{Tr}\left((\bar{\xi}\mathcal{M}\bar{\xi} + \bar{\xi}^\dagger \mathcal{M}^\dagger \bar{\xi}^\dagger)\frac{\lambda^3}{2}\right)\mathcal{H}^3 \\ & + \text{Tr}\left((\bar{\xi}\mathcal{M}\bar{\xi} + \bar{\xi}^\dagger \mathcal{M}^\dagger \bar{\xi}^\dagger)\frac{\lambda^8}{2}\right)\mathcal{H}^8, \end{aligned} \tag{3.43}$$

where the singlet perturbation to the Hamiltonian

$$\mathcal{H}^0 = \frac{1}{\sqrt{6}}\langle \mathcal{B}'|(\bar{q}q)_{\text{QCD}}|\mathcal{B}\rangle, \tag{3.44}$$

and the octet $a=3,8$ Hamiltonian perturbations

$$\mathcal{H}^a = \left\langle \mathcal{B}' \left| \left(\bar{q}\frac{\lambda^a}{2}q\right)_{\text{QCD}} \right| \mathcal{B}\right\rangle. \tag{3.45}$$

Note that terms containing the pseudoscalar mass combination $(\bar{\xi}\mathcal{M}\bar{\xi} - \bar{\xi}^\dagger \mathcal{M}^\dagger \bar{\xi}^\dagger)$ are subleading in the $1/N_c$ expansion and have been neglected. These terms are suppressed by one factor of $1/N_c$ relative to the terms involving $(\bar{\xi}\mathcal{M}\bar{\xi} + \bar{\xi}^\dagger \mathcal{M}^\dagger \bar{\xi}^\dagger)$ since baryon matrix elements of the pseudoscalar QCD quark operators are $O(1/N_c)$.

The explicit symmetry breaking perturbations to the baryon Hamiltonian have expansions in $1/N_c$. The general expansion of the singlet perturbation has the same form as Eq. (3.13) and reduces to

$$\mathcal{H}^0 = b^{0,1}_{(0)} N_c \mathbb{1} + b^{0,1}_{(2)}\frac{1}{N_c}J^2 \tag{3.46}$$

for $N_c=3$. The general expansion for the $(0,8)$ perturbation was derived in Ref. [17],

$$\mathcal{H}^a = \sum_{n=1}^{N_c} b^{0,8}_{(n)}\frac{1}{N_c^{n-1}}\mathcal{D}^a_n, \tag{3.47}$$

where $\mathcal{D}^a_1 = T^a$, $\mathcal{D}^a_2 = \{J^i, G^{ia}\}$ and $\mathcal{D}^a_{n+2} = \{J^2, \mathcal{D}^a_n\}$. Equation (3.47) reduces to

$$\mathcal{H}^a = b^{0,8}_{(1)}T^a + b^{0,8}_{(2)}\frac{1}{N_c}\{J^i, G^{ia}\} + b^{0,8}_{(3)}\frac{1}{N_c^2}\{J^2, T^a\} \tag{3.48}$$

for $N_c=3$.

Vacuum realignment affects the quark mass terms, resulting in baryon-pion couplings which violate strong CP. Equations (2.11) and (2.13) imply that the mass combination appearing in Eq. (3.43) is replaced by [32]

$$\begin{aligned}(\xi\bar{\mathcal{M}}\xi + \xi^\dagger \bar{\mathcal{M}}^\dagger \xi^\dagger) = {} & (\xi\mathcal{M}(\theta)\xi + \xi^\dagger \mathcal{M}^\dagger(\theta)\xi^\dagger) \\ & + i\frac{a\bar{\theta}}{2b}\frac{1}{N_c}(\Sigma - \Sigma^\dagger), \end{aligned} \tag{3.49}$$

where $\mathcal{M}(\theta)$ and $\bar{\theta}$ are defined in Sec. II. The term proportional to $\bar{\theta}$ violates strong CP.

Planar QCD flavor symmetry constrains the coefficients of $\mathcal{L}^{\mathcal{M}}_{\text{baryon}}$. At leading order in the $1/N_c$ expansion, the coefficients of the singlet perturbation are related to the $a=3,8$ octet coefficients:

$$\bar{b}^{0,1}_{(0)} = \frac{1}{\sqrt{6}}b^{0,8}_{(1)},$$

$$\bar{b}^{0,1}_{(2)} = \frac{1}{\sqrt{6}}(2b^{0,8}_{(2)} + 2b^{0,8}_{(3)}). \tag{3.50}$$

The normalization of the singlet perturbation deviates from nonet symmetry at relative order $1/N_c$.

The final version of the leading $1/N_c$ chiral Lagrangian containing explicit symmetry breaking is as follows. The Lagrangian

$$\begin{aligned}\mathcal{L}^{\mathcal{M}}_{\text{baryon}} = {} & \text{Tr}\left([\xi\mathcal{M}(\theta)\xi + \xi^\dagger \mathcal{M}^\dagger(\theta)\xi^\dagger]\frac{\lambda^a}{2}\right)\mathcal{H}^a \\ & + \frac{1}{N_c}\text{Tr}\left([\mathcal{M}(\theta)\Sigma + \mathcal{M}^\dagger(\theta)\Sigma^\dagger]\frac{I}{\sqrt{6}}\right)\mathcal{H}^0 \end{aligned} \tag{3.51}$$

for $a=3,8,9$, respects CP. The strong-CP violating Lagrangian is given by

$$\begin{aligned}\mathcal{L}^{\bar{\theta}}_{\text{baryon}} = {} & i\frac{a\bar{\theta}}{2b}\frac{1}{N_c}\text{Tr}\left((\Sigma - \Sigma^\dagger)\frac{\lambda^a}{2}\right)\mathcal{H}^a \\ & + i\frac{a\bar{\theta}}{2b}\frac{1}{N_c^2}\text{Tr}\left((\Sigma - \Sigma^\dagger)\frac{I}{\sqrt{6}}\right)\mathcal{H}^0 \end{aligned} \tag{3.52}$$

for $a=3,8,9$. Both of these Lagrangians exhibit nonet symmetry at leading order in the $1/N_c$ expansion. The second terms in Eqs. (3.51) and (3.52) represent planar QCD flavor breaking of relative order $1/N_c$ for the singlet perturbation.

1. Comparison with octet and decuplet chiral Lagrangian

The quark mass terms of the $1/N_c$ baryon chiral Lagrangian can be compared to the quark mass terms of the octet and decuplet chiral Lagrangian with no $1/N_c$ expansion. Strong CP violation is neglected in the following comparison.

To first order in the quark mass matrix, the chiral Lagrangian for the octet and decuplet baryons is given by

$$\begin{aligned}\mathcal{L}^{\mathcal{M}} = {} & \sigma\text{Tr}[\mathcal{M}(\Sigma + \Sigma^\dagger)]\text{Tr}(\bar{B}B) - \tilde{\sigma}\text{Tr}[\mathcal{M}(\Sigma+\Sigma^\dagger)]\bar{T}^\mu T_\mu \\ & + b_D \text{Tr}\bar{B}\{(\xi^\dagger\mathcal{M}\xi^\dagger + \xi\mathcal{M}\xi), B\} + b_F\text{Tr}\bar{B}[(\xi^\dagger\mathcal{M}\xi^\dagger \\ & + \xi\mathcal{M}\xi), B] + c\bar{T}^\mu(\xi^\dagger\mathcal{M}\xi^\dagger + \xi\mathcal{M}\xi)T_\mu, \end{aligned} \tag{3.53}$$

where σ and $\tilde{\sigma}$ are the singlet quark mass parameters of the octet and decuplet, respectively. The parameters b_D and b_F describe the flavor octet quark mass splittings of the baryon octet, whereas the parameter c describes the flavor octet quark mass splittings of the baryon decuplet.

There is a one-to-one correspondence between the parameters of the octet and decuplet chiral Lagrangian and the coefficients of the $1/N_c$ baryon chiral Lagrangian at $N_c=3$. The singlet quark mass parameters are related to the $1/N_c$ singlet coefficients by

$$\sigma=\frac{1}{\sqrt{6}}\left(3b_{(0)}^{0,1}+\frac{1}{4}b_{(2)}^{0,1}\right),$$

$$\tilde{\sigma}=\frac{1}{\sqrt{6}}\left(3b_{(0)}^{0,1}+\frac{5}{4}b_{(2)}^{0,1}\right). \tag{3.54}$$

The octet quark mass parameters are related to the $1/N_c$ octet coefficients by

$$b_D=\tfrac{1}{4}b_{(2)}^{0,8},$$

$$b_F=\tfrac{1}{2}b_{(1)}^{0,8}+\tfrac{1}{6}b_{(2)}^{0,8}+\tfrac{1}{12}b_{(3)}^{0,8}, \tag{3.55}$$

$$c=-\tfrac{3}{2}b_{(1)}^{0,8}-\tfrac{5}{4}b_{(2)}^{0,8}-\tfrac{5}{4}b_{(3)}^{0,8}.$$

Notice that the leading octet coefficient $b_{(1)}^{0,8}$ is pure F and does not contribute to b_D.

Relations (3.54) and (3.55) are valid for N_c set equal to three. For arbitrary N_c, the $1/N_c$ expansions of the singlet and octet perturbations can be truncated for baryons with spins of order unity. The leading singlet truncation

$$\mathscr{H}^0=b_{(0)}^{0,1}N_c\mathbb{1} \tag{3.56}$$

implies the parameter relation

$$\sigma=\tilde{\sigma} \tag{3.57}$$

up to a correction of relative order $1/N_c^2$. For the $a=8$ perturbation, the leading truncation is

$$\mathscr{H}^8=b_{(1)}^{0,8}T^8, \tag{3.58}$$

up to a correction of order $1/N_c$ since the mass splittings produced by the operators T^8 and $\{J^i,G^{i8}\}$ are both order unity in the $1/N_c$ expansion [18]. Equation (3.58) leads to the parameter relations

$$b_F=-\tfrac{1}{3}c,\quad b_D=0, \tag{3.59}$$

which are valid at order unity in the $1/N_c$ expansion. The subleading truncation

$$\mathscr{H}^8=b_{(1)}^{0,8}T^8+b_{(2)}^{0,8}\frac{1}{N_c}\{J^i,G^{i8}\} \tag{3.60}$$

is valid up to a correction of order $1/N_c^2$. One linear combination of the two parameter relations Eq. (3.59) survives at this order:

$$(b_D+b_F)=-\tfrac{1}{3}c. \tag{3.61}$$

The correction to this relation is order $1/N_c^2$. Equation (3.61) is in excellent agreement with the experimental values $b_Dm_s\sim30$MeV, $b_Fm_s\sim-95$MeV, and $2cm_s/3\sim150$MeV extracted[6] in Ref. [42].

Planar QCD flavor symmetry relates the singlet quark mass parameters to the octet mass parameters:

$$\sigma\to\tfrac{1}{3}(3b_F-b_D),\quad \tilde{\sigma}\to-\tfrac{1}{3}c. \tag{3.62}$$

The consistency of (3.62) with Eq. (3.50) can be checked using Eqs. (3.54) and (3.55).

2. The proton matrix element $\langle p|m_s\bar{s}s|p\rangle$

Nonet symmetry among the linear quark mass splittings of the baryons has implications for the analysis of the proton matrix element $\langle p|m_s\bar{s}s|p\rangle$. The analysis of the linear in m_s contribution to this matrix element is discussed in this subsection. The affect of contributions to the proton mass which are nonlinear in quark masses can be computed using the methods of Sec. IV.

The proton matrix element of the strange quark mass operator is obtained by differentiation of the proton mass with respect to m_s:

$$\langle p|m_s\bar{s}s|p\rangle=m_s\frac{\partial m_p}{\partial m_s}. \tag{3.63}$$

The standard chiral Lagrangian expansion of the proton mass to linear order in the quark masses is

$$m_p=m_B-2\sigma(m_u+m_d+m_s)+2(b_F-b_D)m_s$$
$$-2(b_F+b_D)m_u+\text{ nonlinear}, \tag{3.64}$$

which implies that

$$\langle p|m_s\bar{s}s|p\rangle=2(-\sigma+b_F-b_D)m_s+\text{nonlinear}. \tag{3.65}$$

Substitution of the nonet symmetry relation Eq. (3.62) leads to an exact cancellation of the b_F term in the nonet symmetry limit.

It is instructive to study this cancellation in the $1/N_c$ expansion. Expanding the proton mass to linear order in quark masses using the $1/N_c$ chiral Lagrangian, and differentiation with respect to m_s leads to

[6]A similar analysis applies for the $\Delta S=1$ weak Lagrangian which is responsible for hyperon nonleptonic decay. The octet and decuplet $\Delta S=1$ weak Lagrangian involves three parameters h_D, h_F, and h_C which are in one-to-one correspondence with the three coefficients of the $1/N_c$ expansion for $\mathscr{H}_{\text{weak}}^{\Delta S=1}$ [17]. The analogues of Eqs. (3.59) and (3.61) for h_D, h_F, and h_C are obtained. Equation (3.59) for hyperon nonleptonic decay was originally predicted in the chiral quark model [43]. The experimental values of these parameters extracted from the s-wave decays at one-loop order in chiral perturbation theory [40,41] are consistent with these parameter relations.

$$\langle p|m_s\bar{s}s|p\rangle = -\frac{2}{\sqrt{6}}\left(N_c b^{0,1}_{(0)} + \frac{3}{4}\frac{1}{N_c}b^{0,1}_{(2)}\right)m_s + \frac{1}{3}\left(N_c b^{0,8}_{(1)} + \frac{3}{2}\frac{1}{N_c}b^{0,8}_{(2)} + \frac{3}{2}\frac{1}{N_c}b^{0,8}_{(3)}\right)m_s + \text{nonlinear}, \tag{3.66}$$

where the relations

$$T^8 = \frac{1}{2\sqrt{3}}(N_c - 3N_s),$$
$$G^{i8} = \frac{1}{2\sqrt{3}}(J^i - 3J^i_s), \tag{3.67}$$

have been used to evaluate proton matrix elements. The nonet symmetry conditions Eq. (3.50) result in an exact cancellation among the singlet and octet quark mass contributions. Thus, the linear contribution to $\langle p|m_s\bar{s}s|p\rangle$ is produced entirely by violation of nonet symmetry at order $1/N_c$ in the $1/N_c$ expansion,

$$\langle p|m_s\bar{s}s|p\rangle = O\left(\frac{1}{N_c}\right) + \text{nonlinear}, \tag{3.68}$$

where the $O(1/N_c)$ term represents $1/N_c$-breaking of nonet symmetry in the singlet channel. The above remarks generalize to arbitrary N_c if the proton is identified with the strangeness-zero baryon of the spin-$\frac{1}{2}$ large-N_c flavor representation. The $1/N_c$ suppression of Eq. (3.68) occurs because the proton contains no strange quarks, so that the leading contribution to $\langle p|m_s\bar{s}s|p\rangle$ comes from diagrams with a single quark loop (Fig. 5) [23] in violation of planar QCD flavor symmetry.

It is conventional to rewrite Eq. (3.65) in terms of the sigma term

$$\sigma_{\pi N} \equiv \hat{m}\langle p|\bar{u}u + \bar{d}d|p\rangle$$
$$= -2\hat{m}(2\sigma + b_D + b_F) + \text{nonlinear}, \tag{3.69}$$

so that

$$\langle p|m_s\bar{s}s|p\rangle = m_s(3b_F - b_D) + \frac{1}{2}\left(\frac{m_s}{\hat{m}}\right)\sigma_{\pi N} + \text{nonlinear}. \tag{3.70}$$

Nonet symmetry among the linear quark couplings implies a significant cancellation between the first term and the sigma term. This cancellation explains sensitivity of central value of the proton matrix element to the precise value of $\sigma_{\pi N}$.

IV. NONANALYTIC CORRECTIONS

The procedure for calculating nonanalytic pion-loop corrections using the $1/N_c$ baryon chiral Lagrangian is examined in this section. Aspects of this problem have been treated previously in Refs. [13–15,19,22,24]. The calculation of nonanalytic corrections to baryon amplitudes in the $1/N_c$ expansion at finite N_c introduces a number of issues which have not been addressed before. A sample calculation of the flavor **27** nonanalytic contribution to the baryon masses is presented in detail to illustrate the technique.

The Feynman rules for baryon-pion couplings can be obtained from the $1/N_c$ chiral Lagrangian $\mathscr{L}_{\text{baryon}}$ and $\mathscr{L}^{\mathscr{M}}_{\text{baryon}}$. The baryon propagator is given by inversion of the quadratic terms in the Lagrangian. This inversion is complicated by the presence of the hyperfine and quark mass splittings. In the chiral limit $m_i \to 0$, the baryon propagator is diagonal in spin, and can be written as

$$\frac{i\mathscr{P}_{\mathrm{J}}}{(k^0 - \Delta_{\mathrm{J}})}, \tag{4.1}$$

where $\mathscr{P}_{\mathrm{J}}$ is a spin projection operator for spin $J = \mathrm{J}$ and

$$\Delta_{\mathrm{J}} = M_{\text{hyperfine}}\big|_{J^2 = \mathrm{J}(\mathrm{J}+1)} - M_{\text{hyperfine}}\big|_{J^2 = \mathrm{J}_{\text{ext}}(\mathrm{J}_{\text{ext}}+1)} \tag{4.2}$$

is the difference of the hyperfine mass splitting for spin $J = \mathrm{J}$ and the external baryon. For p-wave pion emission, Δ_{J} is given by

$$\Delta_{\mathrm{J}} = \begin{cases} \frac{1}{N_c} 2\mathrm{J}\, m^{0,1}_{(2)}, & \mathrm{J}_{\text{ext}} = \mathrm{J} - 1, \\ 0, & \mathrm{J}_{\text{ext}} = \mathrm{J}, \\ -\frac{1}{N_c} 2\mathrm{J}\, m^{0,1}_{(2)}, & \mathrm{J}_{\text{ext}} = \mathrm{J} + 1, \end{cases} \tag{4.3}$$

at leading order $1/N_c$ in the $1/N_c$ expansion, with subleading terms beginning at order $1/N_c^3$. Equation (4.1) solves the inversion problem in the chiral limit in terms of spin projection operators.

For arbitrary finite N_c, the baryon tower consists of spins $J = \frac{1}{2}, \frac{3}{2}, \ldots, N_c/2$. Each spin projection operator must satisfy

$$\mathscr{P}^2_{\mathrm{J}} = \mathscr{P}_{\mathrm{J}},$$
$$\mathscr{P}_{\mathrm{J}'}\mathscr{P}_{\mathrm{J}} = 0, \quad \mathrm{J}' \neq \mathrm{J}, \tag{4.4}$$

by definition. An explicit realization of these conditions is given by

$$\mathscr{P}_{\mathrm{J}} = \frac{\Pi_{\mathrm{J}' \neq \mathrm{J}}(J^2 - J^2_{\mathrm{J}'})}{\Pi_{\mathrm{J}' \neq \mathrm{J}}(J^2_{\mathrm{J}} - J^2_{\mathrm{J}'})}, \tag{4.5}$$

where the projection operator for spin J_{J} is given by the product over all $J_{\mathrm{J}'} = \frac{1}{2}, \frac{3}{2}, \ldots, N_c/2$ not equal to J_{J}. For example, the spin-$\frac{1}{2}$ and $\frac{3}{2}$ projectors are given by

$$\mathscr{P}_{\frac{1}{2}} = \frac{\left(J^2 - \frac{15}{4}\right)\left(J^2 - \frac{35}{4}\right)\cdots\left[J^2 - \frac{N_c}{2}\left(\frac{N_c}{2}+1\right)\right]}{\left(\frac{3}{4} - \frac{15}{4}\right)\left(\frac{3}{4} - \frac{35}{4}\right)\cdots\left[\frac{3}{4} - \frac{N_c}{2}\left(\frac{N_c}{2}+1\right)\right]},$$

$$\mathscr{P}_{\frac{3}{2}} = \frac{\left(J^2 - \frac{3}{4}\right)\left(J^2 - \frac{35}{4}\right)\cdots\left[J^2 - \frac{N_c}{2}\left(\frac{N_c}{2}+1\right)\right]}{\left(\frac{15}{4} - \frac{3}{4}\right)\left(\frac{15}{4} - \frac{35}{4}\right)\cdots\left[\frac{15}{4} - \frac{N_c}{2}\left(\frac{N_c}{2}+1\right)\right]}. \tag{4.6}$$

Each of the projection operators Eq. (4.5) is a polynomial of degree $(N_c-1)/2$ in J^2.

For $N_c=3$, there are only two spins in the baryon tower. The spin projectors reduce to

$$\mathcal{P}_{\frac{1}{2}}=-\tfrac{1}{3}(J^2-\tfrac{15}{4}),$$
$$\mathcal{P}_{\frac{3}{2}}=\tfrac{1}{3}(J^2-\tfrac{3}{4}), \tag{4.7}$$

and the baryon propagator has the form Eq. (4.1) with

$$\Delta_{\frac{1}{2}}=\begin{cases} 0, & \jmath_{\rm ext}=\frac{1}{2}, \\ -\Delta, & \jmath_{\rm ext}=\frac{3}{2}, \end{cases}$$
$$\Delta_{\frac{3}{2}}=\begin{cases} \Delta, & \jmath_{\rm ext}=\frac{1}{2}, \\ 0, & \jmath_{\rm ext}=\frac{3}{2}, \end{cases} \tag{4.8}$$

where

$$\Delta=\frac{3}{N_c}m^{0,1}_{(2)}. \tag{4.9}$$

Away from the chiral limit, quark mass splittings must be considered in the inversion of the baryon quadratic terms. Baryon mass splittings which are comparable to the pion octet masses are to be retained in the baryon propagator. For large-N_c baryons, the leading hyperfine baryon mass splitting is order Λ/N_c whereas the leading quark mass splittings are order[7] m_i. For QCD with $N_c=3$, these splittings satisfy the hierarchy

$$m_u, m_d \ll m_s < \frac{\Lambda}{N_c}. \tag{4.10}$$

Only the leading quark mass splittings proportional to $b^{0,8}_{(1)}T^8$ and the hyperfine mass splitting are comparable to the pion octet masses in QCD. Keeping these two splittings amounts to the neglect of isospin-breaking quark mass splittings and subleading quark mass splittings of order m_i/N_c. The T^8 operator leads to spin-independent baryon mass splittings which are linear in the number of strange quarks. The baryon propagator is diagonal in spin and strange quark number and is given by

$$\frac{i\mathcal{P}_\jmath \mathcal{P}_{n_s}(\jmath)\mathcal{P}_\jmath}{(k^0-\Delta_\jmath-\Delta_{n_s})}, \tag{4.11}$$

where $\mathcal{P}_{n_s}(\jmath)$ is the $N_s=n_s$ strange quark projection operator of the spin-$\jmath$ flavor representation, and (neglecting strong-CP violation)

$$\Delta_{n_s}=\tfrac{1}{2}b^{0,8}_{(1)}(m_u+m_d-2m_s)(n_s-n_{s\,\rm ext}), \tag{4.12}$$

is the T^8 quark mass difference of the propagating baryon and the external baryon. Equation (4.11) solves the inversion problem in terms of strange quark projection operators.

The spin $J=\jmath$ large-N_c flavor representation contains baryons with $N_s=0,1,\ldots,(N_c+2\jmath)/2$ strange quarks. Strange-quark number projection operators can be defined for the spin-$\jmath$ flavor representation in analogy to the spin projection operators,

$$\mathcal{P}_{n_s}(\jmath)=\frac{\Pi_{n_s'\neq n_s}(N_s-n_s')}{\Pi_{n_s'\neq n_s}(n_s-n_s')}, \tag{4.13}$$

where the projection operator for n_s strange quarks is given by the product over all $n_s'=0,1,\ldots,(N_c+2\jmath)/2$ not equal to n_s. For example, the zero and one strange quark projection operators for the spin-$\jmath$ baryons are given by

$$\mathcal{P}_0(\jmath)=\frac{(N_s-1)(N_s-2)\cdots\left[N_s-\left(\frac{N_c+2\jmath}{2}\right)\right]}{(-1)(-2)\cdots\left[-\left(\frac{N_c+2\jmath}{2}\right)\right]},$$
$$\mathcal{P}_1(\jmath)=\frac{N_s(N_s-2)\cdots\left[N_s-\left(\frac{N_c+2\jmath}{2}\right)\right]}{(1)(-1)\cdots\left[1-\left(\frac{N_c+2\jmath}{2}\right)\right]}. \tag{4.14}$$

Note that the projectors are different for each flavor representation with a definite spin $J=\jmath$, since the allowed strangeness sectors of a large-N_c flavor representation depends on its spin. Each strange-quark number projection operator for spin $J=\jmath$ is a polynomial of degree $(N_c+2\jmath)/2$ in N_s.

For $N_c=3$, the spin-$\frac{1}{2}$ flavor representation contains baryons with 0, 1, and 2 strange quarks, while the spin-$\frac{3}{2}$ flavor representation contains baryons with 0, 1, 2, and 3 strange quarks. The $J=\frac{1}{2}$ strange quark projection operators reduce to

$$\mathcal{P}_0(\tfrac{1}{2})=\frac{1}{2}(N_s-1)(N_s-2),$$
$$\mathcal{P}_1(\tfrac{1}{2})=-N_s(N_s-2), \tag{4.15}$$
$$\mathcal{P}_2(\tfrac{1}{2})=\frac{1}{2}N_s(N_s-1),$$

whereas the $J=\frac{3}{2}$ strange quark projection operators reduce to

$$\mathcal{P}_0(\tfrac{3}{2})=-\tfrac{1}{6}(N_s-1)(N_s-2)(N_s-3),$$
$$\mathcal{P}_1(\tfrac{3}{2})=\tfrac{1}{2}N_s(N_s-2)(N_s-3),$$
$$\mathcal{P}_2(\tfrac{3}{2})=-\tfrac{1}{2}N_s(N_s-1)(N_s-3), \tag{4.16}$$
$$\mathcal{P}_3(\tfrac{3}{2})=\tfrac{1}{6}N_s(N_s-1)(N_s-2).$$

[7] The leading $O(N_c)$ terms in the quark mass perturbations $\mathcal{H}^a$, $a=0,3,8$ are proportional to the baryon identity operator $\mathbb{1}$ and do not result in baryon mass splittings. All $O(N_c)$ mass terms must be removed from the Lagrangian by the heavy field redefinition.

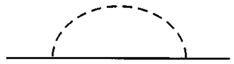

FIG. 6. Feynman diagram responsible for flavor-**27** baryon mass splittings at leading order in the flavor breaking and $1/N_c$ expansions.

The baryon propagator has the form Eq. (4.11) with Δ_{n_s} given by plus and minus

$$\Delta_s = \tfrac{1}{2} b^{0,8}_{(1)}(m_u + m_d - 2m_s) \tag{4.17}$$

for $\Delta S = \pm 1$ transitions.

The generalization of the baryon propagator Eq. (4.11) to include all subleading quark mass splittings is provided in Appendix A for completeness.

A. Flavor-27 baryon mass splittings

Flavor singlet and octet baryon mass splittings are present in the $1/N_c$ baryon chiral Lagrangian. The flavor-**27** mass splittings of the octet and decuplet are calculable and nonanalytic in the quark masses and baryon hyperfine mass splitting at leading order in chiral perturbation theory. This mass splitting arises from the Feynman diagram Fig. 6. The computation of the flavor-**27** component of Fig. 6 for finite N_c, $N_c = 3$, is presented in detail in this section. Computations at larger N_c are less interesting physically and more complicated to extrapolate to $N_c = 3$ because unphysical baryons participate as intermediate states in loop diagrams [13,14], and there are higher partial wave meson-baryon couplings which occur at subleading orders [17].

The loop diagram Fig. 6 involves π, K, and η emission and reabsorption. The η' meson is not included in the loop since it is not soft relative to the baryons in QCD. For degenerate heavy hadrons interacting with mesons, the diagram Fig. 6 depends on a function $F(m)$ of the meson mass m, which is obtained by performing the Feynman loop integration. Neglecting isospin breaking, i.e., the $(m_d - m_u)$ quark mass difference, the diagram depends on the function $F(m)$ for three meson mass values, $F(\pi)$, $F(K)$, and $F(\eta)$, where the meson mass is denoted by its particle label. Any meson loop integral with the exchange of a single meson, in which a meson of flavor a is emitted and a meson of flavor b is reabsorbed, can be written as a symmetric tensor with two adjoint (octet) indices a and b. This symmetric tensor decomposes into flavor singlet, adjoint (**8**) and $\bar{s}s$ (**27**) representations:

$$\begin{aligned}\Pi^{ab} &= \tfrac{1}{8}[3F(\pi) + 4F(K) + F(\eta)]\delta^{ab} \\ &\quad + \frac{2\sqrt{3}}{5}[\tfrac{3}{2}F(\pi) - F(K) - \tfrac{1}{2}F(\eta)]d^{ab8} \\ &\quad + [\tfrac{1}{3}F(\pi) - \tfrac{4}{3}F(K) + F(\eta)] \\ &\quad \times(\delta^{a8}\delta^{b8} - \tfrac{1}{8}\delta^{ab} - \tfrac{3}{5}d^{ab8}d^{888}),\end{aligned} \tag{4.18}$$

where the flavor singlet, octet, and **27** tensors in Eq. (4.18) are proportional to flavor singlet, octet and **27** linear combinations of $F(\pi)$, $F(K)$ and $F(\eta)$.

For the Feynman diagram Fig. 6 with propagator i/k^0,

$$F(m) = \frac{m^3}{24\pi f_\pi^2}. \tag{4.19}$$

The flavor-**27** combination of this function,

$$[\tfrac{1}{3}F(\pi) - \tfrac{4}{3}F(K) + F(\eta)], \tag{4.20}$$

is highly suppressed relative to the singlet and octet combinations, and is numerically very small, of the order 4 MeV. For comparison, the flavor singlet combination of this function is about 126 MeV, while the flavor octet combination is about -38 MeV. This suppression of the flavor-**27** combination is generic in chiral perturbation theory. It explains the small violation of the Gell-Mann–Okubo formula for baryon masses, which is[8] 6.5 MeV, as well as the small flavor-**27** mixing of the vector mesons [44]. The suppression mechanism also applies to other meson-loop corrections involving Π^{ab}, such as flavor-**27** chiral logarithmic corrections to vertices with

$$F(m) = \frac{m^2}{24\pi^2 f_\pi^2}\ln\left(\frac{m^2}{\mu^2}\right). \tag{4.21}$$

The flavor-**27** chiral logarithm is numerically 0.035. Note that the μ dependence of the chiral logarithm cancels at leading order in the **27** combination using the Gell-Mann–Okubo formula for meson masses [45,46]

$$\tfrac{1}{3}m_\pi^2 - \tfrac{4}{3}m_K^2 + m_\eta^2 = 0. \tag{4.22}$$

The computation of Fig. 6 is complicated significantly by the inclusion of baryon mass splittings in the baryon propagator. In this case, the Feynman integral is a nonanalytic function of the baryon mass splitting Δ as well as the meson mass squared. The function $F(m,\Delta)$ is defined by the integral

$$i\delta^{ij}F(m,\Delta) = \frac{1}{f^2}\int \frac{d^4k}{(2\pi)^4}\frac{i^2(\mathbf{k}^i)(-\mathbf{k}^j)}{(k^2 - m^2)(k^0 - \Delta)}. \tag{4.23}$$

The precise formula for $F(m,\Delta)$ is given in Appendix B. The computation of Fig. 6 is performed in this section using the baryon propagator Eq. (4.1), which neglects baryon flavor mass splittings, since the generalization to the propagator Eq. (4.11) can be obtained immediately from this formula. With baryon propagator Eq. (4.1), the flavor-**27** component of the meson tensor is given by

$$\Pi^{ab}_{27} = (\delta^{a8}\delta^{b8} - \tfrac{1}{8}\delta^{ab} - \tfrac{3}{5}d^{ab8}d^{888})I(\pi,K,\eta,\Delta), \tag{4.24}$$

where

$$I(\pi,K,\eta,\Delta) = [\tfrac{1}{3}F(\pi,\Delta) - \tfrac{4}{3}F(K,\Delta) + F(\eta,\Delta)]. \tag{4.25}$$

[8] This suppression mechanism was originally noted in the study of the octet and decuplet masses [42], neglecting the pion mass. The suppression of the flavor **27** combination is more significant when the pion mass is retained.

Neglect of the T^8 quark mass splitting Δ_s affects only the kaon loop graphs which involve $\Delta S=\pm 1$ transitions. The generalization of $I(\pi,K,\eta,\Delta)$ to $I(\pi,K,\eta,\Delta,\Delta_s)$ is obtained by the replacement

$$F(K,\Delta)\rightarrow \tfrac{1}{2}[F(K,\Delta+\Delta_s)+F(K,\Delta-\Delta_s)] \quad (4.26)$$

in Eq. (4.25).

The diagram Fig. 6 is given by the product of a baryon operator times the pion flavor tensor. Using the baryon propagator Eq. (4.1), Fig. 6 is given by

$$\frac{1}{N_c}\sum_{\mathrm{J}} (A^{ia}\mathscr{P}_{\mathrm{J}}A^{ib})\Pi^{ab}_{27}(\Delta_{\mathrm{J}}), \quad (4.27)$$

for arbitrary N_c where the baryon axial vector current operator A^{ia} has a $1/N_c$ expansion Eq. (3.15). The explicit factor of $1/N_c$ occurs from the rescaling $f_\pi \rightarrow \sqrt{N_c}f$. Equation (4.27) reduces to

$$\frac{1}{N_c}[A^{i8}\mathscr{P}_{\frac{1}{2}}A^{i8}I(\Delta_{\frac{1}{2}})+A^{i8}\mathscr{P}_{\frac{3}{2}}A^{i8}I(\Delta_{\frac{3}{2}})] \quad (4.28)$$

for $N_c=3$, where A^{i8} has the $1/N_c$ expansion Eq. (3.16) and the baryon operator is understood to be a $(0,\mathbf{27})$, so that subtraction of flavor singlet and octet components of the baryon operator is implicit in the present notation. The function $I(\pi,K,\eta,\Delta_{\mathrm{J}})$ is abbreviated as $I(\Delta_{\mathrm{J}})$ in Eq. (4.28).

The evaluation of Eq. (4.28) raises an important issue. The baryon operator product of the two axial currents generates n-body operators with $n>N_c$ which are not operators in the operator basis at finite N_c. In order to make sense of this operator product, all of these higher body operators must be rewritten as linear combinations of operators in the operator basis with $n\leqslant N_c$. Since the operator basis is complete and independent [17], this reduction is always possible. In practice, however, this operator reduction is formidable even for the product of two axial vector currents. The problem is solved in this work using spin projection operators. The introduction of spin projection operators makes operator reduction tractable and straightforward. The details are presented in Appendix B.

There are two flavor-**27** combinations of baryon masses, the Gell-Mann–Okubo combination of octet baryon masses

$$\tfrac{3}{4}\Lambda+\tfrac{1}{4}\Sigma-\tfrac{1}{2}(N+\Xi), \quad (4.29)$$

and the decuplet equal spacing rule combination [18],

$$-\tfrac{4}{7}\Delta+\tfrac{5}{7}\Sigma^*+\tfrac{2}{7}\Xi^*-\tfrac{3}{7}\Omega. \quad (4.30)$$

Violation of the Gell-Mann–Okubo formula is given by

$$\frac{1}{N_c}[\mathscr{P}_{\frac{1}{2}}A^{i8}\mathscr{P}_{\frac{1}{2}}A^{i8}\mathscr{P}_{\frac{1}{2}}I(0)+\mathscr{P}_{\frac{1}{2}}A^{i8}\mathscr{P}_{\frac{3}{2}}A^{i8}\mathscr{P}_{\frac{1}{2}}I(\Delta)], \quad (4.31)$$

whereas violation of the flavor-**27** equal spacing rule is given by

$$\frac{1}{N_c}[\mathscr{P}_{\frac{3}{2}}A^{i8}\mathscr{P}_{\frac{3}{2}}A^{i8}\mathscr{P}_{\frac{3}{2}}I(0)+\mathscr{P}_{\frac{3}{2}}A^{i8}\mathscr{P}_{\frac{1}{2}}A^{i8}\mathscr{P}_{\frac{3}{2}}I(-\Delta)]. \quad (4.32)$$

Evaluation of the baryon operators yields

$$\begin{aligned}\tfrac{3}{4}\Lambda&+\tfrac{1}{4}\Sigma-\tfrac{1}{2}(N+\Xi)\\ &=\frac{1}{N_c}\left[\left(\frac{1}{16}a_1^2+\frac{3}{4}\frac{1}{N_c}a_1b_2+\frac{9}{16}\frac{1}{N_c^2}b_2^2+\frac{3}{8}\frac{1}{N_c^2}a_1b_3\right.\right.\\ &\quad\left.+\frac{9}{4}\frac{1}{N_c^3}b_2b_3+\frac{9}{16}\frac{1}{N_c^4}b_3^2\right)I(0)\\ &\quad\left.+\left(\frac{1}{8}a_1^2+\frac{9}{8}\frac{1}{N_c^2}a_1c_3+\frac{81}{32}\frac{1}{N_c^4}c_3^2\right)I(\Delta)\right], \quad (4.33)\end{aligned}$$

and

$$\begin{aligned}-\tfrac{4}{7}\Delta&+\tfrac{5}{7}\Sigma^*+\tfrac{2}{7}\Xi^*-\tfrac{3}{7}\Omega\\ &=\frac{1}{N_c}\left[\left(\frac{5}{8}a_1^2+\frac{15}{4}\frac{1}{N_c}a_1b_2+\frac{45}{8}\frac{1}{N_c^2}b_2^2+\frac{75}{4}\frac{1}{N_c^2}a_1b_3\right.\right.\\ &\quad\left.+\frac{225}{4}\frac{1}{N_c^3}b_2b_3+\frac{1125}{8}\frac{1}{N_c^4}b_3^2\right)I(0)\\ &\quad\left.-\left(\frac{1}{4}a_1^2+\frac{9}{4}\frac{1}{N_c^2}a_1c_3+\frac{81}{16}\frac{1}{N_c^4}c_3^2\right)I(-\Delta)\right], \quad (4.34)\end{aligned}$$

where a_1, b_2, b_3, and c_3 are the coefficients of the $1/N_c$ expansion for the baryon axial vector current Eq. (3.16). Equations (4.33) and (4.34) can be compared with the expressions obtained in chiral perturbation theory with no $1/N_c$ expansion [42]:

$$[-\tfrac{3}{4}(D^2-3F^2)\bar{I}(0)+\tfrac{1}{8}\mathscr{C}^2\bar{I}(\Delta)] \quad (4.35)$$

and

$$[\tfrac{5}{18}\mathscr{H}^2\bar{I}(0)-\tfrac{1}{4}\mathscr{C}^2\bar{I}(-\Delta)], \quad (4.36)$$

respectively, where the function $\bar{I}(\Delta)=I(\Delta)/N_c$ is proportional to $1/f_\pi^2$ rather than $1/f^2$. Equations (4.33) and (4.34) agree with these expressions for $N_c=3$, using the identifications Eq. (3.32).

Reference [18] showed that the two flavor-**27** baryon mass splittings are described by the $1/N_c$ operators

$$c^{27,0}_{(2)}\frac{1}{N_c}\{T^8,T^8\}+c^{27,0}_{(3)}\frac{1}{N_c^2}\{T^8,\{J^i,G^{i8}\}\}, \quad (4.37)$$

so that one of the flavor-**27** mass splittings is order $1/N_c$ in the $1/N_c$ expansion, whereas the other is order $1/N_c^2$. This behavior is most easily seen for $\Delta=0$, where the $c^{27,0}_{(2)}$ mass combination is given by

$$\begin{aligned}5[\tfrac{3}{4}\Lambda&+\tfrac{1}{4}\Sigma-\tfrac{1}{2}(N+\Xi)]-(-\tfrac{4}{7}\Delta+\tfrac{5}{7}\Sigma^*+\tfrac{2}{7}\Xi^*-\tfrac{3}{7}\Omega)\\ &=\frac{9}{16}\frac{1}{N_c}a_1^2I(\pi,K,\eta,0)+O\left(\frac{1}{N_c^3}\right), \quad (4.38)\end{aligned}$$

while the $c^{27,0}_{(3)}$ mass combination is given by

$$-2[\tfrac{3}{4}\Lambda+\tfrac{1}{4}\Sigma-\tfrac{1}{2}(N+\Xi)]+(-\tfrac{4}{7}\Delta+\tfrac{5}{7}\Sigma^*+\tfrac{2}{7}\Xi^*-\tfrac{3}{7}\Omega)$$
$$=\frac{9}{4}\frac{1}{N_c^2}a_1b_2I(\pi,K,\eta,0)+O\left(\frac{1}{N_c^3}\right), \qquad (4.39)$$

so that

$$c_{(2)}^{27,0}=\tfrac{1}{8}a_1^2I(\pi,K,\eta,0)+O\left(\frac{1}{N_c^2}\right),$$
$$c_{(3)}^{27,0}=\tfrac{1}{2}a_1b_2I(\pi,K,\eta,0)+O\left(\frac{1}{N_c}\right). \qquad (4.40)$$

Notice that there is no a_1b_2 contribution to the two-body flavor-**27** coefficient and no a_1^2 contribution to the three-body coefficient for $\Delta=0$. For nonvanishing Δ, it is still true that the first flavor-**27** mass splitting is order $1/N_c$, while the second is order $1/N_c^2$. The $1/N_c$ counting is not explicit, however, since Δ is implicitly order $1/N_c$. The expressions for the coefficients for $\Delta\neq 0$ are more complicated:

$$c_{(2)}^{27,0}=\tfrac{1}{3}a_1^2[-\tfrac{5}{24}I(0)+\tfrac{5}{12}I(\Delta)+\tfrac{1}{6}I(-\Delta)]+O\left(\frac{1}{N_c^2}\right),$$
$$c_{(3)}^{27,0}=\tfrac{1}{18}N_ca_1^2[2I(0)-I(\Delta)-I(-\Delta)]+\tfrac{1}{2}a_1b_2I(0)$$
$$+O\left(\frac{1}{N_c}\right). \qquad (4.41)$$

As before, the coefficient $c_{(2)}^{27,0}$ does not depend on a_1b_2. The coefficient $c_{(3)}^{27,0}$, however, now appears to have an order N_c contribution proportional to a_1^2 which changes the $1/N_c$ counting for this mass splitting. This appearance is illusory. Recall that the function $I(\pi,K,\eta,\Delta)$ depends on the the meson masses and Δ through the function $F(m,\Delta)$ defined in Appendix B. For $|\Delta|\leqslant m$, the linear combination $2F(m,0)-F(m,\Delta)-F(m,-\Delta)$ is order $1/N_c^2$, so the term proportional to a_1^2 is $O(1/N_c)$ and can be neglected relative to leading a_1b_2 term. For $|\Delta|>m$, the same linear combination reduces to $2F(m,0)$, which is smaller than an effect of order $1/N_c^3$ since, by assumption, $m^3<\Delta^3$. Thus, the three-body coefficient reduces to the expression given in Eq. (4.40) even for nonvanishing Δ.

The theoretical calculation of the flavor-**27** mass splittings can be compared with experiment. The experimental value of the Gell-Mann–Okubo mass splitting is 6.53 MeV with negligible uncertainty. The flavor-**27** equal spacing rule mass splitting depends on the unmeasured Δ^- mass which enters the isospin zero mass $\Delta_0\equiv(\Delta^{++}+\Delta^{+}+\Delta^0+\Delta^-)/4$. The Δ^- mass can be determined from the mass relation

$$\Delta^{++}-3\Delta^{+}+3\Delta^0-\Delta^-=0, \qquad (4.42)$$

which is satisfied to order $1/N_c^2$ in the $1/N_c$ expansion and to second order in isospin-breaking parameters [18]. Numerically, the $I=3$ mass difference of the Δ is at most of order 10^{-3} MeV, so neglect of this mass difference introduces negligible error in the determination of Δ^-. Using the value for Δ^- extracted with Eq. (4.42) and Particle Data Group values [48] for the remaining Δ masses[9] yields $\Delta_0=1231.3\pm 1.1$ MeV. Evaluation of the flavor-**27** equal spacing rule mass splitting using Δ_0 and Particle Data Group values for the remaining decuplet masses gives 6.75 MeV, with an uncertainty of 0.004 MeV.

The theoretical formulas depend on the two mass splittings Δ and Δ_s, and the flavor symmetric baryon-pion couplings a_1, b_2, b_3, and c_3 (or equivalently, D, F, $\mathscr{C}$, and $\mathscr{H}$). The mass splittings can be defined precisely [18]:

$$\Delta=\tfrac{1}{10}(4\Delta_0+3\Sigma_0^*+2\Xi_0^*+\Omega)-\tfrac{1}{8}(2N_0+3\Sigma_0+\Lambda+2\Xi_0) \qquad (4.43)$$

and

$$\Delta_s=\tfrac{1}{10}(\Delta_0-\tfrac{1}{2}\Xi_0^*-\tfrac{1}{2}\Omega)-\tfrac{1}{8}(6N_0-3\Sigma_0+\Lambda-4\Xi_0), \qquad (4.44)$$

where the zero subscripts refer to the $I=0$ mass combinations defined in Ref. [18]. Evaluation of the mass splittings yields $\Delta=230.7\pm 0.1$ MeV and $\Delta_s=225.7\pm 0.03$ MeV. The baryon axial couplings were extracted from experiment in Ref. [19]. The Gell-Mann–Okubo mass combination depends primarily on the coefficients a_1 and b_2, which are fairly well determined. The flavor-**27** equal spacing rule mass combination, however, is sensitive to the value of b_3 which is difficult to extract from experiment.[10] This sensitivity is due to the large numerical constants of the b_3 terms appearing in Eq. (4.34), which in turn is a reflection of the fact that the spin-$\frac{3}{2}$ flavor representation is at the top of the baryon tower for $N_c=3$ so the presumption $J/N_c\sim O(1/N_c)$ is breaking down.

In addition to the uncertainty of the baryon axial vector couplings, numerical evaluation of the theoretical formulas for the flavor-**27** mass splittings is further complicated by the sensitivity of the numerics to the precise formulas which are used. For example, imposition of the Gell-Mann–Okubo formula for the meson masses changes the numerical value of the function $I(\pi,K,\eta,\Delta)$ considerably. In addition, the generalization of the theoretical formulas to include Δ_s through Eq. (4.26) changes the numerics significantly. Note that for nonvanishing Δ_s, the function $I(\Delta)$ depends on Δ_s through Eq. (4.26) even for $\Delta=0$. Although the μ dependence of the flavor-**27** combination of $F(m,\Delta,\mu)$ cancels at leading order using the Gell-Mann–Okubo formula for the meson masses if $\Delta_s=0$, there is additional μ dependence from the term $\Delta\Delta_s^2\ln m_K^2/\mu^2$ when $\Delta_s\neq 0$ and $\Delta\neq 0$. This μ dependence is canceled by a finite counterterm proportional to $\Delta\Delta_s^2$. At leading order in chiral perturbation theory, it is possible to drop the $\Delta\Delta_s^2\ln m_K^2/\mu^2$ chiral logarithm from the function I to obtain a μ-independent quantity. It is also possible to drop

[9] There are three different measurements listed for Δ^{++} and Δ^0. These measurements are averaged with errors added in quadrature.

[10] Because of this uncertainty one could consider the alternative of extracting the baryon axial couplings from the flavor-**27** mass splittings, as suggested recently in Ref. [45]. However, the theoretical formula receives sizable (unknown) corrections at higher orders in chiral perturbation theory, so error bars on the extracted couplings are significant.

the finite $\Delta\Delta_s^2$ and $\Delta_s m^2$ terms in I. Note that Δ_s still appears in the last term of Eq. (B1).

The mass combination in Eq. (4.39) does not depend sensitively on c_3 or on the μ-dependence of the $\Delta\Delta_s^2$ chiral logarithm. Numerical evaluation of Eq. (4.39) using $a_1=2(0.791\pm0.007)$ and $b_2=6(-0.058\pm0.011)$, as extracted in Ref. [19], yields -5.3 MeV, whereas the experimental value is -6.2 MeV. The agreement is consistent with a correction of relative order $1/N_c$.

V. PLANAR QCD FLAVOR SYMMETRY AND SU(3) FLAVOR SYMMETRY BREAKING

The two-flavor version of planar QCD flavor symmetry has implications for the structure of SU(3) symmetry breaking at leading order in $1/N_c$. Planar QCD for three light flavors exhibits an approximate U(3) flavor symmetry, which is broken explicitly by the quark mass matrix. In the planar limit, the flavor symmetry breaking due to the quark mass matrix transforms as the $a=3,8,9$ components of a nonet. Neglecting isospin breaking, there is an unbroken U(2) flavor symmetry, which includes the diagonal generators σ^3 and I in the 2×2 subspace. This residual U(2) flavor symmetry can constrain the form of SU(3) breaking in the $1/N_c$ expansion. The constraints must be satisfied at each and every order in SU(3) flavor symmetry breaking.

The relevance of U(2) symmetry for SU(3) breaking is illustrated by the baryon axial vector currents. The $F=2$ version of planar QCD flavor symmetry constrains the leading coefficients of the isosinglet axial vector current A^i relative to the coefficients of the isovector axial vector current A^{ia},a=1,2,3:

$$\bar{b}^{1,1}_{(1)}=\tfrac{1}{2}(a^{1,3}_{(1)}+b^{1,3}_{(2)}),$$
$$\bar{b}^{1,1}_{(3)}=\tfrac{1}{2}(2b^{1,3}_{(3)}). \tag{5.1}$$

Equation (5.1) is the two-flavor analogue of Eq. (3.25). The isosinglet and isovector axial currents couple to the pion quartet $\mathbf{\Phi}=\pi^a\sigma^a/2+\tilde{\eta}I/2$, where $\tilde{\eta}$ is an admixture of η and η'. The η (η') couplings of baryons with zero strangeness are each proportional to the $\tilde{\eta}$ couplings. Thus, the η (η') couplings of strangeness-zero baryons are normalized relative to the pion couplings in the presence of SU(3) breaking by Eq. (5.1) at leading order in $1/N_c$.

Reference [17] derived the flavor-octet baryon axial vector current to linear order in SU(3) symmetry breaking ϵ and leading order in $1/N_c$. The first constraint of Eq. (5.1) can be imposed on this current in the strangeness-zero sector. For strangeness-zero baryons, the $1/N_c$ expansion of the baryon axial vector current to linear order in SU(3) symmetry breaking reduces to

$$A^{ia}+\delta A^{ia}=\left(aG^{ia}+b\frac{1}{N_c}J^iT^a\right)+\epsilon d^{ab8}\left(c_1G^{ib}+c_2\frac{1}{N_c}J^iT^b\right)+\epsilon c_6\delta^{a8}J^i, \tag{5.2}$$

up to terms of relative order $1/N_c^2$. Note that the coefficients a and b contain contributions of order ϵ and reduce to a_1 and b_2 only in the absence of SU(3) breaking. The coefficients a and b automatically satisfy the first constraint of Eq. (5.1) due to the SU(3) symmetry of their operators. The remaining terms contribute the following to the coefficients of the $F=2$ isosinglet and isovector $1/N_c$ expansions when evaluated for strangeness-zero baryons:

$$\delta b^{1,1}_{(1)}=\sqrt{3}\epsilon[-\tfrac{1}{6}(c_1+c_2)+c_6],$$
$$\delta a^{1,3}_{(1)}=\frac{1}{\sqrt{3}}\epsilon c_1, \tag{5.3}$$
$$\delta b^{1,3}_{(2)}=\frac{1}{\sqrt{3}}\epsilon c_2.$$

The factor of $\sqrt{3}$ in the first equation occurs because $\tilde{\eta}$ is proportional to $\eta/\sqrt{3}$. The first quartet symmetry constraint of Eq. (5.1) implies that

$$\bar{c}_6=\tfrac{1}{3}(c_1+c_2), \tag{5.4}$$

up to a correction of relative order $1/N_c$. This is the same constraint on c_6 obtained in Ref. [17]. The above derivation shows that this constraint follows from $F=2$ planar QCD flavor symmetry.

The second constraint in Eq. (5.1) applies to spin-diagonal order $1/N_c^2$ terms which have not been included in Eq. (5.2). These neglected terms reduce to

$$b'\frac{1}{N_c^2}\{J^i,\{J^j,G^{ja}\}\}+\epsilon d^{ab8}d_1\frac{1}{N_c^2}\{J^i,\{J^j,G^{jb}\}\}+\epsilon c_7\delta^{a8}\frac{1}{N_c^2}\{J^2,J^i\}, \tag{5.5}$$

in the strangeness-zero sector, where the coefficient b' is equal to b_3 in the SU(3) flavor symmetry limit, but contains a contribution of order ϵ at linear order in SU(3) breaking. The coefficient b' automatically satisfies the second constraint of Eq. (5.1) due to the SU(3) symmetry of the operator. The remaining terms are constrained to satisfy

$$\bar{c}_7=\tfrac{2}{3}d_1 \tag{5.6}$$

up to a correction of relative order $1/N_c$.

The above analysis shows that SU(3) flavor symmetry breaking does not alter the relationship between pion and η couplings of strangeness-zero baryons from exact SU(3) flavor symmetry to leading order in $1/N_c$. Any violation of this SU(3) normalization requires SU(3) flavor symmetry breaking in a $1/N_c$-suppressed quark loop. This result was originally reported in Ref. [17].

VI. CONCLUSIONS

A $1/N_c$ chiral Lagrangian for baryons is formulated which correctly implements planar QCD flavor symmetry and (spin$\otimes$flavor) symmetry for baryons. The constraints of planar QCD flavor symmetry on the baryon $1/N_c$ expansion have not been realized previously, and are presented in detail in this work. These constraints are valid to leading order in the $1/N_c$ expansion operator by operator in the baryon $1/N_c$

expansion. Thus, planar QCD flavor symmetry constrains the operator coefficients at leading order in $1/N_c$. The symmetry implies that baryon flavor octet and singlet amplitudes form nonets at leading order in $1/N_c$. Specific examples of nonet symmetry include the formation of a flavor nonet axial vector current from the flavor singlet and flavor octet axial vector baryon currents at leading order in $1/N_c$, as well as the formation of a nonet among the flavor singlet and octet baryon mass terms with linear dependence on the quark masses m_i.

The formulation of the baryon chiral Lagrangian in terms of operators with definite $1/N_c$ dependence enables one to study the precise $1/N_c$ structure of the chiral expansion. In specific instances, such as the proton matrix element $\langle p|m_s\bar{s}s|p\rangle$, the leading $1/N_c$ terms cancel exactly, so the $1/N_c$ expansion explains the suppression of the quantity.

The calculation of nonanalytic meson-loop corrections in $1/N_c$ baryon chiral perturbation theory at finite N_c is addressed. The $1/N_c$ and group theoretic structure of the loop corrections is manifest using the method described in this work. The introduction of spin projection operators simplifies the formidable problem of operator reduction, making calculations tractable. A specific example of the flavor-**27** meson-loop contribution to the baryon mass splittings is presented in detail. The $1/N_c$ computation provided in this work generalizes the formulas obtained previously in ordinary baryon chiral perturbation theory with $N_c=3$ to include the leading flavor octet mass splitting Δ_s of the baryons. The $1/N_c$ formulae reveal the $1/N_c$ and flavor-breaking structure of the flavor-**27** baryon mass splittings at leading order in chiral perturbation theory.

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APPENDIX A: BARYON PROPAGATOR

The generalization of the baryon propagator Eq. (4.11) to include all subleading quark mass splittings is provided in this appendix for completeness. Isospin-breaking quark mass splittings are neglected in the discussion for simplicity. These splittings can be included at the expense of introducing additional projection operators.

Including all hyperfine and $a=0,8$ quark mass splittings, the baryon propagator is given by

$$\frac{i\mathcal{P}_{\mathrm{J}}\mathcal{P}_{n_s}(\mathrm{J})\mathcal{P}_{\imath}(\mathrm{J},n_s)\mathcal{P}_{n_s}(\mathrm{J})\mathcal{P}_{\mathrm{J}}}{(k^0-\Delta_{\mathrm{J}}-\Delta_0-\Delta_8)}, \tag{A1}$$

where $\mathcal{P}_{\imath}(\mathrm{J},n_s)$ is the projection operator for isospin $I=\imath$ in the $N_s=n_s$ strange quark sector of the spin $J=\mathrm{J}$ flavor representation, and Δ_0 and Δ_8 are the $a=0$ and $a=8$ quark mass splittings of the propagating baryon relative to the external baryon. The $a=0$ baryon quark mass operator

$$M^0=-\frac{2}{\sqrt{6}}(m_u+m_d+m_s)\mathcal{H}^0 \tag{A2}$$

depends only on polynomials of J^2, so

$$\Delta_0=M^0|_{J^2=\mathrm{J}(\mathrm{J}+1)}-M^0|_{J^2=\mathrm{J}_{\rm ext}(\mathrm{J}_{\rm ext}+1)} \tag{A3}$$

is already diagonalized by the spin projection operators. The $a=8$ baryon quark mass operator

$$M^8=-\frac{1}{\sqrt{3}}(m_u+m_d-2m_s)\mathcal{H}^8 \tag{A4}$$

involves two operator series, generated by T^8 and $\{J^i,G^{i8}\}$ times polynomials in J^2. The operator series involving T^8 times polynomials in J^2 is diagonalized by the spin and strange quark number projection operators. The operator series involving $\{J^i,G^{i8}\}$ requires the introduction of isospin projection operators.

For arbitrary N_c, the structure of the baryon multiplets is such that (i) the isospin of a baryon is equal to the total angular momentum (spin) of the up and down quarks, $I=J_{ud}$ and (ii) the total angular momentum (spin) of the strange quarks is equal to one-half the number of strange quarks, $J_s=N_s/2$. Since $J=J_{ud}+J_s$, it therefore follows that spin-J baryons can only have isospins $I=|N_s/2+\mathrm{J}|,|N_s/2+\mathrm{J}|-1,\ldots,|N_s/2-\mathrm{J}|$, where all possible isospins are allowed for $2\mathrm{J}\leqslant N_s\leqslant(N_c-2\mathrm{J})/2$, but only a subset of the isospins are allowed for $0\leqslant N_s<2\mathrm{J}$ and $(N_c-2\mathrm{J})/2<N_s\leqslant(N_c+2\mathrm{J})/2$. For $N_s=0$, only the largest isospin is allowed. For $N_s=1$, only the two largest isospins are allowed. This pattern of one additional allowed isospin as N_s increases by one unit continues for the interval $0\leqslant N_s<2\mathrm{J}$ until the full set of isospins is allowed for $N_s=2\mathrm{J}$. Similarly, for $N_s=(N_c+2\mathrm{J})/2$, only the smallest isospin is allowed. For $N_s=(N_c+2\mathrm{J})/2-1$, only the two smallest isospins are allowed. This pattern of one additional allowed isospin as N_s decreases by one unit continues for $(N_c-2\mathrm{J})/2<N_s\leqslant(N_c+2\mathrm{J})/2$ until the full set of isospins is allowed for $N_s=(N_c-2\mathrm{J})/2$.

It is easier to digest this pattern of isospins if one specializes to the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ flavor representations with the weight diagrams displayed in Figs. 3 and 4. For $J=\frac{1}{2}$, there are two allowed isospins $I=(N_s+1)/2$ and $I=(N_s-1)/2$ for $1\leqslant N_s\leqslant(N_c-1)/2$. Both $N_s=0$ and $N_s=(N_c+1)/2$ are exceptions: for $N_s=0$, $I=\frac{1}{2}$, whereas for $N_s=(N_c+1)/2$, $I=(N_c-1)/4$. For $J=\frac{3}{2}$, there are four allowed isospins $I=(N_s+3)/2,(N_s+1)/2,(N_s-1)/2,(N_s-3)/2$ for $3\leqslant N_s\leqslant(N_c-3)/2$. The three smallest and largest strangeness sectors $N_s=0,1,2$ and $N_s=(N_c-1)/2$, $(N_c+1)/2$ and $(N_c+3)/2$ are special cases: for $N_s=0$, there is one allowed isospin $I=\frac{3}{2}$; for $N_s=1$, there are two allowed isospins, $I=2$ and $I=1$; for $N_s=2$, there are three allowed isospins $I=\frac{5}{2},\frac{3}{2}$, and $\frac{1}{2}$; while for $N_s=(N_c-1)/2$, there are three allowed isospins, $I=(N_c-7)/4$, $(N_c-3)/4$, and $(N_c+1)/4$; for $N_s=(N_c+1)/2$, there are two allowed isospins $I=(N_c-5)/4$ and $(N_c-1)/4$; and for $N_s=(N_c+3)/2$, there is one allowed isospin $I=(N_c-3)/4$.

The remaining components of the baryon propagator Eq. (A1) can now be defined. Isospin projection operators for the spin-$\jmath$ baryons with n_s strange quarks are given by

$$\mathcal{P}_\imath(\jmath,n_s)=\frac{\Pi_{\imath'\neq\imath}(I^2-I_{\imath'}^2)}{\Pi_{\imath'\neq\imath}(I_\imath^2-I_{\imath'}^2)}, \tag{A5}$$

where the projection operator for isospin $I_\imath$ is given by the product over all $I_{\imath'}$ not equal to $I_\imath$. For $2\jmath\leqslant n_s\leqslant(N_c-2\jmath)/2$, $I_{\imath'}=|(n_s+2\jmath)/2|$, $|(n_s+2\jmath)/2|-1$, $\ldots|(n_s-2\jmath)/2|$. The allowed isospins for the smallest and largest n_s values of the spin-$\jmath$ flavor representations vary according to the pattern described above. The operator series involving $\{J^i,G^{i8}\}$ depends on the operator $J\cdot J_s=\frac{1}{2}(J^2+J_s^2-I^2)$, which in turn depends on isospin. It is easy to evaluate $J\cdot J_s$ for each of the allowed isomultiplets of the spin-$\jmath$ flavor representation. For example, for the spin-$\frac{1}{2}$ representation, $J\cdot J_s$ is equal to $-N_s/4$ for isomultiplets with $I=(N_s+1)/2$, and $(N_s+2)/4$ for isomultiplets with $I=(N_s-1)/2$. For spin-$\frac{3}{2}$, $J\cdot J_s$ equals $-3N_s/4$, $-(N_s-6)/4$, $(N_s+8)/4$ and $3(N_s+2)/4$ for the $I=(N_s+3)/2$, $(N_s+1)/2$, $(N_s-1)/2$ and $(N_s-3)/2$ isomultiplets, respectively. Thus, the baryon mass difference

$$\Delta_8=M^8|_{\jmath,n_s,\imath}-M^8|_{\jmath_{\rm ext},n_{s{\rm ext}},\imath_{\rm ext}} \tag{A6}$$

is diagonalized by the spin, strange quark number, and isospin projection operators.

APPENDIX B: FLAVOR-27 NONANALYTIC MASS SPLITTINGS

This appendix provides additional formulae for the calculation of the flavor-**27** baryon mass splittings from Fig. 6. The function $F(m,\Delta)$ defined in Eq. (4.23) is given by [47]

$$24\pi^2f^2F(m,\Delta,\mu)=\begin{cases}\Delta\left(\Delta^2-\frac{3}{2}m^2\right)\ln\frac{m^2}{\mu^2}-\frac{8}{3}\Delta^3-\frac{7}{2}\Delta m^2+2(m^2-\Delta^2)^{3/2}\left[\frac{\pi}{2}-\arctan\left(\frac{\Delta}{\sqrt{m^2-\Delta^2}}\right)\right], & |\Delta|\leqslant m,\\ \Delta\left(\Delta^2-\frac{3}{2}m^2\right)\ln\frac{m^2}{\mu^2}-\frac{8}{3}\Delta^3-\frac{7}{2}\Delta m^2-(\Delta^2-m^2)^{3/2}\ln\left(\frac{\Delta-\sqrt{\Delta^2-m^2}}{\Delta+\sqrt{\Delta^2-m^2}}\right), & |\Delta|>m,\end{cases} \tag{B1}$$

where finite terms with mass dependence Δ^3 and $m^2\Delta$ have been retained. For $\Delta_s\neq0$, there is a chiral logarithmic contribution to the flavor-**27** combination proportional to $\Delta\Delta_s^2$, as well as a finite term. For $\Delta=0$, the function reduces to

$$F(m,0)=\frac{m^3}{24\pi f^2}. \tag{B2}$$

The remainder of the Appendix is devoted to evaluating the flavor-**27** baryon operator

$$\frac{1}{N_c}[A^{i8}\mathcal{P}_{\frac{1}{2}}A^{i8}I(\Delta_{\frac{1}{2}})+A^{i8}\mathcal{P}_{\frac{3}{2}}A^{i8}I(\Delta_{\frac{3}{2}})], \tag{B3}$$

using spin projection operators.

For $N_c=3$, the baryon axial current A^{i8} has a $1/N_c$ expansion in terms of the four operators of Eq. (3.16). The flavor-**27** baryon operator product contains n-body operators, $n>N_c$, which are complicated anticommutators of the one-body operators. Each of these operators contains two flavor octet one-body operators, which each may be either T^8 or G^{i8}; all remaining one-body operators in the operator product are spin operators. Operator reduction of these spin operators is immediately possible using spin projection operators. The following observations facilitate this reduction: (i) $A^{i8}\mathcal{P}_\jmath A^{i8}$ is purely diagonal; (ii) J^i is purely diagonal; (iii) $\{J^i,G^{i8}\}$ is purely diagonal, and $\{J^i,G^{i8}\}=2J^iG^{i8}=2G^{i8}J^i$ since $[J^i,G^{i8}]=0$; and (iv) $(\{J^2,G^{i8}\}-\frac{1}{2}\{J^i,\{J^j,G^{j8}\}\})$ is purely off-diagonal, since the second term subtracts off the diagonal component of $\{J^2,G^{i8}\}$. Thus, this operator reduces to $\mathcal{P}_{\frac{1}{2}}\{J^2,G^{i8}\}\mathcal{P}_{\frac{3}{2}}+\mathcal{P}_{\frac{3}{2}}\{J^2,G^{i8}\}\mathcal{P}_{\frac{1}{2}}$, which can be replaced by $\frac{9}{2}(\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}+\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}})$.

Using the above observations, it is straightforward to show that

$$\begin{aligned}\frac{1}{N_c}A^{i8}\mathcal{P}_{\frac{1}{2}}A^{i8}=&\frac{1}{N_c}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}\left(a_1^2+6\frac{1}{N_c^2}a_1b_3+9\frac{1}{N_c^4}b_3^2\right)\\&+\frac{1}{N_c}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}\left[a_1^2+9\frac{1}{N_c^2}a_1c_3+\left(\frac{9}{2}\right)^2\frac{1}{N_c^4}c_3^2\right]+\frac{1}{N_c}\mathcal{P}_{\frac{1}{2}}T^8\mathcal{P}_{\frac{1}{2}}T^8\mathcal{P}_{\frac{1}{2}}\left(\frac{3}{4}\frac{1}{N_c^2}b_2^2\right)\\&+\frac{1}{N_c^2}(\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}J^iT^8\mathcal{P}_{\frac{1}{2}}+\mathcal{P}_{\frac{1}{2}}J^iT^8\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}})\left(a_1b_2+3\frac{1}{N_c^2}b_2b_3\right)\end{aligned} \tag{B4}$$

and

$$\frac{1}{N_c}A^{i8}\mathcal{P}_{\frac{3}{2}}A^{i8}=\frac{1}{N_c}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}\left(a_1^2+30\frac{1}{N_c^2}a_1b_3+225\frac{1}{N_c^4}b_3^2\right)$$
$$+\frac{1}{N_c}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}\left[a_1^2+9\frac{1}{N_c^2}a_1c_3+\left(\frac{9}{2}\right)^2\frac{1}{N_c^4}c_3^2\right]+\frac{1}{N_c}\mathcal{P}_{\frac{3}{2}}T^8\mathcal{P}_{\frac{3}{2}}T^8\mathcal{P}_{\frac{3}{2}}\left(\frac{15}{4}\frac{1}{N_c^2}b_2^2\right)$$
$$+\frac{1}{N_c^2}(\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}J^iT^8\mathcal{P}_{\frac{3}{2}}+\mathcal{P}_{\frac{3}{2}}J^iT^8\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}})\left(a_1b_2+15\frac{1}{N_c^2}b_2b_3\right). \tag{B5}$$

The flavor-**27** mass combinations, the Gell-Mann–Okubo combination of spin-$\frac{1}{2}$ octet baryons,

$$\tfrac{3}{4}\Lambda+\tfrac{1}{4}\Sigma-\tfrac{1}{2}(N+\Xi) \tag{B6}$$

and the equal spacing rule of the spin-$\frac{3}{2}$ baryons,

$$-\tfrac{4}{7}\Delta+\tfrac{5}{7}\Sigma*+\tfrac{2}{7}\Xi*-\tfrac{3}{7}\Omega \tag{B7}$$

are given by

$$-\frac{1}{N_c}[\mathcal{P}_{\frac{1}{2}}A^{i8}\mathcal{P}_{\frac{1}{2}}A^{i8}\mathcal{P}_{\frac{1}{2}}I(0)+\mathcal{P}_{\frac{1}{2}}A^{i8}\mathcal{P}_{\frac{3}{2}}A^{i8}\mathcal{P}_{\frac{1}{2}}I(\Delta)] \tag{B8}$$

and

$$-\frac{1}{N_c}[\mathcal{P}_{\frac{3}{2}}A^{i8}\mathcal{P}_{\frac{3}{2}}A^{i8}\mathcal{P}_{\frac{3}{2}}I(0)+\mathcal{P}_{\frac{3}{2}}A^{i8}\mathcal{P}_{\frac{1}{2}}A^{i8}\mathcal{P}_{\frac{3}{2}}I(-\Delta)], \tag{B9}$$

respectively, where the overall minus signs occur because the baryon mass term in the Lagrangian is negative. Each of the baryon operators appearing in Eqs. (B8) and (B9) can be obtained from Eqs. (B4) and (B5).

The (0,**27**) operator basis consists of two operators, $\{T^8,T^8\}$ and $\{T^8,\{J^i,G^{i8}\}\}$. The above reduction has left one additional operator structure, the product of two G^{i8}'s. The flavor-**27** mass splittings are evaluated using the relations Eq. (3.67) for T^8 and G^{i8}. For the spin-$\frac{1}{2}$ baryons,

$$\mathcal{P}_{\frac{1}{2}}\{T^8,T^8\}\mathcal{P}_{\frac{1}{2}}=-\tfrac{3}{2},$$
$$\mathcal{P}_{\frac{1}{2}}\{T^8,\{J^i,G^{i8}\}\}\mathcal{P}_{\frac{1}{2}}=-\tfrac{3}{2}, \tag{B10}$$

on the Gell-Mann–Okubo combination. The product of two G^{i8}'s with an intermediate spin-$\frac{1}{2}$ baryon,

$$\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}=-\tfrac{1}{16}, \tag{B11}$$

is obtained by evaluating the operator

$$\mathcal{P}_{\frac{1}{2}}J^iG^{i8}\mathcal{P}_{\frac{1}{2}}J^jG^{j8}\mathcal{P}_{\frac{1}{2}}=\tfrac{3}{4}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}} \tag{B12}$$

on the Gell-Mann–Okubo combination. The remaining operator, the product of two G^{i8}'s with an intermediate spin-$\frac{3}{2}$ baryon is readily obtained using the (0,**27**) operator identity [17]

$$\{G^{i8},G^{i8}\}=\tfrac{1}{4}\{T^8,T^8\}, \tag{B13}$$

which implies that

$$\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}+\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}=\tfrac{1}{4}\mathcal{P}_{\frac{1}{2}}T^8T^8\mathcal{P}_{\frac{1}{2}}. \tag{B14}$$

Thus,

$$\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}=-\tfrac{1}{8}. \tag{B15}$$

Using these matrix elements, one obtains the nonanalytic contribution to the Gell-Mann–Okubo mass splitting, Eq. (4.33).

The evaluation of the operator products for the spin-$\frac{3}{2}$ baryons is similar. For the spin-$\frac{3}{2}$ baryons,

$$\mathcal{P}_{\frac{3}{2}}\{T^8,T^8\}\mathcal{P}_{\frac{3}{2}}=-3,$$
$$\mathcal{P}_{\frac{3}{2}}\{T^8,\{J^i,G^{i8}\}\}\mathcal{P}_{\frac{3}{2}}=-\tfrac{15}{2}, \tag{B16}$$

on the equal spacing rule flavor-**27** combination. The matrix element

$$\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}=-\tfrac{5}{8}, \tag{B17}$$

follows from the evaluation of

$$\mathcal{P}_{\frac{3}{2}}J^iG^{i8}\mathcal{P}_{\frac{3}{2}}J^jG^{j8}\mathcal{P}_{\frac{3}{2}}=\tfrac{15}{4}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}} \tag{B18}$$

on the equal spacing rule mass combination. The remaining operator is determined using the identity

$$\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}+\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}=\tfrac{1}{4}\mathcal{P}_{\frac{3}{2}}T^8T^8\mathcal{P}_{\frac{3}{2}}, \tag{B19}$$

to be

$$\mathcal{P}_{\frac{3}{2}}G^{i8}\mathcal{P}_{\frac{1}{2}}G^{i8}\mathcal{P}_{\frac{3}{2}}=\tfrac{1}{4}. \tag{B20}$$

Using these matrix elements, one obtains the nonanalytic contribution to the flavor-**27** equal spacing rule mass splitting, Eq. (4.34).

[1] See, for example, S. Weinberg, Physica **96A**, 327 (1979); J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984).
[2] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[3] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[4] For a pedagogical review, see S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985).
[5] M. Gell-Mann and G. Veneziano, as cited in G. Veneziano [6]. See also Peris [36].
[6] G. Veneziano, Nucl. Phys. **B159**, 213 (1979).
[7] P. Di Vecchia and G. Veneziano, Nucl. Phys. **B171**, 253 (1980).
[8] E. Witten, Ann. Phys. (N.Y.) **128**, 363 (1980).
[9] C. Rosenzweig, J. Schechter, and G. Trahern, Phys. Rev. D **21**, 3388 (1980); P. Nath and R. Arnowitt, *ibid.* **23**, 473 (1981).
[10] P. Di Vecchia, F. Nicodemi, P. Pettorino, and G. Veneziano, Nucl. Phys. **B181**, 318 (1981).
[11] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[12] E. Witten, Nucl. Phys. **B156**, 269 (1979); G. Veneziano, *ibid.* **B159**, 213 (1979); P. Di Vecchia, Phys. Lett. **85B**, 357 (1979).
[13] R. Dashen and A.V. Manohar, Phys. Lett. B **315**, 425 (1993); **315**, 438 (1993).
[14] E. Jenkins, Phys. Lett. B **315**, 431 (1993); **315**, 441 (1993); **315**, 447 (1993).
[15] R. Dashen, E. Jenkins, and A.V. Manohar, Phys. Rev. D **49**, 4713 (1994).
[16] E. Jenkins and A.V. Manohar, Phys. Lett. B **335**, 452 (1994).
[17] R. Dashen, E. Jenkins, and A.V. Manohar, Phys. Rev. D **51**, 3697 (1995).
[18] E. Jenkins and R.F. Lebed, Phys. Rev. D **52**, 282 (1995).
[19] J. Dai, R. Dashen, E. Jenkins, and A.V. Manohar, Phys. Rev. D **53**, 273 (1996).
[20] C. Carone, H. Georgi, and S. Osofsky, Phys. Lett. B **322**, 227 (1994).
[21] C. Carone, H. Georgi, L. Kaplan, and D. Morin, Phys. Rev. D **50**, 5793 (1994).
[22] M. Luty and J. March-Russell, Nucl. Phys. **B426**, 71 (1994).
[23] M. Luty, Phys. Rev. D **51**, 2322 (1995).
[24] M. Luty, J. March-Russell, and M. White, Phys. Rev. D **51**, 2332 (1995).
[25] W. Broniowski, Nucl. Phys. **A580**, 429 (1994).
[26] A.V. Manohar, Phys. Lett. B **336**, 502 (1994).
[27] N. Dorey, J. Hughes, and M.P. Mattis, Phys. Rev. D **50**, 5816 (1994); Phys. Rev. Lett. **73**, 1211 (1994).
[28] N. Dorey and M.P. Mattis, Phys. Rev. D **52**, 2891 (1995).
[29] A. Takamura, S. Sawada, and S. Kitakado, Prog. Theor. Phys. **93**, 771 (1995); A. Takamura, S. Sawada, Y. Matsui, and S. Kitakado, Report No. DPNU-95-12 [hep-ph/9506275], 1995 (unpublished).
[30] J.-L. Gervais and B. Sakita, Phys. Rev. Lett. **52**, 87 (1984); Phys. Rev. D **30**, 1795 (1984).
[31] P. Di Vecchia, Acta Phys. Austriaca Suppl. **XX**II, 341 (1980).
[32] A. Pich and E. de Rafael, Nucl. Phys. **B367**, 313 (1991).
[33] V. Baluni, Phys. Rev. D **19**, 2227 (1979).
[34] R.J. Crewther, P. Di Vecchia, G. Veneziano, and E. Witten, Phys. Lett. **88B**, 123 (1979).
[35] H. Georgi, Phys. Rev. D **49**, 1666 (1994).
[36] S. Peris, Phys. Lett. B **324**, 442 (1994).
[37] E. Jenkins and A.V. Manohar, Phys. Lett. B **255**, 558 (1991); **259**, 353 (1991).
[38] E. Eichten and B. Hill, Phys. Lett. B **234**, 511 (1990).
[39] H. Georgi, Phys. Lett. B **240**, 447 (1990).
[40] E. Jenkins, Nucl. Phys. **B375**, 561 (1992).
[41] R. Springer, Report No. DUKE-95-95 [hep-ph/9508324], 1995 (unpublished).
[42] E. Jenkins, Nucl. Phys. **B368**, 191 (1994).
[43] A.V. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984).
[44] E. Jenkins, A.V. Manohar, and M.B. Wise, Phys. Rev. Lett. **75**, 2272 (1995).
[45] M.K. Banerjee and J. Milana, Phys. Rev. D **52**, 6451 (1995).
[46] M.J. Savage, Phys. Lett. B **359**, 189 (1995).
[47] E. Jenkins and A.V. Manohar, in *Proceedings of the Workshop on "Effective Field Theories of the Standard Model,"* edited by U. Meissner (World Scientific, Singapore, 1992).
[48] Particle Data Group, L. Montanet *et al.*, Phys. Rev. D **50**, 1173 (1994).